Review of "Teleparallel Newton-Cartan Gravity", by Philip K. Schwartz (*Classical and Quantum Gravity* 40, 105008, 2023)

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Newton-Cartan theory (NCT) is a non-relativistic theory of space and time in which—like general relativity (GR)—gravitational effects are a manifestation of spacetime curvature (for an introduction to NCT, see [17]). NCT can be understood to be equivalent to 'standard' potential-based Newtonian gravitation theory (NGT) (see [21]). NCT and its torsionful generalisations (on which see e.g. [6]) have, in recent decades, found a number of fruitful and important applications in physics—for example, to non-relativistic holography (see e.g. [9, 10]) and to condensed matter physics (especially the fractional quantum Hall effect—see e.g. [11, 20, 23]).

Just as NCT is the curved-spacetime equivalent of NGT in the non-relativistic setting, when one moves to relativistic theories one finds that GR is the curved-spacetime equivalent of another theory, known as *teleparallel gravity* (TPG) (for introductions, see [1, 3]), in which gravity (a) as in NGT acts a force, in the sense that it leads to test bodies traversing non-geodesic trajectories, and (b) is a manifestation of spacetime torsion. This invites the question (to my knowledge first raised in [16]): to what extent does the analogy between these two theory equivalences hold up? In [19], it was shown that the analogy is exact, in the sense that one can bring to bear the machinery of teleparallelisation (i.e., the trading of curvature for torsion degrees of freedom via the Cartan equations) in order to show that NGT just is the teleparallel equivalent of NCT: here, the gravitational potential of NGT is understood to be a manifestation of the torsion of the mass gauge field obtained when one gauges the Bargmann algebra (on such gauging, see [2]).

But, these observations notwithstanding, at this point many questions remain. Perhaps most obviously: can one take the non-relativistic limit of the GR-TPG correspondence in order to obtain the NCT-NGT correspondence? In [19], this question was approached via taking the null reduction (see [6, 7]) of a Bargmann-Eisenhart solution to 5D vacuum GR and its teleparallel equivalent. However, it is perhaps more natural to ask: if one takes a $1/c$ expansion of

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GR-TPG, what does one obtain? (On such $1/c$ expansions, see e.g. [15, §6]).

It is exactly this question which is answered in the article under review here. After a welcome and very clear review of the extant literature on these topics (§2), Schwartz demonstrates that, taking a $1/c$ expansion of TPG and imposing by hand that the non-relativistic clock form τ be closed (thereby ensuring a standard of 'absolute time'—see [4]), one obtains—in agreement with [19]—NGT (§§3–4). As Schwartz writes of this $d\tau = 0$ assumption,

> for this to hold, the assumption of absolute time is crucial: otherwise, the limiting geometric objects would transform not under the Bargmann algebra, but under a certain Lie algebra expansion of the Poincaré algebra, and would instead define what has been termed a 'torsional Newton-Cartan type II' (TNC type II) geometry. (p. 16)

Quite so—indeed, although the study of 'Type II NCT' is still in its nascency (see [13, 14, 15]), it is quite natural to ask whether there is an extended, 'Type II NGT' obtained via a Type II limit of TPG; Schwartz is completely correct to point to this (p. 25) as an obvious avenue for future study.

But in any case, securing NCT-NGT from a $1/c$ expansion of GR-TPG is to be commended; in so doing, this article constitutes a very valuable contribution to the literature. In the remainder of this review, I'll make one technical point, before remarking upon some of the avenues for future pursuit which Schwartz identifies (§5).

The technical point is this. As Schwartz writes (p. 20), after taking the $1/c$ expansion of TPG, one must 'gauge fix' the connection to vanishing spatial torsion, $T^{a}{}_{bc} = 0$ in order to obtain NGT. In [19], my co-author and I asserted mistakenly that this follows from the Bianchi identities alone; I'm grateful for Schwartz' correction (p. 20). But perhaps there is more to say about this condition—after all, in [18], considerations on the convergence of derivative operators in the non-relativistic limit seem to underwrite this result automatically. I'll leave to future work a careful assessment of this result.

On Schwartz' identified future directions: I've already remarked upon (and concurred with Schwartz on the importance of) the possibility of an extension to the 'Type II' setting. But there is another important possible extension: in fact, GR and TPG are known to be but two nodes in a relativistic 'geometric trinity' of gravity—the other node is *symmetric teleparallel gravity* (STPG), in which gravitational effects are a manifestation not of curvature or of torsion, but of non-metricity (see [5] for a recent review). This invites the obvious question: what would one obtain if one were to take a $1/c$ expansion of STPG? In fact, since the publication of Schwartz' article arguably the question has already been answered (at least in the Type I context) in [22], in which said limit is taken and a 'purely non-metric' equivalent of NGT and NCT is constructed, thereby completing a 'non-relativistic geometric trinity'.

Schwartz also raises the possibility of applying such non-relativistic limits to modified teleparallel theories. I agree that this would be interesting: for example, one could apply such limits to $f(R)$, $f(T)$ and $f(Q)$ theories; although the standard line is that the geometric trinity breaks down in this setting, in

[8] it has been shown that this is perhaps too quick—so, could one take the non-relativistic limit of the correspondences identified in that article also, in order to construct yet more correspondences between extended non-relativistic theories? Or—to take a different line entirely—what would arise if one were to take the *ultra*-relativistic limit of the geometric trinity (for GR, this leads to *Carrollian gravity* [12], but to my knowledge for the other nodes of the trinity the answer is completely unknown)? Clearly, research questions such as these abound; it's not much exaggeration to say that Schwartz' paper provides the key to unlocking them.

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