The model of thin shell in General Relativity[1]

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Abstract: Models are of central importance in many scientific contexts and scientists spend significant amounts of time in building, testing, comparing, and revising models. The study of models and how they are used in scientific practice is a widely debated topic in the actual philosophy and history of science. An example of a model that we will consider is that of thin shell that is now widely used both in General Relativity and in Astrophysics.

Keywords: General Relativity, Model, Thin Shell.

1. Introduction

Models are of central importance in many scientific contexts and scientists spend significant amounts of time in building, testing, comparing, and revising models. The study of models and how they are used in scientific practice is a widely debated topic in the actual philosophy and history of science. An example of a model that we will consider is that of thin shell - that is now widely used both in General Relativity and in Astrophysics - of which we will look for the essential characteristics.

Models are important in scientific practice because it is through the study of them and their application to real cases that new generations of scientists are formed; in physics courses - for example - one is not interested in the historical process that led to the identification of a particular model, but only in its final structure and how to apply it to real cases (Giere 1988). Models are also essential for the acquisition and organization of scientific knowledge, as by studying a model one discovers the characteristics of the object it represents. But how does this representation take place? The approach that we will follow proposes that representation is effective insofar as a relationship of similarity is established between the model and the object represented.

According to Frigg the similarity relationship can be described in its simplest form with the statement: “A scientific model **M** represents a target **T** iff **M** and **T** are similar in relevant respects and to the relevant degrees” (Frigg, Nguyen 2017). It is the scientist's task - through the determination of theoretical hypotheses - to establish which aspects of the model are reflected in reality and to what degree.

A theoretical hypothesis is a linguistic entity – a statement – that asserts some kind of similarity relationship between a model and a real system. Thus, for example, to state that “the position and velocity of the earth and moon in the earth-moon system are very close to those of a Newtonian model of two particles with a central force inversely proportional to the square of the distance between them” (Giere 1988) is a classic example of a theoretical hypothesis that establishes which aspects of the model (position and velocity) have a real basis and to what degree this happens (very close). It is therefore possible to refine our definition of model by asserting: “A scientific model **M** represents a target system **T** if and only if a theoretical hypotheses **H** asserts that **M** and **T** are similar in certain respects and to certain degrees” (Frigg, Nguyen 2017).

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It now remains to clarify what M consists of. In order to describe the properties of a model in General Relativity, we will have to deal with an abstract model whose components we are now going to describe following Giere's approach (Giere 2006). The starting point consists in choosing the mathematical structure of which the model is the implementation. The mathematical structure gives us the tools to build the models. If we consider the classic example of the harmonic oscillator, the mathematical structure used is that of second degree differential equations and their solution.

Below it is necessary to indicate the Theoretical Principles which are the basis of the construction of the model. Giere defines the Theoretical Principles – of which the Principles of Dynamics are an example – as extremely general models whose task is to establish the relationships between the elements of the model; they are also defined as 'templates' to be used for the construction of more specific models. In the case of Newton's Principles of Dynamics, they establish a relationship between previously defined quantities such as the position and acceleration of an object and the concepts of force and mass which are not determined[2].

To build models that have a physical value, it is necessary to specify Special Conditions. In the example of Classical Mechanics that we are following, we have to indicate the correct force functions; so if Hooke's force is indicated as a special condition we obtain the model of the harmonic oscillator, while if we choose the universal gravitational force we determine the classical models of gravity. It is by choosing the special conditions that the models assume their explanatory power.

The models defined by the Theoretical Principles and by the Special Conditions are however still abstract structures, in which the relationship of similarity with the target is missing. To achieve this goal it is necessary to give an interpretation to the components of the model (Giere 1988). Thus in the example of the harmonic oscillator the interpretation suggested by experience is that of linking the variable x with the displacement of a particle from its equilibrium position and the constant k with a specific property of the model. It is clear that the interpretation process in the case of models built on the basis of General Relativity will be decidedly more complex and problematic.

2. Models in General Relativity

The mathematical structures that are used today in General Relativity are those made available by differential geometry. Of particular importance is the concept of pseudo-Riemanian manifold M which is locally similar to an Euclidean vector space, equipped with a pseudo-Riemanian metric[3]. The manifold is covered with charts (A, x^i) which allow to define the metric as $g = g_{ij}dx^i dx^j$. So we have a generic 'mathematical' model that we can represent as:

$$< M, g_{ij} >.$$

Differential geometry in General Relativity was introduced starting from the 60s of the last century with what took the name of the 'geometric' or 'coordinate free' approach (Norton 1993). In the previous period, Einstein and the physicists used the absolute calculus of Ricci-Levi Civita without a precise geometric interpretation[4]. However, it was Minkowski in 1908/09 who introduced the first geometric methods in relativity; following the line indicated by the Erlangen program – which suggested determining the geometries on the basis of their characteristic transformation groups – he set out to identify the geometry generated by the Lorentz transformations[5]. Minkowski realized that the new geometry was a structure

[2] For Giere the principles of dynamics are definitions.
[3] The pseudo-Riemannian metric is equivalent to the inner product of the vectors of the tangent space at each point of the manifold, with indefinite signature.
[4] The 'non-geometric' approach to General Relativity characterizes the historical process of the thin-shell model construction.
[5] Lorentz transformations are used in Special Relativity in the transition between inertial reference systems.

that linked space and time in an inseparable entity, to which he gave the name of space-time. From this point of view, Minkowski's work must be seen as the first process of interpretation of the mathematical model of Relativity: a manifold represents (is in correspondence with) a physical space-time.

If we now consider the Theoretical Principles underlying the models in General Relativity, Giere himself [6] (Giere 1999 pag.51) shows us the principle of general relativity as a possible example; this principle asserts that any reference system is equivalent for the formulation of physical laws. The validity of the principle of general relativity in Einstein's theory has given rise to a long dispute (Norton 1993) which led to extreme positions such as that of Synge who asserted "… we need not bother about the name, for the word 'relativity' now means primarily Einstein theory and only secondarily the obscure philosophy which may have suggested it originally" (Synge 1960 p.IX). Indeed Einstein gave several definitions of the principle of general relativity and in some of them he deduced it from the principle of general covariance. For example, in a 1916 review article he states

> The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates. that is, are co-variant with respect to any substitutions whatever (generally covariant). It is clear that a physical theory which satisfies this postulate will also be suitable for the general postulate of relativity (Einstein 1916).

It is clear that this definition includes transformations not only between physical reference systems, but also between 'mathematical' coordinate systems such as those between Cartesian coordinates and Polar coordinates. To overcome the difficulty and following considerations on the 'point-coincidence argument' (Norton 1993) Einstein arrived at the definition that we find in the 1918 article, where the principle of relativity takes the form "The laws of nature are only assertions of timespace coincidences; therefore, they find their unique, natural expression in generally covariant equations" (Einstein 1918). The principle thus expressed has a physical content in its first part (space-time coincidence) and a formal content in the second one. If a Theoretical Principle has the task of establishing the relationships between the elements of the model, almost as if it were a template, the principle of relativity thus defined is an excellent candidate as it establishes a class of admissible point-events and the form of the physical laws that relate them. We have therefore built an extremely abstract model, made up of a mathematical structure (manifolds) and a general principle that informs us about how the laws, that act on it, should be (covariance principle).

To make this modeling more concrete, it is necessary to introduce the Special Conditions. At this end Einstein provided an equation implementing the general covariance which has the following form:

$$R_{ij} + \frac{1}{2} g_{ij}\mathrm{R} = \frac{8\pi \mathrm{G}}{c^4} T_{ij}$$

On the left side there is a purely geometric quantity described by the metric g_{ij} and by components representing the curvature of the manifold (R_{ij})[7]. On the right-hand side there is the stress-energy tensor which describes the energy content associated with the manifold. The structure of our model now becomes $< M, g_{ij}, T_{ij} >$ and the properties of the metric are determined by the tensor T. Note that we are now in the presence of a class of physical models, in fact by modifying the tensor T – which is equivalent to choosing the typology of matter/fields in which we are interested – we generate a specific model of corresponding space-time[8]. If we represent the special condition in our model we get

$$< \mathrm{M}, g_{ij}, T_{ij}, \mathrm{f}(g, T) >$$

[6] In his writings Giere does not elaborate why the principle of general relativity is a theoretical principle.

[7] Which include the first and second derivatives of the metric.

[8] The reciprocal is also valid: by modifying the geometry of the manifold, one can ask which is the corresponding stress-energy tensor.

where $f(g,T)$ represents Einstein equations.

Finally, we come to the interpretation phase which consists in the correct association of model structures with target elements. In our case the interpretation phase starts with the determination of the Special Condition; in fact Einstein identifies the correct correspondences in the development of the equations that bear his name. The standard interpretation that is taught in Relativity courses is that the manifold represents space-time and has a curved geometry, determined by the presence of the stress-energy tensor which describes the mass-energy content.

A final consideration must be made for the theoretical hypotheses. As we indicated in the previous chapter, they are statements that establish the similarity link between the model and the target. If we examine the articles of the physicists who built the thin shell model at the beginning of the 900s, we find similar sentences, as evidenced by the following passage by Darmois:

> If we consider a set of masses in motion, and the four-dimensional manifold which corresponds to them […], to each of the masses corresponds a tube of universe limited by a certain three-dimensional boundary. Between these universe tubes extends the representation of empty regions of matter. To a system comprising n masses, the solar system for example, corresponds a scheme with n universe tubes. This four-dimensional representation can be considered as a means […] of representing observations. (Darmois 1927).

Here we are in the presence of a theoretical hypothesis because there is the establishment of a similarity between a scheme of n universe tubes with our solar system (even if the level of this relationship is not specified).

3. Thin Shell Model

The thin shell in General Relativity represents a 3-dimensional surface that separates two manifolds with different metrics. In the case of the thin shell, a quantity of mass-energy is distributed on it, but a limiting case that had historically precedence is when the surface has no energy content; in this case we are dealing with a boundary surface. In the following we study first the case of boundary surface and then that of thin shell; anyway, in both situations the mathematical structure that is implemented is the same and will consist of two 4-dimensional manifolds (V_1 e V_2) separated by a 3-dimensional sub-manifold (S), as shown in Fig.1. In the figure we see some fundamental properties of S, among which it is important to remember the metric of the hypersurface $h = h_{\alpha\beta} d\xi^\alpha d\xi^\beta$ which describes its geometric properties and its external curvature K, defined by the equation $\nabla_i \vec{n} = K_i^j \vec{e_j}$, which describes its immersion respectively in V_1 e V_2.[9]

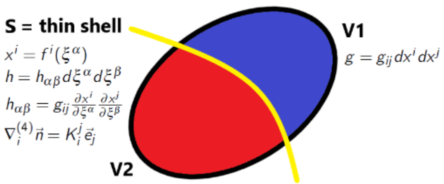

Fig. 1. Mathematical model of thin-shell

[9] $\vec{n}$ is the normal vector to S; ∇_i is the 4-dimensional covariant derivative; $\vec{e_j}$ are base vectors on V. We will use Latin indices for 4-dimensional properties and Greek indices for 3-dimensional properties.

According to the definition in the previous chapter, we therefore have two models:

$$M_k =< \mathrm{V_k}, g_{ij}^{(k)}, T_{ij}^{(k)}, \mathrm{f}\left(g^{(k)}, T^{(k)}\right) >$$

where index $k = 1,2$ represents the two space-time. We now have to build the model associated with the separation surface of the two manifolds $\mathrm{S} = V_1 \cap V_2$ which has the form:

$$\mathrm{N} =< \mathrm{S}, h_{\alpha\beta}, T_{\alpha,\beta}, \sigma\left(g^{(1)}, g^{(2)}\right) >$$

where $\sigma\left(g^{(1)}, g^{(2)}\right)$ is the special condition that characterizes the model and is a function of the two metrics that are 'glued' on S.

To identify the special conditions of our model, suppose that the stress-energy tensor presents a discontinuity as it passes through the surface S. In this case, the curvature also undergoes a discontinuity which corresponds to a discontinuity of the second derivatives of the metric. We therefore find ourselves with the problem of studying the continuity properties of the elements that make up our model.

More generally, a mathematical physics problem cannot be solved completely by writing the solutions of partial differential equations, but we need also to specify the boundary conditions and the discontinuity conditions on the surfaces on which some unknown quantities can be discontinuous as well as their derivatives.

If one considers, for example, the classical model of a spherical shell of matter – which is the classical analogue of the relativistic thin shell – we need to impose the condition that the potential tends to zero at infinity and that the derivative of the potential with respect to the normal to the shell is continuous. We find ourselves in a similar situation in General Relativity and in fact Synge and O'Brien in a 1952 article (O'Brien, Synge 1952) ask themselves “We think of a 3-space S in space time across which some of the component of $\mathrm{T}_{\mu\nu}$ are discontinuous (e.g. the history of the surface of the earth)” and try to determine the junction conditions of the metric and its partial derivatives, and of the stress-energy tensor. Using Gaussian coordinates[10], a 'boundary layer' through which the quantities change continuously and making the thickness of the layer tend to zero, Synge and O'Brien obtain the following conditions for the mathematical physics problem:

$$C_{OS} \Rightarrow g_{ik}\ , \frac{\partial g_{\mu\nu}}{\partial x^4}\ , T_k^4 \text{ are continuous through S.}$$

These conditions are called O'Brien - Synge junction conditions and are the special conditions that characterize our boundary surface model:

$$\mathrm{N_{OS}} =< \mathrm{S}, h_{\alpha\beta}, T_{\alpha,\beta}, \mathrm{C_{OS}} >$$

A different approach was taken by Darmois (Darmois 1927). Starting from the form of Einstein's equations that admit gravitational waves (proposed by Einstein with a perturbative method) Darmois obtains a series of physical results based solely on properties intrinsic to the manifolds. Studying the case of boundary surfaces he uses the external curvature to determine the continuity of the metric and its derivatives on the surface S, obtaining as junction conditions:

$$C_D \Rightarrow g_{ij} \text{ e } K_{ij} \text{ are continuous through S.}$$

The Darmois conditions represent a new type of special conditions for our boundary surface model:

$$\mathrm{N_D} =< \mathrm{S}, h_{\alpha\beta}, T_{\alpha,\beta}, \mathrm{C_D} >$$

Lichnerowicz (Lichnerowicz 1955), a student of Darmois and Cartan, also determines his junction conditions for boundary surfaces. Unlike what has been seen up to now, the French mathematician introduces the concept of admissible coordinates, which are those particular coordinates with respect to which the components of the metric and its first derivative are continuous. Thus we have the Lichnerowicz conditions:

$$C_L \Rightarrow \text{S is covered by admissible coordinates}$$

[10] In the Gaussian coordinate system chosen by Synge the surface equation is $x_4 = 0$

that are related to the following boundary surface model:

$$\mathrm{N_L} =< \mathrm{S}, h_{\alpha\beta}, T_{\alpha,\beta}, \mathrm{C_L} >$$

We therefore have 3 boundary surface models which are identified by as many special conditions. Bonnor and Vickers (Bonnor, Vickers 1981) showed that the three conditions under consideration are equivalent to each other, therefore we can define a single model of the boundary surface using the Darmois condition [11]

$$\mathrm{N_{BoundarySurface}} =< \mathrm{S}, h_{\alpha\beta}, T_{\alpha,\beta}, [\mathrm{K}_{\alpha\beta}] = 0 >$$

where $[K_{\alpha\beta}] = K_{V_2} - K_{V_1}$ represents the jump of K in passing through S.

Finally, we come to deal with the contribution of Israel (Israel 1966) which has the merit of dealing not only with the case of boundary surface but also with that of thin shell. In fact, he obtains Darmois's result for boundary surfaces and identifies thin shells as those particular surfaces in which only the metric is continuous, but not its first derivatives; this condition is expressed by the jump of external curvature $[K_{\alpha\beta}] \neq 0$.

The abrupt change of discontinuity of the metric through S is caused by the presence of mass-energy distributed on S which Israel identifies in the stress-energy tensor of the surface, determined by the jump of the external curvature:

$$8\pi S_{\alpha\beta} = -[K_{\alpha\beta}] + h_{\alpha\beta}[K_\gamma^\gamma]$$

With this equation we can finally determine the correct thin shell model:

$$\mathrm{N_{ThinShell}} =< \mathrm{S}, h_{\alpha\beta}, S_{\alpha\beta}, [\mathrm{K}_{\alpha\beta}] \neq 0 >$$

3. Conclusions

In this article we presented a generic definition of models in General Relativity and then we implemented it in the case of boundary surfaces and thin shells.

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