Is the metric signature really electromagnetic in origin?

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Abstract

The programme of 'pre-metric electromagnetism', developed by Hehl and collaborators, seeks to derive certain aspects of the spacetime geometry of the world (in particular, metrical signature) from elementary, empirically-informed axioms regarding electromagnetic fields. The programme should, therefore, be of profound interest to both empiricist and relationalist philosophers. Up to this point, however, pre-metric electromagnetism has received very little attention within the philosophy of physics; this paper seeks to rectify the situation, by engaging in a detailed foundational study of the programme. In particular, in this article, we (a) present a streamlined version of the programme, identifying key input assumptions, (b) consider the connections between this programme and other notable projects in the foundations of spacetime theories, e.g. (i) the dynamical approach to spacetime and (ii) constructive axiomatics, and (c) consider in detail the extent to which this programme can be generalised beyond electromagnetism. In achieving these tasks, we hope to open up to philosophers an extraordinarily fecund—but lamentably little-known!—project within the foundations of physics.

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1 Introduction

No debate in the foundations of spacetime theories has more prominence than that between substantivalists and relationalists. According to the former, space and time are fundamental and exist independently from material bodies; according to the latter, space and time are (in some way or another) ontologically subordinate to material bodies (and the relations which obtain between those bodies).[1]

Debates between substantivalists and relationalists often have a metaphysical flavour somewhat removed from the details of the dynamics of specific physical theories. To take two examples: (i) according to Maudlin's 'Newtonian relationalism', the relationalist can shirk a commitment to Newtonian absolute space by countenancing *trans-temporal* relations between material bodies; this equips the relationalist with the tools to account for Newton's thought experiment of the globes as presented in the *Scholium* (say) [74];[2] (ii) Sklar [98] proposes that one needn't commit oneself to pieces of spatiotemporal structure in order to underwrite (say) absolute notions of acceleration, for one can instead take such accelerations to be primitive, monadic properties of material bodies.

This general 'pure metaphysics' emphasis notwithstanding, some recent versions of relationalism have been more sensitive to the details of particular physical theories. One prominent example is the 'dynamical approach' to spacetime theories of Brown and Pooley [19, 20, 21], according to which spatiotemporal geometrical structure *just is* a codification of the symmetries of the dynamical laws for material bodies: the laws of physics feature more prominently on this approach than on other, more traditional versions of relationalism.[3] That being said, all is not necessarily well with the dy-

[1] Of course, there is a variety of ways of making more precise the difference between substantivalism and relationalism—see e.g. [11, 33, 39, 78, 84]—but for now, this characterisation is sufficient.

[2] Recall that the challenge here is this: how is the relationalist to account for different tensions in the string yoking the two globes, when *ex hypothesi* those are the only material bodies in the universe? The Newtonian relationalist can ground facts about the string tension in these trans-temporal distance relations.

[3] The term 'relationalism' is ambiguous between attempts to reduce to material bodies (and the relations between those bodies) (i) manifold points, versus (ii) the distance relations obtaining between those points. In this article, we understand 'relationalism' to take broad scope; on this understanding, attempts to reduce (ii) to material bodies (and the relations between those bodies) also count as relationalist projects. In this sense,

namical approach, for it has faced charges of obscurity or circularity (or both): see e.g. [36, 47, 80].[4]

One approach to subordinating spatiotemporal (geo)metrical structure to properties of the equations governing material bodies which one finds in the physics literature, and which has on occasion been mentioned in the same breath as the dynamical approach (see in particular [47, 89]), is the programme of *pre-metric electromagnetism* of Hehl and collaborators (see in particular [54, 56, 57, 58, 62]). The question to be addressed by these authors is this: is there any way of reformulating or reconceptualising Maxwell's equations so that they in fact do not presuppose *ab initio* metrical structure, but rather come themselves to fix uniquely that structure? In fact, the question has a long pedigree: it was first raised by Kottler in 1922, and was also famously taken up by Peres in 1962 (see [54] for historical discussion, original sources, and critical assessment). If successful, the programme *derives* certain aspects of the metric field of relativistic spacetime theories—in particular, its signature—such that (philosophically speaking) these aspects can be regarded as being subordinate to the electromagnetic fields. Thereby, the approach appears to meet many of the desiderata of the dynamical approach to spacetime, *qua* relationalist thesis, for it indicates that the distinction between space and time—typically taken to be encoded in the metric signature[5]—need not be regarded as being fundamental, but rather can be taken to be derivative on material fields and their properties.[6] Clearly, if correct, this would be a significant insight into the nature of space and time, which would and should be of broad philosophical interest.

As of yet, however, the philosophical and foundational implications of the programme of pre-metric electromagnetism *à la* Hehl *et al*. have not been scrutinised; exactly the goal of the present article is to fill that lacuna. The structure of the article, then, is this. In §2, we introduce the programme of pre-metric electromagnetism. In §3, we explain how on this programme the metrical signature can be regarded as being derivative on material fields. In §4, we subject the programme to a detailed philosophical appraisal, in four respects: (i) its ability to liberate electromagnetism of (arguably) problematic commitments to 'background' structures (in the standard formulation, that of a Lorentizan metric signature), (ii) its connection with the dynamical approach to spacetime theories (as mentioned above), (iii) its connection to the programme of 'constructive axiomatics', which was promulgated by Reichenbach in 1924 [91], and (iv) its connection to the *à la mode* agenda of 'spacetime functionalism'. In §5, we assess the prospects for generalising this programme beyond electromagnetism. There is also an appendix, in which we consolidate some relevant results from differential geometry.

To answer the title question: while the programme of pre-metric electromagnetism offers *prima facie* a technically sound means of reducing metrical signature to material bodies, its advantages over the standard approach are on reflection unclear. For

the dynamical approach to spacetime, as well as the programme of pre-metric electromagnetism (which constitutes the focus of this article), qualify as relationalist agendas.

[4]Another example of relationalism in this spirit is the programme of 'shape dynamics' of Barbour and his collaborators. For a book-length introduction, see [76].

[5]See [25, §6.2.1] for discussion.

[6]See footnote 3 for the sense in which the dynamical approach counts as a version of relationalism (cf. also [84]).

example, this approach is not any more ontologically parsimonious than the 'standard' formulation of electromagnetism which it seeks to replace. It is also not clear that this approach has achieved more 'background independence' than the standard formulation of electromagnetism. In addition, there are significant obstacles for generalizing the approach beyond electromagnetism, although there remains interesting and important technical work to be undertaken on this front. While overall we remain sceptical regarding the conceptual merits of pre-metric electromagnetism, nevertheless we would like to emphasize that we regard the programme as being of significant intrinsic interest and importance of in the foundations of physics in its attempt to develop empirically and conceptually perspicuous physical theories and to reduce their undue dependence upon the spacetime metric.

2 Pre-metric electromagnetism

In this section, we first remind the reader of the relevant aspects of standard, textbook electromagnetism (§2.1), before turning our attention to pre-metric electromagnetism (§2.2).

2.1 Textbook electromagnetism

We begin with standard electromagnetism. When written in terms of differential forms, Maxwell's equations for electromagnetism read as follows:

$$d * F = J, \tag{1}$$

$$dF = 0; \tag{2}$$

here and throughout, d denotes the exterior derivative acting on differential forms; F is the Faraday tensor, which encodes facts about the electric and magnetic fields and which is a spacetime 2-form (by the Poincaré lemma, which states that every closed form is locally exact, it follows from (2) that locally one can write $F = dA$ for some 1-form A, which is the electromagnetic potential); J represents a current, and is a spacetime 3-form; d denotes the exterior derivative operator, and $*$ denotes the Hodge star operator.[7] For our purposes, the crucial point to note is that the Hodge star operator presupposes a metric field on the manifold—intuitively, given an n-dimensional (pseudo)-Riemannian manifold (M, g_{ab}), the Hodge star operator 'completes' a p-form to the volume form of (M, g_{ab}), and thereby defines an $(n-p)$-form on M.[8] The point is that

[7]Note that in the covariant tensor formulation $\partial_\mu F^{\mu\nu} = J^\nu$, the current J^ν is a 4-vector. To convert the 3-form J in (1) to J^ν, we use the Hodge star to convert the 3-form to a 1-form, and then use the metric to raise the index. The differential form formulation of Maxwell's equations has the advantage of minimizing the role of the metric, and is therefore a good starting point for the pre-metric programme. For further details on the inter-conversion between these two different formulations of Maxwell's equations, see [10].

[8]One might reasonably ask here: *which* volume form, since any top-ranked form can in principle serve to define a volume element? Typically in these discussions, one means the volume form $\sqrt{-g}dx^1 \wedge \ldots \wedge dx^n$, which is the top-ranked form 'adapted' to the Lorentzian metric g_{ab} (and in standard electromagnetism, this is the Minkowski metric, which is a non-dynamical entity).

presupposition of a metric field is baked into the standard formulation of electromagnetism.[9]

In standard presentations of electromagnetism, in addition to Maxwell's equations, one has the Lorentz force law, which, in the language of differential forms, can be written

$$f = qu \lrcorner F. \tag{3}$$

Here, $\lrcorner$ denotes the interior product operation on differential forms. (3) is the equation of motion for a point charge with charge q, describing the electromagnetic force f on said charge. On the right hand side, $u \lrcorner F$ is the interior product of the tangent vector u to the worldline of the charge with the 2-form F.[10]

2.2 The pre-metric programme

The question to be addressed, then, is this: is there any way of reformulating or reconceptualising Maxwell's equations such that they in fact do *not* presuppose primitive metrical structure? Although—as already mentioned—the question has a venerable pedigree, it has over the past twenty or so years been taken up in a particularly intriguing manner by Hehl and his collaborators. It is upon this modern version of the pre-metric approach which we focus in this article; the presentation of [62] will prove to be particularly congenial and clear for our purposes, and indeed we will focus upon that article in the remainder of this subsection, although where it is helpful to do so we will also discuss other entries in the series.

How, then, do Hehl *et al.* reformulate electromagnetism so as to be independent of metrical notions? We begin with a 4-dimensional differentiable manifold M which satisfies 'hyperbolicity', in the sense that it is foliable into singly-indexed 3-dimensional submanifolds—this is a purely topological assumption (i.e., no (geo)metrical assumptions required), made precise in the following axiom (the first of five), deployed in [62]:[11]

Axiom 1. *Spacetime is a* 4*-dimensional differentiable manifold M that admits a foliation into codimension-1 hypersurfaces* Σ*.*

Let the hypersurfaces be labelled by a smooth parameter $\sigma \in (-\infty, \infty)$, which is called 'topological time'.[12] Given σ, we additionally choose a vector field on the manifold v such that $v \lrcorner d\sigma = 1$.[13] The above is an axiom about manifold topology: it is a pre-

[9] For background on the notions of differential forms introduced in this paragraph, see the appendix.

[10] There are well-known foundational issues regarding the consistency of electromagnetism when taken as this complete package—see [50]. We'll pass over these issues almost entirely in this article, since they're not relevant for the points which we seek to make.

[11] For the sake of clarity, we have made minor modifications to and comments upon the axioms of [62]; the greatest differences can be found in the context of **Axiom 5**, but we'll explain in due course our reasons for deviating there from the presentation of [62].

[12] 'Topological time' at this stage has no particular presumed chronometric significance. Since topological time is just a label, the assumption of smoothness here is innocuous.

[13] In [62], v is not chosen explicitly. This obscures the fact that in a bare differentiable manifold, we cannot obtain the vector field v from the scalar field σ alone. (In the standard approach with a background metric, v, which equals $\partial/\partial\sigma$, can be derived from $d\sigma$ simply by raising the index.) With both σ and v, we are able to uniquely 'dissect' any differential form into a tangential part and an orthogonal part with respect

supposition of the pre-metric approach, which must be taken for granted if one is to proceed.[14]

Having laid down this axiom, so as to achieve independence from any background metric we next use differential forms in order to formalize all mathematical objects representing the electromagnetic field. The foliation of the manifold postulated in **Axiom 1** allows for convenient decompositions of differential forms. These will play a central role in the formalism of pre-metric electromagnetism. More specifically, for an n-form α, we have

$$\alpha = \beta \wedge d\sigma + \gamma, \tag{4}$$

where β is an $(n-1)$-form and γ is an n-form, which is a unique decomposition if we require β to be 'tangential' and γ to be 'orthogonal' to the direction of 'time'.[15]

The next axiom deployed is physical in nature (regarding, as it does, the charge current density), and reads as follows:[16]

Axiom 2. *The charge current density is a conserved twisted* 3*-form* J*, so that when* J *is integrated over a closed* 3D *submanifold* $C \subset M$ *(so that* $\partial C = \emptyset$*), one has* [17]

$$\oint_C J = 0. \tag{5}$$

From this, it follows that $dJ = 0$, via Stokes' theorem. Intuitively, **Axiom 2** states that the rate of the change of the electric charge ρ (i.e., $\dot{\rho}$) in a region is equal to the

to time (see footnote 15). This issue is recognised at [56, p. 115]. Note that both are mere 'coordinates' that do not have physical significance. Note also that the postulation of a differential manifold ensures that the existence of such a vector field, so that no additional postulation is needed.

[14]The axiom imposes a precursor to global hyperbolicity, the latter of which is very often assumed when general relativity (say) is applied to concrete physical scenarios. It is important to recognise, however, that the assumption is not innocuous: it is a significant restriction on the class of spacetime models which can be considered. It's also worth flagging that the assumption of global foliability is strong, and arguably not strictly necessary—one might choose to weaken to a local notion of foliability. Nevertheless, to retain continuity with the work of Hehl *et al.*, we will continue to assume global foliability in what follows.

[15] In more detail, given a coordinate system x, y, z, σ, a p-form can be expressed as a linear combination of p-wedge products of dx, dy, dz, and $d\sigma$. For example, a 2-form can be written as $adx \wedge dy + bdy \wedge dz + cdz \wedge d\sigma + rd\sigma \wedge dx$ for some a, b, c, r. This can be rearranged into the decomposition form specified by (4). A decomposition satisfying the form of (4) alone is not unique—since, for example, it does not prohibit γ to contain $d\sigma$. The further constraint that γ is 'orthogonal' to the direction of time means that $\gamma(v) = 0$, which ensures the uniqueness of the decomposition. We say $\beta \wedge d\sigma$ is 'tangential' to time since it contains $d\sigma$, and by extension we also say that β is 'tangential' to time since it is wedged with $d\sigma$. Thus, we may call β and γ respectively the (spatio)temporal and spatial component of α.

[16]Twisted differential forms are defined in the appendix.

[17]Note that by 'manifold' we always mean the general notion of 'manifold with boundary', which refers to a Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n_+ := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_1, ..., x_n \geq 0\}$ for some n. Relatedly, by 'boundary', we mean the notion of 'boundary of a manifold' instead of that of 'topological boundary'. The boundary of an n-dimensional manifold M is the complement of the interior of M in M, where the interior of M is the set of all points in M that have neighborhoods homeomorphic to open subsets of $\mathbb{R}^n$.

It is important to note that the 'closed' 3D submanifold C here refers to a compact boundaryless manifold, such as a 3-sphere, rather than one of the folia (which is not compact). The paradigmatic case in this context is a 3D 'cylinder' with its height along the time dimension and its top and bottom surface being a 2-sphere. The total flux into or out of this region being zero is an elegant formulation of charge conservation, as explained in the text. Note that if the region in question is one of the folia, then the integral of the flux may not be zero (but rather equals the total charge of the region).

electric flux $\underline{d}j$ into the region ($\underline{d}$ is the spacelike exterior derivative and dots denote derivatives with respect to σ)—this is clear once one recalls that J can be decomposed into components as (ρ, j): ρ and j result from the decomposition of J with respect to 'time', where ρ is a 3-form considered as the charge density in 'space' and j is a 2-form considered as the current density of ρ. We can derive from $dJ = 0$ (along with other plausible conditions, which have to do with the numerical coefficients in the time-space decomposition of J—which, however, can be normalised away) that $\underline{d}j + \dot{\rho} = 0$, which amounts to the usual continuity equation for the conservation of electric charge.

By de Rham's theorem (which has to do with the theory of de Rham cohomology groups—see [56, pp. 96-101]), one can write[18]

$$J = dH, \tag{6}$$

where H is a twisted 2-form called the 'electromagnetic excitation'; this can be decomposed using (4) into two parts $\mathscr{D}$ and $\mathscr{H}$, respectively (these are called the 'electric' and 'magnetic' excitations, respectively).[19] Of course, H is not unique: the addition to H of any exact form would still satisfy (6). The physical significance of $\mathscr{H}$ and $\mathscr{D}$ is this: both can be measured by ideal electric conductors and superconductors, respectively [56, p. 116]—as Hehl and Obukhov state, "the electric excitation $\mathscr{D}$ can be *measured* by means of Maxwellian double plates as charge per unit area, the magnetic excitation $\mathscr{H}$ by means of a small test coil, which compensates the $\mathscr{H}$-field to be measured, as current per unit length. [...] In other words, the extensive quantities $\mathscr{D}$ and $\mathscr{H}$—and thus the 4-dimensional excitation H—have an operationally [*sic*] significance of their own, since they are related to charge at rest or in motion, respectively" [55, p. 7]. In this sense, **Axiom 2** can be regarded as being imbued with empirical content—the significance of this will be explored further below, in §4.3.

We turn now to the third axiom of the pre-metric approach, which has to do with the behaviour of point charges:

Axiom 3. *The Lorentz force density—a twisted covector-valued* 4*-form—is*

$$f_\alpha = (e_\alpha \lrcorner F) \wedge J, \tag{7}$$

[18] To be clear, de Rham cohomology addresses (at least in part) the question: when can a spacetime n-form be considered the curvature of some $(n-1)$-form potential? The Poincaré lemma—which we also consider below—regards all closed n-forms, and finds them to be locally exact (i.e., of the form $d\eta$, for some $(n-1)$-form η). In the pre-metric approach, when we are dealing with a closed 2-form F (given (2)), both de Rham theorems and the Poincaré lemma are potentially relevant. If the conditions of de Rham theorems are satisfied, F is globally of the form dA; otherwise, given the Poincaré lemma and the fact that F is closed, F can still be regarded as being locally of the form dA. For the twisted 3-form J, dynamics do not impose that $dJ = 0$ (unlike the case of F); therefore, only the de Rham theorems are relevant. In this case, however, the conditions of those theorems are satisfied, and so the above result obtains.

[19] That is, $H = -\mathscr{H} \wedge d\sigma + \mathscr{D}$ [62, p. 66]. This is analogous to the decomposition of F into E and B, namely $F = E \wedge dt + B$ (see footnote 22); note here that B is to be understood as a 2-form which encodes the components of the magnetic vector field. Note, however, that the role between the magnetic and the electric components of H is somewhat exchanged when compared to those of F: $\mathscr{H}$ is temporal and $\mathscr{D}$ spatial while E is temporal and B spatial—this is perhaps not surprising since H is standardly the Hodge dual of F (note, though, that this condition has yet to be imposed in the pre-metric approach). We can derive that $\mathscr{D}$ and $\mathscr{H}$ satisfy inhomogeneous Maxwell equations $d\mathscr{D} = \rho$ and $d\mathscr{H} - \partial_\sigma \mathscr{D} = j$, which are respectively the Gauss law and the Ampère Law, although these are standardly formulated in terms of the electric and magnetic fields. See [57] for more details.

where e_α is a frame and F is an untwisted 2-form.

This is an alternative formulation of the Lorentz force law (3) that appeals to the frame field.[20] Note that no metric is required in order to write down this law; thus, treating it as an axiom is consistent with the pre-metric methodology. In the pre-metric approach, **Axiom 3** is treated as imbuing the electromagnetic field strength tensor F with empirical content (see [56, p. 123]); accordingly, its conceptual status is in fact quite subtle—we'll return to this in §4.3.[21] Once again, one can decompose F into (spatio)temporal and spatial parts—these are the electric and magnetic fields, E and B, respectively.[22]

The fourth axiom of pre-metric electromagnetism is again physical in nature, and involves the conservation of magnetic flux:[23]

Axiom 4. *Magnetic flux is conserved, so that for any 2D closed submanifold $C \subset M$ (such that $\partial C = \emptyset$),*[24]

$$\oint_C F = 0. \tag{8}$$

This entails $dF = 0$ under Stokes' theorem. By de Rham's theorem, the above result yields $F = dA$, where A is an untwisted 1-form, the electromagnetic potential.[25] As with **Axiom 2**, this axiom has fairly direct operational meaning, since the electric and magnetic components of F can be measured in the lab.[26]

So far, pre-metric electromagnetism has yielded, *inter alia*, the equations $dH = J$ and $dF = 0$. At this point, the fields H and F remain independent of one another, so we should now consider the ways in which they might be coupled together. In particular, we require that this 'constitutive relation for vacuum' be *local*, which means that H at a certain spacetime point depends only upon F at that point (and not upon other points in addition); moreover, we require the relation to be *linear*, in the sense that if $H = \kappa(F)$, then $\kappa(a\Phi + b\Psi) = a\kappa(\Phi) + b\kappa(\Psi)$, for a, b arbitrary constants and Φ, Ψ arbitrary 2-forms. In addition, κ is *symmetric*, such that $\kappa(\Phi) \wedge \Psi = \Phi \wedge \kappa(\Psi)$. These conditions on the constitutive relation can be imposed as a further axiom:

[20] For the details of the interconversion between (3) and (7), see [56, pp. 123-124]. Operations such as the interior product and wedge product are defined in the appendix.

[21] It would be pushing matters too far, however, to suggest (in the absence of any explicit operational procedure) that **Axiom 3** constitutes an *operational definition* of F.

[22] Namely, $F = E \wedge d\sigma + B$. This is entirely analogous to the decomposition $F = E \wedge dt + B$ (note that here, as in footnote 19, B is to be understood as a 2-form). Just as the latter in the case of a curved spacetime requires selecting a local frame, choosing σ and v amounts to choosing a local frame. See the discussion in §4.3. See also footnote 19.

[23] We follow the lead of e.g. [56] in calling this an axiom to do with the 'conservation of magnetic flux'. But since the axiom expresses a conservation law for the Faraday tensor F, arguably it would be better to call this an axiom to do with the conservation of *electro*magnetic flux.

[24] As in **Axiom 2**, C is a compact manifold without boundary. However, note that since C is here a $2D$ submanfold, it can be entirely spatial, unlike the submanifolds referred to in **Axiom 2**. As a result, unlike the electric flux, the magnetic flux should be zero even for a closed *spatial* region, which is related to the closed lines of the magnetic field.

[25] Even if the conditions of the de Rham theorems do not obtain, one will still have this result locally, in light of the Poincaré lemma—see [56, p. 132]. Recall also footnote 18.

[26] Although see our concerns regarding the operational significance of the electric and magnetic fields expressed in §4.3.

Axiom 5. *$H = \kappa(F)$, with κ being local, linear, and symmetric.*

Note that this differs from Axiom 5 as presented in [62]—this, indeed, is the only point at which we deviate substantially from the presentation of that article. Axiom 5 as presented in [62] can be understood as a corollary to **Axiom 5** presented above; Hehl and Itin therefore take **Axiom 5** as presented above to be a further postulate. Since it is the above condition which will ultimately be required in order to derive the Lorentzian metric signature, we have elected to present this condition as an axiom directly.

Axiom 5 is used, first of all, to derive an expression for a quantity Σ_α, called the 'energy-momentum current' of the electromagnetic field, which is a covector-valued 3-form (the object is defined using a frame field e_α, hence the index). This quantity is introduced as a generalized potential of the Lorentz force—that is, the Lorentz force describes how the energy-momentum current varies under spatiotemporal variation of the electromagnetic field:

$$d\Sigma_\alpha = f_\alpha + X_\alpha, \tag{9}$$

where X_α involves the derivative of the frame field e_α (for explicit expressions, see [62, p. 69]).[27] Then, substituting (7) from **Axiom 3** into (9) and using the properties of the constitutive relation from **Axiom 5**, a somewhat lengthy algebraic calculation yields (see [57, p. 5])

$$\frac{1}{2}\left[d\left(e_\alpha \lrcorner H\right) \wedge F - d\left(e_\alpha \lrcorner F\right) \wedge H\right] = X_\alpha, \tag{10}$$

which gives rise to

$$\Sigma_\alpha = \frac{1}{2}\left[\left(e_\alpha \lrcorner H\right) \wedge F - \left(e_\alpha \lrcorner F\right) \wedge H\right]. \tag{11}$$

This also means that the trace $T := \vartheta^\alpha \wedge \Sigma_\alpha$ (where ϑ^α is the coframe field associated to e_α) of the energy-momentum current Σ_α vanishes, which ensures that the photon be massless (see [62, p. 70]). This trace would not vanish were it not for the conditions imposed via **Axiom 5**; therefore, this can be considered (at least some of) the extra empirical content delivered by **Axiom 5**, over and above that of **Axiom 3** alone.

From these axioms, there follows the 'electric-magnetic reciprocity relation' that preserves the energy-momentum current, which is

$$H \to \zeta F, \tag{12}$$

$$F \to -\frac{1}{\zeta}H. \tag{13}$$

where ζ is a twisted 0-form (which is a scalar function save that it changes signs under parity inversion—see appendix) [62, p. 71].

[27] The frame field e_α consists of four tangent vectors at each point. Note that $d\Sigma_\alpha$ is not defined explicitly in [62], which is problematic because Σ_α is not an ordinary differential form. We can infer from the source that the $d\Sigma_\alpha$ is implicitly defined as $d(e_\alpha \lrcorner \Sigma_\alpha)$, namely that we first evaluate Σ_α at each e_α and then apply d to each of them, resulting in a 4-tuple of 4-forms. Without this explicit definition, the role of the frame field e_α is obscured. Similarly, 'the derivative of the frame field' referred to in [62] does not literally mean the exterior derivative of the field since the latter is not a form, but rather the operator $d(e_\alpha \lrcorner)$, which first contracts a form with the frame field and then applies d to the result.

3 Emergence of metric signature

So far, we have recapitulated the axiomatic version of pre-metric electromagnetism, as presented by Hehl *et al*. (and with particular attention to the presentation in [62]). Supposedly, in this approach no metric occurs in the fundamental posits.[28] But—perhaps surprisingly!—the theory already contains the seeds of metrical information; in particular, the seeds of the signature.[29] With the work of the previous section in hand, we are now ready to derive the metric signature from **Axioms 1–5** of pre-metric electromagnetism.[30] It is worth flagging that we intend this section to be fairly uncritical; a thoroughgoing foundational and philosophical assessment of this supposed derivation of the metric signature is deferred to §4.

Given the reciprocity relation (12) and (13) and the constitutive relation $H = \kappa(F)$, it is not hard to derive that κ is proportional to the identity operator, and in particular $\kappa^2 = -\zeta^2\mathbb{I}$ (see [62, §4.3]). In fact, we can now replace ζ by an ordinary (rather than twisted) scalar function λ, since the differences between ordinary scalars and twisted scalars disappear when the two are squared. We can assimilate the scalar factor by defining $\mathbb{J} = \kappa/\lambda$ so that $\mathbb{J}^2 = -\mathbb{I}$. This reminds us straightforwardly of the Hodge star operator $*$ (see appendix), which has the property that $*^2 = (-1)^{\mathrm{ind}}\mathbb{I}$ when acting on 2-forms in 4-dimensional spacetime, where 'ind' denotes the number of minus signs in the metric signature. We hereby now reverse the order of derivation: instead of deriving the Hodge star from the metric signature, we derive the metric signature $\mathrm{sig}\,(g_{ab})$ from $\mathbb{J}$, which plays the formal role of the Hodge star up to a scalar factor. In particular, we have:

$$\mathrm{sig}\,(g_{ab}) = \begin{cases} (+,-,-,-) \quad \text{or} \quad (-,+,+,+) \quad \Longleftarrow \qquad \mathbb{J}^2 = -\mathbb{I} \\ (+,+,-,-) \quad \text{or} \quad (+,+,+,+) \quad \Longleftarrow \qquad \mathbb{J}^2 = \mathbb{I} \end{cases} \tag{14}$$

It follows that the $\mathrm{sig}\,(g_{ab})$ is Lorentzian, given **Axioms 1–5** of pre-metric electromagnetism as presented in the previous section.

Of course, the difference between signatures $(+,-,-,-)$ and $(-,+,+,+)$ is typically (and quite reasonably) regarded as being a conventional (rather than factual) one; by contrast, the difference between signatures $(+,+,-,-)$ and $(+,+,+,+)$ is far from trivial or a matter of convention alone.[31] For example, in the former case, one (perhaps surprisingly) has well-posed Cauchy problems (see [30, 103] for the mathematical details) and so in a straightforward sense can model temporal evolution; this is not so in

[28] We will assess in §4 whether, strictly, this is true.

[29] We mean this in the sense of [14], according to which the Lorentizan signature is the 'seed' of spacetime. Le Bihan and Linnemann argue that (again, perhaps surprisingly) this 'seed' can be found in many approaches to quantum gravity (despite claims in the philosophy literature that spacetime disappears from the fundamental commitments of such theories—on which see e.g. [61]).

[30] It is possible that all of the background metric—not just the signature—can be derived. But we are silent in this paper about whether the other parts of a metric can be derived from electrodynamics or other matter theories.

[31] For presentational clarity, we're setting aside other possibilities in the second of the above categories—e.g., $(-,-,+,+)$, $(-,+,+,-)$, or $(-,-,-,-)$. For some brief philosophical discussion of these issues, see [89, fn. 35] and (separately) [25, §6.3].

the latter case.[32] This is fortuitious, since the different choices of signature associtated with $\mathbb{J}^2 = -\mathbb{I}$ thereby turn out to be physically immaterial; not so for the case $\mathbb{J}^2 = \mathbb{I}$.[33]

While the Lorentzian metric signature is thereby determined uniquely from assumptions regarding the constitutive relation encoded in **Axiom 5**, we may ask the question as to whether other metric signatures are derivable from pre-metric electromagnetism given different assumptions regarding the constitutive relation. The answer is affirmative: indeed, we have already seen that if $\mathbb{J}^2 = \mathbb{I}$, i.e., if $\kappa^2 = -\zeta^2\mathbb{I}$, then it is underdetermined whether we have $(+,+,-,-)$ or $(+,+,+,+)$, both of which are non-Lorentzian signatures. Relatedly, in the recent work of [43], Hodge star operators associated with Galilean spacetime (which is obtained by widening the lightcones of a Lorentzian spacetime subject to certain integrability conditions) and Carrollian spacetime (which is obtained by narrowing the lightcones of a Lorentizan spacetime) are constructed; in both cases, they are nilpotent operators, as one would expect, given $*^2 = (-1)^{\text{ind}}\mathbb{I}$. This opens up further possibilities which could arise from pre-metric electromagnetism: if κ were nilpotent, one could in principle situate the theory in Galilean or Carrollian spacetime. As in the $\mathbb{J}^2 = \mathbb{I}$ case considered previously, however, in this case the signature (and, hence, the spacetime structure) would not be fixed uniquely by this particular form of the constitutive relation—in this regard, a Lorentzian signature appears to be special.[34]

4 Philosophical appraisal

Technically, the approach to pre-metric electromagnetism of Hehl *et al.* appears to have achieved the following. First, it is (or at least, purports to be) a reformulation of classical electromagnetism without presupposing a metric or any structure that is conceptually equivalent to, or derivative upon, metrical structure.[35] Second, the Lorentzian signature is *derived* from the theory. These points, however, are not completely obvious or indubitable. The derivation of the signature from the theory hinges upon the constitutive relation between F and H. One may object from this single step of derivation that the constitutive relation selected by Hehl *et al.* is conceptually *equivalent* to choosing a Lorentzian signature along with the standard formula $H = *F$. In this sense, the thought would go, we already introduce the metric signature in disguise via **Axiom 5**, and therefore this is not metric-free electromagnetism after all. But this criticism seems unfair, since there is no *a priori* conceptual connection between F, H, and metrical sig-

[32] That being said, the connection between metric signature and the possibility of temporal evolution in fact turns out to be quite subtle—see [90] and, relatedly, [73].

[33] There is, however, a potential problem for the programme of pre-metric electromagnetism in this vicinity which, to our knowledge, has thus far gone unacknowledged: even supposing that $\mathbb{J}^2 = -\mathbb{I}$, why should it be the case that the 'time' direction identified in the resulting Lorentzian signature be the same as that of the vector field v, orthogonal to $d\sigma$? That the two coincide appears to be 'miraculous'—cf. [89, fn. 35]. (Our thanks to Jeremy Butterfield for discussion on this point.)

[34] To be clear: the assumptions regarding the constitutive relation which are encoded in **Axiom 5** suffice to derive a Lorentzian metric signature; in principle, however, a different set of assumptions might still yield a Lorentzian metric signature. In practice, however, different assumptions regarding the constitutive relation seem to yield different spacetime geometric results: see [55].

[35] One might question whether it's really possible to understand the approach as having proceeded without presupposing a metric; we'll return shortly to this worry.

nature. The former are well-defined independently of any geometric concept, and their relation being local, linear and symmetric are reasonable requirements independent of any considerations to do with a metric.

Although this response seems to us to be reasonable, it's worth expanding upon some conceptual puzzles which arise out of the derivation presented in §3. One significant concern is this: as written, (14) implies that that there is some antecedently-given piece of ontology—the metric field g_{ab}—with some associated property (i.e., ideology, in the sense of Quine [87]; cf. [84])—its signature $\mathrm{sig}\left(g_{ab}\right)$—not to be regarded as being fundamental, but rather to be regarded as having been derived via the programme of pre-metric electromagnetism.[36] This would seem to imply that the pre-metric programme regards there as existing *ab inito* a metric field after all; it is just that certain properties of said metric field are to be derived on the basis of dynamical considerations (cf. our discussion of the related programme of Schuller *et al.* in §4.4). This strikes us as an acceptable understanding of what is going on in the pre-metric derivation of §3—although it is an understanding the discussion of which is often not emphasised in the writings of Hehl *et al.* on this topic.[37,38]

It is also worth noting that there seem to be at least two other alternative ways in which one could interpret the derivations presented in the previous two subsections. First: one could make some attempt to separate out explicitly the signature degrees of freedom of g_{ab} from all others (this proposal is discussed in greater detail in §4.1); if successful, one could thereby treat those degrees of freedom as derivative *ontology*. Second, one could continue to think of $\mathrm{sig}\left(g_{ab}\right)$ as a piece of derived ideology, but reject the view that this need be a property of anything in the model: rather, in this case, one would regard that derived property as having modal import: if there *were* a metric in the model, then it *would* have this signature. We confess that we do not find this latter option particularly compelling, for if there were indeed a metric in the model (not to mention the possibility of multiple metrics, which would complicate the story yet further), what would compel it to have that particular derived signature, rather than some other?[39] But in any case, our purpose here isn't to defend one of these options, but rather to chart the space of interpretative questions which arise for Hehl *et al.*, having presented the derivation of the metric signature within the context of pre-metric electromagnetism which we have recapitulated in §§2–3.[40]

[36]In the foregoing, we've plumped for an understanding of the metric field g_{ab} on which it is a piece of ontology—a field 'living' on the manifold M. But there is an alternative understanding of g_{ab} according to which it too is a piece of ideology, for it encodes properties of the spacetime points comprising M, in particular distances and angles between said points. For the sake of simplifying the narrative, in the above we have elected to identify g_{ab} as a piece of ontology.

[37]This might be the preferred position of Hehl *et al.*, since they want to leave room for a separate theory of gravity that interacts with other matter fields including the electromagnetic field (see [56, ch. F]).

[38]Continuing on the theme of footnote 36: if the metric field g_{ab}—*qua* object encoding properties of points of M—were indeed to be regarded as a piece of ideology, then one would, in analogy with our discussion in the main text, wonder what it could mean to retain a commitment to g_{ab} in general relativity, yet reject a commitment to the manifold of points M—this being the interpretative stance urged by Earman and Norton in light of the hole argument [41] (see [86] for a recent review of the hole argument). In turn, this might motivate one to reformulate general relativity in terms of e.g. algebraic fields—on which see e.g. [40, 93] (and, from an angle more related to the 'dynamical relativity' of §4.2, [28, 75]).

[39]This is related closely to concerns regarding 'miracle resolution' (see [89]) which we'll discuss in §4.1.

[40]On top of—but related to—these questions, there is another: why take it to be that case that metric

Even setting aside these issues (at least for the time being), there remains a great deal more to explore regarding the philosophical significance of pre-metric electromagnetism. In this section, we take up these matters with respect to: (i) the question as to whether it is possible to liberate electromagnetism from a commitment to 'background' structures (§4.1), (ii) the significance of pre-metric electromagnetism for the version of spacetime relationism associated with the 'dynamical approach' of Brown and Pooley [19, 20, 21] (some of the issues here were outlined already in §1) (§4.2), (iii) the relation between pre-metric electromagnetism and the programme of 'constructive axiomatics' promulgated by Reichenbach [91] (§4.3), and (iv) the connections between this work and the *à la mode* topic of spacetime functionalism (§4.4).

4.1 Background independence

It is sometimes claimed that general relativity occupies a privileged point in the space of spacetime theories by virtue of its 'substantive general covariance' or 'background independence' (for reviews of such issues, see e.g. [79, 85, 88]; we use the two terms interchangeably in what follows, and will for simplicity tend to use the latter term).[41] Clearly, this presupposes several things:

1. A precise definition of these terms.
2. A clear assessment that general relativity satisfies said definitions.
3. A clear assessment that other theories do not satisfy said definitions.

Now, when it comes to (1), a cornucopia of definitions present themselves—see [88] for a survey and assessment—however, we will focus here on a definition in terms of 'absolute objects' (going back to Anderson [5], and subject to significant critical scrutiny in [82]), since (a) it is one of the most popular and plausible of such definitions, and (b) as we will see, it interacts in particularly interesting ways with the above-presented results from pre-metric electromagnetism.

The 'no absolute objects' definition of background independence says this: a theory is background independent just in case it has no absolute objects in its formulation, where an absolute object is an object fixed (up to isomorphism) in all dynamically possible models of that theory.[42] To illustrate the thought behind this proposal, consider the following two versions of special relativity, which Pooley dubs in [85] **SR1** and **SR2**. **SR1** has kinematical possibilities given by triples $\langle M, \eta_{ab}, \varphi \rangle$ for some fixed Minkowski metric field η_{ab} (i.e., η_{ab} fixed identically in all kinematically possible models of the theory) and scalar field φ, and dynamical possibilities (a subset, of

signature derived in the pre-metric approach really is the signature *of spacetime*? This is akin to the 'hard problem of spacetime' (itself akin to the 'hard problem of consciousness' raised in the philosophy of mind: see [26]) which has been discussed in the content of the (supposed) emergence of spacetime in quantum gravity: see [13, 27, 64, 70].

[41] For an invitation to philosophers to explore the space of spacetime theories, see [68].

[42] For a more formal presentation of Anderson's account, also commenting on the differences between Anderson's original presentation [5, p. 83] and Friedman's later reformulation [49, pp. 58–60], see [82, pp. 348–353]. Cf. also [59].

course, of the kinematical possibilities) given by the Klein-Gordon equation

$$\eta_{ab}\nabla^a\nabla^b\varphi = 0, \tag{15}$$

where ∇ is the Levi-Civita derivative operator of η_{ab}. By contrast, **SR2** has kinematical possibilities given by triples $\langle M, g_{ab}, \varphi\rangle$ for some generic Lorentzian metric field g_{ab} in this case not fixed identically across kinematical possibilities, and dynamical possibilities given by

$$g_{ab}\nabla^a\nabla^b\varphi = 0, \tag{16}$$

$$R^a{}_{bcd} = 0, \tag{17}$$

where now ∇ is the Levi-Civita derivative operator of g_{ab}. (In both **SR1** and **SR2**, we take it to be the case that $M = \mathbb{R}^4$.)

The point of introducing these two different versions of special relativity is this: intuitively, **SR2** is no more 'dynamical' in any interesting sense than **SR1**, for in essence all that one has done is convert the metric field's being fixed at the level of kinematics to its being fixed at the level of dynamics, by way of introduction of a suitable additional dynamical equation (imposing flatness of the associated derivative operator). The 'no absolute objects' definition captures this intuition correctly, for it adjudicates that neither **SR1** not **SR2** are background independent: both theories fail the definition.

Turn now to general relativity. At first blush, one's verdict might be that this theory passes the 'no absolute objects' definition, for in this case the metric field is dynamical, and varies from dynamical possibility to dynamical possibility. However, Pitts demonstrates correctly in [82] that this verdict is false, for consider the fact that a Lorentzian metric field g_{ab} can generically be decomposed as

$$g_{ab} = \hat{g}_{ab}\sqrt{-g}^{2/n}, \tag{18}$$

where g is the determinant of the metric, $\hat{g}_{ab}$ is the conformal metric density, and n is the dimension of the metric manifold under consideration (for us, $n = 4$). Note, in particular, that locally the scalar density g is indeed fixed (up to isomorphism) in all models of general relativity—so, in fact, general relativity *does* have an absolute object! (For further detailed discussion of this point, see [88, §3.3].)

This is very closely related to the fact that the signature of the metric in general relativity is fixed (up to isomorphism) in all dynamical (indeed, kinematical) possibilities. To see this, note that the metric determinant encodes this signature: if $g < 0$, one can infer that the metric has Lorentzian signature (recall again our discussion of these issues in §3; note also that $g < 0$ underdetermines the particular Lorentzian signature convention). Of course, g encodes more than just the metric signature: the magnitude of g encodes in addition facts about global volume which, alongside the conformal structure encoded in $\hat{g}_{ab}$ (which essentially fixes the lightcone structure at each point), together via (18) suffice to fix the metric g_{ab}. But the point is that one can see straightforwardly how the metric signature is the same (up to isomorphism) in all dynamical possibilities of general relativity, even if the metric itself is not so fixed.

If one could identify a means of modifying general relativity so as to incorporate dynamical signature change, then this concern regarding the exhibition of absolute objects even in general relativity—insofar as it is indeed a concern!—could be overcome.

There is, indeed, a precedent for considering signature change in (modified versions of) general relativity: see e.g. [15, 17, 53**?**]. Arguably though, such approaches are lacking, in the sense that they really seem to be changing the subject, by moving to theories *distinct* from general relativity. What would seem to be preferable would be an account as to how the metric signature need not be regarded as being fundamental in a relativistic theory: in this way, concerns regarding any commitment to the fundamentality of this particular absolute object could be avoided.[43]

This is exactly what the machinery of pre-metric electromagnetism seems to provide: at least within the framework of something like Einstein-Maxwell theory, the thought would be that the metric signature can be derived from the electromagnetic sector. But is it really so? At this point, it might again help to think through the issues in terms of kinematical and dynamical possibilities. For clarity of presentation separating out the two sub-constituents g and $\hat{g}_{ab}$ of g_{ab}, one could write the kinematical possibilities of standard Einstein-Maxwell theory as $\langle M, g, \hat{g}_{ab}, F_{ab}, J^a \rangle$; dynamical possibilities are given in terms of the Maxwell and Einstein equations written in terms of the relevant objects. This won't exactly do for our purposes, however, since we've already seen that g encodes *more* than merely signature degrees of freedom.[44]

What we require, therefore, is a cleaner decomposition of a Lorentzian metric field, which allows us to isolate exclusively the signature degree(s) of freedom. One way to proceed would be to use frame fields $e_a{}^{\mathsf{a}}$ in the combination

$$g_{ab} = \eta_{\mathsf{ab}} e_a{}^{\mathsf{a}} e_b{}^{\mathsf{b}}, \tag{19}$$

through which the Lorentzian metric field $g_{\mu\nu}$ can be taken to be *defined* in terms of the frame fields; $\eta_{\mathsf{ab}} = \mathrm{diag}\,(-1,1,1,1)$ is here to be regarded as being a mere matrix. Thus, kinematical possibilities of electromagnetism set in a relativistic spacetime can be construed such that it is the frame fields which are primitive; thereby, the models can be written $\langle M, \eta_{\mathsf{ab}}, e_a{}^{\mathsf{a}}, F_{ab}, J_{abc} \rangle$.[45] The merit of this particular decomposition over that discussed before is that η_{ab} encodes *all and only* signature degrees of freedom—so, on the pre-metric approach, it is exactly this degree of freedom which is expunged.[46] In this case, dynamics are picked out by (a) the constitutive relations discussed in §3 which suffice to fix a Lorentzian signature and thereby the metric signature (now understood as, in part, a derived object), and (b) the Einstein-Maxwell equations (where now, of course, some constituents of such equations are to be regarded as being derived, rather than fundamental).[47]

[43] Of course, there's still the open possibility that the theory turns out to have other absolute objects. It's also important to be clear that, in viewing said absolute object as being non-fundamental, one has not thereby *excised* said object (one is, in other words, not being an eliminativist about that object, but rather simply reducing it to something else); nevertheless, the thought would be that one has made some progress by not having to regard this object as being fundamental. (For further discussion of issues regarding the fundamentality of absolute objects, see [88, ch. 3].)

[44] In addition, since $\hat{g}_{ab}$ encodes lightcone structure, it also speaks to the metric signature.

[45] Recall from §2 that, in the absence of a metric with respect to contract indices, the elementary current J must be understood to be a (twisted) 3-form.

[46] The tradeoff, of course, is that by using frame fields we have introduced *surplus* degrees of freedom: see [83]. Note also that this decomposition will work only when the manifold M is parallelizable.

[47] As usual in foundational discussions of electromagnetism, there are questions to be asked regarding whether to take F or its potential A (*mutatis mutandis* J or its potential H) as ontological primitives. It's principally for the sake of simplicity that we elect to formulate our theories above in terms of F and J.

How does this compare with the case of the move from **SR1** to **SR2**? There, one retains the same ontology (of a Lorentzian metric field) *and* ideology (flatness of the derivative operator compatible with said metric field) in both theories—it is simply that the ideology is imposed at the level of kinematics in **SR1**, while at the level of dynamics in **SR2**.[48] By contrast, in the move from 'standard' electromagnetism to pre-metric electromagnetism, one in a certain sense trades ontology for ideology: the metric signature in the former theory is encoded in extra relations imposed upon the remaining geometric objects which comprise the kinematical possibilities of the latter theory.[49] These differences notwithstanding, an analogy with the case of **SR1** and **SR2** is fairly plain: in both cases, one has moved kinematical impositions to dynamical ones. By analogy, then, if the original version of the theory qualifies as not being background independent in virtue of its possession of an absolute object, one might worry that this reformulated version, appealing to pre-metric electromagnetism, should also fail to be background independent. But there are ways to push back against this: one might argue that because there is an *ontological* reduction in the case of electromagnetism in a way that there is not in the case of **SR1** and **SR2**, there is a better case to be made that there is genuine liberation from background structures here. On the other hand, if one notes that the metric signature and g are still *implicitly definable* (cf. [12]), then, as already indicated, one can levy the charge of a *lack* of background independence (understood in terms of the absence of absolute objects) in both cases.

We'll close this subsection with two more criticisms in the vicinity. First: at [89, fn. 34], it is argued that in theories such as Einstein-Maxwell (standardly formulated), it's a miracle (by which the authors of [89] mean, something surprising and unexplained) why the signature associated with the Lorentizan metric field which codifies the symmetries of the dynamical equations governing matter coincides with the signature of the primordial metric field g_{ab}. One might think that the move to pre-metric electromagnetism can help to resolve this issue, for one could argue that the signature of g_{ab} is *nothing other* than that derived from the equations governing material fields. In fact, however, it's at least not obvious to us that there would be a reduction in 'miracles' on making this move, for it still appears to be outstanding why the signature obtained from the pre-metric approach should interact with the other metrical degrees of freedom (encoded—redundantly—in the frame fields) just so as to yield a well-behaved Lorentizan metric field g_{ab}.

Second: in standard electromagnetism, one has the constitutive relation $H = *F$, where $*$ is the Hodge star operator associated with g_{ab}. Given this, one needn't commit to H or F as being primitive: both can be derived from J. By contrast, in pre-metric electromagnetism, one must retain a commitment to H (or J, as presented above) and F as primitive ontological posits, in order to derive the Hodge operator for a Lorentzian metric field from the constitutive relation between these objects. Thus, ontological parsimony and Occamist norms (cf. [34]) do not favour pre-metric electromagnetism

[48]For more on the kinematics/dynamics distinction in modern theories of physics, see [32, 73].

[49]We say 'in a certain sense', because one could argue that the metric signature is not an *entity*, but a *property* of the metric field to begin with. Fair enough: in this case, the correct thing to say is that one moves from primitively-stipulated ideological commitments to ones which are derived on the basis of further dynamical considerations, much (again) as in the case of the move from **SR1** to **SR2**. (Cf. our discussion in the introduction to this section.)

in the way that one might have expected. Since Occamist motivations are some of the strongest militating in favour of relationalism in general (cf. [9, 84]), this point is, therefore, arguably a blow to some of the foundational motivations underlying the pre-metric programme.

4.2 Dynamical relativity

One of the central tenets of the 'dynamical approach' to spacetime theories of Brown and Pooley [19, 20, 21] is that spacetime structure is to be understood as nothing more than a codification of the symmetries of the dynamical laws governing material fields: it is for this reason that Brown and Pooley declared famously that the Minkowski metric of special relativity is a 'glorious non-entity'.[50] Of course, this is a relationalist thesis—however, when one gets into the weeds of their positions, different projects appear to be driving the ambitions of Pooley versus Brown. For the former, it is an Occamist ambition to expunge redundant structure from one's roster of metaphysical commitments (see [84]).[51] For Brown, by contrast, what is problematic are, in particular, objects which violate the 'action-reaction principle': which act on other bodies, but which are not acted back upon in turn (see the discussion in [19], as well as [67] for a wider-ranging discussion of this principle).

When it comes to the Minkowski metric field of special relativity, Brown and Pooley's ambitions align, and both authors seek to reduce the metric field to symmetries of the dynamical equations via the dynamical approach. However, when it comes to the metric field g_{ab} of general relativity, the two authors diverge: Pooley is still motivated to expunge a commitment to (what's represented by) this structure on Occamist grounds; by contrast, since g_{ab} is dynamical, Brown is explicitly uninterested in such a reduction (see [19, ch. 9]). *This being said*, Brown recognises that the *signature* of the metric in general relativity is indeed still fixed, non-dynamical, and action-reaction principle violating—so, his reductionist ambitions continue to track this more modest target.

On the face of it, pre-metric electromagnetism may seem to avail Brown of exactly the resources he requires in order to expunge any commitment to the fundamentality of the metric signature in general relativity. However, for the reasons already articulated in the previous subsection, one mustn't be too hasty here, for (a) this programme clearly requires very particular material content, so cannot be fully general in the sense of being applicable to all possible worlds regardless of their material content (see §5 for prospects of generalisation), and (b) as already discussed, arguably the suspect absolute objects are 'waiting in the wings' even when this programme can be applied—so it is at least not obvious that Brown should be satisfied with it. Note also the the reduction of the metric *signature* to dynamical considerations will not satisfy authors

[50] Actually, this is equivocal between *eliminativism* about the Minkowski metric—according to which said object simply does not exist in one's roster of fundamental ontological commitments—and *reductionism* about the Minkowski metric—according to which said object exists, but is reducible to other entities in one's ontology. In our view, it is better to view this version of the dynamical approach as a form of reductionism rather than eliminativism; this squares with our comments below on functionalism.

[51] Note that Pooley's views have evolved over time; in fact, he does not currently endorse the dynamical approach at all. In [84], he argues that the strongest motivation for this version of relationalism which he can identify is an Occamist one.

(such as Pooley, on the above presentation) who strive for such a reduction of the *full* metrical structure of spacetime (for this, some other strategy—perhaps a generalisation of Huggett's 'regularity relationalism' [60] (see [84, 99])—may be more effective). Indeed, this is especially so given that the Occamist stripes of pre-metric electromagnetism, as we have just seen, are not as evident as one might think initially.

Finally, even granting that the metric signature can be derived from electromagnetic considerations, this result *could*, in principle, be co-opted for the purposes of those with a more 'geometrical' outlook (see e.g. [22] for more on the dynamical/geometrical debate in the philosophy of spacetime). In particular, having derived said signature, one could argue subsequently that it acts as a constraint on the metrical signature associated with all *other* material fields—in this sense, the signature as it is manifest in all physics would indeed be specifically *electromagnetic* in origin. On the other hand, those with a dynamical outlook may prefer a more egalitarian stance, according to which each set of material fields yields its own distinct metrical signature—these may or may not agree, but this is ultimately a contingent matter, or at the very least is dependent upon the interactions between those fields. (Of course, this assumes that the pre-metric approach can be generalised to other material fields—more on this in §5.) This point is consistent with that of Geroch [51], who stresses that each set of material fields may come equipped with its own light cone structure.

4.3 Connection to constructive axiomatics

The dynamical approach discussed in the previous subsection occupies one corner of the literature on what is known as the 'epistemology of spacetime' (of course, being an ontological reduction account, the approach is also metaphysical in nature—but the point is that, having secured said reduction, one would thereby also have come to settle an epistemological issue—see [37]). In another corner of this literature, one finds the programme of what's known as 'constructive axiomatics'. The idea of constructive axiomatics, which is an agenda going back to Reichenbach in 1924 [91], is that one should build up the entire edifice of one's theories (including in many theories *inter alia* a commitment to a spacetime metric) from axioms which (supposedly) have indubitable empirical content. Thereby, the thought goes, one comes to set one's theories on solid empirical footing. The most famous constructive axiomatisation of general relativity is the 1972 approach of Ehlers, Pirani, and Schild (henceforth EPS) [42],[52] who purport to build up the Lorentzian manifold of general relativity from the trajectories of light rays (yielding, the claim goes, conformal structure) and freely-falling particles (yielding, the claim goes, projective structure). The EPS axiomatisation has recently been subjected to detailed foundational scrutiny in [1, 72]; in this subsection, we will assume some familiarity with the approach—the reader is referred to the cited articles for more background.

The fact that Hehl *et al.* formulate *axioms* for pre-metric electromagnetism also affords their approach the flavour of a project in constructive axiomatics; Hehl and Obukohov, indeed, explicitly identify the approach as such in [58]. But the very different nature of the axioms of the pre-metric approach as compared with those of EPS

[52] Reichenbach's own axiomatisation was somewhat of a flop: see [92, 94] for the history.

means that the former warrants some further consideration *qua* constructivist project. The first question to settle is this: *which* of the axioms of pre-metric electromagentism are in fact empirical in nature? Tackling each in turn, the empirical content of the pre-metric axioms can—very briefly!—be summarised as follows:

Axiom 1: This is an axiom about the foliability of the manifold into hypersurfaces of constant (topological) time. Insofar as *locally* we think we can distinguish (even phenomenologically) a time direction, it seems to have empirical status; however, the *global* assumption seems to go beyond anything which we think we can access empirically.[53]

Axiom 2: The empirical significance of $\mathscr{H}$ and $\mathscr{D}$ (which, recall, are the magnetic and electric components of H, which in turn is the potential for the current J) is this: both can be measured by ideal electric conductors and superconductors in the lab [56, p. 116]. J is measured indirectly through these objects—so this is the operational significance of the axiom. This, indeed, allows the axiom to make contact with the empirical, albeit in a much more theory-laden and indirect manner than e.g. the axioms of EPS.[54]

Axiom 3: The motions of charged test bodies can be considered observable (albeit perhaps indirectly); such is the empirical justification for this axiom.[55] In turn, this is used to afford F with empirical meaning.[56]

Axiom 4: The empirical content of this axiom—which regards the conservation of F—derives from measuring in the lab the electric and magnetic parts of F, which are the usual electric (E) and magnetic (B) fields, respectively.

Axiom 5: As discussed in §2, the empirical content of this axiom is encoded in the energy-momentum trace being zero, which corresponds to the photon being massless.[57]

[53]The weakening of **Axiom 1** to 'local' foliations presented in footnote 14 thereby appears to be better justified on empirical/phenomenological grounds than the 'global' version of this axiom. Note that (empirical/phenomenological, local) evidence for there being *some* foliation does not imply uniqueness—-so there is no tension here with e.g. footenote 27.

[54]Cf. 'theoretical constructivism', discussed in [1].

[55]Of course, there's more work to be done if this is to be compelling—one might appeal to e.g. (a) the fact that we can 'track' the trajectories of outgoing bodies in particle scattering experiments (such as those at the LHC), or (b) the fact that an operational prescription for 'tracking' test bodies using radar coordinates is given in [42] (see [72] for discussion).

[56]It might not, however, be sufficient to afford an operational *definition* of F—see footnote 21.

[57]By 'massless', we mean that the field in question is dispersion-free—i.e., propagates on null geodesics. There may ultimately be some complications to this understanding, given that it has recently been argued in [8, 71] that electromagnetic fields in curved spacetimes *can* exhibit dispersion; nevertheless, we'll set aside these complications here, on the grounds that (a) such dispersion relations remain to be fully understood, and (b) these dispersion relations appear only to be non-vanishing in the context of particular spacetimes (e.g. Gödel, Kerr) in any case.

It is worth noting here, however, that the characterisation of masslessness in terms of dispersion-freeness is not necessarily completely unproblematic, given that the latter notion makes sense only once one has to hand a particular metric field (with its associated signature)—but this is what one is supposed to be constructing in the pre-metric programme! The right thing to say here, it seems to us, is this: the motions of photons are empirically detectable in any case (at least granting the points raised in footnote 55), but can

It is important to note that it is not necessarily possible to isolate the empirical content of each of **Axioms 1–5** from that of the rest. In fact, one can summarise the inter-relations between the axioms as follows. **Axiom 1** is basic; the application of **Axiom 3** relies on this in the sense that to decompose the F field featuring in (7) into its electric and magnetic components presupposes the 'space'/'time' split of **Axiom 1**; **Axiom 2** and **Axiom 4** both presuppose both **Axiom 1** and **Axiom 3**, for they both presuppose a 'space'/'time' split and (in order to be operationalised via e.g. the motions of test bodies) the Lorentz force law; finally, **Axiom 5** comes last, and relies on the other axioms as it presupposes the existence of a constitutive relation which follows only one those other axioms are deployed. These inter-relations are not necessarily problematic for the pre-metric programme *qua* constructive axiomatisation—indeed, even the axioms of the EPS approach [42] build on one another in a similar manner;[58] nevertheless, they are worth keeping in mind.

What one learns here is that although all of the axioms of pre-metric electromagnetism have empirical content, this is often not as direct as that of the EPS axioms (which pertain, as already mentioned, to the observable motions of free particles and light rays). Indeed, in this regard there are two further points which we wish to highlight. First, regarding **Axiom 2** and **Axiom 4**: without a background metric, we cannot decompose H or F into spatial and temporal components *uniquely*—but it is these very components which are the things supposed to have operational significance, on the above presentation![59] This raises the concern that the empirical facts encoded in these axioms in fact presuppose metrical structure after all. Second, and relatedly: a frame field occurs in **Axiom 3** (as well as in the construction of Σ_α after the introduction of **Axiom 5**—this object being used to draw out the empirical content of that latter axiom), but (i) this field is completely arbitrary, and (ii) the concept defined in its basis does not obviously have physical meaning. In response to these concerns, one could argue that the field is supposed to represent a fiducial observer; insofar as we are reasoning from the empirical observations of *some* observer with *some* rest frame, this should be regarded as being unproblematic.[60]

In any case, let us now make a further point of comparison between the pre-metric approach and that of EPS. From the axioms, one derives (in the manner surveyed in §2) various equations of motion, which one then constrains via further theoretical relations, e.g. the reciprocity relation (which in this case follow from the axioms directly). The point is that, in this approach, the end-point of the axiomatisation is a *dynamical theory*, rather than merely a set of kinematical *states* as in the case of EPS—in this sense, this approach bears a clear conceptual advantage over the latter.

Insofar as the pre-metric approach derives (or at least purports to derive) the metric signature from certain dynamical equations denuded of metrical structure (subject to

only be charcterised *as* dispersion-free once one has to hand this additional structure. But ultimately, it is the *motions* which are empirically-detectable, and which are underwritten by **Axiom 5**—*not* their status as dispersion-free/those of massless bodies.

[58] For further discussion in the context of EPS, see [1].

[59] Recall footnote 13.

[60] Fletcher argues in [48] that to introduce a frame field is to step outside of the theoretical architecture of a given theory, e.g. general relativity or Einstein-Maxwell theory. In light of the responses offered here, we're unconvinced by the claim.

the relevant additional strictures already outlined)—*viz.*, $dH = J$, $dF = 0$, and the constitutive relation between H and F—it is also (as already discussed) clearly a version of the dynamical approach. To our knowledge, therefore, it is unique in lying at the intersection of constructivism and dynamicism. *Qua* constructive axiomatisation, there are also perhaps some reasons for a proponent of the dynamical approach to favour it over that of EPS: in the case of the latter, Brown (for example) complains that, given its impoverished empirical base, it is "too operational" (personal communication; see [1] for further discussion). Given that the pre-metric electromagnetism has a much richer empirical base (potentially encompassing a range of empirical phenomena to do with electromagnetism), a proponent of the dynamical view need not levy the same complaint in this case.

Hehl and Obukhov's own main complaint against EPS, articulated in [58], is that the latter appeal to the notion of a point particle, which they claim is no longer tenable in light of e.g. quantum mechanics. Here, they may have failed to appreciate that point particles and light rays are meant to be empirical *starting points* rather than fundamental, theoretical descriptions of matter that predict these empirical outcomes, which is not the concern of EPS.[61] But still the complaint against EPS hints at a perhaps legitimate concern about the paradoxical nature of their starting points. How we formulate the observational statements is not devoid of theoretical postulates: we neither observe a *point* particle nor that it follows a *smooth* trajectory, as assumed by EPS. It is unclear why these theoretical postulates are less problematic, or more *indubitable*, than (say) the waves and the dynamical equations of electromagnetism, especially considering that the latter has the virtue of guiding and correcting the empirical observations and may thereby have an elevated epistemic status.[62]

4.4 Connection to functionalism

Spacetime functionalism is sometimes presented in the contemporary philosophy of physics literature as the thesis that "spacetime is as spacetime does" [65]. Absent further details, this is of course vague: Butterfield and Gomes put flesh on the bones by understanding "functionalism as a species of reduction" [24] in the sense of Lewis [69], so that if the primitive posits of one's theory fix uniquely some piece of (putatively) spatiotemporal structure, then the latter is reduced to the former, and (in this sense) one can be a functionalist about said latter structure. Butterfield and Gomes give several examples of this approach to spacetime functionalism, including *inter alia* the functional reduction of the temporal metric in the 'Machian relationalist' theories of Barbour and collaborators (see [76] for an introduction). The case which they discuss which is most relevant for our purposes is the programme of Schuller *et al.* [38, 96] (as in previous subsection *vis-à-vis* the EPS axiomatisation, here we assume some basic familiarity with this work of Schuller *et al.*; the above-cited work of Butterfield and Gomes provides an accessible introduction).

In Schuller's programme, one begins with a Lagrangian theory of matter coupled to *some* tensorial geometrical structure, the exact nature of which (e.g., index structure)

[61] Although note that authors such as Lämmerzahl have sought to modify the EPS approach by using more realistic—but theoretical!—quantum mechanical models of matter as inputs: see [66], and [1] for discussion.

[62] For more on different degrees of theoretical inputs in constructive axiomatic approaches, see [1].

is left unspecified. By imposing certain constraints on such theories (e.g., 'predictivity' and 'quantisability': note that these constraints are yet more theoretical—i.e., less directly connected to empirical phenomena—than the axioms deployed in Hehl's pre-metric approach; cf. again [1]), Schuller *et al*. can e.g. rule out birefingence (according to which there are multiple possible lightcone structures at each spacetime point),[63] and ultimately arrive at the conclusion that the geometrical structure postulated at the outset must in fact be that of a Lorentzian metric field (with its associated unique lightcone structure at each spacetime point) satisfying the Einstein equation.[64]

The (Lewisian) functionalist strategy of Schuller *et al*. does indeed seem to afford another means of not regarding the metric field of (say) general relativity as being fundamental. However, as already indicated in parentheses above, the assumptions which it makes are more theoretical than those of Hehl's approach; moreover, the approach *begins* by assuming that there is some geometrical structure before proceeding to pin down its properties, whereas the pre-metric electromagnetism approach of Hehl *et al.* does not (at least explicitly—although recall the concerns raised at the beginning of this section that this may in fact be the case after all) begin with an ontological assumption of this form. This being said, on the other hand the payoff from the approach of Schuller *et al*. is arguably larger, insofar as the entirety of the metrical structure—rather than just the metric signature—is derived alongside the salient dynamics.[65] Moreover, the approach of Schuller *et al.* is (at least at present) more general than that of Hehl *et al*., in the sense that it is not wedded to electromagnetism *in particular*. (Though, again, see below for the prospects for generalising Hehl's approach.) And finally: insofar as the approach of Schuller *et al*. proceeds in terms of action principles rather than in terms of equations of motion, it might ultimately have the merit of being more amenable to, say, path integral quantisation.

5 Prospects for generalisation

There are compelling motivations for seeking to extend the pre-metric approach to electromagnetism to other matter theories, such as non-abelian Yang-Mills theories.[66] One important such motivation is to avoid a dependence of the metric field upon the electromagnetic field *in particular*—the issue here is that if other matter theories necessarily were to involve the metric field, then they would also be counterfacturally dependent

[63] Notably, some of Hehl's axioms can also be understood as ruling out birefrigence—see [81].

[64] Schuller considers the conditions which he imposes to be "classically hardly negotiable necessary requirements for the field equations and their geometric optical limit" [95, p. 685]. But arguably there is a parallel between the approach of Schuller *et al*. on the one hand, and that of Hehl *et al*. on the other, in the sense that just as one might argue that the latter's postulates regarding the relationships obtaining between H and J bake in what one is after (and so are no more conceptually perspicuous than simply postulating that the metric signature be Lorentzian), one might also argue that the former's primitive postulates again bake in what one is after.

[65] Hehl *et al.* in fact regard the metric field as being on a par ontologically with all material fields. As outlined in their book [56], Hehl and Obukhov aim to purge the metric from electromagnetism so that the electromagnetic current density can be a pure matter contribution to the Einstein field equations, interacting with the metric field.

[66] Yang-Mills theories are gauge theories based upon a special unitary gauge group $SU(N)$. For $N = 1$, we have electromagnetism.

on the electromagnetic field; however, there is little reason to think that the electromagnetic field should play such a privileged role amongst all matter fields. Additionally, the desire to isolate and minimise the role of the metric field in the formulation of electromagnetism applies to other matter theories as well. By reasoning analogous to that of Hehl *et al.* in the case of electromagnetism, this leads to a conceptually clearer formulation of those theories.

In this section, we investigate the feasibility of generalising the pre-metric approach to non-abelian Yang-Mills theories. We demonstrate that while this is possible in principle, the resulting formulation is less straightforward or attractive due precisely to the non-abelian nature of a generic Yang-Mills theory. We close this section with some more general reflections on the prospects for a philosophy of non-abelian Yang-Mills theories.

5.1 Preliminaries for Yang-Mills theories

A generic Yang-Mills theory is characterised by two equations of motion which look very similar to the Maxwell equations of electromagnetism (1) and (2) (see e.g. [97]):

$$d_A F = 0, \tag{20}$$

$$d_A * F = J; \tag{21}$$

Despite this apparent similarity, there are two crucial differences between (20) and (21) and the Maxwell equations of electromagnetism. The first is that the field strength F, current J, and 'connection' A are here not ordinary differential forms, but rather *Lie algebra-valued* differential forms (for technical background here, see the appendix).[67] The second is that instead of using ordinary exterior derivative operators on physical quantities, in the above equations one uses a 'covariant exterior derivative' d_A associated with a connection A, which is a $\mathfrak{g}$-valued 1-form (where $\mathfrak{g}$ denotes the Lie algebra in question). For any $\mathfrak{g}$-valued form P, we have $d_A P = dP + [A, P]$, where d is the exterior derivative applied to such forms (see appendix) and $[A, P]$ is the Lie bracket defined by that of $\mathfrak{g}$ (again, see appendix). Furthermore, F is a $\mathfrak{g}$-valued 2-form, which is also called a 'curvature form', but physically represents the field strength of a given Yang-Mills field. Technically, this means that the curvature can be obtained by applying the covariant exterior derivative with respect to the connection A to A itself, namely $F = d_A A = dA + [A, A]$ (which also amounts to $F = dA + A \wedge A$[68]).

5.2 The pre-metric approach to Yang-Mills theories

In generalising the pre-metric approach to generic Yang-Mills theories, we notice first that, as with electromagnetism, all the primary physical quantities (in this case $\mathfrak{g}$-valued forms) and all the operators other than the Hodge star are independent of the metric.

[67] It is easiest to understand the objects under consideration here and the relations obtaining between them via the formalism of principle fibre bundles—on which see e.g. [77]. The terminology of 'connection' is appropriate to that setting. For our purposes in this article, the presentation in this paragraph will suffice.

[68] The $A \wedge A$ term does not vanish in the non-abelian case; this is straightforward to see one one restores suppressed Lie algebra indices.

Thus, to apply the methodology of Hehl *et al.*, we mainly need to reconstrue the Hodge star in (21) as a relation between two physical fields, and from there to derive the metric signature. Furthermore, in order to implement fully the pre-metric methodology, we need to reformulate the differential equations in integral forms to highlight their empirical meaning.[69] However, as we explain below, these tasks cannot be carried out straightforwardly in the case of a generic Yang-Mills theory because the techniques to which Hehl *et al.* appeal rely on the abelian property of electromagnetism. That said, we can make use of some less well-known techniques for non-abelian theories in order to make progress here.

We proceed now by imitating the axioms in §2 in formulating the axioms of a generic non-abelian Yang-Mills theory. For reasons which will become clear, however, this presentation does not follow the original order.

5.2.1 Derivation of the signature

Begin with **Axiom 1**, which concerns the topology of spacetime. This axiom stays essentially the same here:

YM Axiom 1. *Spacetime is a* 4*-dimensional differentiable manifold M that admits a foliation into codimension-*1 *hypersurfaces* Σ.

This is the background arena of our theory, on which we shall define all the primitive physical quantities and operators for Yang-Mills theories.

Let's jump ahead to the final axiom of §2.2 (i.e., **Axiom 5**), which postulates the spacetime constitutive relation between the current potential H and the field strength F, from which we derive the metric signature. Assume for the time being that the existence of F and H are legitimized by other axioms. The good news is that this axiom also remains essentially the same for a generic Yang-Mills theory, with F and H now being Lie algebra-valued (twisted) differential forms:

YM Axiom 5. $H = \kappa(F)$*, with κ being local, linear and symmetric.*

As with Hehl *et al.*, we would like to derive from this axiom that the energy-momentum current Σ_α (which, in the absence of a force f, can be defined directly by H and F as before) is anti-symmetric with respect to H and F, namely $\Sigma_\alpha \propto (e_\alpha \lrcorner H) \wedge F - (e_\alpha \lrcorner F) \wedge H$. Fortunately, the derivation is essentially the same as in electromagnetism because all the operators involved essentially 'ignore' the Lie algebra values (see appendix). This gives us the same reciprocity relation between H and F, namely $H \to \zeta F, F \to -\frac{1}{\zeta} H$, which is a symmetry of Σ_α. The derivation of the metric signature from the reciprocity relation is also the same as in electromagnetism. That is, we obtain that $\kappa = -\zeta^2 \mathbb{I}$, which plays the role of the Hodge star and gives rise to the Lorentzian signature in the same manner as before. Therefore, we have successfully obtained the Lorentzian signature from the theory so formulated.

[69] Our point here is that what we can measure more directly is the accumulative effect of the electromagnetic field over a region rather than its value at a point. In this sense, the integral form of the equations is more empirically-grounded. Cf. [31, p. 45].

5.2.2 From an abelian to a non-abelian Stokes' theorem

The impact of the non-abelian feature of a generic Yang-Mills theory arises from the point that one seeks to generalise **Axiom 4**, which (recall) states that magnetic flux is conserved, so that $\oint_C F = 0$ for any closed submanifold C, from which we can deduce $dF = 0$ by applying Stokes' theorem as usual. There are (at least) two problems with generalizing this axiom to generic Yang-Mills theories:

1. Recall that $d_A F = dF + [A, F]$. In the generic case, the commutator $[A, F]$ does not vanish because the Lie algebra in which A and F take their values is non-abelian. Thus, we cannot apply Stokes' theorem in order to generate a conservation law in the style of Hehl *et al*.

2. The ordinary integral is not even well-defined in the case of non-abelian Lie algebra-valued forms. For one thing, the ordinary integral is largely insensitive to the order of integration (the order in which one 'adds up' the integrand quantity along a path or a region does not matter, except for the overall direction). But the integration order does matter in the non-abelian case since the Lie group elements do not commute. Besides, we also need a connection in order to compare Lie algebra elements at different spacetime points. So the ordinary integral is clearly inadequate.

These problems can be addressed to some extent by appealing to a non-abelian version of Stokes' theorem.[70] Let Ξ be a simply-connected hypersurface of dimension $d < n$, where n is the dimension of the manifold. Let F and A be Lie algebra-valued d-forms and $(d-1)$-forms (respectively), satisfying $F = d_A A$. $\mathscr{F}, \mathscr{A}$ are appropriate terms built out of F and A (which will be illustrated by a concrete example below). One non-abelian version of Stokes' theorem then implies [45, eq. 1.1]:

$$P_{d-1} e^{\int_{\partial \Xi} \mathscr{A}} = P_d e^{\int_{\Xi} \mathscr{F}}. \tag{22}$$

Here, $P_n e^{\int \cdot}$ denotes a path-ordered integral which integrates over Lie algebra elements and produces a Lie group element, with n referring to the dimension of the integral (e.g., P_1 is a path-ordering and P_2 is a surface-ordering). Each P_n introduces n parameters through maps from loops $\{\gamma : S^{n-1} \to M | \gamma(0) = \text{a fixed base point}\}$. To illustrate, consider a concrete lower-dimensional instance of (22) (Σ is a 2-dimensional surface while its boundary $\partial\Sigma$ is a closed curve) [45, eq. 3.1]:[71]

$$P_1 e^{-\oint_{\partial\Sigma} d\sigma A_\mu \frac{dx^\mu}{d\sigma}} = P_2 e^{\int_\Sigma d\tau d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}. \tag{23}$$

Here, σ is a parameter on the closed curve $\partial\Sigma$ introduced by the path-ordering P_1; τ is a second parameter on the surface Σ introduced by the surface-ordering P_2, which

[70] There is a large literature here—see e.g. [3, 4, 6, 18, 29, 44, 45, 46].

[71] Note that we have omitted the integration constant W_R in this equation for simplicity, which is the value of W at the base point $\sigma = \tau = 0$. The negative sign results from the differential equation relating W and A_μ—see [45, eq. 2.1]. See also footnote 72. The negative sign in this equation is a matter of convention adopted in [45] which is related to how we define the relation between transported quantities and the connection that transports them. The negative sign convention implies that for a positive connection, the quantities decrease along the direction of transport.

roughly amounts to scanning Σ with τ-indexed loops, which in turn consist of σ-indexed points. In this way, the order of integration is specified fully. On the LHS of (23), since A_μ is a 1-form, we feed it the tangent vectors $\frac{dx^\mu}{d\sigma}$ on the path $\partial\Sigma$ in integrating it over the path. This results in a Lie group element, since Lie algebra elements specify 'infinitesimal paths' in the principal bundle space and adding them up produces a path which amounts to a Lie group element. The RHS of (23) is similar except for the addition of W and its inverse W^{-1}. The role of these objects is to parallel transport $F_{\mu\nu}(x)$ to the base point in the outmost loop.[72] Intuitively, we parallel transport all $F_{\mu\nu}(x)$ with $x \in \Sigma$ to a point using A, and then integrate them up in the order specified by P_2. The result again is a Lie group element. It is important to note that (23) and more generally (22) are invariant under reparametrization [45, p. 344].[73] Otherwise, the equations would be unphysical, since the parameters introduced by the P_n-ordering are arbitrary.

Note that under the non-abelian Stokes' theorem, (23) and more generally (22) are equivalent to their associated differential equations in the form of $d_A H = F$ for appropriate H, F (the derivation works in both directions). In this sense, they are integral forms of these equations, which we will utilize below.

With (22) in hand, we are now ready to derive the integral form of the conservation laws. If the hypersurface in the RHS of (22) is a closed submanifold, and namely has no boundary, then on the LHS the integral over the boundary is trivial—it yields the identity element in the Lie group. In this case, we have

$$P_d e^{\int_\Xi \mathscr{F}} = \mathbb{1}, \tag{24}$$

where $\mathbb{1}$ is the identity element in the Lie group. It is easy to see that the curvature of a curvature vanishes by applying the non-abelian Stokes' theorem to the LHS of (24) twice in different directions (one involving $d-1$ and one involving $d+1$). Note that this is different from claiming the covariant derivative d_A is nilpotent, which is false, since two different connections are involved. As a special case, however, we do have $d_A^2 A = 0$.

Based on this, it is straightforward to formulate the integral form of the conservation law of the Lie algebra-valued curvature 2-form F in Yang-Mills theory equivalent to (20) ($d_A F = 0$), and analogous to **Axiom 4** in the case of electromagnetism:

YM Axiom 4. *For any closed two-dimensional submanifold* Ξ*,*

$$P_2 e^{\int_\Xi \mathscr{F}} = \mathbb{1}. \tag{25}$$

There cannot be a net flux into the region Ξ, thus the flux is conserved. Thus we have obtained a version of **Axiom 4** suitable to non-abelian Yang-Mills theories. Note that, unlike **Axiom 4**, the above axiom is expressed in terms of $\mathscr{F}$; since, however, this object is built from F, one can still argue that **YM Axiom 4** has empirical significance—albeit perhaps not quite as direct a manner as **Axiom 4**. In short: if F has empirical significance, then $\mathscr{F}$ will inherit this.

[72] Let L be a path joining x and the base point. Then W is defined as $P_1 e^{-\oint_L d\sigma A_\mu \frac{dx^\mu}{d\sigma}}$.
[73] On this point, see also [3, 4, 44].

Similarly, we can give an analogous version of **Axiom 2**, which concerns the conservation of electric current J. In Yang-Mill theories, J is a Lie-algebra-valued twisted 3-form. Let $\mathscr{J}$ be an appropriate term constructed from J,

$$\mathscr{J} := d\zeta d\tau d\sigma W^{-1} J_{\mu\nu\lambda} W \frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\zeta}. \tag{26}$$

We then postulate:

YM Axiom 2. *For any closed three-dimensional submanifold* Ξ,

$$P_3 e^{\int_{\Xi}\mathscr{J}} = \mathbb{1}. \tag{27}$$

Here, however, there is an important caveat: to derive the covariant derivative equation $d_A J = 0$ from the integral form, we need to apply the non-abelian Stokes' theorem to a 4-dimensional path-ordered integral. While we do not see an obvious reason why the 4-dimensional case does not hold,[74] we are not aware of any explicit proof for this in the literature (note that d in (22) is less than the dimension of spacetime and therefore at most $d = 3$). This technical lacuna may be attributed to the fact that the integral version of the standard Hodge star formulation of Yang-Mills theories does not require a higher-dimensional non-abelian Stokes' theorem (the integral version of equation $d_A * F = J$ involves only up to a 3-dimensional volume-ordered integral) as opposed to our pre-metric formulation in the spirit of Hehl *et al*.

5.2.3 From an abelian to a non-abelian Poincaré lemma

But even if we can successfully derive $d_A J = 0$ from **YM Axiom 2**, the second difficulty that we face is that we cannot apply the Poincaré lemma to this equation in order to infer $J = d_A H$—namely, that there is a potential or excitation field H for J. Recall that the field H plays a central role in Hehl *et al*.'s approach—the constitutive relation between F and H gives rise to the metric signature, as we have already seen. But this reasoning does not hold in the Yang-Mills theory because the Poincaré lemma does not apply to the non-abelian covariant exterior derivative.[75]

As with Stokes' theorem, generalizing the Poincaré lemma to the non-abelian case has been a sustained research focus, and there are various strategies offered in the literature to varying degrees of rigor [2, 7, 52, 63, 100]. For example, Gross proves that if we formulate non-abelian Lie algebra-valued forms as path-dependent forms called 'lasso forms', then we can apply the non-abelian Poincaré lemma to derive the connection A from $d_A L(\sigma) = 0$ where $L(\sigma)$ is the path-dependent lasso form for F [52, pp. 38–39]. The virtue of this approach is that we do not need to appeal to A explicitly in the Yang-Mills equations, which as Gross argues results in a more analogous

[74] Intuitively, since Ξ can be any closed $3D$ region, we can shrink it to an infinitesimal region (a 4-hypercube spanned by 4 tangent vectors), so the integral in **YM Axiom 2** would become the desired covariant derivative. But we should be very cautious about such intuitions given that the high-dimensional case is notorious non-amenable to intuitions, not to mention the additional obstruction of non-abelianness. Even the simpler notion of exterior derivative is difficult to picture in the higher-dimensional case.

[75] Recall that in the abelian case, we do not need a separate covariant exterior derivative since $[A,\cdot] = 0$ and thus $d_A = d$.

formulation to electromagnetism. Unfortunately, Gross's non-abelian Poincaré lemma applies only to 1-forms and 2-forms [52, p. 32], and therefore remains inadequate for our purposes, since the current J is a (twisted) 3-form. There are other results pertaining to inverting the covariant exterior derivative (e.g., [63, 100]), but the conditions for applying those methods are also not satisfied by our problem.

Of course, one can choose to dodge these technical difficulties by postulating the existence of H in an axiom. For example, one can postulate an axiom in the following form ($\mathscr{H}$ is the appropriate term associated with H—not to be confused with the 'magnetic' excitation of H introduced previously):

$$P_2 e^{\int_{\partial\Xi}\mathscr{H}} = P_3 e^{\int_{\Xi}\mathscr{J}} . \tag{28}$$

But this axiom would be highly unattractive, especially in the program of Hehl *et al*. For one thing, is much less simple conceptually and foundationally than the conservation law for J. Moreover, it postulates another primitive field H in addition to F and J, and thus renders this formulation ontologically extravagant. This problem can be mitigated by postulating H *instead of* J if H has more significant empirical meaning. In this case, J can be straightforwardly derived by $J = d_A H$.

In summary: although the standard formulation of Yang-Mills theory appear beguilingly similar to Maxwell equations for electromagnetism, a closer examination reveals several as-yet unresolved technical difficulties which prevents us from following through exactly with Hehl *et al*.'s approach. We therefore conclude that the pre-metric programme extends only partially to generic Yang-Mills theories.[76]

5.3 Outlook

What we have achieved here—modulo the substantial caveats mentioned above—is a metric-free reformulation of a generic Yang-Mills theory, from which we derive the Lorentzian signature. In this sense, the Lorentzian signature is not just grounded in electromagnetism, but in more general matter theories. But we should be completely explicit about the above-mentioned substantial caveats:

1. A satisfactory reformulation of the conservation laws hinges on technical developments on the non-abelian generalized Stokes' theorem in higher dimensions.

2. The potential field H for the conserved current J may need to be posited as an axiom. This is undesirable because it is less intuitive as a physical axiom than conservation laws, and moreover because the resulting theory is thereby committed to more ontological primitives.

What can we conclude from these points? Barring further technical developments, this reformulation amounts to less than that to which Hehl *et al*.'s programme aspired. Still, we may (a) accept this reformulation and conclude that the Lorentzian signature is grounded in a general matter theory and is therefore subject neutral. Or, we may (b)

[76] In our discussion here, we haven't mentioned the non-abelian version of **Axiom 3**—i.e., a non-abelian Lorentz force law. Suffice it to say that such a law does exist in the non-abelian case: see e.g. [16]. So, as far as we can see, this axiom would remain unproblematic in the non-abelian case.

reject this reformulation and conclude that metrical signature is in nature electromagnetic and electromagnetic alone, precisely because the electromagnetic field commutes with the connection while a generic Yang-Mills theory does not.

Ad (a): Following this route seems to lead to further philosophical puzzles, for example: why is it, then, that the signatures dervied from all different material Yang-Mills fields happen to coincide? (Recall §4.2, in which this question was already raised.) *Ad* (b): this route can lead to further pessimism about the pre-metric programme in general—for why should we prescribe a special ontological status to the electromagnetic field (namely, providing the fabric of spacetime for *all other* material fields), just because of its seemingly conceptually insignificant feature of having this commutation relation? This may suggest that Hehl's program of reducing the metric signature to electromagnetism in particular is dialectically unstable. We may either achieve a fully satisfactory generalization, or conclude that the metric signature is not reducible to matter fields in the way Hehl *et al*. envision after all.

Stepping back somewhat, what this section reveals is that there is much work to be done towards developing a systematic philosophy of non-abelian Yang-Mills theories—*inter alia*, a prospectus would include: (i) the field copy problem—when does the curvature of a gauge field underdetermine its potential, and to what degree?; (ii) non-abelian Stokes' theorem—how to generalize this theorem to arbitrary Yang-Mills fields in arbitrary dimensions?; (iii) non-abelian Poincaré lemma—how to generalise this result to arbitrary Yang-Mills fields in arbitrary dimensions?; (iv) non-abelian Lorentz force law—what are the properties of such a law in the general case?; etc. All of these questions have deep philosophical significance. *Ad* (i): if the curvature of a gauge field underdetermines its potential, then this may lead to further issues of underdetermination in gauge theories beyond those raised by the familiar gauge orbits of these theories ($U(1)$ in the case of electromagnetism). *Ad* (ii): answers to this question invite comparison with the notoriously vexed issue of conservation of gravitational energy in general relativity (on which see e.g. [35] for a recent masterly review). *Ad* (iii): *inter alia*, this raises questions for theoretical equivalence (on which see e.g. [101, 102] for recent surveys)—if a field strength tensor cannot always be written as the exterior derivative of some potential, does this thereby stand in the way of the two formulations' being equivalent? And *ad* (iv): one might ask, do such laws raise similar issues of inconsistency as those presented for the case of electromagnetism in [50]? It goes without saying that answering any and all of these questions will have to remain a task for another day; nevertheless, the forgoing discussion should already make evident the philosophical and physical (as well as, of course, mathematical) significance of doing so.

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A Mathematics of differential forms

In this appendix, we review some basics of differential forms, twisted differential forms, and Lie algebra-valued differential forms.

A.1 Differential forms

Definition 1. *A* differential r-form *is a totally antisymmetric tensor of rank* $(0,r)$.

As a tensor of type $(0,r)$, a differential form takes r vectors and yields a number. As a totally antisymmetric tensor, it changes sign under the exchange of any two of its indices (for example, a 3-form T_{ijk} is such that $T_{ijk} = -T_{jik} = T_{jki} = -T_{kji} = \ldots$). Geometrically, an r-form in an n-dimensional space indicates the density and alignment of $(n-r)$-dimensional surfaces. For example, a 2-form in $3D$ space can be represented by aligned lines. (For some beautiful diagrams illustrating this visualisation of differential forms, see [23].)

Definition 2. *The* interior product *of an r-form ω with a vector a, written as $a \lrcorner \omega$, is the $(r-1)$-form such that $(a \lrcorner \omega)(a_2, a_3, \ldots a_r) = \omega(a, a_2, \ldots, a_r)$.*

In particular, the interior product (sometimes called the 'contraction') of a 1-form with a vector amounts to evaluating the 1-form on the vector. For any scalar function f (which is a 0-form), we have that $a \lrcorner f\omega = f a \lrcorner \omega = f(a \lrcorner \omega)$.

Definition 3. *An* exterior product $\wedge$ *(also called the 'wedge product') is a mapping from any p-form ω and any q-form γ to a $(p+q)$-form such that it satisfies associativity, and for any vector a, we have $a \lrcorner (\omega \wedge \gamma) = (a \lrcorner \omega) \wedge \gamma + (-1)^p \omega \wedge (a \lrcorner \gamma)$.*

Such a mapping can be shown to be unique. It is useful to note that we have $\omega \wedge \gamma = (-1)^{pq} \gamma \wedge \omega$, due to the antisymmetry of forms. Geometrically, as a simple example, the exterior product of 1-forms in $2D$ space can be visualized by arrays of points with an orientation (like screws).

Definition 4. *An* exterior derivative *d is a mapping from an r-form to an $(r+1)$-form such that for any 0-form f, any p-form ω, and any q-form γ, (i) df is the differential of f, and $d(df) = 0$; (ii) $d(\omega \wedge \gamma) = d\omega \wedge \gamma + (-1)^p \omega \wedge d\gamma$.*

Such a mapping can be shown to be unique. The integration of differential forms, which plays a central role in Hehl *et al.*'s formalism, is defined as follows. Let Γ be an embedding of $\mathbb{R}^k$ into M, and let Σ be the region of $\mathbb{R}^k$ such that $\Gamma(\Sigma)$ is the integration region. Γ^* is the pullback of Γ. Then we have:

Definition 5. $\int_{\Gamma(\Sigma)} \omega := \int_\Sigma (\frac{\partial}{\partial x^k} \ldots \lrcorner \frac{\partial}{\partial x^1} \lrcorner (\Gamma^* \omega)) dx^1 dx^2 \ldots dx^k$.

This requires assigning an orientation to the integration region given by $dx^1 dx^2 \ldots dx^k$, which one might find unnatural. This is an important reason for favoring twisted differential forms (see [23, pp. 197-198]).

Regarding the integration of forms on manifolds, there is the following famous result (where Γ is a region and $\partial\Gamma$ is its boundary):

Theorem 1 (Stokes' theorem)**.** $\int_{\partial\Gamma}\omega = \int_{\Gamma} d\omega$.

In physics, we are often concerned with the differential forms which can be understood to be the potentials (i.e., exterior derivatives) of other forms. For such matters, the following definitions and results are relevant.

Definition 6 (Exact form)**.** *A differential form ω is* exact *if there is a form H such that $\omega = dH$.*

The study of when a differential form is exact has to do with the theory of de Rham cohomology.

Definition 7 (Closed form)**.** *A differential form ω is* closed *if it is such that $d\omega = 0$.*

For closed differential forms, even if they cannot be understood globally to be the potential of some other form, the following result underwrites this result locally:

Theorem 2 (Poincaré lemma)**.** *In a space contractible to a single point, every closed form is exact.*

Finally, on a manifold with metric, we can define the following important operations on differential forms:

Definition 8. *On manifold with metric g_{ab}, the* sharp operator $\sharp$ *maps any* 1*-form α into a tangent vector $\sharp\alpha$ such that, for arbitrary vector Y^a, $g_{ab}(\sharp\alpha)^a Y^b = \alpha_a Y^a$.*

Definition 9. *In an n-dimensional space, a* Hodge star *operator $*$ is a linear operator from a p-form to an $(n-p)$-form such that, for β any r-form and α a* 1*-form, $*(\beta \wedge \alpha) = \sharp\alpha \wedge *\beta$.*

The Hodge star operator can be shown to be unique. For the above definition to be well-defined, we have to define the Hodge star of a 0-form—so pick $*1$ to be an n-form of unit density and positive orientation [23, p. 160]. The Hodge star contains information about the metric and the orientation of the space. Geometrically, we may visualise the Hodge star as mapping p-dimensional surfaces to $(n-p)$-dimensional orthogonal surfaces. As stated in the main text, the Hodge star operator 'completes' a p-form to the metric-compatible volume form of (M, g) (where g is the metric on M), and thereby defines an $(n-p)$-form on M.

A.2 Twisted differential forms

With the above basics of differential forms established, we turn now to twisted differential forms. To prepare, an *orientation* of a vector space is given by an equivalence class of all volume forms that have the same sign.

Definition 10. *A* twisted form *is an equivalence class of pairs consisting of an ordinary form ω and an orientation of whole space Ω under the equivalence relation $(\omega, \Omega) \equiv (-\omega, -\Omega)$. The twisted form represented by (ω, Ω) is said to have an orientation Ω_ω which satisfies $\Omega_\omega \wedge \omega = \Omega$.*

For example, the orientation of the twisted form $(dx, dx \wedge dy \wedge dz)$ is given by $dy \wedge dz$. Suppose we represent dx visually by the surface spanned by the y and z axis. Then we can consider the orientation of dx as pointing out from the surface. We can call this an 'outer orientation'. Now, the twisted form $(dx, dx \wedge dy \wedge dz)$ has the orientation given by $dy \wedge dz$, which can be visualized as pointing counterclockwise (or clockwise depending on the convention) inside the surface. We can call this an 'inner orientation'. Thus, a twisting amounts to changing an outer orientation to an inner orientation. Note that it is often more natural to represent physical quantities using twisted forms than ordinary ones. For example, a twisted 3-form electric current in $4D$ spacetime can be visualized as directed lines, namely lines with inner orientations. (For more illustrations of twisting, see [23, pp.183ff.].)

As Burke writes of twisted differential forms,

> Do not let the fact that twisted forms are being represented in terms of ordinary forms lead you to think that twisted forms are less fundamental. The situation here is symmetric: we could define an ordinary vector as a pair consisting of a twisted vector and an orientation under the same equivalence relation. [23, pp. 189–90]

The point here is that one can define 'ordinary' differential forms in terms of twisted forms or *vice versa*, but ultimately one can regard either as being the more primitive notion; in this sense, twisted differential forms need not be construed as being mathematically derivative upon 'ordinary' differential forms, despite potential notational implications to the contrary. Under the operations defined in the previous subsection, twisted forms behave in the expected ways:

Definition 11 (Wedge product (twisting))**.** $(\omega \wedge \gamma, \Omega) = (\omega, \Omega) \wedge \gamma = \omega \wedge (\gamma, \Omega)$.

Definition 12 (Exterior derivative (twisting))**.** $d(\omega, \Omega) = (d\omega, \Omega)$.

Since the Hodge star operation itself presupposes an orientation, it too can be twisted, which is what allows for the possibility of a twisted Hodge star operator:

Definition 13 (Hodge star (twisting))**.** $(*, \Omega)(\omega, \Omega) = *\omega$; $(*, \Omega)\omega = (*\omega, \Omega)$

As illustration: the twisted Hodge star maps an ordinary 1-form in $3D$ space to a *twisted* 2-form, and a twisted form to an ordinary form. Ironically, because the Hodge star depends upon an orientation, the twisted Hodge star in fact does *not* depend on the orientation of the space, despite its more complicated notation.

Despite its appearance, a twisted differential form can be integrated over in a space without orientation, e.g. the Möbius strip. For twisted differential forms, we define

Definition 14 (Integration (twisted))**.** $\int_{(\Gamma, \Omega)} (\omega, \Omega) = \int_\Gamma \omega$.

Here, Ω is an orientation for the whole space, and (Γ, Ω) is a subspace that has an outer orientation [23, p. 197]. Given this, we then have the following result pertaining to the integration of twisted forms (see [23, pp. 199-200]):

Theorem 3 (Stokes' theorem (twisted))**.** $\int_{(\Gamma, \Omega)} d(\omega, \Omega) = \int_{(\partial\Gamma, \Omega)} (\omega, \Omega)$.

A.3 Lie algebra-valued differential forms

Finally, the formalism of non-abelian Yang-Mills theories mandates recourse to Lie algebra-valued differential forms. To introduce these, first recall the definition of a Lie algebra:

Definition 15 (Lie algebra)**.** *A* Lie algebra *is a vector space* $\mathfrak{g}$ *over a field* F*, together with a binary operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *called the* Lie bracket*, satisfying the following axioms:*

1. *Bilinearity:* $[ax+by,z] = a[x,z]+b[y,z]$; $[z,ax+by] = a[z,x]+b[z,y]$ *for all scalars* $a,b \in F$ *and all elements* $x,y,z \in \mathfrak{g}$.
2. *Alternativity:* $[x,x] = 0$*, for all elements* $x \in \mathfrak{g}$.
3. *Jacobi identity:* $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$*, for all elements* $x,y,z \in \mathfrak{g}$.

Before proceeding further, we must define the following elementary notion:

Definition 16. *Let* S_n *be the symmetric group of degree n. Then a permutation* $s \in S_{k+l}$ *is called a* (k,l)-shuffle *if it satisfies* $s(1) < \cdots < s(k)$*, and* $s(k+1) < \cdots < s(k+l)$.

Now, a Lie algebra-valued differential form—naturally!—takes values in some Lie algebra. On such objects, we can define the following operations:

Definition 17 (Wedge product ($\mathfrak{g}$-valued))**.** *If* α *is a* $\mathfrak{g}$*-valued k-form on M and* β *is a* $\mathfrak{g}$*-valued l-form on M, then their wedge product* $\alpha \wedge \beta$ *is a* $(k+l)$*-form given by:*

$$(\alpha \wedge \beta)(u_1,u_2,...,u_{k+l}) = \sum_s \mathrm{sgn}(s)[\alpha(u_{s(1)},...,u_{s(k)}),\beta(u_{s(k+1)},...,u_{s(k+l)})],$$

where the sum is over all (k,l)*-shuffles s.*

Here, $\mathrm{sgn}(s) := (-1)^n$, with n being the number of swappings of which s consists, which can be shown to be well-defined. Note that the Lie bracket on the right-hand side above is that equipped by the Lie algebra $\mathfrak{g}$.

Definition 18 (Exterior derivative ($\mathfrak{g}$-valued))**.** *The exterior derivative d is an anti-derivation with respect to the Lie bracket. If* α *and* β *are* $\mathfrak{g}$*-valued forms on a manifold, then we have* $d[\alpha,\beta] = [d\alpha,\beta]+(-1)^{\deg(\alpha)}[\alpha,d\beta]$.

Intuitively, we can consider a $\mathfrak{g}$-valued form as a tensor product of an ordinary form and $\mathfrak{g}$. Then, the exterior derivative on the form is obtained by simply 'ignoring' the Lie algebra part. That is, we first apply the operation to the ordinary form and then reattach the Lie algebra part—hence an apparent similarity between Definition 4 and Definition 18.

References

[1] Emily Adlam, Niels Linnemann, and James Read. Constructive axiomatics in spacetime physics part II: Constructive axiomatics in context, November 2022.

[2] Qusay S.A. Al-Zamil and James Montaldi. Witten-Hodge theory for manifolds with boundary and equivariant cohomology. *Differential Geometry and its Applications*, 30(2):179–194, 2012.

[3] Orlando Alvarez, Luiz A. Ferreira, and J. Sánchez-Guillén. A new approach to integrable theories in any dimension. *Nucl. Phys. B*, 529:689–736, 1998.

[4] Orlando Alvarez, Luiz A. Ferreira, and J. Sánchez-Guillén. Integrable theories and loop spaces: fundamentals, applications and new developments. *International Journal of Modern Physics A*, 24(10):1825–1888, 2009.

[5] James L. Anderson. *Principles of relativity physics*. Academic Press, New York, 1967.

[6] Irina Aref'eva. Non-abelian Stokes formula. *Theor. Math. Phys.*, 43:353, 1980.

[7] Akira Asada. Non abelian Poincaré lemma. In Antonio M. Naveira, Angel Ferrández, and Francisca Mascaró, editors, *Differential Geometry Peñíscola 1985*, pages 37–65, Berlin, Heidelberg, 1986. Springer Berlin Heidelberg.

[8] Felipe A Asenjo and Sergio A Hojman. Do electromagnetic waves always propagate along null geodesics? *Classical and Quantum Gravity*, 34(20):205011, 2017.

[9] Joshua Babic. *Equivalence and relationism*. PhD thesis, University of Geneva, 2023.

[10] J. C. Baez and J. P. Muniain. *Gauge Fields, Knots And Gravity*. Series On Knots And Everything. World Scientific Publishing Company, 1994.

[11] David John Baker. Knox's inertial spacetime functionalism (and a better alternative). *Synthese*, 199(2):277–298, 2021.

[12] Thomas William Barrett. What do symmetries tell us about structure? *Philosophy of Science*, (4):617–639, 2017.

[13] Baptiste Le Bihan. Spacetime emergence in quantum gravity: Functionalism and the hard problem. *Synthese*, 199(2):371–393, 2021.

[14] Baptiste Le Bihan and Niels Linnemann. Have we lost spacetime on the way? narrowing the gap between general relativity and quantum gravity. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 65:112–121, 2019.

[15] Martin Bojowald and Suddhasattwa Brahma. Signature change in two-dimensional black-hole models of loop quantum gravity. *Phys. Rev. D*, 98:026012, Jul 2018.

[16] A. Boozer. Classical Yang-Mills theory. *American Journal of Physics - AMER J PHYS*, 79:925–931, 09 2011.

[17] Suddhasattwa Brahma. Emergence of time in loop quantum gravity. In Nick Huggett, Keizo Matsubara, and ChristianEditors Wüthrich, editors, *Beyond Spacetime: The Foundations of Quantum Gravity*, page 53–78. Cambridge University Press, 2020.

[18] Ninoslav E. Bralić. Exact computation of loop averages in two-dimensional Yang-Mills theory. *Phys. Rev. D*, 22:3090–3103, Dec 1980.

[19] Harvey R Brown. *Physical Relativity*. Oxford University Press, Oxford, 2005.

[20] Harvey R Brown and Oliver Pooley. The origin of the space-time metric: Bell's 'Lorentzian pedagogy' and its significance in general relativity. In Craig Callender and Nick Huggett, editors, *Philosophy Meets Physics at the Planck Scale*. Cambridge University Press, Cambridge, 2001.

[21] Harvey R Brown and Oliver Pooley. Minkowski space-time: a glorious nonentity. *Philosophy and Foundations of Physics*, 1:67–89, 2006.

[22] Harvey R. Brown and James Read. The dynamical approach to spacetime theories. In Eleanor Knox and Alistair Wilson, editors, *The Routledge Companion to Philosophy of Physics*, pages 70–85. 2021.

[23] William L. Burke. *Applied Differential Geometry*. Cambridge University Press, 1985.

[24] Jeremy Butterfield and Henrique Gomes. Functionalism as a species of reduction, July 2020.

[25] Craig Callender. *What Makes Time Special?* Oxford University Press, Oxford, 2017.

[26] David J. Chalmers. *The Conscious Mind: In Search of a Fundamental Theory (2nd edition)*. Oxford University Press, 1996.

[27] David J. Chalmers. Finding space in a nonspatial world. In Christian Wüthrich, Baptiste Le Bihan, and Nick Huggett, editors, *Philosophy Beyond Spacetime*. Oxford University Press, 2021.

[28] Lu Chen and Tobias Fritz. An algebraic approach to physical fields. *Studies in History and Philosophy of Science Part A*, 89(C):188–201, 2021.

[29] C. P. Constantinidis, L. A. Ferreira, and G. Luchini. Direct test of the integral Yang-Mills equations through SU(2) monopoles. *Phys. Rev. D*, 96:105024, Nov 2017.

[30] Walter Craig and Steven Weinstein. On determinism and well-posedness in multiple time dimensions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465(2110):3023–3046, 2009.

[31] Erik Curiel. The constraints general relativity places on physicalist accounts of causality. *Theoria (Madrid, Spain)*, 15(1(37)):33–58, 2000.

[32] Erik Curiel. Kinematics, dynamics, and the structure of physical theory. 2016.

[33] Shamik Dasgupta. The bare necessities. *Philosophical Perspectives*, 25:115–160, 2011.

[34] Shamik Dasgupta. Symmetry as an epistemic notion. *British Journal for the Philosophy of Science*, 67(3):837–878, 2016.

[35] Sebastian de Haro. Noether's theorems and energy in general relativity. In James Read and Nicholas J. Teh, editors, *The Philosophy and Physics of Noether's Theorems: A Centenary Volume*, page 197–256. Cambridge University Press, 2022.

[36] Neil Dewar. General-relativistic covariance. *Foundations of Physics*, 50(4):294–318, 2020.

[37] Neil Dewar, Niels Linnemann, and James Read. The epistemology of spacetime. *Philosophy Compass*, 17(4), 2022.

[38] Maximilian Düll, Frederic P. Schuller, Nadine Stritzelberger, and Florian Wolz. Gravitational closure of matter field equations. *Phys. Rev. D*, 97:084036, Apr 2018.

[39] John Earman. Leibnizian space-times and Leibnizian algebras. In *Historical and Philosophical Dimensions of Logic, Methodology and Philosophy of Science*, pages 93–112. Springer, 1977.

[40] John Earman. *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time*. MIT Press, Cambridge, MA, 1989.

[41] John Earman and John Norton. What price spacetime substantivalism? the hole story. *British Journal for the Philosophy of Science*, pages 515–525, 1987.

[42] J. Ehlers, F.A.E. Pirani, and A. Schild. Republication of: The geometry of free fall and light propagation. *General Relativy and Gravity*, 44:1587–1609, 2012.

[43] Marián Fecko. Galilean and Carrollian invariant Hodge star operators, 2022. https://arxiv.org/abs/2206.09788.

[44] L. A. Ferreira and G. Luchini. Integral form of Yang-Mills equations and its gauge invariant conserved charges. *Phys. Rev. D*, 86:085039, 2012.

[45] L.A. Ferreira and G. Luchini. Gauge and integrable theories in loop spaces. *Nuclear Physics B*, 858(2):336–365, 2012.

[46] Paul M. Fishbane, Stephen Gasiorowicz, and Peter Kaus. Stokes' Theorems for Nonabelian Fields. *Phys. Rev. D*, 24:2324, 1981.

[47] Samuel C. Fletcher. Relativistic spacetime structure. In Eleanor Knox and Alastair Wilson, editors, *The Routledge Companion to the Philosophy of Physics*, pages 160–180. Routledge, 2021.

[48] Samuel C. Fletcher. *Foundations of general relativity*. Cambridge University Press, 2023.

[49] Michael Friedman. *Foundations of space-time theories: Relativistic physics and philosophy of science*. Princeton University Press, Princeton, 1983.

[50] Mathias Frisch. *Inconsistency, Asymmetry and Non Locality: A Philosophical Investigation of Classical Electrodynamics*. Oxford University Press, 2005.

[51] Robert Geroch. Faster than light?, 2010. https://arxiv.org/abs/1005.1614.

[52] Leonard Gross. A Poincaré lemma for connection forms. *Journal of Functional Analysis*, 63(1):1–46, 1985.

[53] J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28:2960–2975, Dec 1983.

[54] Friedrich W. Hehl, Yakov Itin, and Yuri N. Obukhov. On Kottler's path: Origin and evolution of the premetric program in gravity and in electrodynamics. *International Journal of Modern Physics D*, 25(11):1640016, 2016.

[55] Friedrich W Hehl and Yuri N Obukhov. How does the electromagnetic field couple to gravity, in particular to metric, nonmetricity, torsion, and curvature? In *Gyros, Clocks, Interferometers...: Testing Relativistic Graviy in Space*, pages 479–504. Springer, 2001.

[56] Friedrich W. Hehl and Yuri N. Obukhov. *Foundations of classical electrodynamics: charge, flux, and metric*. Birkhäuser, 2003.

[57] Friedrich W. Hehl and Yuri N. Obukhov. To consider the electromagnetic field as fundamental, and the metric only as a subsidiary field. *Foundations of Physics*, 35(12):2007–2025, 2005.

[58] Friedrich W. Hehl and Yuri N. Obukhov. Spacetime metric from local and linear electrodynamics: A new axiomatic scheme. In Jürgen Ehlers and Claus Lämmerzahl, editors, *Special Relativity: Will it Survive the Next 101 Years?*, pages 163–187. Springer, Berlin, 2006.

[59] Anne L. D. Hiskes. Space-time theories and symmetry groups. *Foundations of Physics*, 14(4):307, 1984.

[60] Nick Huggett. The regularity account of relational spacetime. *Mind*, 115(457):41–73, 2006.

[61] Nick Huggett and Christian Wüthrich. Emergent spacetime and empirical (in)coherence. *Studies in History and Philosophy of Modern Physics*, 44(3):276–285, 2013.

[62] Yakov Itin and Friedrich W. Hehl. Is the Lorentz signature of the metric of spacetime electromagnetic in origin? *Annals of Physics*, 312(1):60–83, 2004.

[63] Radosław Antoni Kycia and Josef Šilhan. Inverting covariant exterior derivative, 2022. https://arxiv.org/abs/2210.03663.

[64] Vincent Lam and Christian Wüthrich. Spacetime is as spacetime does. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, https://doi.org/10.1016/j.shpsb.2018.04.003, 2018.

[65] Vincent Lam and Christian Wüthrich. Spacetime is as spacetime does. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 64:39–51, 2018.

[66] Claus Lämmerzahl. Minimal coupling and the equivalence principle in quantum mechanics. *Acta Physica Polonica B*, 29(4):1057–1070, 1998.

[67] Dennis Lehmkuhl and Harvey Brown. Einstein, the reality of space, and the action-reaction principle. In Partha Ghose, editor, *Einstein, Tagore and the Nature of Reality*. Routledge, 2016.

[68] Dennis Lehmkuhl, Gregor Schiemann, and Erhard Scholz. *Towards a Theory of Spacetime Theories*. New York, NY: Birkhauser, 2016.

[69] David Lewis. How to define theoretical terms. *Journal of Philosophy*, 67(13):427–446, 1970.

[70] Niels Linnemann. On the empirical coherence and the spatiotemporal gap problem in quantum gravity: And why functionalism does not (have to) help. *Synthese*, 199(S2):395–412, 2020.

[71] Niels Linnemann and James Read. Comment on 'Do electromagnetic waves always propagate along null geodesics?'. *Classical and Quantum Gravity*, 38, 2021.

[72] Niels Linnemann and James Read. Constructive axiomatics in spacetime physics part I: Walkthrough to the Ehlers-Pirani-Schild axiomatisation. *arXiv:2112.14063*, 2021.

[73] Niels Linnemann and James Read. On the status of Newtonian gravitational radiation. *Foundations of Physics*, 51(2):1–16, 2021.

[74] Tim Maudlin. Buckets of water and waves of space: Why spacetime is probably a substance. *Philosophy of science*, pages 183–203, 1993.

[75] Tushar Menon. Algebraic fields and the dynamical approach to physical geometry. *Philosophy of Science*, 86(5):1273–1283, 2019.

[76] Flavio Mercati. *Shape Dynamics: Relativity and Relationalism*. Oxford University Press, 2018.

[77] Mikio Nakahara. *Geometry, topology and physics*. CRC Press, Boca Raton, FL, 2003.

[78] Jill North. *Physics, structure, and reality*. Oxford University Press, 2021.

[79] John D. Norton. General covariance and the foundations of general relativity: Eight decades of dispute. *Reports of Progress in Physics*, 56:791–861, 1993.

[80] John D. Norton. Why constructive relativity fails. *The British journal for the philosophy of science*, 59(4):821–834, 2008.

[81] Volker Perlick. On the hyperbolicity of Maxwell's equations with a local constitutive law. *Journal of Mathematical Physics*, 52(4):042903, 2011.

[82] J. Brian Pitts. Absolute objects and counterexamples: Jones–Geroch dust, Torretti constant curvature, tetrad-spinor, and scalar density. *Studies in History and Philosophy of Modern Physics*, 37:347–71, 2006.

[83] J Brian Pitts. Gauge-invariant localization of infinitely many gravitational energies from all possible auxiliary structures. *General Relativity and Gravitation*, 42(3):601–622, 2010.

[84] Oliver Pooley. Substantivalist and relationalist approaches to spacetime. In Robert Batterman, editor, *The Oxford Handbook of Philosophy of Physics*. Oxford University Press, 2013.

[85] Oliver Pooley. Background independence, diffeomorphism invariance, and the meaning of coordinates. In Dennis Lehmkuhl, editor, *Towards a Theory of Spacetime Theories*. Birkhäuser, 2017.

[86] Oliver Pooley. The hole argument. In *The Routledge Companion to Philosophy of Physics*, pages 145–159. Routledge, 2021.

[87] Willard Van Orman Quine. Ontology and ideology. *Philosophical Studies*, 2(1):11–15, 1951.

[88] James Read. *Background Independence in Classical and Quantum Gravity*. Oxford University Press, 2023.

[89] James Read, Harvey R Brown, and Dennis Lehmkuhl. Two miracles of general relativity. *Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics*, https://doi.org/10.1016/j.shpsb.2018.03.001, 2018.

[90] James Read and Bryan Cheng. Euclidean spacetime functionalism. *Synthese*, 200(6):1–22, 2022.

[91] Hans Reichenbach. *Axiomatization of the Theory of Relativity*. University of California Press, 1969.

[92] Thomas Ryckman. *The Reign of Relativity: Philosophy in Physics 1915?1925*. New York: Oxford University Press, 2005.

[93] Robert Rynasiewicz. Rings, holes and substantivalism: On the program of leibniz algebras. *Philosophy of Science*, pages 572–589, 1992.

[94] Robert Rynasiewicz. Weyl vs. Reichenbach on Lichtgeometrie. In A. J. Kox and Jean Eisenstaedt, editors, *The Universe of General Relativity*, pages 137–156, Boston, MA, 2005. Birkhäuser Boston.

[95] Frederic P. Schuller. Constructive gravity: Foundations and applications. In *The Fifteenth Marcel Grossmann Meeting*, pages 677–697.

[96] Frederic P. Schuller. All spacetimes beyond Einstein (Obergurgl lectures), 2011. https://arxiv.org/abs/1111.4824.

[97] David Skinner. Non-abelian gauge theory. 2022. University of Cambridge.

[98] Lawrence Sklar. *Space, time, and spacetime*. University of California Press, Berkeley and Los Angeles, CA, 1977.

[99] Syman Stevens. Regularity relationalism and the constructivist project. *British Journal for the Philosophy of Science*, 71(1):353–372, 2020.

[100] Theodore Voronov. On a non-abelian Poincaré lemma. *Proceedings of the American Mathematical Society*, 140, 05 2009.

[101] James Owen Weatherall. Part 1: Theoretical equivalence in physics. *Philosophy Compass*, 14(5):e12592, 2019.

[102] James Owen Weatherall. Part 2: Theoretical equivalence in physics. *Philosophy Compass*, 14(5):e12591, 2019.

[103] Steven Weinstein. Multiple time dimensions, 2008. https://arxiv.org/abs/0812.3869.