On the geometric trinity of gravity, non-relativistic limits, and Maxwell gravitation

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Abstract

We show that the common core of the recently-discovered non-relativistic geometric trinity of gravity is Maxwell gravitation. Moreover, we explain why no such dynamical common core exists in the case of the better-known relativistic geometric trinity of gravity.

Contents

1 Introduction

Inter alia, the following questions surely count as mainstream in contemporary philosophy of spacetime physics:

1. What is the 'correct' spacetime setting for Newtonian gravity, especially in light of Newton's Corollary VI?[1] (On this topic, see e.g. Dewar (2018), Knox (2014), Teh (2018), Wallace (2020), and Weatherall (2016, 2018).)

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[1]Recall that Newton's Corollary VI reads as follows: "If bodies are moving in any way whatsoever with respect to one another and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces." (Newton 2014, p. 99).

2. Are there spacetime theories which are in some sense or other 'equivalent' to general relativity, and what would be the philosophical significance of such theories, were they to exist? (On this topic, see e.g. Bain (2006), Duerr and Read (2023), Knox (2011), Rosenstock et al. (2015), Wolf and Read (2023a), and Wolf, Sanchioni, et al. (2023).)

3. How is one to take the non-relativistic limit of general relativity, and what is the resulting theory? (On this topic, see e.g. Fletcher (2019) and Malament (1986).)

Until now, discussions of these questions have, broadly speaking, been quarantined from one another. Our purpose in this article is to show that these questions (and answers to said questions) are in fact related to one another in intimate and significant ways.

To explain what we mean here, begin with question (1). (For the time being a qualitative account will suffice; the mathematics to substantiate the claims made here will follow later in this article.) Typically, Newtonian gravitation theory (NGT) in its potential-based formulation *chez* Laplace and Poisson is taken to be set in a flat spacetime; gravitational effects in this spacetime are encoded in the gravitation potential which leads to test bodies not traversing geodesics of the flat, compatible connection. This being said, NGT has a hidden symmetry (sometimes referred to as 'Trautman symmetry' (Teh 2018)): if one (a) subjects *all* material bodies to an additional constant gravitational field, and (b) changes one's standard of straightness (i.e. one's derivative operator) to compensate for this, then in fact no physical change ensues. (This is related to Newton's Corollary VI, as we will explain below; cf. Read and Teh (2022).) When one moves to a new formalism purged of this additional symmetry,[2] one arrives at the structure of Newton-Cartan theory (NCT): a non-relativistic spacetime theory in which gravitational effects are, just as in the case of general relativity (GR), manifestations of spacetime curvature.

This much is well-known. But there remains some ambiguity in the literature as to how NGT relates to another spacetime theory known as 'Maxwell gravitation' (MG), also developed in light of Newton's Corollary VI. (On this, see Chen (2023), Dewar (2018), March (2023), and Saunders (2013)) Moreover, recently NGT has been shown to in fact admit of an interpretation whereby it is a theory with a *torsionful* geometry, in the sense that the gravitational potential can be associated with the torsion of the 'mass gauge field' which arises when one gauges the Bargmann algebra (for more on the mass gauge field, see Andringa et al. (2011), Read and Teh (2018), Teh (2018), and Wolf, Read, and Teh (2023)). Even less well-known (indeed, we might say, almost unknown!) is that both NGT and NCT are equivalent to an alternative non-relativistic theory, recently dubbed 'symmetric Newtonian gravitation theory' (SNGT), in which gravitational effects are manifestations neither of curvature (as in NCT)

[2] Here, we set aside the differences between what are known as 'reduction' and 'internal sophistication' about symmetires: see Dewar (2018) and Martens and Read (2021).

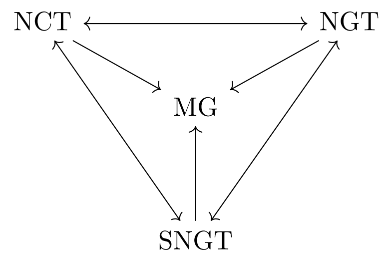

Figure 1: Maxwell gravitation as the common core of the non-relativistic geometric trinity of gravity.

nor of torsion (as in NGT), but of spacetime non-metricity.[3] Here, we demonstrate that these pieces fit together in the following way: NGT, NCT and SNGT constitute a 'non-relativistic geometry trinity'; the structure common to all said theories (the 'common core', in the sense of Le Bihan and Read (2018)) *just is* the structure of MG (see Figure 1).[4]

Already, this illuminates quite substantially the connections between these four non-relativistic theories of spacetime and gravity. And yet, that is only the beginning of the story. Taking now together questions (2) and (3) in our above list, it is becoming increasingly well-known to philosophers of physics that there exists a 'geometric trinity' of relativistic theories of gravitation, of which GR constitutes but one node (see e.g. Jiménez et al. (2019) for a recent review in the physics literature). The other two nodes are 'teleparallel gravity' (TEGR), in which gravitational effects are a manifestation of exclusively spacetime torsion, and 'symmetric telelparallel gravity' (STGR), in which gravitational effects are a manifestation of exclusively spacetime non-metricity. In Wolf and Read (2023b), it was shown that the above-discussed non-relativistic trinity (*sans* any mention of MG) is indeed the non-relativistic limit (in the sense of a $1/c$ expansion *à la* Schwartz (2023)) of this relativistic geometry trinity. The web of connections is, therefore, as presented in Figure 2 (in that figure, for clarity, we omit MG).

What we add to this discussion in the present paper is an answer to the following question: does there exist a 'common core' of the relativistic geometric trinity in the same sense that MG is the common core of the non-relativistic trinity, and if so is it the case that MG is the non-relativistic limit of said relativistic common core? In fact, we do not need to answer the second part of this question, for we will answer the first part in the negative: there is *no* suitable common core of the relativistic geometric trinity to begin with.

We should be clear about what 'suitable' means here. Although it is true

[3]To remind the reader: 'curvature' quantifies the extent to which parallel transport of a vector along a closed loop doesn't preserved angles; 'torsion' quantifies the extent to which parallel transport in two directions doesn't commute; 'non-metricity' quantifies the extent to which parallel transport of a vector along a closed loop doesn't preserve the length of that vector. For further background, see e.g. Hehl et al. (1995).

[4]The same notion of a common core is also discussed in e.g. De Haro and Butterfield (2021).

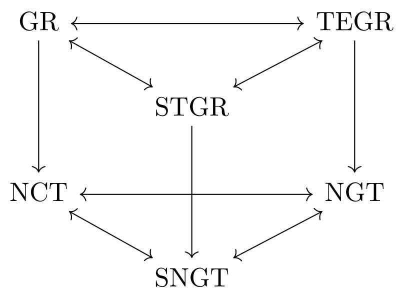

Figure 2: The relativistic geometric trinity and the non-relativistic geometric trinity as its non-relativistic limit.

that there is some common core to the *kinematical* structure of the relativistic geometric trinity—indeed, in Wolf, Sanchioni, et al. (2023) this was argued to be the conformal structure common to all three of GR, TEGR, and STGR—said kinematical structure is insufficient to build up dynamics such that the resulting theory is equivalent all three original nodes of the trinity. By contrast, in the non-relativistic case there is again a kinematical common core—this time, it consists in what has been dubbed in the recent philosophical literature a 'standard of rotation' (see Weatherall (2018))—; however—and quite differently to the relativistic case!—that common core *is* sufficient to construct dynamics (naturally, the dynamics of MG) which *are* equivalent to the dynamics of each of the nodes of the non-relativistic geometric trinity. Later in this article, we will explain exactly how this comes to be the case: it turns out that the crucial ingredients are, as it were, 'injected' on taking the non-relativistic limit of the relativistic geometric trinity.

To summarise, then, in this article we (a) identify MG as the dynamical common core of the recently-discovered non-relativistic geometric trinity of gravity, and (b) explain how it can be that no analogous common core exists in the case of the relativistic geometric trinity of gravity. In so doing, we (i) clarify questions in (1) regarding the 'correct' spacetime setting for Newtonian gravity, (ii) connect that entire literature up to the geometric trinity of gravity and its Newtonian limit, which has also aroused recent philosophical interest. More specifically, the structure of the article is this. In §2, we remind the reader of the mathematical details of both the relativistic geometric trinity and the non-relativistic geometric trinity. In §3, we present MG as the common core of the non-relativistic trinity, and connect our discussion to that of the 'correct' spacetime setting for Newtonian gravity. In §4, we address the matter of the existence (or otherwise) of a relativistic common core. We close in §5.

2 Geometric trinities

In this section, we recall the mathematical details underlying the existence of the relativistic geometric trinity of gravitation theories (§2.1) and the non-relativistic geometric trinity of gravitation theories (§2.2).

2.1 The relativistic geometric trinity

The 'geometric trinity' of gravity refers to a family of three relativistic theories of gravitation: general relativity (GR), the 'teleparallel equivalent to general relativity' (TEGR), and the 'symmetric teleparallel equivalent to general relativity' (STGR). These theories are all 'equivalent' to each other in the sense that they share equivalent dynamical equations of motion, but distinct in the sense that these shared dynamics result from entirely different geometric degrees of freedom that manifest in each respective theory (see e.g. Capozziello et al. (2022) and Jiménez et al. (2019)).

Kinematical possibilities of general relativity are typically presented as tuples of the form $\langle M, g_{ab}, \Phi\rangle$, where M is a four-dimensional differentiable manifold, g_{ab} is a Lorentzian metric field on M, and Φ represents material fields. The dynamical possibilities of the theory are encoded by the Einstein field equations, which govern the behavior of these spacetime and material fields. However, the geometric degrees of freedom responsible for sourcing the dynamics of the respective theories in the geometric trinity are properties of the affine connection. We will thus take GR to be a theory given by models of the form $\langle M, g_{ab}, \overset{c}{\nabla}, \Phi\rangle$, where $\overset{c}{\nabla}$ refers to the familiar Levi-Civita derivative operator with non-vanishing *curvature*. Typically, $\overset{c}{\nabla}$ is not included explicitly in the models of GR, for it is fixed uniquely by g_{ab}.

Spacetime curvature is defined in the following way:

$$R^{a}{}_{bcd}\xi^{b} := -2\nabla_{[c}\nabla_{d]}\xi^{a}, \tag{1}$$

where ξ^a is a smooth vector field. However, curvature is not the only geometric property that a connection can manifest. An affine connection can also possess *torsion* or *non-metricity*. The torsion tensor is given by

$$T^{c}{}_{ab}\nabla_{c}\alpha := 2\nabla_{[a}\nabla_{b]}\alpha, \tag{2}$$

where α is a smooth scalar field; torsion thereby encodes the antisymmetry of a connection. Non-metricity is given by the non-vanishing of the covariant derivative of the metric tensor

$$Q_{abc} := \nabla_{a}g_{bc}. \tag{3}$$

Heuristically, curvature measures the rotation of a vector when it is parallel transported along a closed curve, torsion measures of the non-closure of the parallelogram formed by two vectors being parallel transported along each other, and non-metricity measures how the length of a vector changes when parallel transported (see e.g. Figure 1 in Jiménez et al. (2019) or Hehl et al. 1995).

The Levi-Civita connection of GR is unique in the sense that it is the unique derivative operator which is both torsion-free (i.e. $T^a{}_{bc} = 0$) and metric-compatible (i.e. $Q_{abc} = 0$), but with generically non-vanishing curvature (i.e. $R^a{}_{bcd} \neq 0$). However, in order to build a viable relativistic spacetime theory, it is not necessary to use $\overset{c}{\nabla}$. Indeed, one can decompose a general affine connection as

$$\nabla = (\overset{c}{\nabla}, K^a{}_{bc} + L^a{}_{bc}), \tag{4}$$

where $K^a{}_{bc}$ is known as the 'contorsion tensor' and $L^a{}_{bc}$ is known as the 'distortion tensor' (here, we use the notation of Malament (2012, p. 53)). The contorsion tensor can be understood as the difference tensor between the Levi-Civita connection and the torsionful (but flat and metric-compatible) connection of TEGR. The disortion tensor can be understood as the difference tensor between the Levi-Civita connection and non-metric (but flat and torsionless) connection connection of STGR.

If—as above—we take GR to be a theory with kinematical possibilities of the form $\langle M, g, \overset{c}{\nabla}, \Phi\rangle$, then TEGR can be taken to be a theory with kinematical possibilities given by $\langle M, g, \overset{t}{\nabla}, \Phi\rangle$, where $\overset{t}{\nabla} = (\overset{c}{\nabla}, K^a{}_{bc})$ refers to the TEGR connection with non-vanishing torsion. Likewise, STGR is a theory with kinematical possibilities given by $\langle M, g, \overset{n}{\nabla}, \Phi\rangle$, where $\overset{n}{\nabla} = (\overset{c}{\nabla}, L^a{}_{bc})$ refers to the STGR connection with non-vanishing non-metricity.

One can use (4) as a dictionary by which to translate between these theories. That is, one can rewrite the geometric objects of interest in one theory in terms of the geometric objects of one of the other trinity theories, and thereby witness their equivalence. For example, one can take the curvature scalar R of the Levi-Civita connection, and use (4) to express it in terms of the TEGR connection and the contorsion tensor, or in terms of the STGR connection and the distorsion tensor. One finds that

$$-R = T + B_T = Q + B_Q, \tag{5}$$

where T is the torsion scalar, Q is the non-metricity scalar, and $B_{T/Q}$ refers to boundary terms of the respective theories. This also illustrates that these theories are dynamically equivalent, as the Lagrangian expressions for all of these theories can be written using the geometric scalars (in the case of GR, recall the Einstein-Hilbert action). Upon utilizing standard variational procedures, the boundary terms that arise in (5) vanish, resulting in the standard Einstein field equations for all theories (but of course expressed in their particular geometric languages).[5]

[5]For more on the significance of these boundary terms, see Wolf and Read (2023a) for philosophical discussion on concerning their implications for theory equivalence and theory structure and see Oshita and Wu (2017) for further physics discussion.

2.2 The non-relativistic geometric trinity

It was shown recently by Wolf and Read (2023b) that there is a non-relativistic analogue of the geometric trinity, whereby standard Newtonian gravity can likewise be reconceptualised and/or reformulated as a theory of curvature, torsion, or non-metricity. The three nodes of the non-relativistic geometric trinity of gravity are 'Newton-Cartan theory' (NCT), 'Newtonian gravitation theory' (NGT), and 'symmetric Newtonian gravitation theory' (SNGT).

Following the presentation of the relativistic theories above, we take NCT to be a theory with kinematical possibilities of the form $\langle M, t_a, h^{ab}, \overset{c}{\nabla}, \Phi\rangle$. As before, M is a four-dimensional differentiable manifold, Φ represents material fields, and $\overset{c}{\nabla}$ is a torsion-free and compatible (now in the sense that $\nabla_a t_b = \nabla_a h^{bc} = 0$) derivative operator with non-vanishing curvature. However, there are some important differences:

1. The metrical structure of non-relativistic theories is notably different from that of relativistic theories, because t_a and h^{ab} refer to degenerate temporal and spatial metrics on M: see Malament (2012, ch. 4).[6] Metric compatibility applies separately to both metrics; in addition, t_a and h^{ab} are orthogonal to each other, so that $t_a h^{ab} = 0$. Loosely speaking, t_a is supposed to represent Newtonian absolute time, and h^{bc} is supposed to represent Newtonian absolute space.

2. The dynamical possibilities for NCT are encoded in the 'geometrised Poisson equation':
$$R_{ab} = 4\pi\rho t_a t_b, \tag{6}$$
where R_{ab} is the Ricci curvature of the NCT connection $\overset{c}{\nabla}$; moreover, one typically includes the following curvature conditions in one's presentation of NCT (we will discuss these curvature conditions more later in the article[7]):
$$R^{a\ \ c}_{\ b\ \ d} = R^{c\ \ a}_{\ d\ \ b}, \tag{7}$$
$$R^{ab}_{\ \ cd} = 0. \tag{8}$$

NGT is a theory given by models of the form $\langle M, t_a, h^{ab}, \overset{t}{\nabla}, \Phi\rangle$, where $\overset{t}{\nabla}$ again is a flat, metric-compatible derivative operator with (in a certain quite non-obvious sense which we will explain!) non-vanishing torsion. One can translate between NCT and NGT by introducing the difference tensor $\overset{t}{\nabla} = (\overset{c}{\nabla}, t_b t_c \overset{t}{\nabla}{}^a \phi)$, where ϕ is a scalar field representing the gravitational potential. This allows one to cast the dynamics of this theory in its familiar form

[6] Through this article, we assume temporal orientability, in the sense of Malament (2012, ch. 4).

[7] See also Malament (2012) and Teh (2018) for further discussion of the physical significance and meaning of these conditions.

given by the standard Poisson equation:

$$\overset{t}{\nabla}_a \overset{t}{\nabla}^a \phi = 4\pi\rho. \tag{9}$$

Although the translation between NCT and NGT (in the form of the the famous 'geometrisation' and 'recovery' theorems—(see also Malament (2012, ch. 4)) has been known since the work of Trautman (1965), it was much more recently shown by Read and Teh (2018) that standard Newtonian gravity can be understood as the 'teleparallel equivalent' of NCT in much the same way that TEGR is the teleparallel equivalent of GR. Here, the gravitational field follows from the torsion of the 'mass gauge field' m_a, which is obtained by gauging the Bargmann algebra: one has $T(M) := dm_a$; gauge fixing $m_a = \phi t_a$ yields (9).

Even more recently, SNGT has been constructed by Wolf and Read (2023b). This theory is given by models of the form $\langle M, t_a, h^{ab}, \overset{n}{\nabla}, \Phi\rangle$, where $\overset{n}{\nabla}$ is flat and torsion-free but possesses non-vanishing non-metricity, so that in particular

$$\begin{aligned} \overset{n}{\nabla}_a t_b &= \sigma_a t_b, \\ \overset{n}{\nabla}_a h^{bc} &= \sigma_a h^{bc}, \end{aligned} \tag{10}$$

where $\sigma_a = \alpha t_a$ is an exact, spatially constant one-form such that that encodes the non-metricity of the theory. Similarly, one can translate between NCT and SNGT by introducing the difference tensor $\overset{n}{\nabla} = (\overset{c}{\nabla}, \sigma_{(b}\delta^a{}_{c)})$ and obtain an equivalent equation to (6) and (9) cast now in terms of non-metricity degrees of freedom:

$$-\frac{3}{2}\overset{n}{\nabla}_b \sigma_c + \frac{3}{4}\sigma_b\sigma_c = 4\pi\rho t_b t_c. \tag{11}$$

This 'non-relativistic geometric trinity' mirrors the more familiar relativistic geometric trinity in that one can present three gravitational theories that are dynamically equivalent to familiar Newtonian gravitational theory, formulated in the geometric languages of curvature, torsion, and non-metricity. While in the relativistic case this is apparent at the level of the action, NCT, NGT, and SNGT by contrast cannot be formulated using an action principle (for the reasons underlying this, see Hansen et al. (2019)), so we can only demonstrate their equivalence via Trautman-style geometrisation and recovery theorems.[8] However, the non-relativistic trinity bears another important relationship to the relativistic trinity. All of the theories in the non-relativistic trinity can be obtained by taking an appropriate non-relativistic limit (typically in terms of a $1/c$ expansion in the style of Schwartz (2023)) of their corresponding curvature, torsion, or non-metricity based relativistic analogues: see Wolf and Read (2023b). This completes all the legs of Figure 2.

[8] Off-shell non-relativistic equivalence would require recourse to the 'Type II' versions of these theories, which we discuss in §5.

3 Maxwell gravitation and the non-relativistic trinity

As emphasised in the previous section, the three nodes of each of the relativistic and non-relativistic geometric trinities are all empirically equivalent theories, which nevertheless appear to disagree fundamentally on the geometrical structure which they attribute to the world. For instance, according to GR the gravitational behaviour of matter is to be understood as a manifestation of spacetime curvature, whereas according to TEGR and STGR spacetime is necessarily flat. This means that the relativistic and non-relativistic theories present a case of strong underdetermination—distinct theories between which no possible evidence could be expected to decide.

Faced with such cases of strong underdetermination, philosophers have suggested several approaches to dealing with the problem (on this see e.g. Le Bihan and Read (2018)). Famously, one of these is the common core approach.[9] The common core approach advocates identifying the invariant kinematical structure of the theories, and then showing that this structure is sufficient to formulate a distinct, ontologically viable theory in its own right; one which, moreover, retains the empirical content of the original theories. Moving to this new interpretative framework alongside a judicious invocation of Occamist norms (on which see Dasgupta (2016)) then allows one to 'break' the underdetermination by interpreting the theories in such a way that they completely agree on the structure they attribute to the world. The aim of this section is to show that in the case of the non-relativistic geometric trinity, such a common core theory exists, and it is a theory known as 'Maxwell gravitation': a theory which has quite independently attracted philosophical interest (for reasons to do with (1) as presented in the introduction).

To do so, we begin by recalling some facts about Maxwellian spacetime. This is a structure $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$, where t_a, h^{ab} are orthogonal temporal and spatial metrics as introduced in the previous section, and $\circlearrowleft$ is a standard of rotation compatible with t_a and h^{ab}. This was introduced originally by Weatherall (2018): if t_a, h^{ab} are compatible temporal and spatial metrics on M, then a standard of rotation $\circlearrowleft$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, antisymmetric rank-$(2,0)$ tensor fields $\circlearrowleft^b \xi^a$ on M, such that

1. $\circlearrowleft$ commutes with addition of smooth vector fields;
2. Given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowleft^a (\alpha\xi^b) = \alpha \circlearrowleft^a \xi^b + \xi^{[b} d^{a]}\alpha$;
3. $\circlearrowleft$ commutes with index substitution;
4. Given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = 0$ then $\circlearrowleft^a \xi^b$ is spacelike in both indices; and

[9]Alternatively, Duerr and Read (2023) suggest that the relativistic geometric trinity invites a certain conventionalism about geometry; we'll return to this in §5.

5. Given any smooth spacelike vector field σ^a, $\circlearrowleft^a \sigma^b = D^{[a}\sigma^{b]}$,

where D is the unique Levi-Civita connection induced by h^{ab} on each spacelike hypersurface. We will say that a connection and a standard of rotation are *compatible* iff $\circlearrowleft^a \eta^b = \nabla^{[a}\eta^{b]}$ for all vector fields η^a on M. It follows that any compatible, torsion-free connection ∇ on M determines a unique compatible standard of rotation—namely, the map $\circlearrowleft: \eta^a \to \nabla^{[a}\eta^{b]}$ (Weatherall 2018).

However, in light of the discussion of the previous section, this invites a natural further question: are there non-metric or torsionful connections which are *also* associated with metric compatible standards of rotation in the above sense? It turns out that the answer to this question is 'yes', and a partial characterisation of such connections is given by the following two propositions:[10]

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla\rangle$ be a non-relativistic spacetime, where t_a and h^{ab} are compatible, and where $\nabla_a t_b = \sigma_a t_b$ and $\nabla_a h^{bc} = \sigma_a h^{bc}$. Then the map $\circlearrowleft: \eta^a \to \nabla^{[a}\eta^{b]}$ is a standard of rotation compatible with t_a and h^{ab} iff $h^{an}\sigma_n = 0$.*

Proof. First, suppose that $h^{an}\sigma_n = 0$. That $\circlearrowleft$ satisfies conditions (1)-(3) is immediate from properties of derivative operators. For condition (4), note that if $d_a(\eta^n t_n) = 0$ we have

$$0 = \nabla_a(\eta^n t_n) = t_n \nabla_a \eta^n + \sigma_a t_n \eta^n, \tag{12}$$

so that

$$\begin{aligned} t_n(h^{m[n}\nabla_m \eta^{a]}) &= -\frac{1}{2}h^{ma}t_n\nabla_m\eta^n \\ &= \frac{1}{2}h^{ma}\sigma_m t_n \eta^n \\ &= 0 \\ &= t_n(h^{m[a}\nabla_m\eta^{n]}) \end{aligned}$$

and $\circlearrowleft^a \eta^b$ is spacelike in both indices. Finally, consider condition (5). We know that $\nabla^a h^{bc} = h^{an}\sigma_n h^{bc} = 0$. So let ξ^a be a unit timelike vector field on M, $\hat{h}_{ab}$ the spatial metric relative to ξ^a,[11] and D the unique spatial derivative operator such that $D_a h^{bc} = 0$. Then for any spacelike vector field σ^a on M, $D_a\sigma^b = \hat{h}_{an}\hat{h}^b{}_m \nabla^n \sigma^m$ (Weatherall 2018) so that

$$h^{n[a}D_n\sigma^{b]} = h^{n[a}\hat{h}_{nm}\hat{h}^{b]}{}_r\nabla^m\sigma^r = h^{n[a}\nabla_n\sigma^{b]},$$

where we have used the fact that since $h^{an}\sigma_n = 0$, $\nabla^a(t_n\eta^n) = t_n\nabla^a\eta^n$ for any smooth vector field η^a on M.

Conversely, suppose that the map $\circlearrowleft: \eta^a \to \nabla^{[a}\eta^{b]}$ is compatible with t_a and h^{ab}.

[10] Partial, because Proposition 1 considers only the case of Weylian non-metricity, where $Q_{abc} = \sigma_a g_{bc}$. Given Proposition 1 though, the generalisation is obvious.

[11] That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = 0$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b\xi^a$.

Then we must have for all smooth vector fields η^a on M that if $d_a(\eta^n t_n) = 0$, $\circlearrowright^a \eta^b$ is spacelike in both indices. If $d_a(\eta^n t_n) = 0$ then (12) holds with respect to η^a so that $t_n(h^{m[n}\nabla_m \eta^{a]}) = 1/2 h^{ma}\sigma_m t_n \eta^n = 0$. But this can only be the case for arbitrary η^a if $h^{an}\sigma_n = 0$. □

Proposition 2. *Let $\langle M, t_a, h^{ab}, \nabla\rangle$ be a non-relativistic spacetime, where t_a and h^{ab} are compatible, and ∇ is compatible with the metrics but possibly torsionful. Then the map $\circlearrowright\colon \eta^a \to \nabla^{[a}\eta^{b]}$ is a standard of rotation compatible with t_a and h^{ab} iff $T^{abc} = 0$.*

Proof. First, note that since ∇ is compatible with t_a and t_a is closed, $t_n T^n{}_{ab} = 0$. That the map $\circlearrowright\colon \eta^a \to \nabla^{[a}\eta^{b]}$ satisfies conditions (1)-(3) is again immediate from properties of derivative operators. (4) follows from the fact that ∇ is compatible with t_a, using that $d_a\alpha = \nabla_a\alpha$ for any 0-form field α. Finally, consider (5). Let ξ^a be a unit timelike vector field on M, and $\hat{h}_{ab}$ the spatial metric relative to ξ^a. We know that the action of D on spacelike vector fields is defined as follows: $D_a\sigma^b = \hat{h}_{an}\hat{h}^b{}_m \nabla'^n\sigma^m$, where ∇' is an arbitrary torsion-free derivative operator such that $\nabla'^a h^{bc} = 0$. Moreover, we know that $\nabla = (\nabla', K^a{}_{bc})$ for some such ∇', where $K^a{}_{bc} = 1/2 T^a{}_{bc} + T_{(b}{}^a{}_{c)} + h^{an} t_{(b} f_{c)n}$ is the Newton-Cartan contorsion (Schwartz 2023). Hence

$$\begin{aligned} D^{[a}\sigma^{b]} &= \hat{h}^{[a}{}_n \hat{h}^{b]}{}_m \nabla'^n \sigma^m \\ &= \hat{h}^{[a}{}_n \hat{h}^{b]}{}_m (\nabla^n\sigma^m - h^{nr} K^m{}_{rs}\sigma^s) \\ &= \nabla^{[a}\sigma^{b]} - K^{[ab]}{}_n \sigma^n \\ &= \nabla^{[a}\sigma^{b]} - \frac{1}{2} T^{nab}\sigma_n \end{aligned}$$

for some covector σ_n, where we have used the fact that σ^a is spacelike and $t_n T^n{}_{ab} = 0$. Thus if $T^{abc} = 0$, then (5) is satisfied. Conversely, if the map $\circlearrowright\colon \eta^a \to \nabla^{[a}\eta^{b]}$ satisfies (5), it follows that $T^{abc} = 0$. □

Propositions 1 and 2 are tantalising, because they show that non-relativistic affine connections which exhibit either torsion or non-metricity may—under certain conditions—be associated with a compatible standard of rotation, just as with curvature based connections. This raises the prospect that Maxwellian spacetime might be the invariant kinematic structure of the non-relativistic geometric trinity. We isolate the sense in which this is so in the following proposition:

Proposition 3. *Consider the triple $\langle M, t_a, h^{ab}\rangle$ with t_a and h^{ab} defined as above. Consider three connections: a curvature based connection $\overset{c}{\nabla}$, a non-metricity based connection $\overset{n}{\nabla}$ with $\overset{n}{\nabla}_a t_b = \sigma_a t_b$ and $\overset{n}{\nabla}_a h^{bc} = \sigma_a h^{bc}$, and a torsion based connection $\overset{t}{\nabla}$. Let $\overset{c}{\nabla} = (\overset{n}{\nabla}, L^a{}_{bc}) = (\overset{t}{\nabla}, K^a{}_{bc})$, where $L^a{}_{bc} = -\sigma_{(n}\delta^a{}_{c)}$ and $K^a{}_{bc}$ is as in Proposition 2. Then $\overset{c}{\circlearrowright}$, $\overset{n}{\circlearrowright}$, $\overset{t}{\circlearrowright}$ are standards of rotation compatible with the metrics and $\overset{n}{\circlearrowright} = \overset{c}{\circlearrowright} = \overset{t}{\circlearrowright}$ iff $h^{an}\sigma_n = 0$, $T^{abc} = 0$, and $f_{ab} = t_{[a}\phi_{b]}$ for some ϕ_a.*

Proof. That $\overset{c}{\circlearrowleft}$ is a standard of rotation compatible with the metrics follows immediately from Proposition 1 of Weatherall (2018). The remainder of the proof just involves some calculation. Let η^a be an arbitrary vector field on M. We know (Proposition 1) that $\overset{n}{\circlearrowleft}$ is compatible with the metrics iff $h^{an}\sigma_n = 0$, so for the first half of the equality we just need to verify that if $h^{an}\sigma_n = 0$,

$$\begin{aligned}\overset{c}{\nabla}{}^{[a}\eta^{b]} - \overset{n}{\nabla}{}^{[a}\eta^{b]} &= -L^{[ab]}{}_n\eta^n \\ &= h^{m[a}\sigma_{(m}\delta^{b]}{}_{n)}\eta^n \\ &= \frac{1}{2}\sigma_m h^{m[a}\eta^{b]} \\ &= 0.\end{aligned}$$

For the second part of the equality, let ξ^a be a unit timelike vector field, and $\hat{h}_{ab}$ the spatial metric relative to ξ^a. Given Proposition 2, we need to show that if $T^{abc} = 0$, then $\overset{n}{\circlearrowleft} = \overset{t}{\circlearrowleft}$ iff $f_{ab} = t_{[a}\phi_{b]}$ for some ϕ_a. We have

$$\begin{aligned}\overset{c}{\nabla}{}^{[a}\eta^{b]} - \overset{t}{\nabla}{}^{[a}\eta^{b]} &= -K^{[ab]}{}_n\eta^n \\ &= -\frac{1}{2}(T_n{}^{ab} + t_n f^{[ab]})\eta^n \\ &= -\frac{1}{2}(\hat{h}_{nm}T^{mab} + t_n f^{ab})\eta^n \\ &= -\frac{1}{2}t_n f^{ab}\eta^n,\end{aligned}$$

where we have used that f_{ab} is antisymmetric. Since t_a is nowhere vanishing and η^a is arbitrary, this last term will vanish just in case $f^{ab} = 0$. But this is equivalent to the requirement that $f_{ab} = t_{[a}\phi_{b]}$ for some covector ϕ_a. □

The significance of Proposition 3 stems from two facts. First, $\sigma_a = \alpha t_a = d_a\lambda$ in any model of SNGT. Meanwhile, the homogeneous Trautman conditions (7) and (8) ensure that given any model of NCT, we can always guarantee that $T^{abc} = 0$, $f_{ab} = t_{[a}\phi_{b]}$ in the associated models of our teleparallel theory through a judicious 'gauge fixing' of the torsion and frame (see Read and Teh (2018) and Schwartz (2023) for details). The result is that each model of SNGT, and its corresponding model of NCT, and associated family of Trautman recoveries all pick out a unique Maxwellian spacetime.

Not only this, but the condition (8) guarantees in addition that this Maxwellian spacetime is unique, up to isomorphism, across the entire space of models.[12] So Proposition 3 in fact gives us the stronger result: SNGT, NCT, and NGT can all be understood as theories built on the very same Maxwellian spacetime background.

[12] This follows since (8) holds with respect to $\overset{c}{\nabla}$ iff $\overset{c}{\nabla}$ is rotationally equivalent to some flat compatible torsion-free connection (Malament 2012, proposition 4.2.4; Weatherall 2018, proposition 1), and flat compatible torsion-free connections are unique up to isomorphism.

This agreement is striking. It is also to be expected. Stepping back for a minute, all three nodes of the non-relativistic geometric trinity are supposed to be empirically equivalent to standard potential-based Newtonian gravity (i.e. NGT *sans* a torsionful interpretation). *That* theory is one in which spacetime is flat, metric, and (spatiotemporally) torsion-free. Moreover, the dynamics of Newtonian gravitation pick out a privileged class of frames which are non-rotating with respect to that connection.[13] Any theory which is empirically equivalent to Newtonian gravitation must be able to distinguish this same class of frames, which requires agreement on the Maxwellian spacetime structure.

We can see this in more detail by considering the conditions which ensure agreement of the rotation standard. As remarked before, in the case of NCT and NGT, these amount to the homogeneous Trautman conditions. It is well known that only one of these conditions (7) emerges in the non-relativistic limit of GR. The second Trautman condition (8) must be put in by hand as an additional assumption once the non-relativistic limit has been taken.[14] In light of this, a well-known result by Künzle and Ehlers proceeds to generalise the Trautman recovery theorem by dropping the condition (8) (see Malament (2012, ch. 4) for discussion). The resulting theory, however, is *not* empirically equivalent to standard potential-based Newtonian gravity, but leads to ineliminable coriolis force terms in the recovered models.

A similar story plays out, *mutatis mutandis*, for SNGT and NCT. There, the relevant condition which ensures agreement of the rotation standard is that $h^{an}\sigma_n = 0$. Moreover, since models of NCT satisfy the homogeneous Trautman conditions, we also require that σ_a is closed (on which, see Wolf and Read (2023b)). The first of these conditions emerges naturally in the non-relativistic limit of STGR (Wolf and Read 2023b), but the second, which ensures that the metrics can be conformally rescaled to be compatible with $\overset{n}{\nabla}$, does not. Without the assumption that σ_a is closed, however, the recovered models of NCT (and by extension, NGT) will not be empirically equivalent to standard potential-based Newtonian gravity, via the same reasoning applied above.

To summarise the results of this section so far, we've isolated a Maxwellian spacetime structure as the common core of the non-relativistic geometric trinity. We've also seen that the constraints which guarantee agreement on the Maxwellian spacetime structure are precisely the constraints which are standardly imposed after taking the non-relativistic limit to ensure empirical equivalence with the potential-based formulation of Newtonian gravity. The next point to note is that this structure is sufficient to formulate the dynamics of Newtonian gravity. This was first done in Dewar (2018), and recently given an 'intrinsic' formulation by Chen (2023) and March (2023); the resulting theory—'Maxwell gravitation'—has models of the form $\langle M, t_a, h^{ab}, \circlearrowleft, \Phi\rangle$. Together with propositions 1, 2, and 3, this substantiates our earlier claim that MG constitutes

[13] Here we set aside substantivalist-relationalist questions of whether this is a correct prediction of Newtonian theory, on which see e.g. Barbour (2010) and Pooley (2013).

[14] Although one can recover this condition by restricting attention to spacetimes which are, in a certain weak sense, asymptotically flat when taking the non-relativistic limit—on which, see e.g. Malament (2012, ch. 4.5).

the dynamical common core of the non-relativistic geometric trinity.[15] It also paves the way for an interpretation of Newtonian gravity on which the structure it attributes to the world is strictly *less* than that of a connection.

This takes us to the connection with Corollary VI and the Trautman symmetry, to which we alluded in Section 1. As articulated by Jacobs (2023), the 'dynamic shift' symmetry of potential-based Newtonian gravity *à la* Corollary VI and the Trautman symmetry in which the connection and gravitational potential are altered simultaneously can be understood as being two sides of the same coin: both consequences of the invariance of the Newtonian dynamics under the Maxwell group. But Maxwell transformations produce a linear, time-dependent acceleration of the matter content of the original solution. *Prima facie*, one might think that purging the theory of this symmetry would involve excising the structure needed to make sense of such linear accelerations—to wit, a connection—leaving only the standard of rotation.

In that sense, that MG should be the dynamical common core of the non-relativistic geometric trinity was already suggested by the dynamical symmetries of Newtonian gravity. On the one hand, the irrotational degrees of freedom of the connection were already known to be superfluous to the internal dynamics of the matter distribution. On the other hand, agreement on the rotation standard is necessary for empirical equivalence to standard Newtonian gravity.

What, then, to make of the fact that one can also use Trautman symmetry to motivate the move to NCT? One way to think about this is that while we are always free to *define* a connection from the standard of rotation and matter distribution by coupling the degrees of freedom of the connection to the matter distribution, there is necessarily a certain amount of slack in how this connection is constructed. This is because the projective degrees of freedom of an affine connection far outstrip the degrees of freedom of the matter distribution. Taking up this slack in different places allows us to express Newtonian gravity as a theory of curvature, or torsion, or non-metricity—in some cases, we can even specify the connection uniquely! That we can specify the curvature-based connection uniquely under certain weak conditions on the mass density field is what ensures that $\overset{c}{\nabla}$, as well as the rotation standard, is also an invariant of Trautman gauge symmetry. But the fact remains *viz-à-viz* Corollary VI that the full structure of an affine connection is not needed to support the dynamics, and so, if one introduces such a connection, one has to reckon with the fact that—again, necessarily—there are multiple distinct ways of doing so.

4 Prospects for a relativistic common core

Recall that the projective structure of a spacetime theory identifies a certain subset of worldlines the (unparameterised) geodesics; the conformal structure of a given relativistic spacetime theory specifies a lightcone at every spacetime

[15]In the following section, we'll discuss further the distinction between 'kinematical' and 'dynamical' common cores.

point (see e.g. Matveev and Scholz (2020)). Famously—a result going back to Weyl (1921)—a Lorentzian spacetime is fixed by its associated projective and conformal structure: see Malament (2012, ch. 2); the corresponding existence result was proved by Ehlers et al. (2012), and is discussed further by Linnemann and Read (2021a). In Wolf, Sanchioni, et al. (2023), it was identified that one can move between nodes of the relativistic geometric trinity by modifying projective structure while leaving conformal structure unchanged (for all three theories leave lightcone structure unmodified); in a similar manner (one ultimately irrelevant to our purposes here, but perhaps nevertheless worth pointing out) one can move to non-relativistic theories by 'widening the lightcone', thereby changing conformal structure (this constitutes a geometrical way of thinking about taking the non-relativistic limit: see Malament (1986)).

This means that, at the level of kinematics at least, the 'common core' of the geometric trinity is conformal structure. Unlike the non-relativistic geometric trinity, the kinematical common core in the relativistic trinity cannot consist in a standard of rotation, for the three geometric trinity connections agree on their associated standards of rotation iff ${Q^{[ab]}}_c = {T_c}^{ab} = 0$, which is certainly not true in general. But conformal structure alone is, of course, insufficient to recover the predictions of all models of GR (*mutatis mutandis* TEGR, STGR), as (for example) there are many solutions of that theory which are *not* conformally invariant, and so which make recourse to structure over and above conformal structure. Thus, although there is a *kinematical* common core to the relativistic geometric trinity, there is no obvious *dynamical* common core: by which we mean, some alternative theory the equations of which advert only to the kinematical objects common to each of the models of the elements of the trinity, yet which can nevertheless recover all of the empirical predictions of the models of the original theories.[16]

This situation differs strikingly from that of the non-relativistic geometric trinity. In that case—as we have already seen in the previous section—in order to obtain each of the nodes of the trinity from its relativistic counterpart, one must impose additional geometrical restrictions; it is precisely said restrictions which guarantee the empirical equivalence of the resulting theories. So, in the case of the non-relativistic trinity, not only is there a kinematical common core (namely, a standard of rotation), but, in addition, there is a dynamical common core—namely, the dynamics of Maxwell gravitation.

5 Conclusions

In this article, we've shown that Maxwell gravitation constitutes the mathematical common core of the recently-discovered non-relativistic geometric trinity of gravity (first presented by Wolf and Read (2023b)); we've also explained why there exists no such common core of the relativistic geometric trinity. In un-

[16]Here, we're eliding to some extent the fact that equation-like statements are often also taken to be part of the kinematical content of a theory: see (Curiel 2016; Linnemann and Read 2021b) for further discussion.

dertaking this work, we take ourselves to have made good on the exhortation of Lehmkuhl (2017) to explore and chart the ‘space of spacetime theories’—at least with respect to this small (albeit philosophically important!) corner of the landscape. Along the way, we have developed certain tools—e.g. the distinction between a ‘kinematical common core’ and ‘dynamical common core’ of spacetime theories—which we hope might find broader application.

Our work has interesting philosophical implications. For example, the absence of a common core in the case of the relativistic geometric trinity, in contrast to the case of the non-relativistic geometric trinity, could be taken to imply that there is a stronger case to be made for geometric conventionalism—i.e., the systematic and conscious refusal to assign truth values to propositions about geometry—in the relativistic case than in the non-relativistic case (cf. Duerr and Read (2023), in which the absence of a common core is stated explicitly to militate in favour of geometric conventionalism). This stands in contrast to the verdict of Weatherall and Manchak (2014), for whom the case for conventionalism is *stronger* in the non-relativistic case than in the relativistic case.[17]

There also are many future prospects; here we mention just two. First: one might be interested in whether (a) there exists an extended non-relativistic geometric trinity for the off-shell Newtonian limit presented by Hansen et al. (2020) (this question was also raised by Wolf and Read (2023b)), and (b) if so, whether there exists a dynamical common core to this non-relativistic trinity. And second: one might wonder whether (a) there exists an *ultra*-relativistic geometric trinity obtained by taking the ultra-relativistic (i.e. roughly speaking, $c \to 0$) limit of the relativistic geometric trinity, and (b) again whether there exists a dynamical common core to that trinity also.

Acknowledgements

E.M. acknowledges support from Balliol College, Oxford, and the Faculty of Philosophy, University of Oxford. W.W. acknowledges support from the Center for the History and Philosophy of Physics at St. Cross College, Oxford and the British Society for the Philosophy of Science. J.R. acknowledges the support of the Leverhulme Trust.

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[17]We concur with the critique of Weatherall and Manchak (2014) given by Duerr and Ben-Menahem (2022). One limitation of the analysis of Weatherall and Manchak is that they set aside explicitly consideration of torsionful reformulations of GR.

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