Demarcating Descartes's Geometry with Clarity and Distinctness

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Abstract

Descartes's doctrine of clarity and distinctness states that whatever is clearly and distinctly perceived is true. This paper looks at his early doctrine from *Rules for the Direction of the Mind*, and its application to the demarcation problem of curves in Descartes's *Geometry*. This paper offers and defends a novel account of the demarcation criterion of curves: a curve is geometrical just in case it is clearly and distinctly perceivable. This account connects Descartes's rationalist epistemological programme with his ontological views about mathematics.

Keywords

Descartes, *Clarity and Distinctness*, *Geometry*, *Demarcation Problem*

1 Introduction

Descartes's doctrine of clarity and distinctness is generally stated as 'whatever we perceive clearly and distinctly is true'. On its face, it is a general scheme for arriving at truths. For example, if I can clearly and distinctly perceive that I am thinking, then it must be true that I am thinking. But, importantly, it is also a *way* of knowing certain truths: e.g., if I can clearly and distinctly perceive that I am thinking, then I *know for certain* that I am thinking. In addition, Descartes applies this perception to objects, not only to judgements or propositions. Generally, clear and distinct perception is understood as a perception of the mind, independent of the body, by which we arrive at knowledge (Descartes also calls it 'certainty').[1] For Descartes, these certainties are the first principles or the foundation that grounds all knowledge.

The doctrine continues to interest philosophers and historians of philosophy today. For instance, Paul (2020, p.1) begins his paper by asserting that '[c]lear and distinct perception is the centerpiece of Descartes's philosophy'. There is also a recent paper by Nawar (2022) concerning the medieval history of Descartes's clear and distinct perception. However, much of the discussion on the doctrine is focused on Descartes's later works, such as the *Meditations* and the *Principles*. This paper examines the origin of the doctrine of clarity and distinctness, by focusing on Descartes's very first philosophical work, *Regulae ad Directionem Ingenii* (*Rules for the Direction of the Mind*).

In the *Regulae*, Descartes provides the rules by which one can arrive at all certain knowledge. Although it is an unpublished work, it is a good insight to Descartes's earlier and more raw thoughts, which are less visible in his later work, in particular with its relation to the development of Descartes's mathematics. Especially given the publication of the previously unknown edition of the *Regulae* (i.e. Serjeantson and Edwards (2023)), it is more important now than ever to understand Descartes's philosophical ideas from it.[2] As Gaukroger (1992) shows, the mathematical work in the *Regulae* is not fully developed as the work we see in the *Geometry*. However, as the philosophical origins of Descartes's sophisticated mathematics, the *Regulae* shows the strongest connection.

Given this, this paper considers the application of the doctrine to a problem Descartes introduces in the *Geometry*: the demarcation of geometrical curves from mechanical curves. The demarcation problem asks *what is the condition that demarcates curves in Descartes's Geometry?* That is, what is the best way to characterise geometrical curves? Descartes tried to answer this question in Book II of the *Geometry*. He begins by criticising the ancients for their 'incorrect' inclusion and exclusion of certain curves in geometry. Then he offers several remarks on the conditions which characterise certain geometrical or non-geometrical curves. Descartes called these non-geometrical curves "mechanical", following (according to Descartes) the ancients' naming of the curves. Among the mechanical curves are the Archimedean spiral and the quadratrix, which, Descartes points out, the ancients correctly excluded from geometry. Despite his criticism, there is an ongoing debate about what Descartes's fundamental condition was for demarcating curves.

Surprisingly, many historians of and philosophers of mathematics who are interested in the demarcation problem look at Descartes *only* as a mathematician. That is, their focus is to understand the mathematics of his time in order to analyse Descartes's demarcation problem. The two most well-known interpretations of the demarcation conditions are given by *construction* (Mancosu, 2007) and by *algebraic equations* (Bos, 1981, 2001). Although they agree that their demarcation conditions will lead to extensionally equivalent collection of geometrical curves, they disagree on the role of clear and distinct perception. Bos (2001) hints that the constructions for geometrical curves must be clear and distinct, but, he does not explain this in detail. Mancosu (2007) goes even

[1] This, as I show, is the key to the demarcation of curves in his *Geometry*.

[2] I only focus on the previously known manuscript of the *Regulae* in this paper, primarily from the Adam-Tannery edition (Descartes, 1996).

further to reject that clarity and distinctness can provide a demarcation condition for curves.

I will show that once we understand Descartes's earlier doctrine, it becomes clear that the demarcation problem ought to be understood from Descartes's philosophical writings. Furthermore, although the demarcation problem is considered to be a metaphysical one (e.g. Domski (2009)), the use of Descartes's doctrine suggests that it is also an epistemological problem, highlighting the epistemological nature of Descartes's mathematics.

Understanding the demarcation problem better continues to be of importance in recent works in history and philosophy of mathematics. For example, Crippa (2017, p.1253) writes that one of Descartes's achievements in the *Geometry* is his demarcation of curves. And the influence of Descartes's epistemology and mathematics on our contemporary philosophy of mathematical practice is highlighted by Rabouin (2018). In this paper, I aim to show that there is indeed a strong connection between Descartes's epistemology and mathematics, and that Descartes's mathematical project in the *Geometry* is a foundational one (see, e.g., Panza (2011, p.44)), similar to his epistemological projects. That is, Descartes's fundamental mathematical achievements in the *Geometry* – e.g., his transformation of ancients' notion of proportions via algebraic operations (see Blaszczyk (2022) for more on this) – are founded in his epistemology.

I argue that a curve is geometrical just in case it *can* be clearly and distinctly perceived. There are two ways of understanding clear and distinct perception of geometrical curves. One reading relies on clear and distinct perception of the *construction* of the curve from given geometrical figures, while the other concerns clear and distinct perception of the *relations* involved between the given geometrical figures. This is different from the existing interpretations that rely on construction and algebraic equation as the fundamental conditions, which are both *mathematical* conditions, as opposed to a philosophical one. The former condition characterises geometrical curves to be those constructed by a 'continuous motion', while in the latter, they are expressed as polynomial equations in two variables, i.e. $F(x, y) = 0$ – I'll discuss these in more detail in section 2.

The *Regulae* is Descartes's first, though incomplete, philosophical work, believed to have been written some time before 1628. In the present paper, I focus on the early doctrine of clarity and distinctness as found in the *Regulae*. This early doctrine relies heavily on Descartes's notion of 'deduction'. I show (in section 3) that deduction comes in two kinds: deduction by memory, and deduction by imagination. I further argue (in section 3.5) that deduction by memory has *explanatory superiority* over deduction by imagination, while the latter has *intuitive superiority* over the former. While they seem to be independent of each other, I show that it is only when both kinds are considered, deduction can be reduced to clear and distinct perception[3], and that both are involved in Descartes's method of solving geometrical problems.

In section 4, I demonstrate that these two kinds of deduction together provide an *explanation* for Mancosu's and Bos's readings of Descartes, respectively. Hence, clarity and distinctness is the

[3]By 'reduction' here, I mean that deduction can be shortened to be as instantaneous as clear and distinct perception. This does not mean the latter is inferior to the former.

most fundamental condition for demarcating curves, and unlike the other two conditions, clarity and distinctness arises from Descartes's philosophy. When this philosophical doctrine is applied to a mathematical domain, the boundary between what is geometrical and what is not can be made more precise.

Another upshot is that what can be clearly and distinctly perceived in Descartes's early philosophy must be independent of empirical observations. Since geometrical objects are pure, what is clearly and distinctly perceived must also remain pure. This means, even though our observation of a construction of a curve could be described to be clear and distinct, if what makes it clear and distinct is the construction's empirical features, such curve cannot be included in geometry. Some mechanical curves cannot be included in geometry because our understanding of them depend on empirical or observational facts. This is a philosophical insight which is not offered in characterising the demarcation criterion as based on construction or on algebraic equations. I will discuss this in more detail in section 5, as I defend my demarcation criterion from possible objections as well as a challenge by Mancosu (2007).

2 The Demarcation Problem

In this section, I describe the background to Descartes's demarcation of curves, and explain Mancosu's and Bos's demarcation conditions. Then I argue that circles and lines have a special role in Descartes's geometry, just as they did for the ancients.

2.1 The Demarcation of Curves

In Book I of the *Geometry*, Descartes uses algebra to solve geometrical problems. Then in Book II, he puzzles over the demarcation of *problems* by the ancients. In particular for the 'linear problems', Descartes argues that some of the solutions ought to be geometrical, unlike the ancients who called all the solutions 'mechanical'. Since the solutions to these problems can be geometrically given, he characterises geometrical curves in the following way:

> ... all points of those curves which we may call "geometric," that is, those which admit of [*tombent sous*] precise and exact measurement [*mesure précise & exacte*], must bear a definite relation [*rapport*] to all points of a straight line, and [...] this relation must be expressed by means of a single equation (S&L p.48; *Geometry* p.319; AT 6:392)[4].

Here, Descartes gives three conditions for geometrical curves: (i) the curves 'admit of precise and exact measurement', (ii) all points of the curve 'bear a definite relation to all points of a straight

[4] Throughout this paper, I will abbreviate the English translation of the *Geometry* (Descartes, 1637) as 'S&L', referring to the translators Smith and Latham. The original page number from the 1637 publication will be cited after '*Geometry*'. The Adam-Tannery edition (Descartes, 1996) will be cited as 'AT' followed by the volume and the page numbers.

line', and (iii) this definite relation must be 'expressed by means of a single equation'. By the 'curves that admit of precise and exact measurement', Descartes is referring to that the objects of geometry (i.e. figures) are actually about 'exact measurement' or 'exact magnitudes', and so that geometrical curves must admit of exact measurements. From this condition, I claim that the next two conditions follow as properties of geometrical curves in virtue of the exact measurements. However, it is not immediately obvious what it means (ii) for a point to 'bear a definite relation to all points of a straight line'. As I will show, condition (ii) can be clarified as follows:

> (ii') A curve C is geometrical just in case there are a straight line L and a relation R between points, such that for every point P on the curve C, there is a point X on L where P is R-determined by X.

Since P refers to an arbitrary point on the curve C, this means every point on the curve C can be determined by how it R-relates to the points on L. Similarly, we can start with an arbitrary point Y on L, to R-determine some point Q on C. That is, the relation R shows how the points on the curve C and the line L are related. This is what Descartes means when he says that all points on the curve must bear a definite relation to the points on L. Then the third condition (iii) claims that this definite relation R can be expressed in an equation (e.g.) $F(x, y) = 0$, where x denotes the shortest distance between a fixed point on C and the arbitrary point P, while y denotes the shortest distance (i.e. along the line L) between a fixed point on L and the point X.

The demarcation condition refers to the condition by which we, the subjects, can determine that the curve is geometrical or not. There are two important existing interpretations of the demarcation condition, one by Mancosu and another by Bos. In the next section, I will explain their interpretations.

2.2 Interpretations of Descartes's Demarcation

Mancosu claims that curves can be demarcated based on their constructions. This is based on Descartes's remarks that certain curves must be included in geometry if they were described by a 'continuous motion or by several successive motions':

> ... we have no more right to exclude the more complex curves than the simpler ones, provided they can be conceived of as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede; since for in this way an exact knowledge of the magnitude of each is always obtainable.[5] (S&L p.43, *Geometry* p.316, AT 6:389)

Thus, a curve is geometrical if and only if it can be constructed by a single continuous motion or by several successive motions. Mancosu explains that it is because Descartes had an explicit

[5] Even in this remark, Descartes suggests that this demarcation criterion is suitable *because* exact knowledge can be obtained.

belief that 'the quadrature of the circle is impossible geometrically' (Mancosu, 2007, pp.119-120). The solution to the quadrature of the circle involves constructing a curve called quadratrix. The construction of the quadratrix, and the Archimedean spiral, involves two or more distinct motions. Thus, generalising from these curves, no curve involving two or more distinct motions could be geometrical.

However, Bos disagrees with Mancosu, and argues that the demarcation condition ought to be based on '[Descartes's] conviction that proportions between curved and straight lines cannot be found exactly' (Bos 1981, pp.314-5; 2001, verbatim p.342). This conviction is not concerned about the proportions between *any* curved lines and *any* straight lines. But it refers to the relation (per condition (ii)) between the points on the curve and the given line, which is expressed in an algebraic equation (per condition (iii)). If this relation cannot be found between a given curve and any appropriate straight line, then the proportion between the curve and the straight line cannot be found exactly.

What Bos is appealing to then is Descartes's remark about the proportions between 'string-like' curves and straight lines. Descartes claims that for string-like curves, 'the relation [between the line and the curve] does not admit of exact determination' (S&L p.44, *Geometry* p.317, AT 6:392). Hence, Bos's demarcation condition is more centred around the algebraic equations, rather than constructions:

> "geometrical" curves were those that, with respect to rectilinear coordinates, had an algebraic equation. (Bos, 2001, p.335)

That is, a curve is geometrical if there is a relation (in particular concerning the proportion) between the curve and a known straight line, which can be expressed in two variables x and y, such that $F(x, y) = 0$.

Note that Bos's claim is not that the curves *are* algebraic equations, but that they are tools for demarcating the geometrical and non-geometrical curves. A stronger view which claims that geometrical curves *are* indeed the algebraic equations can be found in Giusti (1987, 1999). I do not discuss Giusti's view further, but I direct interested readers to Giusti (1987, 1999) and Mancosu (1996, p.82).

Both Mancosu and Bos agree that their demarcation conditions lead to the extensionally equivalent collection of curves. That is, a curve that is constructed by a single continuous motion or successive continuous motions can be expressed in an algebraic equation $F(x, y) = 0$, and vice versa.[6] However, neither give a *philosophical* explanation for *why* that is the case. In one occasion, Bos (2001, pp.282, 405) refers to Kempe's Universality Theorem (Kempe, 1875) to explain the extensional equality, while noting that Descartes's would not have been able to prove this.[7] Neither

[6]According to Boyer (1956, p.130), the term 'algebraic curves' was given by Leibniz precisely for Descartes's geometrical curves. See also Bos (2001, p.336)

[7]However, Kempe's original 'proof' was flawed, and a correct proof was only offered in 2002 (Kapovich and Millson, 2002).

of them seems to think Descartes *actually* had reasons to believe that they are equivalent, despite claiming that they are.

My demarcation condition actually offers a philosophical explanation for the extensional equivalence between Mancosu's and Bos's conditions. Rather than relying on a mathematical result, which was not available to Descartes, look at Descartes's own doctrine of clarity and distinctness.

I will show shortly, the doctrine of clarity and distinctness in its early form concerns *intuition* and *deduction* as found in the *Regulae*. I argue that the requirement for 'continuous motion' in geometrical constructions follows from *deduction by imagination*. Similarly, the 'algebraic expressibility' follows from the fact that *deduction by memory* as an inference in which each proposition must be intuited. Conversely, since, by Descartes's own account, geometrical curves are those which admit of exact magnitudes, if the curve (and its relevant magnitudes) is clearly and distinctly perceived then they must be geometrical. In the next section, I explain the two kinds of deduction, and argue that we come to know of the magnitudes of geometrical curves through deduction. Thus, I argue that geometrical curves are those whose magnitudes can be deduced, i.e., the curves whose magnitudes can be clearly and distinction perceived.

3 Deduction, and Clear and Distinct Perception

The goal of this section is to show that (1) deduction can be reduced to clear and distinct perception, and (2) that there are two kinds of deductions, each of which gives rise to each of Mancosu's and Bos's readings. I will focus on two examples from the *Regulae* in order to explain the two kinds of deduction. We will see then that Descartes's method of solving geometrical problems requires the full concept of deduction from the *Regulae*.

3.1 The Early Doctrine of Clarity and Distinctness

Descartes's doctrine of clarity and distinctness states that 'whatever is clearly and distinctly perceived is true' (*Meditations* AT 7:35, CSM 2:24[8]; Verbatim the *Discourse* Rule 1). On the face of it, it only applies to propositions which can be assigned a truth value. But we see in the *Meditations*, Descartes refers to *ideas* (in particular, of God) as being true as well as clear and distinct: 'This is enough to make the idea that I have of God the truest and most clear and distinct of all my ideas' (AT 7:47, CSM 2:32). An idea, for Descartes, is like an *image* of things:

> Some of my thoughts are as it were the images of things, and it is only in these cases that the term 'idea' is strictly appropriate (AT 7:37, CSM 2:25)

So an *idea* of God is *like* an image of God. Although Descartes is not entirely clear on what these ideas are exactly, from such an idea, we can make a proposition about God: e.g., that God is

[8] I will abbreviate the two volumes of *The philosophical writings of Descartes* (Descartes, 1985) as 'CSM', referring to the translators, followed by the volume and page numbers.

'supremely perfect' (AT 7:65, CSM 2:45). What the doctrine tells us is that these propositions must be true if they are perceivable clearly and distinctly. And thus, according Descartes, the idea is true because the propositions made from the idea are clearly and distinctly perceived. The philosophical discussion on why Descartes uses ideas in this way is immense, and this would require an extensive study on the *Meditations* and the *Principles*. For the purpose of this paper, it is sufficient to say that ideas of things *resemble* images of those things, and we can make propositions from these images, and such propositions can be clearly and distinctly perceived when the ideas themselves are clear and distinct.

While most scholars focus on the *Meditations* in analysing Descartes's doctrine, I concentrate on the earlier version of the doctrine that can be found in the *Regulae* and the *Discourse*. Unlike the later doctrine where Descartes considers more abstract ideas, e.g. of God, in the earlier doctrine, the ideas he studies are more visual. In the *Regulae* particularly, many of his examples are mathematical. This is not surprising since Descartes's doctrine is derived from his view that only the demonstrations in arithmetic and geometry can arrive at certainty (*Regulae* AT 10:366, CSM 1:13; *Meditations* AT 7:65, CSM 2:45; also see Mancosu (2007); Gaukroger (1995)).

In its early form, the doctrine concerns intuition and deduction. Intuition is a mental act that is 'evident' [*evidentia*] and 'certain' [*certitudo*] (*Regulae* AT 10:369, CSM 1:15) and grasps a clear and distinct idea (*Regulae* AT 10:366-8, also see Gaukroger (1992, p.589)). Deduction, on the other hand, is different from intuition, since it consists of an inference 'from true and known principles through a *continuous and uninterrupted movement of thought* in which each individual proposition is clearly intuited' (AT 10:369, CSM 1:15, my emphasis). But, as Gaukroger claims, deduction is 'ultimately modelled on intuition' and 'in the limiting case becomes intuition' (Gaukroger, 1992, p.589). For Descartes, these two mental acts are the only ways to arrive at certain knowledge (Rule 3, *Regulae*, AT 10:366, CSM 1:13). Thus the doctrine is a bi-implicational statement such that whatever we can intuit or deduce (i.e. clearly and distinctly perceive) must be true, and any certainty, truth or knowledge must be obtained from intuition or deduction alone.[9]

It is important not to confuse Descartes's notion of deduction with either contemporary formal logic or scholastic syllogistic deduction. These are formal systems of rules; Descartes's deduction is meant as an innate capacity of minds. Now I will look at particular examples of Descartes in the *Regulae* to explain what intuition and deduction are.

[9]One could question how 'true belief' could be possible for Descartes as opposed to knowledge. For Descartes, it is possible to arrive at truth beliefs from experience, but what makes it *belief* rather than *knowledge* is that its truth comes from the bodily experience rather than the pure intellect. There is a difference between *cognition* and *scientia* in Descartes, which are both translated as *knowledge* in English. In that sense, a true belief is a *cognition* rather than *scientia*, as it is only the latter that can be known by the pure intellect. Thus, any certainty or knowledge [scientia] is obtained from intuition or deduction alone.

3.2 Intuition, and Clarity and Distinctness

Intuition is 'the conception of a clear and attentive mind, which is so easy and distinct that there can be no room for doubt about what we are understanding.' (AT 10:368, CSM 1:14) So, whatever truths we can intuit must be certain, but also these truths are grasped entirely independently of experience. As such, the truths we intuit are 'so clearly and distinctly [known] that they cannot be divided by the mind into others which are more distinctly known' (AT 10:418, CSM 1:44). The examples Descartes offers include 'that he exists, that he is thinking, that a triangle is bounded by just three lines, and a sphere by a single surface...' (AT 10:368, CSM 1:14). Thus, whatever Descartes intuits does not have to be mathematical, but can be applied generally. As I will show in a moment, this is because it is entirely dependent on the *pure intellect*, a faculty of the rational mind. So let us take a small detour and discuss Descartes's four faculties of the mind. These faculties are important in understanding Descartes's deduction.

The embodied mind, for Descartes, has four faculties: pure intellect, memory, imagination, and sense-perception[10]. It is easier to understand what the latter three are, so let me explain those first. The memory simply refers to our capacity to remember facts or things. Imagination is an ability to pictorially visualise things in our mind, e.g. ideas. Sense-perception refers to our five senses: seeing, touching, tasting, hearing and smelling. Importantly for Descartes, the three faculties depend on the body, which is distinct from the mind, and *therefore* they are prone to error if not used appropriately.[11] The pure intellect, on the other hand, is the capacity of the mind that is independent of the body. And so, it is only the pure intellect that can arrive at certainty. This capacity of arriving at truth is referred to as *perception*, similar to seeing in sense-perception. But the former consists of the perception by the *mind*, rather than the body. These four faculties together refer to the innate abilities of any embodied mind: where the faculties of memory, imagination and sense-perception serve as assistants to the pure intellect.[12]

We will see shortly that the two kinds of deduction, one by imagination and another by memory, refer to the two faculties mentioned above.[13] We will see later that deduction, which is a process

[10] By the 'embodied mind', I do not mean to suggest that every faculty depends on the body.

[11] As I will show later, deduction is such method in which we make use of the faculties.

[12] See Simmons (2003, p.564ff) for a discussion on the different perception of the faculties.

[13] As Descartes develops this further, he seems to distinguish the picture or image in one's *mind* from one in *imagination*. For example, in a letter to Mersenne, he says that there is a difference between the ideas in imagination and in the mind. The latter can be conceived without an 'image' (i.e. a picture), while the other cannot be (AT 3:395). See Simmons (2003) for more on this. In the earlier work, however, this distinction is not as clear. In Book II of the *Regulae*, Descartes talks in detail about abstracting and picturing bare figures in imagination for intuiting facts about mathematics. But when it comes to the non-mathematical examples, it is unclear how that 'he exists' or that 'he is thinking' can be pictured as such. This is perhaps why Descartes distinguishes the ideas in the mind and in the imagination in his later doctrine of clarity and distinctness, so the doctrine generalises to more abstract truths beyond mathematics. According to Hatfield (2017, p.444), Descartes could not accept the view that the pure intellect relied on imagination, as that would suggest God could not be known clearly and distinctly. While this is true, this is no longer a problem when we focus on the mathematical examples. This might even be the reason why Descartes could not finish the *Regulae* – Book III, which was never written, was to consider non-mathematical

rather than an immediate perception, can be reduced to intuition with the help of imagination. So the reduction of deduction allows the pure intellect to intuit the truth without further relying on other bodily states.[14] [15] For now, let us further explicate what Descartes's 'intuition' is.

In the *Regulae*, intuition concerns having a picture in one's *mind* and seeing the truth from it. (See Gaukroger (1992) for more details.) Thus, any truth we intuit, we ought to be able to have a picture or an image of it. Consider the following example of intuition. Descartes claims he 'intuits' [*intuendum*] that $2 + 2 = 4$ and that $3 + 1 = 4$ (AT 10:369, CSM 1:15). In perceiving these facts and grasping the clear and distinct ideas of the facts, we must be able to picture them as images. Given that Descartes pictures numbers as dots (AT 10:450, CSM 1:64), we can imagine the equality $2 + 2 = 4$ by putting two dots and two dots together, and seeing that it is equal to four dots, as in the left diagram in figure 1. And we imagine that $3 + 1 = 4$ as in the right diagram. Since we can imagine these in single images, we can make the relevant propositions immediately, thereby intuiting that $2 + 2 = 4$ and that $3 + 1 = 4$, respectively.

Figure 1: Intuition

As mentioned before, in general, if we 'intuit' a truth, we must have an image of the things from which we can make a judgement of its truth. Of course, it is not clear how one could see immediately that (e.g.,) $52 + 73 = 125$, and frustratingly, Descartes does not tell us how either.[16] I will put these problems aside, and take that if we are able to have a single image or an idea or the proposition, then it is possible to see it clearly and distinctly.

certainties. Thus, narrowing our domain of certainty to be that of mathematics, I claim that if we *can* intuit a truth p, then we *could* form a picture of p in imagination, and conversely, if we *can* form a picture of p, then we *could* intuit an appropriate truth.

[14] Although whether the perception of pure intellect and of imagination ought to be treated the same for Descartes is not entirely clear, certain passages and letters from Descartes indicate that they should be treated differently – see Simmons (2003, p.565) for more discussion on this. The usual contrast between them seem to be whether the figures are *imagined* or understood, rather than whether the ideas are the same. In that sense, having the image alone does not consist in understanding, i.e. intuiting the truth by the pure intellect. So we can contrast intuition, i.e. the understanding and seeing the truth, from just seeing the mere *picture*.

[15] Another problem is that Descartes is not clear himself how exactly deduction, which relies on other faculties, guarantees that the ideas are indeed of the pure intellect. This problem is beyond the scope of this paper.

[16] It is not even clear whether Descartes would intuit $52 + 73 = 125$, but only equations that are simpler.

For examples of intuition, it is not difficult to provide a picture from which Descartes intuits the appropriate truth based on the imagination. But for deduction, it is different. Although Descartes offers a few examples of deduction in the *Regulae*, these examples are not thoroughly explained and have been left for the readers to interpret. I will first explain what Descartes could mean by deduction based on the same example as intuition. I will also briefly explain two kinds of deduction we find in Descartes: by *memory* or by *imagination*.

3.3 Two Kinds of Deduction

Deduction is an inference that is 'from true and known principles through a continuous and uninterrupted movement of thought in which each individual proposition is clearly intuited' (AT 10:370, CSM 1:15). We distinguish deduction from intuition in that 'we are aware of a movement or a sort of sequence in [deduction,] but not in [intuition]' (AT 10:370, CSM 1:15). Let me clarify what this means by looking at Descartes's example of the deduction of $2+2=3+1$. I give two explanations for deduction which rely on two distinct faculties of the embodied mind: memory and imagination.

It's important to note that deduction is not an operation of the pure intellect alone. It is a method in which we use our faculties to arrive at certainties. In deduction, we not only employ the intuition of the pure intellect, but also make use of memory and imagination. However, one must be careful when using memory and imagination, which are the faculties of the embodied mind. The view that deduction can be as clear and distinct as intuition is common in the literature. (see e.g. Gaukroger (1995) and Hatfield (1998, 2017)) My interpretation is unique in the following sense: I show *how* deduction can be reduced to intuition, and thus *how* deduction becomes clear and distinct perception.

Before I show that deduction can be reduced to clear and distinct perception, let me explain the two kinds of deduction we can find in Descartes's writing. After understanding them both, I will explain how the two kinds can be interpreted as the demarcation conditions of Mancosu and Bos, respectively.

Deduction by Memory

For the first kind of deduction, we must use memory, since the *certainty* of deduction comes from the use of memory:

> [...] we are distinguishing mental intuition from certain deduction on the grounds that we are aware of a movement or a sort of sequence in the latter but not in the former, and also because immediate self-evidence is not required for deduction, as it is for intuition; deduction in a sense gets its certainty from memory. (AT 10:370, CSM 1:15)

So the deduction of $2+2=3+1$ can be understood as follows:

(i) Intuit $2+2=4$.

(ii) Intuit $3 + 1 = 4$. And remember the previous truth.

(iii) Substitute $3 + 1 = 4$ into $2 + 2 = 4$.

(iv) So $2 + 2 = 3 + 1$.

This inference requires holding either of the intuited propositions (e.g. $2+2 = 4$) in memory. Then by substitution we arrive at $2 + 2 = 3 + 1$. Importantly, this is a 'continuous movement of thought that was not interrupted,' since we can explain each step of the argument from the previous steps. Furthermore, each proposition considered at each step of the deduction is intuited: that $2 + 2 = 4$ and that $3 + 1 = 4$. Then it necessarily follows from the two that $2 + 2 = 3 + 1$. Thus, if we deduce a truth by *memory*, each proposition intuited from the previous one can be listed as each step of the inference.

Deduction by Imagination

The second kind of deduction uses imagination rather than memory. It is easier to see how deduction can *become* intuition in the case of deduction by imagination: by imagining the conclusion of deduction, we can *intuit* the truth. Descartes says:

> For if we have deduced one proposition from another immediately, then provided the inference is evident, it already comes under the heading of true intuition. (AT 10:389, CSM 1:26)

Since this kind of deduction becomes intuition, it might seem superior to the deduction by memory. I will show that this is not the case shortly. For now, I explain deduction by imagination using the same example as above.

After intuiting that $2 + 2 = 4$ and $3 + 1 = 4$, we can further imagine a picture that shows $2+2 = 4 = 3+1$. As we move from the image of $2+2 = 4$ to combine it with the image of $3+1 = 4$, we need to move the first image continuously onto the second image without interruptions. Then we can remove the middle four dots to see that $2 + 2 = 3 + 1$ (see figure 2). In this interpretation, the continuous motion occurs *in* imagination, rather than in memory (AT 10:388, CSM 1:25). This is an important distinction: the kind of deduction used depends on the mental faculty in which the continuous motion happens.

In the next section, I will consider another example that Descartes uses to explain how deduction can be reduced to intuition. In this example, Descartes is explicit that the 'continuous movement' happens in 'imagination' (AT 10:388, CSM 1:25). I will argue that neither interpretation is superior to the other, but they must be considered for their unique properties. Namely, deduction by memory serves to explain *how* or *why* certainties can be found. By contrast, deduction by imagination fails to provide that. But deduction by imagination reduces the long inferential process to a single image to be intuited. Hence, both kinds of deduction must be understood in order for us to clearly and distinctly perceive those which can be deduced.

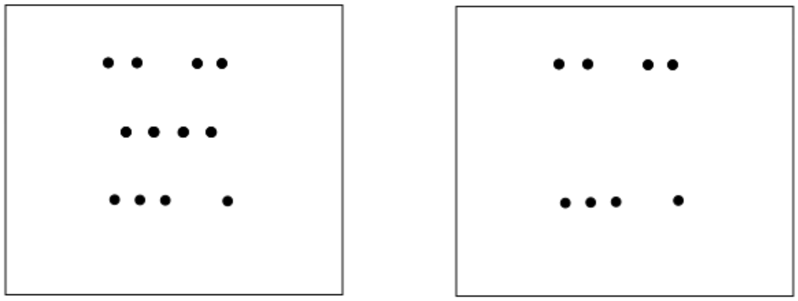

Figure 2: Intuition of a non-immediate deduction

3.4 Geometrical Problem Solving

I now show that the method of solving problems in geometry concerns deduction, and explain how both kinds of deduction are used to clearly and distinctly perceive geometrical truths.

In Book II, Descartes writes:

> If [...] we wish to solve any problem, we first consider it already done, and give names to all the lines that seem needful for its construction,–to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must parse the difficulty in any way that shows most naturally that they [the lines] mutually depend on each other, until we find it possible to express the same quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other. (S&L pp.6-7 with my modification, *Geometry* p.300, AT 6:372)

So Descartes's method of solving geometrical problems can be summarised in four simple steps:

(i) Suppose that the problem is already solved.

(ii) Assign letters to all straight lines which seem necessary for constructing the solution.

(iii) Consider how the lines depend on each other, until we find two ways to express the same quantity.

(iv) Reduce the equation to its simplest to have one single equation.

When the method is presented this way, it is not hard to see how the method relies on both kinds of deduction. The first step requires imagination, in which the solution can be pictured. To do so, we must imagine how the curve (i.e. the solution) is traced by continuous motions or successive motions. Then we can assign letters to each straight line in the imagination. Then, Descartes uses the properties of similar triangles to express equality between different proportions. These equations allow us to express certain quantities algebraically, based on how they related to other quantities in the imagination. We can then simplify the relevant equation by intuiting the other algebraic equations, holding them in memory and substituting appropriately.

Let us turn to an example from the *Regulae*, to explain how the two kinds of deduction are used in solving the geometrical problem. Given five magnitudes A, B, C, D and E, Descartes claims he can find the relations between A and B, B and C, C and D, and D and E. But to find the relation between A and E, he must use *memory* in order to recall all the relations:

> If, for example, by way of separate operations, I have come to know first what the relation between the magnitudes A and B is, and then between B and C, and between C and D, and finally between D and E, that does not entail my seeing what the relation is between A and E; and I cannot grasp what the relation is just from those I already know, unless I recall all of them. (AT 10:387-8, CSM 1:25)

Similar to the inference given for the previous example, we can reformulate this in a way, so we can see each step of the inference and deduce (by memory) the relation between A and E by memory. I will give the deduction by memory first, before showing how the conclusion could be deduced by imagination.

Deduction by Memory

Interpreting the 'relation between two magnitudes' to be referring to the ratio between them, and also that 'knowing' the relation to mean knowing *that* the relation is constant, we can express the inference as follows: let a, b, c, d and z be some constants, then

i) Intuit $A : B = a : z$.

ii) Intuit $B : C = b : z$. Remember the previous one.

iii) Intuit $C : D = c : z$. Remember the previous two.

iv) Intuit $D : E = d : z$. Remember the previous three.

These equalities between ratios can be expressed as fractions, as Descartes does in the *Geometry*. This is how Descartes expresses the relations as an algebraic equation, so the four steps of intuition can be re-expressed as below. Henceforth, I will use $|\cdot|$ to denote the length of a line as opposed to the line itself.[17]

[17] For Descartes, the line itself would have been the magnitude. That is, the line A is the magnitude $|A|$. However, the modern notation differentiates the line from the 'magnitude' of the line. I am using the modern notation here for clarification.

i, ii, iii, iv) $\frac{|A|}{|B|} = \frac{a}{z}$, $\frac{|B|}{|C|} = \frac{b}{z}$, $\frac{|C|}{|D|} = \frac{c}{z}$, and $\frac{|D|}{|E|} = \frac{d}{z}$.

The first equation $\frac{|A|}{|B|} = \frac{a}{z}$ can be rearranged as $|B| = \frac{|A|z}{a}$, and similarly the other equations can also be rearranged to express $|C|, |D|$ and $|E|$.

v, vi, vii, viii) $|B| = \frac{|A|z}{a}$, $|C| = \frac{|B|z}{b}$, $|D| = \frac{|C|z}{c}$ and $|E| = \frac{|D|z}{d}$.

Then we can substitute $|B| = \frac{|A|z}{a}$ into the equation $|C| = \frac{|B|z}{b}$ in place of $|B|$. This gives us $|C| = \frac{|A|z^2}{ab}$. Repeating this process for the next equations to obtain the following:

ix) $|C| = \frac{|B|z}{b} = \frac{\frac{|A|bz}{a}z}{b} = \frac{|A|z^2}{ab}$.

x) $|D| = \frac{|C|z}{c} = \frac{\frac{|A|z^2}{ab}z}{c} = \frac{|A|bcz^3}{abc}$, by remembering the above equality and substituting.

xi) $|E| = \frac{|D|z}{d} = \frac{\frac{|A|z^3}{abc}z}{d} = \frac{|A|z^4}{abcd}$, by remembering the above equality and substituting.

Then we can re-arrange the equation in order to express $\frac{|A|}{|E|}$, which is equivalent to the ratio between A and E:

xii) $A : E = abcd : z^4$.

Since a, b, c, d and z are all constants, which are used to express the known ratios, we also know the constant ratio $A : E$. The inference above shows how we can deduce $A : E$ by a continuous movement in thought using the faculty of memory. What is necessary in the steps is to remember all the other four ratios. But it is not hard to see how one can make mistakes in remembering, and thus result in arriving at an incorrect answer for $A : E$. So Descartes explains that for this problem, he can 'simulatenously intuit the relations' and leave memory at rest:

> So I shall run through them several times in *a continuous motion of the imagination*, simultaneously intuiting one relation and passing on to the next, until I have learnt to pass from the first to the last so swiftly that *memory is left with practically no role to play*, and I seem to intuit the whole thing at once. (AT 10:388, CSM 1:25, my emphasis)

As the passage above shows, deducing the solution by memory alone is not enough to reach certainty. So we must turn to deduction by imagination.

Deduction by Imagination

I will use triangles to depict the ratio between two magnitudes, as Descartes does so in the *Geometry*. So a triangle can be determined by its two sides A and B and a fixed angle between the sides standing for the ratio $A : B$. Then, we can imagine four triangles for the four given ratios $A : B, B : C, C : D$ and $D : E$, and move the triangles next to each other in a continuous way (see figure 3). That is, move the second triangle (for $B : C$) to meet the first triangle (for $A : B$) along the line B, etc.

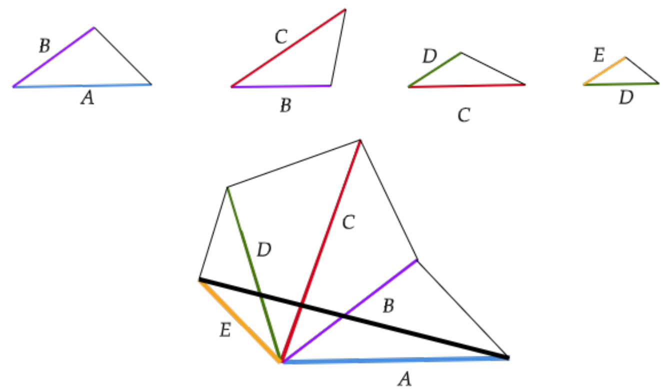

Figure 3: Intuition of $A : E$

Then we have a triangle with sides A and E, from which we intuit the constant ratio $A : E$ between A and E. Hence, Descartes can 'intuit the whole thing at once', as if we were only intuiting the relation $A : E$ without the other relations.

So how does this example show that deduction can be reduced to clear and distinct perception? Recall that clear and distinct perception, or intuition, is the perception of the mind which immediately gets us to see the truth. The deduction by imagination shows that the solution $A : E$ to the problem above can be imagined as pictured and thus intuited. But in order to understand what the ratio $A : E$ is, we must use deduction by memory, which allows us to reduce it to the equation. Hence, both kinds of deductions are involved for the answer that $A : E = abcd : z^4$.

Hence, if a geometrical problem *can* be solved by (both kinds of) deduction, then the solution *can* be clearly and distinctly perceived (and vice versa). And so a curve, which is a solution to a problem, is geometrical if the problem can be solved by deduction.

Before I go on to explain how Mancosu's and Bos's demarcation conditions can be explained from clear and distinct perception, let me briefly discuss how the two kinds of deduction are different from each other and what benefits each kind has. By doing so, I show why using both kinds of deduction was important for Descartes.

3.5 The Differences between Memory and Imagination

While reducing deduction by imagination to intuition makes it easier to find the relation between A and E (as an image), it might not be easier when explaining *how* one arrived at the ratio. In particular, intuition involves understanding 'simultaneously, and not successively' (AT 10:407, CSM 1:37), but deduction involves 'a certain movement of our natural intelligence, a certain movement of inferring one thing from another' (AT 10:407, CSM 1:37). Recall Descartes's criticism of the ancients for failing to explain *why* and *how* they arrive at their solutions. Similarly, intuition might not be enough to explain the *why* and the *how*, since the understanding is gained in one perception, and it needs to simultaneously attend to many things. In demonstrating how one has acquired the knowledge, the use of algebra is important. As Descartes claims that algebra is 'a sort of analysis' which can be applied to solving all problems (AT 10:373, CSM 1:17).

Deduction by memory is explanatorily superior to one by imagination: namely, every intuitive proposition can be stated as an algebraic equation. This allows the reader to follow each step of the deduction. The diagrams themselves are the *ideas* rather than the steps in the deduction. And since deducing the certainties must occur in one's mind, with deduction by imagination, it is much more difficult to do *explain* the steps when the demonstration is given on paper.[18]

But there is another advantage to the deduction by memory: it allows us to treat the problem more generally. For example, if we were asked to bisect a line, I can draw the solution in my imagination for a *particular* given line. However, I can also generalise the solution for *any* given line by assigning a symbol a to represent the magnitude of the line. This is exactly what Descartes did for the solution to the Pappus problem in the *Geometry*. The ancients already had a solution that was based on construction which corresponds to the deduction by imagination, although the ancients only considered three-dimensional cases. But Descartes's use of algebra allowed him to generalise the problem to arbitrarily many given lines and thus the solution also was generalised indefinitely. The algebraic solution and the method of solving the problem can be applied to *any* particular Pappus problem.

Despite these differences, the two kinds of deduction share the most important thing in common: deduction involves continuous movements. What they differ on is *where* this continuous movement occurs. If it is in memory, we have to move from one proposition to the next by holding the propositions in memory. If we interrupt this movement by trying to substitute multiple propositions all at once, we could make mistakes and arrive at a falsity. In contrast, the certainty of deduction by imagination follows from that, instead of an inference between propositions being made, we can reduce the inference to a single intuitive image. This image has to be given in a continuous motion without any interruptions – we cannot focus on multiple movements if they are happening at the same time. When we follow the pictures in a single continuous movement in imagination, the image, which we intuit the conclusion from, will be sufficient for the truth we aimed for. Thus, in both kinds of deduction, a continuous movement in thought or in imagination is involved. The truths obtained at each step can be given as a proposition (e.g. $2 + 2 = 3 + 1$) or as an image. The conclusion of deduction can also be given as a proposition (e.g. $A : E = abcd : z^3$) or as an image (e.g. figure 3).

These two kinds of deduction are important in demarcating the curves by the notion of clarity

[18] It is possible that Descartes was criticising the ancients for *only* using deduction by imagination. I do not discuss this further here.

and distinctness. On one hand, if we use our imagination for clarity and distinctness, the demarcation can be given by the construction; on the other hand, if we follow the propositional inference, then the demarcation can be given by an algebraic equation. According to Gaukroger (1992, 1995), intuition is the most clear and distinct perception, and it is the limit of deduction. Henceforth, I will use 'deduction' and 'clear and distinct perception' interchangeably in the context of considering figures. This is because any deduction by memory *can* be replaced with deduction by imagination, and any deduction by imagination can be reduced to intuition.

The next few sections will focus on the demarcation problem of curves in *Geometry*. I will show that the early doctrine of clarity and distinctness provides an explanation for the extensional equivalence between Mancosu's and Bos's geometrical curves.

4 From Clarity and Distinctness to Other Demarcation Conditions

My characterisation, based on Descartes, of the demarcation of geometrical curves is as follows:

> **Demarcation by C&D:** A curve is geometrical just in case it can be clearly and distinctly perceived.

Since a geometrical curve is a solution to some geometrical problem, we can describe the curve by applying Descartes's method of solving problems. As I showed in the previous section, such method involves the two kinds of deduction, and deduction can be reduced to clear and distinct perception. Hence, a curve is geometrical if it can be clearly and distinctly perceived. And conversely, if a curve can be clearly and distinctly perceived, that means the magnitudes admitted of the curve can be known. Thus, the curve is geometrical.

In order to explain that Mancosu's and Bos's demarcation conditions follow from my demarcation conditions, I focus on demonstrating that a circle with a given radius is geometrical. Then I will turn to the construction of the mesolabum compass and demonstrate how deduction can be used to determine the geometricality of the curve.

4.1 Construction as a Demarcation Condition from Deduction by Imagination

The connection between imagination and the construction of curves is found in the *Geometry*, as well as the *Regulae*. In the *Geometry*, Descartes often *imagines* the tracing of curves in the *Geometry*. See for example:

> I shall consider next the curve CEG, which I imagine [j'imagine] to be described by the intersection of the parabola CKN... (AT 6:393, Descartes (1637, p.84), *Geometry* p.337).

For Descartes, the construction of curves is in *imagination*, rather than drawing of it with pencil and paper (or whatever tools Descartes would have used to draw them physically). And the continuous movement required in the construction of curves in imagination refers to the deduction by imagination. Now let us turn to the construction of a circle of radius r.

Imagine a straight line OR of length r. Fixing the point O, we can rotate this line in a single continuous movement in our imagination. Then the curve traced by the point R is given by a single continuous movement in our imagination. That is, every point of the curve is determined by the line OR as it rotates. Then we can say that the curve is clearly and distinctly perceived, as every point can be intuited as the line OR and the length of OR is known to be r.

Since circles of any given radius can always be described by this continuous motion, they are included in geometry. This explains why the construction of the curve by a single continuous motion is sufficient to show the geometricality of curves as Mancosu would claim.[19]

Let us now turn to Bos's demarcation condition.

4.2 Algebra as a Demarcation Condition from Deduction by Memory

Now, I will explain how the circle can be expressed in an algebraic equation. Once the equation is obtained, it is easy to see what Descartes means by the 'definite relation' between the points of the curve and the points of the line. Finding the equation follows Descartes's method of solving problems. Importantly, Descartes defines $+, -. \times, \div, \sqrt{}$ as geometrical operations. That is, we can add, subtract, multiply and divide any two magnitudes, and find the root of any magnitude *geometrically*. These algebraic operations, while defined on all magnitudes, are asserted only on line segments. Since any geometrical magnitudes, such as lines, surfaces, volumes, can be reduced to line segments, this makes it possible to associate curves to algebraic equations. See Panza (2005, pp.25ff) for more details. The algebraic operations can be pictured as in figure 4.

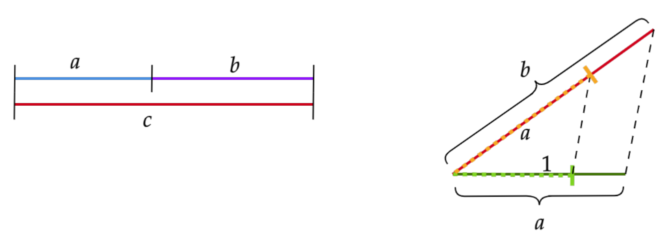

Figure 4: Geometrical Operations

[19]Note that, Mancosu does not explain whether he means construction *in imagination* instead of physical/pencil-paper construction. But since Descartes discusses how the tracing of curves is in his imagination, I assume that Mancosu would agree.

The left diagram shows that the magnitude joining a and b is equal to the magnitude c, or that removing magnitude a from c is equal to the magnitude b. The right diagram shows multiplication and division. Given a triangle with lengths 1 (i.e. the unit) and a, the magnitude b is such that $1 : a = a : b$. That is the two dotted lines joining the two sides of the triangle are parallel. Then the length b is equal to $a \cdot a$, i.e. a^2. If the green line a was c, then b is equal to $a \cdot c$. That is, b is obtained by multiplying the yellow dotted line a with the green line c. Division is understood in the same way.[20] As we can see, these operations rely on the proportionality of similar triangles, which are important for understanding Descartes's geometry. With the geometrical operations expressed diagrammatically, let us return to the circle of radius r.

Suppose the circle is constructed by rotating the line OR about the centre O. Then consider an arbitrary point P on the circle. Let us call the original line OR, 'OA' by fixing the point A as the starting point of the line OR, to distinguish it from any new positions of the line OR. Now for the point P, we can imagine a line XP such that the point X lies on OA, and XP is perpendicular to OA.

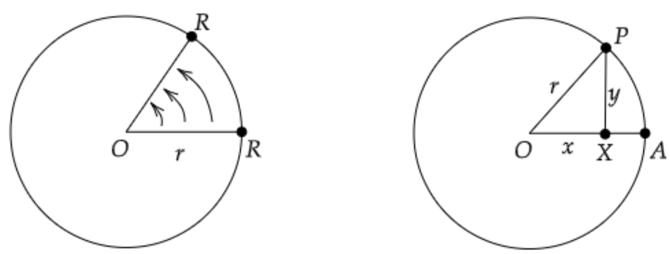

Figure 5: Geometricality of Circles

Now assign letters to certain magnitudes. Let the line OX be assigned x and the line XP assigned y. Since the angle OXP is a right angle, the triangle OXP is a right triangle. Then we can apply the Pythagorean theorem, to obtain the equation $r^2 = x^2 + y^2$. So the polynomial in two variables is $F(x, y) = r^2 - (x^2 + y^2) = 0$.[21] Then, if we are given the magnitude x, we can always deduce the magnitude y. That way, we can determine exactly where the point P is, given the point X.[22]

[20] I omit the figure for the operation $\sqrt{}$ here, since we do not need it for this paper. For those interested, see Bos (2001, pp.293-6).

[21] There are several other ways to find this equation. One way is to use Thales' theorem: the triangle drawn from the diameter and a point on the circle is a right triangle at the point. Then we can use the properties of similar triangles, as Descartes does for solving geometrical problems.

[22] The way we obtained the equation of the circle shows that it satisfies the second and the third conditions of

The above arguments can be generalised to any curves that are geometrical. We first construct the curve by a continuous movement of given geometrical figures in our imagination. Then we can apply Descartes's method to find the appropriate equation. The method of arriving at the equation involves reducing the equation to the simplest, which relies on deduction by memory. Thus, Bos's demarcation condition follows from deduction by memory.

Having explained how Mancosu's and Bos's demarcation conditions follow from the two kinds of deduction, which together can be reduced to clear and distinct perception, I will turn to the mesolabum compass. I use the following example to demonstrate that the doctrine of clarity and distinctness can be applied for the curve described by the mesolabum.

4.3 Mesolabum

Descartes uses the mesolabum compass to explain that any curve constructed by this is geometrical. Let us call them the **mesolabum curves**. The compass can be described as follows. Let rulers XY and YZ be hinged at Y as in figure 6. Then fix another ruler at point B along XY, perpendicular to the ruler XY. Initially, the ruler XY sits on top of YZ, but as XY moves away from YZ, a circle of radius $|BY|$ can be traced by the point B. So we see the circle traced as the curve AB in the figure.

The ruler, fixed at point B, intersects with the ruler YZ. Let the point of intersection be called C. Then C is imagined to be moving along YZ, depending on the angle BYZ makes. At C, attach another ruler perpendicular to YZ such that it intersects XY at a point D. This point D also varies as the angle BYZ increases since D depends on the point C. Again, attach a ruler at the point D perpendicular to XY and intersecting YZ at point E,... . Each of these points $C, D, E, F, G, ...$ depend on the angle made by the lines XY and YZ. In particular, when the XY lies on top of YZ, the points $A, B, C, D, E, F, G, ...$ all coincide. As the angle increases, we can imagine curves being traced by D, F, H, ... Note that each traced curve should have the same concavity.

Descartes remarks then that the curve traced by the point D must be geometrical, since it is as clearly and distinctly described as the circle AB. Let me explain how this the curve is traced as *clearly and distinctly* as the circle. Given the circle AB, the line BC is the tangent of the circle at the point B. As B moves, we can imagine the continuous movement of C, which is the intersection between the tangent at B and the ruler YZ. Then, from the movement of the line BC, we can imagine the movement of CD. Each movement is continuous and uninterrupted as it depends on the previous one. In fact, we only need to consider one motion at a time – the motion of BC, then the motion of CD etc. Hence, the motion of CD is continuous and uninterrupted as this motion can be imagined successively from the motion of BC. Thus, the curve AD is as clearly and distinctly perceived as the circle AB, since from the circle AB, we can trace the curve AD by a continuous and uninterrupted motion in our imagination.

geometricality given by Descartes. Namely, that there is a definite relation between the point P and the point X (ii). This relation is then expressed as an equation $F(x, y)$ in two variables (iii).

The above description shows that the curve is clearly and distinctly perceivable by relying on a geometrical figure, the circle AB whose radius is known to be $|BY|$, and by deduction by imagination. I will now explain how the curve is clearly and distinctly perceivable by deduction by memory.

Note that the angle BYA is equal to the angle BCD. Since these are both right triangles, we can use properties of similar triangles.

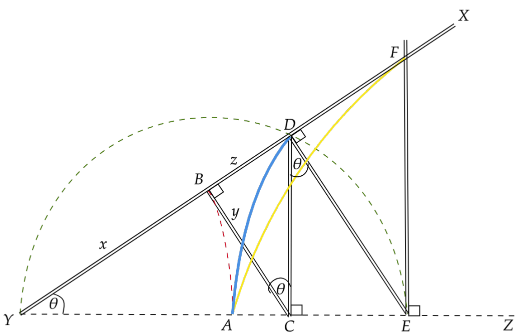

Figure 6: The Mesolabum Compass

From that, we can determine exactly where D lies by deducing the relevant magnitudes. Note that the length $|BY| = x$ is the radius of the circle traced by B, so it is constant. The similar triangles tell us $x : y = y : z$. Then we can express this algebraically and reduce it to obtain that $z = \frac{y^2}{x}$, a polynomial equation expressed in two variables y and z (since x is a constant). So if we know exactly where the point B lies, and if y can be known, then we can determine the point D by the magnitude z. Hence, once we know the magnitude y, we can determine the point D by deduction by memory.

I have shown thus far how a curve which can be clearly and distinctly perceived is geometrical. That is, it satisfies that (i) it is constructed by a continuous and uninterrupted movement in imagination, and also (ii) it can be expressed as a polynomial equation in two variables due to deduction by memory. But how would we show the converse? That is, are all geometrical curves clearly and distinctly perceivable? Simply, the answer is yes, because if a curve is clearly and distinctly perceivable, the magnitudes admitted of the curve can be intuited. And thus, the curve admits of exact magnitudes, hence it is geometrical.

However, one might further worry that the clarity and distinctness criterion does not show that a non-geometrical curve cannot be clearly and distinctly perceived. This is, in fact, the criticism offered by Mancosu (2007): he argues that Descartes claims of a particular curve which is clearly and distinctly perceived is not actually geometrical. In the next section, I defend my demarcation condition by showing that, in those cases, what is clearly and distinctly perceived is not the curve nor any geometrical relations involved, but empirical or non-geometrical facts about the world. Hence, the curve does not satisfy my demarcation condition.

5 Defending Clarity and Distinctness as a Demarcation Condition

In this section, I briefly show that my demarcation condition is philosophically more fundamental than the other two criteria because it shows that geometrical curves can be understood from *rational reasoning alone*, independently of experience. Then, I focus Mancosu's challenge (Mancosu, 2007) against clarity and distinctness before defending against the challenge by focusing on the cycloid, and the 'string-like' curves as examples. I show how the construction of a cycloid involves two distinct motions in imagination, thus the cycloid cannot be clearly and distinctly perceived. Then I argue that the 'string-like' curves cannot be known, so they cannot be clearly and distinctly perceived. Since these are both known to be excluded from geometry, my arguments support the accuracy of my demarcation condition. I further defend against Mancosu (2007) and show that what Descartes claims to perceive clearly and distinctly in his letter to Mersenne is not a geometrical relation/magnitude. Thus, Mancosu's challenge does not hold against my demarcation condition.

Recall that Descartes considered that while both deduction and experience could give us certain knowledge, it is only the former that 'cannot be performed wrongly by an intellect which is in the least degree rational' (AT 10:365; CSM 1:12; Rule 2). He adds that the knowledge from experience can be fallacious because 'men take for granted certain poorly understood observations', but it is not due to 'faulty inferences' (AT 10:365; CSM 1:12; Rule 2). It follows then that 'arithmetic and geometry prove to be much more certain than other disciplines', not only because we arrive at the knowledge by deduction (or pure inference alone) rather than from experience, but also its objects are 'pure and simple' (AT 10:365; CSM 1:12; Rule 2). Thus, what determines a curve as geometrical is that we make no empirical assumptions and that the exact magnitudes admitted of the curve can be deduced starting from the intuited facts of pure and simple geometrical objects.

What does this tell us about the mechanical curves? Note that, according to Descartes, we cannot make faulty *inferences*, but only make errors when we accept certain *observational facts* to be true. What Descartes sees clearly and distinctly in Mancosu's criticism is an *observational fact* rather than a geometrical fact since the objects of observation are not pure and simple.

Similar remarks can be made of other mechanical curves. Although we cannot generally determine that a curve to be mechanical based on one particular construction of the curve, often the canonical construction is one that relies on empirical features of the world. It is not simply about the construction (as Mancosu might claim), but about whether the curve's construction can be clearly and distinctly imagined, independently of the empirical world. What is emphasised by my demarcation condition is the independency of empirical world on our geometrical knowledge.

5.1 Mancosu's Challenge

In his paper, Mancosu (2007) argues that the demarcation condition cannot be given by clarity and distinctness:

> In my opinion, this passage constitutes quite a challenge to all those who would like to use clearness and distinctness as the criteria which together define what can be conceived geometrically in opposition to what can only be analyzed mechanically. (Mancosu, 2007, p.119)

The passage Mancosu targets is from a letter Descartes writes to Mersenne on 27 May, 1638:

> ... you ask me if I think that a sphere which rotates on a plane describes a line equal to its circumference, to which I simply reply yes, according to one of the maxims I have written down, that is that whatever we conceive clearly and distinctly is true. For I conceive quite well that the same line can be something straight and something curved, like a string
>
> (Translation by Mancosu (2007, p.118); AT 2:140-1).

I will first explain what Descartes and Mersenne's problem is. Then I will give a reconstruction of what Mancosu's challenge is. Once the reconstruction is given, it will be apparent that the challenge does not apply to the demarcation by clarity and distinctnesss given in my paper.

As the sphere rotates on the plane, what Descartes sees is similar to a ball rolling on a plane. We can visualise this further and suppose that the ball is painted in green, and the paint is not yet dry. So as the ball makes a full rotation, it draws a line on the plane. What Descartes sees 'clearly and distinctly' is the equality between the line drawn on the plane and the circumference of the ball.

Note that the motion involved in the problem is the same as in the construction of the cycloid, which I argue shortly to be mechanical according to my demarcation criterion. For now, let us focus on the significance of the fact that the equality between the line and the circumference is clearly and distinctly perceived.

Recall that Descartes considers that there cannot be an error in deduction as long as it is done by at least rational mind. In that sense, the error made in clear and distinct perception is not from the continuous reasoning but in the assumption of observational facts as certain truths. In the case of Descartes and Mersenne's problem, the equality between the line traced and the circumference can be clearly and distinctly perceived because of the assumption that there's friction in the plane such that as the sphere rotates at a constant speed, it rolls on the plane by the distance equal to the angular rotation. That is, if the sphere has rotated by the angle θ then it rolls on the surface by the distance θ. Hence, what is being clearly and distinctly perceived is based on *observational facts* rather than *geometrical facts*. Thus, Mancosu's challenge does not target my demarcation criterion based on the doctrine of clarity and distinctness, since my demarcation criterion is based on Descartes's view that geometry is about pure and simple objects, and so the magnitudes of the curves that can be clearly and distinctly perceived by the mind alone.

I will now turn to the example of the cycloid to show how the canonical construction of the cycloid can *only* be clearly and distinctly perceived if we rely on some empirical observation, and that if we attempt to perceive it by the pure intellect alone, then the construction requires two distinct movements. Therefore, this construction, and hence the curve, is mechanical.[23]

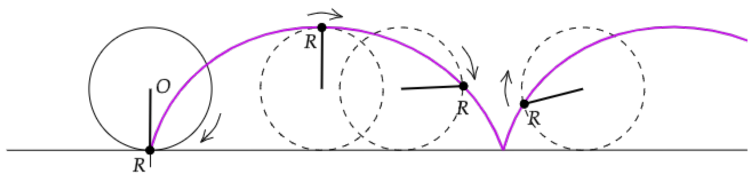

Figure 7: The Cycloid

5.2 The Construction of the Cycloid is Mechanical

A canonical construction of the cycloid is as follows. Given a circle, let R be a point on the circle. Consider the tangent to the circle at R. Now imagine the circle rolling along the given tangent such that R traces a curve as shown in figure 7. This curve is the cycloid.

As Jesseph (2007) shows, Descartes explicitly considers the cycloid mechanical in his correspondences with Mersenne:

> It should be noticed that the curves described by circles [*rouletes*] are lines entirely mechanical, and number among those that I have rejected from my geometry; it is therefore no wonder that their tangents cannot be found by any of the rules I have given (Translation by Jesseph (2007, p.242), AT 2:313)

However, Blaise Pascal 'saw the cycloid as a paradigm of geometric intelligibility' (Jesseph, 2007). In support of Descartes's claim, I will provide two arguments showing that the cyloid is mechanical. The first will show that we can see the construction clearly and distinctly, but it's based on some empirical observation, namely that the circle rolls over a plane with friction. The second argument will show that if we focus on the known geometrical figures (i.e. the radius of the circle), the construction cannot be clearly and distinctly perceived. This is because there are two distinct motions on the radius simultaneously. Hence, the construction is not clearly and distinctly perceivable.

[23] In order to really show that the curve is mechanical by its construction, we must evaluate all possible constructions of the curve. Since this is not humanly possible, one ought to turn to algebraic equations to explicate the curve's mechanicality.

When we imagine the circle rolling along the tangent, we are in fact making an assumption that there's friction on the tangenet line such that the circle would roll. If there was no friction, the circle's rotation will cause it to spin at the point R. This means the curve traced by R would be the circle itself, rather than the cycloid. By assuming that the circle is lying on the tangent which is a plane with friction, the circle moves along the plane as it starts rotating. This is an *observational fact* we have accepted when considering a circle as a ball rolling over a surface, and such claim cannot be deduced from purely geometrical facts.

One could argue that the circle and the line are geometrical figures, and the motion is clearly and distinctly perceivable, so the cycloid must be geometrical. But this is misleading because in this case, the circle in question is not necessarily geometrical. For a curve to be geometrical, we must have deduced each point on the circle from other known magnitudes. So when we know the radius of the circle, as that is an exact magnitude admitted of the circle, we can clearly and distinctly perceive the circle based on its radius. In other words, assuming the circle as a geometrical figure is an observational fact in this case, rather than a geometrical one which we have deduced.

So can we analyse this construction based on geometrical figures? That is, if we only consider the straight lines that are involved in generating the cycloid, would the construction be generated by a single continuous motion or successive motions? Take the radius of the circle as a straight line with known magnitudes. This line rotates around a point O. However, the point O moves along a straight line parallel to the tangent at R. Thus, there are two distinct motions involved in generating the cycloid based on geometrical figures. Hence, this construction is not clearly and distinctly perceivable.

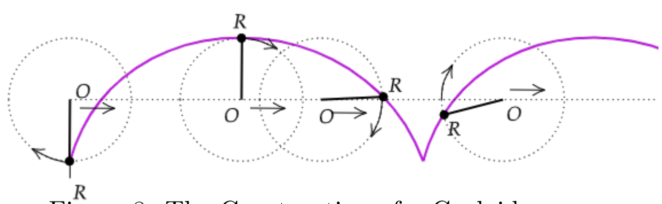

Figure 8: The Construction of a Cycloid

I've shown that the above construction of the cycloid is mechanical because the clear and distinct motion in its construction is not based on geometrical facts but on the observational fact involving friction, and if we focus on the geometrical figures alone, such as the line OR, the motion cannot be clearly and distinctly perceived. However, the demarcation of curves by the continuous motion can be applied to determine a curve geometrical but it cannot be used to determine a curve mechanical. In order to evaluate whether a curve is mechanical, we need to consider *all* possible constructions of the curve and determine that they are all mechanical. Unfortunately, this is not humanly possible, but there are other ways to determine whether a curve is geometrical. So let us turn to Descartes's 'string-like' curves.

5.3 String-like Curves: The Ratio between a Curve and a Straight Line

In *Geometry*, Descartes rejects the spiral and the quadratrix from geometry because their constructions involve two continuous motions. But there is another reason for excluding a curve from geometry which appears in his discussion of 'string-like curves'. He writes,

> [...] geometry should not include lines that are like strings, in that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact. (S&L p.91, *Geometry* p.340, AT 6:411)

Here, Descartes is claiming that the rational mind cannot *discover* the ratio between a length of a curved line and the length of a straight line. The relevance of this remark is as follows: for any geometrical curve, each point of the curve is determined by two unknown magnitudes such that there is a relation between them which can be expressed in an algebraic equation. Hence Descartes's remark that the 'ratios between straight and curved lines are not known' implies that there is no algebraic equation defining the ratio (i.e. a relation) between the straight and curved lines. If there was such an equation, then the relation can be known to us.[24]

A possible philosophical reason for rejecting that knowability of the ratio between a curved line and a straight line could be inferred from Descartes's view on angles. For Descartes, angles are not included in the algebraic equations, and his remarks on angles suggest that angles are always interpreted as ratios between two straight lines:

> ... since all the angles of the triangle ARB are known, the ratio between the sides AB and BR is known. ... the three angles of the triangle DRC are known, and therefore the ratio between the sides CR and CD is determined. (S&L p.29, *Geometry* p.310, AT 6:383)

It seems that for Descartes, any angle θ can be measured exactly, i.e. known, as long as there is a triangle ABC such that the lines AB and BC make the angle θ and the ratio between AB and

[24] One possible historical reason (other than Descartes's recognition that the construction of quadratrix cannot be geometrical) for rejecting that such ratio can be known might be due to Viète's equation, which shows that there is an infinite product expression of π as follows (see Boyer (1968, pp.352-3)):

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

Unlike the Archimedian approximation which is given geometrically, this equation is given algebraically. Descartes, possibly having known that there is an infinite product expression of π, and so that the ratios between a curve and a straight line can only be expressed in infinite products (or sums), which suggests that it would take infinitely long chains of deduction to know the ratio exactly. Thus the ratio cannot be known to the finite human mind.

BC can be known. This is precisely what the cosine rules tell us. Additionally, Descartes's use of similar triangles in his method of solving problems suggests that his treatment of angles were always in terms of similar triangles. Thus, angles are not magnitudes in the same sense as straight lines, but always are ratios between two straight lines.

Given this view on angles, let us return to the remark on the ratio between curved and straight lines. Let us consider a circle as a curved line and its radius r as a straight line, then the arc length l making angle θ can be given as $l = r\theta$. Then we can consider the ratio between the arc length l and the radius r as the angle θ. So knowing the angle θ means we can know the ratio between the two straight lines of a triangle making the angle θ, including one with a line length r. So we can straighten the arc l to a straight line of the same length. And since the construction of straightening a curve will allow one to construct the quadratrix, according to Mancosu (2007) such construction would not have been allowed in geometry by Descartes.

Whatever Descartes's actual reason might have been, it is clear that the ratio between a curved line and a straight line cannot be understood in the same way as we understand the relations between two straight line. Allowing such ratios as geometrical even suggests that we should allow all kinds of curves in geometry as long as their length can be deduced from a straight line based on this ratio.[25]

6 Conclusion

In this paper, I offered a novel criterion for demarcating curves in Descartes's *Geometry* based on his epistemological doctrine of clarity and distinctness. Broadly, the doctrine states that whatever can be clearly and distinctly perceived is true. On the face of it, the doctrine only seems to be applicable to propositions, but I argued that it can be applicable to any object and the truths about the object. I then argued that geometrical curves are those which are clearly and distinctly perceivable.

For a curve to be clearly and distinctly perceivable, its magnitudes have to be deducible. Descartes's deduction, I argued, comes in two types: deduction by imagination and deduction by memory. I also showed that the former being intuitively superior than the latter, and the latter being explanatorily superior than the former, while both are involved in Descartes's method of solving geometrical problems. For demarcating curves, I showed that when a curve's magnitudes are deduced by imagination, the curve can be traced in a single continuous movement, or successive movements, as Mancosu's criterion suggests. On the other hand, when a curve's magnitudes are deduced by memory, the relations between the magnitudes can be expressed in an algebraic equation, as Bos's criterion suggests. Thus, the criterion based on Clarity and Distinctness explains why

[25] In fact, Descartes allowed certain constructions using strings even if they were inexact, as long as they were used 'only to determine lines whose lengths are known' (S&L p.92, *Geometry*, p.341, AT 6:412). For example, as Panza (2011, pp.83ff) shows, Descartes accepts the gardener's constructions of ellipses and hyperbolas using strings, despite the inexactness of the equipment.

there is an extensional equivalence between Mancosu's and Bos's criteria. Thus, my demarcation criterion offers an advantageous view over Mancosu's and Bos's in that it provides a philosophical explanation for the mathematical demarcation conditions.

I further defended my criterion from a possible objection based on Descartes's claim that some clearly and distinctly perceivable curves cannot be included in geometry. On the face of it, it seems as though this directly contradicts the criterion based on Clarity and Distinctness. However, I argued that what Descartes sees clearly and distinctly is not the curve, but some observational facts about the physical reality. Deduction cannot be erroneous according Descartes, but empirical observations, no matter how clear or distinct they are, can lead us to mistakes. Thus, it clarifies that the demarcation criterion based on Clarity and Distinctness shows that geometrical curves must be clearly and distinctly perceivable by the pure intellect.

Descartes's demarcation of curves in *Geometry* is of philosophical interest to historians and philosophers of mathematics. Descartes defines geometrical curves as those which admit of exact measurement. While the demarcation problem is considered to be a metaphysical one, this paper shows that it is also an epistemological and a foundational one. Thus, Descartes's works in mathematics and his philosophy ought to be considered as closely related to each other.

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