On the Relativity of Magnitudes

Delboeuf's Forgotten Contribution to the 19th Century Problem of Space

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Abstract

The argument for the Euclidean nature of space based on the relativity of magnitudes has been overlooked by contemporary philosophers of physics and mathematics.In the present essay, we focus on the relevance of this argument to 19th Century philosophy of geometry. In this context, Delboeuf's contribution has been unduly neglected. Delboeuf's philosophy of geometry is more authentically neo-Kantian than that of Helmholtz, it is connected in curious ways to some of Leibniz's unpublished writings and helps to resolve certain questions in the foundations of Poincaré's philosophy of mathematics. We refute a fallacious argument against Delboeuf's ideas, espoused by Russell, which seems to have gone unchecked since. We conclude with some comments concerning the relevance of the discussion to scientific methodology and present-day cosmology.

Keywords: Delboeuf, Poincaré, Problem of space, Relativity of magnitude, Euclidean

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> Dieser Stoff kann also vorgestellt werden als ein physischer Raum; dessen Punkte sich in dem geometrischen bewegen.
>
> Bernhard Riemann
> *Gravitation und Lichts, 1853*

Introduction

Following the discovery of the mathematical possibility of non-Euclidean geometries by Lobachevsky (1829a,b), the "problem of space", that is, the problem of determining which among the available geometries should be chosen as that which represents the space of our physical world, drew the attention of pre-relativistic physicists, philosophers and mathematicians throughout the 19th Century. The well known contributions of Helmholtz (1870, 1876), Riemann (1854), Poincaré (1898) and Lie (1893) brought together an assemblage of empiricist and neo-Kantian ideas which would give birth to new perspectives, such as geometrical conventionalism. On one hand, neo-Kantian strands of thought suggested that the geometry of space should be something regarded as distinct from the material contents therein; on the other hand, empiricists argued that geometry is only an abstraction from the observed behaviours of material bodies. By considering space as a *condition for the possibility of measurement* rather than a condition for the possibility of experience, Helmholtz developed a form of *empiricist neo-Kantianism*, which had a profound and enduring influence on later neo-Kantians and logical positivists (Ryckman, 2003; Friedman, 2001, 2009).

There are, however, some problems with this standard 19th Century approach to the problem of space: Firstly, Helmholtz's "conditions for the possibility of measurement" rely on an approximately physically instantiated notion of rigid bodies, which was undermined by later developments in special and general relativity. Secondly, while Poincaré's lesser emphasis on empiricism arguably avoids this issue, the Helmholtzian views which he adopts appear to be inconsistent with certain other aspects of his philosophy of space (see sec. 3.3).

The aim of this paper is to introduce English-speaking audiences to Joseph Delboeuf's alternative approach to the problem of space, which has been overlooked by most recent accounts (for instance Dewar and Eisenthal (2020); Heinzmann (2001)). While Delboeuf's work of 1860 is likely one of the earliest examples of neo-Kantianism in the philosophy of geometry, he is not even mentioned in recent neo-Kantian texts in the philosophy of physics such as Friedman (2001, 2014/1983); Ryckman (2005); Bitbol et al. (2009). The most significant acknowledgement of Delboeuf's contribution since the mid 20th Century is by Torretti (2012/1978). While Toretti recognises that Delboeuf was "probably the earliest philosophical writer who had first-hand acquaintance with the works of Lobachevsky" (p.153), and acknowledges Delboeuf's ideas as "interesting" (p.298), he ultimately gives them a disfavorable verdict. Indeed, Torretti revives a fallacious objection to delboeuf's philosophy of geometry which had formerly been touted by Russell as well as Poincaré (see sec. 3.2).

Delboeuf defended the apriority of Euclidean space on the basis of the relativity of magnitudes. Although he developed his ideas largely independently, the central argument is not unique to him, indeed the insight dates back all the way to Wallis (1696) (see sec. 2.2.1), and has recently been revived by Čulina (2020, 2018, 2023). Delboeuf is unique in that he gives this notion of the relativity of magnitude the status of "first postulate", thereby attempting to erect geometry upon new foundations. Interestingly, his account bears a remarkable affinity to some of Leibniz's unpublished writings on the foundations of geometry (see sec. 2.3.2).

One might claim that this discussion of Euclidean apriorism is a purely historical curiosity; after all, we are equipped today with far more sophisticated mathematical tools than were available in the 19th Century. However, I contend that the issues that this essay helps to illuminate are of perennial philosophical import. Moreover, Delboeuf's ideas are directly relevant to physics in two domains: (1) the abundance of relativistic gravitational models that propose a return to flat space for methodological reasons (often by appealing to Poincaré's geometrical conventionalism), and (2) perhaps more interestingly, Delboeuf's ideas may provide philosophical grounds for speculative scale-invariant models of cosmology, which have recently seen rising interest.[1]

Among the novel arguments and findings made in this essay: (1) We uncover the remarkable convergence of thought between Delboeuf's approach and an essay by Leibniz titled *Uniformis locus* which has only been made available relatively recently by De Risi (2005, 2007) (see section 2.3). (2) we elucidate the fundamental difference between Helmholtz's empirically realisable motions, and Delboeuf's symmetry conditions, and determine that only the latter are relevant to the Kantian conception of space (see section 3.1). (3) We refute Bertrand Russell's enduring '*relative angles*' objection to Delboeuf's notion of the relativity of magnitudes (see section 3.2). (4) We propose that Delboeuf's *intuition of homogeneity* may provide a foundation, not merely of Euclidean geometry, but for *synthetic a priori* reasoning in general, and show how it may account for Poincaré's notion of the mathematical infinite (see section 3.3). We conclude in section 4 with some suggestions concerning the relevance of these findings to scientific methodology. In section 1, we set the scene by discussing certain key aspects of Poincaré's philosophy of space that do not directly involve the question of geometry, but which are nonetheless deeply relevant to later discussions.

1. Poincaré on space and mathematical reasoning

While Poincaré is quite famous today for his geometrical conventionalism (which will be discussed in section 2.1), other aspects of his philosophy of space and mathematics are equally significant and will help provide context for our subsequent discussion of geometry. In this section, we will briefly cover some of these aspects: (1) Poincaré's revision of the Kantian notion of synthetic a priori reasoning, (2) Poincaré's empiricist account of the distinction between changes of state and changes of place, (3) the distinction between empirical objects and their mathematical idealisations.

1.1. Mathematical reasoning

By the late 19^{th} Century, Kant's claim that mathematics contains synthetic a priori propositions was being subjected to severe criticism by the logicists. Frege and Russell attempted to show that all true mathematical statements could be derived from a basic set of concepts defined terminologically. Thus all mathematical truths would be reduced to logic, without need of intuitions.[2] Poincaré, on the other hand, did not abandon the notion of synthetic a priori reasoning, but rather, attempted to revise it.[3]

Poincaré expounds his conception of synthetic a priori knowledge in the first chapter of *Science and Hypothesis*, titled *On the Nature of Mathematical Reasoning*. Kant had claimed that the basic propositions of arithmetic, such as $5 + 7 = 12$, are *synthetic*, since nowhere in the concept of the sum of 5 and 7 is contained the concept of 12. Something additional is needed for Kant, that is, an intuition of space in which the two quantities can be placed side-by-side with one another, and the operation of summation can be accomplished (Kant, 2004/1783, p.18). Such a claim is controversial, in part because it depends greatly upon how we define things. It is not too difficult to define our numbers in such a way that basic propositions of arithmetic,

[1] For flat space models of general relativity or similar theories on flat space, see for example: Rosen (1940a,b); Gupta (1954); Kraichnan (1955); Dicke (1957); Thirring (1961); Huggins (1962); Weinberg (1964a,b); Ogievetsky and Polubarinov (1965); Mittelstaedt and Barbour (1967); Nachtmann et al. (1969); Deser (1970); Fang and Fronsdal (1979); Cavalleri and Spinelli (1980); Davies and Falkowski (1982); Logunov and Mestvirishvili (1985); Lasenby et al. (1998); Pitts and Schieve (2001); Arminjon (2002); Broekaert (2005). For the question of scale-invariance in cosmology, see for instance: Barbour (2010); Mercati (2018); Sloan (2018); Gryb and Sloan (2021)

[2] Note that for Frege, this meant that mathematics would be purely analytic, whereas Russell viewed logic as synthetic.

[3] See (Folina, 2016/1992) for an in depth discussion of Poincaré's neo-Kantianism.

such as $5 + 7 = 12$ or $2 + 2 = 4$ appear as analytic truths. Unlike Kant, Poincaré does not argue that there is anything synthetic in these basic propositions; instead, he claims that the so-called "demonstrations" of these sums are really only analytic "verifications". However, these trivial verifications are not the true subject matter of mathematics, on the contrary, Poincaré (2015/1913, p.33) insists:

> It may even be said the very object of the exact sciences is to spare us these direct verifications.

The essence of mathematics, for Poincaré, lies in the ability to generalise across an infinity of cases, using what is called "reasoning by recurrence", or "mathematical induction". The basic structure of a proof by induction proceeds as follows:

1. The theorem is proven for $n = 1$.
2. It is shown that if it is true for $n = a$, it must be true for $n = a + 1$.
3. Therefore we know that it is true for $n = 2$, and likewise $n = 3, 4, 5....$ By induction, we have shown that it must be true for all $n \in \mathbb{Z}$.

This enables one to make generalisations about some theorem over an infinity of cases.[4] It is in this possibility of reasoning by recurrence—which Poincaré calls "the mathematical reasoning *par excellence*"—that he locates the true *synthetic a priori* judgement (Poincaré, 2015/1913, p.39):

> This rule, inaccessible to analytic demonstration and to experience, is the veritable type of the synthetic a priori judgment. [...] Mathematical induction, that is, demonstration by recurrence, [...] imposes itself necessarily because it is only the affirmation of a property of the mind itself.

1.2. Empirical ground of space

In chapter IV of *Science and Hypothesis*, titled *Space and Geometry*, Poincaré takes up the perspective of a naive investigator attempting to make sense of the world present to his senses while lacking any pre-conceived notions about how these ought to be organised and interpreted. How do we come to the idea of space, and in particular, how do we distinguish between changes of position and changes of state (such as changes in colour)? Poincaré presents this problem as follows (Poincaré, 2015/1913, p.70):

> Whether an object changes its state or merely its position, this is always translated for us in the same manner: by a modification in an aggregate of impressions. How then could we have been led to distinguish between the two?

His solution is rather straight forward:

> It is easy to account for. If there has only been a change of position, we can restore the primitive aggregate of impressions by making movements which replace us opposite the mobile object in the same relative situation. We thus correct the modification that happened and we reestablish the initial state by an inverse modification.

A change in spatial position is distinguished from a change in state by the possibility of performing the reverse operation by means of the correlative movement of our own bodies. In the case of sight, this movement may also be performed by the "appropriate movement of the eyeball."

Now the possibility that certain "aggregates of impressions" may be restored through our correlative movements depends upon the existence of "solid bodies", i.e. bodies which retain the relations among their parts while changing position with respect to us. Indeed, it is the observation of solid bodies, Poincaré argues, that has taught us to distinguish between changes of state and changes of position, such that, he concludes:

> *if there were no solid bodies in nature, there would be no geometry.*

1.3. Mathematical idealisation

While this empiricist account of the origin of geometry is persuasive, it does not lead directly to the complete *mathematical* notion of geometric space. While I may observe that as a body recedes from me, I can restore its original size by approaching it once more, I can never infer from experience that this will continue to be true if the body recedes to an arbitrarily large distance. Should my concept of space therefore be limited to distances for which the compensation is practically realisable? Clearly, when we imagine objects in geometrical space, we do not limit ourselves to distances which our bodies are capable of traversing. Rather, we consider space as potentially infinite in extent, and thereby we implicitly imagine an *idealised* observer capable of visiting all parts of this space at will to perform the necessary compensatory motions. The space of geometry differs in this respect from the empirical condition that motivated its invention.

It is even more clear that an *idealisation* is involved when we place our representations in space-time.[5] In space-time we imagine bodies, extended in time as well as in space, and thereby we implicitly invoke the possibility of an idealised observer that may travel to the different parts of this space-time and measure it with ideal rulers and clocks. But these motions are by no means physically realisable. It is not even possible to visit distant points which seem to lie in our plane of simultaneity, let alone to travel into the past. If geometry can, in any sense, be said to have an empirical origin, it must have departed from this empirical conception in order to encompass the notion of time.

[4]In section 3.3, we will argue that this reasoning by recurrence is made possible due to a fundamental a priori intuition of homogeneity.

[5]The present comments concerning space-time are not drawn directly from Poincaré's works, however, they are inspired by his discussions of the issues. We will quote Poincaré explicitly towards the end of the subsection.

A second issue concerning the difference between the empirical and mathematical concepts of space is that no empirically given "natural solid" is absolutely rigid. When we look at any given body close enough we find motion and change in its structure, modifications due to heat, the vibrations of the constituent particles, and so on. These contingencies make it impossible for us to use empirical objects as standards for the definition of a mathematical space. As Poincaré (2015/1913, p.79) puts it:

> Geometry would be only the study of the movements of solids; but in reality it is not occupied with natural solids, it has for object certain ideal solids, absolutely rigid, which are only a simplified and very remote image of natural solids.

To reach the mathematical concept of space, we must substitute our empirical notions of solid bodies with their *ideal* counterparts; and in so doing we substitute the empirically grounded, physical concept of space—which we have no definite knowledge of—with the pure, mathematical concept of space, of which we have absolute knowledge *a priori*. Once we have performed this substitution of the empirical objects with their ideal counterparts we make it possible to apply the mathematical reasoning which Poincaré characterises as *synthetic a priori*: While we may attempt to infer by *physical* induction that a rock remains the same wherever it is placed in relation to the other bodies of the universe, this knowledge will only ever be approximate, contingent and subject to the possibility of being refuted by experience. However, if my object is not a rock, but an ideal rigid body in Euclidean space, I can say with absolute certainty that it will retain the relations amongst its parts no matter where it is placed in this space. In Poincaré's words (Poincaré, 2015/1913, p.40):

> Induction applied to the physical sciences is always uncertain, because it rests on the belief in a general order of the universe, an order outside of us. Mathematical induction, that is, demonstration by recurrence, on the contrary, imposes itself necessarily because it is only the affirmation of a property of the mind itself.

2. The problem of space's geometry

In the first part of this section, we briefly recount the well-known history of Helmholtz and Riemann's canonical 19th Century approach to the problem of spatial geometry which was based on the notion of the free mobility of bodies (2.1). In the latter parts (2.2 and 2.3), we discuss an alternative approach to the problem based on the idea of the relativity of magnitude.

2.1. The axiom of free mobility

The repeated failures to prove the necessity of Euclid's fifth postulate on the basis of the first four culminated in Lobachevsky's construction of a self-consistent geometry based on the denial of the parallel postulate (Lobachevsky, 1829a). Just as theorems concerning shapes in Euclidean geometry can be studied and proven, a corresponding set of theorems pertaining the Lobachevsky's hyperbolic geometry can be proven mathematically. Which set of theorems is, then, true of our space? This glaring ambiguity at the level of mathematics prompted various thinkers, including Riemann and Helmholtz, to seek an empirical ground for the validity of Euclidean geometry, or lack thereof.

2.1.1. Riemann

Following from Gauss' work on the geometry of curved surfaces, Riemann developed the general concept of a "multiply extended manifold" whose curvature may vary from point to point. Since the metrical properties of this manifold should be grounded in empirical facts, this manifold needed to be susceptible of measurement, which implied the mobility of certain quantities in space (Riemann, 1854):

> Measuring involves the superposition of the quantities to be compared; it therefore requires a means of transporting one quantity to be used as a standard for the others.

The first hypothesis that Riemann explores is that "the length of lines is independent of their configuration, so that every line can be measured by every other." This allows for a broad class of possible geometries that we now know as *Riemannian geometries*. Riemann also remarked that if we assume—not only that lines are independent of configuration—but also that the bodies are so, then:

> it follows that the curvature is everywhere constant, and the angle sum in all triangles is determined if it is known in one.

2.1.2. Helmholtz

Helmholtz placed great emphasis on this latter idea, arguing that the mobility of rigid bodies was a necessity for the possibility of measurement, and concluded that only the spaces of constant curvature could properly be considered as geometry (Helmholtz, 1866, 1870). Helmholtz acknowledges, however, that the natural bodies apparent to observation are never identical to our idealisations of these. In his latter paper (Helmholtz, 1870), Helmholtz approaches something like a Kantian view, according to which the notion of a geometric figure would be "formed independently of actual experience". However, Helmholtz insists that:

> we should have to maintain that the axioms of geometry are not synthetic propositions, as Kant held them: they would merely define what qualities and deportment a body must have to be recognised as rigid.

Rather than being a *condition for the possibility of representation*, like Kant thought, space and its geometry become *conditions for the possibility of measurement*, as it were.[6] Helmholtz

[6] This view is elaborated by Russell (1898).

finishes his essay by settling on a conventionalist stance according to which: if taken apart from mechanical propositions, the axioms of geometry "constitute a form into which any empirical content whatever will fit." However, this is not only true of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry (Helmholtz, 1870).[7]

2.1.3. Poincaré

Poincaré further developed this conventionalist standpoint (Poincaré, 1898, 1905/1902), for which he is quite famous today. According to Poincaré any empirical assertion of some given geometry over another is founded on a "disguised definition"; It is a *convention*, and only an appeal to some extra-empirical theory virtue such as "*simplicity*" may allow us to decide between conventions. Poincaré's preference for conventionalism, rather than a purer form of Kantianism was grounded in his *group-theoretic* approach to the problem. For Poincaré space is not a form of the sensability, since "sensations by themselves have no spatial character", rather the "sensible space" must be a *form of our understanding*: "it is an instrument which serves us not to represent things to ourselves, but to reason upon things" (Poincaré, 1898).

However, the geometry of this form cannot be determined a priori for Poincaré, since there are a multiplicity of conceivable forms that we may use to reason on things. These are the *group structures*, which are the objects of study of mathematics. The various transformations on a Euclidean space are only one among a multitude of possible group structures that may be employed if experience warrants it. From this point of view, nothing truly distinguishes Euclidean geometry from the alternatives apart from its simplicity, and the fact that it is at least approximately instantiated in the observable behaviours of natural solids and rays of light.

Despite claiming the conventionality of geometry, Poincaré devotes very little attention in his work to geometries of changing curvature. The bulk of Poincaré's discussion of the conventionality of geometry in part II on *Space* (chapters III, IV and V) of *Science and Hypothesis* concerns the geometries of constant curvature. The reason for his neglect of the former is given in the one passage in which they are briefly discussed (Poincaré, 2015/1913, p.63):

> most of these definitions are incompatible with the motion of a rigid figure, [...] These geometries of Riemann, in many ways so interesting, could never therefore be other than purely analytic and would not lend themselves to demonstrations analogous to those of Euclid.

Like Helmholtz, Poincaré rejects Riemann's geometries of changing curvature on the basis that they are incompatible with the motion of rigid figures. However, for Poincaré the crucial point here is that this incompatibility would undermine the very aim of mathematics. Since the geometrical properties of figures in a space of changing curvature would depend upon the value of the curvature from place to place, it would become hopeless to make those inductive *generalisations* that Poincaré views as so central to mathematical reasoning. Particular propositions about these geometries would not be synthetic, but *analytic*, since they would depend upon how the curvature is defined to change from point to point.

2.2. The relativity of magnitude I: before Delboeuf

Helmholtz and Poincaré's refutations of Kant's Euclidean a priori rest essentially on a single claim: that the geometries of constant positive or negative curvature of Riemann and Lobachevsky respectively may just as well serve as *forms* into which the empirical content of our sensations may be placed. These geometries, they say, have just the same right to be viewed as *transcendental* as that of Euclid.

But is there not some characteristic of Euclidean space, beyond its mere "simplicity", that sets it apart from those of constant non-zero curvature? Indeed there is. It is that *the Euclidean space remains similar to itself at different scales.* In other words, we may *zoom into* some part of this space without changing anything about it. Thinking in terms of figures, rather than space itself, Euclidean space is the only space which allows for the possibility of incongruent *similar* figures (i.e. figures which differ in size but possess the same shape). All other geometries necessarily fail this test since curvature is a scale-dependent property of space. For instance, the sum of the angles of a triangle placed in Lobachevsky's hyperbolic space will shrink as the triangle is enlarged with respect to this space; therefore, two equilateral triangles of different sizes will not be similar.

This criterion, by which Euclidean space can be uniquely determined, seems to have only been considered a handful of times in the history of geometry.[8] For us, it is easy to become conscious of it, since we have knowledge of non-Euclidean geometry, and we can thereby easily identify what characteristic distinguishes Euclidean geometry by contrast. Prior to the development of the theory of non-Euclidean geometries however, it would have been more difficult to deduce the relationship between the absence of an absolute scale and the parallel postulate.

2.2.1. Wallis (1663)

The relationship between the possibility of similarities and the parallel postulate was first recognised by the English mathematician John Wallis, Savilian Chair at Oxford, in 1663 (although his proof was published in 1696 (Wallis, 1696)), over a century prior to the discovery of non-Euclidean geometries. Wallis attempted to show that Euclid's fifth postulate can be deduced from ideas which are self-evident. Though his proof is usually regarded as yet another failed historical attempt to prove the parallel postulate, we will see that his argument is quite significant and profound.[9]

[7]The "pseudospherical" geometry is Helmholtz's term for the hyperbolic geometry of Lobachevsky and Bolyai.

[8]See Čulina (2018) for a modern proof.

[9]See for instance Jammer (2013, p.145) for a characterisation of Wallis's proof as a failed attempt to prove the parallel postulate.

Wallis's proof is in two parts:[10]

1. Firstly, Wallis demonstrates that Euclid's fifth postulate is identical to the possibility of constructing *similar* triangles, that is, triangles which have the same shape though they differ in size.
2. Secondly, Wallis provides a metaphysical argument for the possibility of transformations by similarity. This is due to the distinction between quality and quantity. Whereas, for Wallis, the size of a figure is a *quantity*, the shape of a figure belongs to the category of *quality*. Being different categories, these two must be able to vary independently of one another.

2.2.2. Carnot and Laplace

The next mention of this relationship between the possibility of similar figures and the parallel postulate appears in a note in Carnot's *Géometrie de Position* (Carnot, 1803):[11]

> The theory of parallels depends on a more primary notion which appears to me to be of the same order of clarity as that of the perfect equality or superposition of figures; this is the notion of *similarity*. It seems to me that it can be regarded as a self-evident principle, that that which exists as large, such as a ball, a house, a drawing, can be made in small, and vice-versa; by consequence, whatever figure we may imagine, it is possible to imagine others of all sizes and similar to the first, that-is-to-say of which all the dimensions have amongst themselves the same proportions as that of the first. This notion once admitted, it is easy to establish the theory of parallels, without recourse to the notion of infinity.

Though Carnot asserts that the proof is easy, he does not derive it. Moreover, he does not cite Wallis's proof, so it is not clear whether or not he learned of it from there.

This idea is also mentioned by Laplace in passing amid a discussion of the scale-invariance of the inverse-square law of gravitational attraction (Laplace, 1835, p.471-472). Likewise in a footnote we find:[12]

> The perception of extension contains a special property, self-evident and without which we cannot rigorously establish the properties of parallels. The idea of a limited extension, for example of the circle, contains nothing which depends on its absolute size. But, if we diminish, by thought, its radius, we are inevitably inclined to diminish in the same ratio its circumference and the sides of all the figures inscribed. This proportionality appears to be a much more natural postulate than that of Euclid; it is curious to find it again in the results of universal gravity.

Once again, the work of Wallis is not mentioned.

2.3. The relativity of magnitude II: Delboeuf

Joseph Delboeuf was a Belgian psychologist, mathematician, and philosopher. Although he spent the bulk of his career as an experimental psychologist, he obtained doctoral degrees in both philosophy and mathematics and was deeply concerned with the foundations of geometry in his youth. While he was studying philosophy at the University of Liège, his friend and colleague François Folie had attempted to prove the necessity of Euclid's parallel postulate. Folie's professor had pointed out the questionable proposition involved, and this disappointment led Folie to abandon the endeavour (Delboeuf, 1895, p.346). Delboeuf, on the other hand, did not abandon his youthful ambitions, and some years later published a radical reconception of geometry that would place Euclidean intuitions surely at its foundation (Delboeuf, 1860).

The difficulty with Euclid's fifth postulate draws investigators into a labyrinth from which they can only escape by a total revolution in thinking about geometry. In this respect, two pathways are available; we may either (1) seek new foundations for Euclidean geometry, or (2) we should absorb Euclid's geometry into a more general conception, of which Euclid's is only a special case. The second approach, that of the "neo-geometers", has been favoured by history. Delboeuf, on the other hand, embarks upon the first project (Delboeuf, 1894b, p.122). Lobachevsky's discovery, for Delboeuf, did not disprove the necessity of the parallel postulate in geometry; rather, it only served to help us better understand what our Euclidean intuitions are founded on.

In his *Prolegomenes Philosophiques De La Geometrie Et Solution Des Postulats* of 1860, Delboeuf proposes that the *homogeneity* of space be taken as the *first postulate* of geometry (Delboeuf, 1860, p.129). Homogeneity, for Delboeuf, is a more restrictive criterion than what this word usually means today. Today, people define the homogeneity of a manifold by the criterion that all points stand in the same relation to the whole; the whole can be decomposed into *equal* parts. This condition holds of the circumference of a circle for instance, or the

[10] The original text is written in Latin by Wallis. We will not go through the details of Wallis's demonstration here; the reader can consult this in Hill (1925) for a reconstruction of the proof in English. See also Therrien (2020) for a more detailed discussion of Wallis's proof.

[11] Translation of: "La théorie des parallèles tient à une notion première qui me paroît être à-peu-près du même ordre de clarté que celle de l'égalité parfaite ou de la superposition; c'est la notion de *similitude*. Il me semble qu'on peut regarder comme un principe de première évidence, que ce qui existe en grand, comme une boule, une maison, un dessin, peut être fait en petit et réciproquement; que part conséquent, quelque figure qu'on veille imaginer, il est possible d'en imaginer d'autres de toutes grandeurs et semblables à la première, c'est-à-dire dont toutes les dimensions aient entre elles les mêmes proportions que celles de la première. Cette notion une fois admise, il et facile détablire la théorie des parallèles, sans recourir à la notion de l'infini." (Emphasis in original).

[12] Translation of: "La perception de l'étendue renferme donc une propriété spéciale, évidente par elle-meme et sans laquelle on no peut rigoureusement établir les propriétés des parallèles. L'idée d'une étendue limitée, par exemple du cercle, ne contient rien qui dépende de sa grandeur absolue. Mais, si nous diminuons, par la pensée, son rayon, nous sommes portés invinciblement à diminuer dans le meme rapport sa circonférence et les côtes de toutes les figures inscrites. Cette proportionnalité me parait être un *postulatum* bien plus naturel que celui d'Euclide; il est curieux de la retrouver dans les résultats de la pesanteur universelle."

surface of a sphere. Delboeuf calls this criterion "isogeneity"; *homogeneity*, on the other hand, is a stronger criterion according to which the parts of a homogeneous "quantum" are related by *similarity*, i.e. a homogeneous quantum is similar to itself at different scales.[13] The only examples of homogeneous quanta that can be represented in three dimensional space are: *the straight line*, *the plane*, and *Euclidean space itself* (Delboeuf, 1860, p.145):[14]

> The circumference and the spherical surface are not homogeneous like the line, the plane and our space, but simply isogenous.

An *isogenous* space would permit the possibility of *equal* figures about any point; that is, it allows for the *congruence* of figures. We have seen above how this criterion implies that such a space must be of constant curvature. But a *homogeneous* space, in addition to being isogenous, must permit the possibility of *similar* but incongruent figures, that is, changes in a figure's size do not affect its shape (Delboeuf, 1860, p.133). As Delboeuf puts it:[15]

> *The reciprocal independence of shape and size*, such is the *first postulate* of geometry. We have established *philosophically*, basing ourselves on the homogeneity of space, the need for our mind to conceive and admit this independence.

It is not hard to notice that this is exactly the same axiom that Wallis (1696) had discovered to replace the parallel postulate, indeed for both of these gentlemen, the possibility of similarity is not in need of mathematical proof, but must be asserted philosophically as a foundational principle of geometry. There is no indication that Delboeuf was aware of Wallis's proof since it was never cited in his works. Unlike Wallis, however, who had merely used this principle to replace the parallel postulate, Delboeuf elevates it to the status of *first postulate* and derives the entire edifice of Euclidean geometry from it (Mach, 1906, p.119).

2.3.1. Mathematical definitions

In present-day terms, we may define Delboeuf's notion of isogeneity in terms of the translational symmetry of a metric. Suppose the group action of the translational group T on a manifold M is an exact isometry of the metric g. In that case, we can call the geometry (M, g) *translationally symmetric*, which corresponds to what Delboeuf calls *isogeneity*. It can be shown that non-Euclidean geometries of constant curvature such as the hyperbolic geometry of Lobachevsky and the spherical geometry of Riemann, are translationally symmetric.

Delboeuf's notion of *homogeneity*, or self-similarity corresponds to a special case of *conformal isometry*. Conformal isometries are transformations ϕ which preserve the metric structure up to an overall scale factor Ω^2:

$$\phi^* g = \Omega^2 g \tag{1}$$

If Ω^2 is constant across space, then the transformation produced by ϕ is a dilatation. It can be shown that for a Riemannian manifold, the combination of translational isometry and rescaling conformal isometry uniquely selects Euclidean geometry (Wald, 2010).

2.3.2. The Leibniz connection

One of the most intriguing things about Delboeuf's concepts of homogeneity and isogeneity is that one can find an almost identical discussion in an unpublished essay of Leibniz titled *Uniformis locus* which has so far been discussed solely in the work of Vincenzo de Rizi (De Risi, 2005, 2007, 2015). The essay, which is accessible in De Risi (2007, p.582-585) (Latin) and De Risi (2005) (English), begins with the definitions of the plane, the straight and space in terms of self-simlarity:[16]

> A locus can be called uniform or self-congruent if its congruently bounded parts are congruent. On the other hand, a locus is self-similar if its similarly bounded parts are similar. The only self-similar loci are the straight line, the plane, and space itself. Uniform loci include all self-similar loci and, besides, others—that is to say, among the lines, the arc of a circle and the cylindrical helix and, among the surfaces, the spherical and the cylindrical ones.

The *self-similarity* Leibniz speaks of corresponds to the *homogeneity* of Delboeuf, whereas *self-congruence* corresponds to Delboeuf's *isogeneity*. This is confirmed a little later in the essay, where Leibniz writes:

> As I have discussed similarity and congruence, I have also distinguished between homogeneity and equality. In fact, the loci that can be transformed into similar ones are homogeneous; while the loci that can be transformed into congruent ones are equal.

The resemblance to Delboeuf's writings here is quite striking. Not only does Leibniz share Delboeuf's definition of homogeneity, but indeed, Delboeuf himself had used the notion of the equality of all points to define isogenous quanta. There is no evidence however that Delboeuf was aware of this essay by Leibniz, since it had not been published at the time. In fact, Delboeuf recounts that he was only alerted to the similarity between his work and some of Leibniz's other writings (which had just been made available two years prior in Leibniz (1858)) by his mentor Ueberweg after the publication of his book (Delboeuf, 1895, p.346). While it is plausible that Delboeuf may

[13] In his writings, Delboeuf uses the term "quantum" to refer to geometrical objects such as lines, surfaces or spaces.

[14] Translation of: "La circonférence et la surface sphérique ne sont pas *homogènes* comme la droite , le plan et notre espace , mais simplement *isogènes*"

[15] Translation of: "*L'indépendance réciproque de la forme et de la grandeur*, tel est le *premier postulat* de la géométrie. Nous avons établi *philosophiquement*, en nous basant sur l'homogénéité de l'espace, la nécessité pour notre esprit de concevoir et d'admettre cette indépendance." (Emphasis in original).

[16] The term "locus", which Leibniz uses, corresponds to the term "quantum" in Delboeuf's writings.

have been influenced indirectly by Leibniz through his conversations with Ueberweg, the similarity between Leibniz and Delboeuf should be understood first and foremost as an instance of convergence of thought, suggesting a remarkable affinity between their respective attempts to lay new foundations for geometry in the light of the difficulties concerning the parallel postulate.[17]

2.3.3. Neo-Kantianism

Delboeuf's philosophy of geometry is essentially neo-Kantian. Kant's philosophy is discussed throughout *Prolegomenes*, indeed the central argument of the text is framed in terms of a dialectic between Kantian idealism and Ueberweg's realism. The influence of Kant's thought is perhaps made most clear in book III, chp. 1 during his discussion of the definitions of the straight line and the plane. Delboeuf notes that there are a multitude of available definitions of a straight: viewed from the standpoint of *distance*, it is the shortest path between two points; from that of *direction*, it is a line of constant direction, and so on (Delboeuf, 1860, p.175). Given one of these definitions, the others would appear as *synthetic* truths, but none can be used to demonstrate the others analytically. However, these *synthetic* theorems, Delboeuf argues, are each in fact *analytic* decompositions of the original intuition that gave rise to them all (p.177). If we wish to escape the paradoxes, to overcome the impossible task of deriving one definition from another, we must seek to characterise the fundamental *essence* of the straight or of the plane. This leads Delboeuf to define them in terms of their *homogeneity*; like Euclidean space, the straight and the plane are distinguished by their invariance under dilations (p.180):[18]

> *The plane is a homogeneous surface;*
>
> *The straight is a homogeneous line;*[19]
>
> that is to say that a portion of a plane, *magnified*, generates the same plane; that a portion of a straight, *magnified*, reproduces the straight. We can therefore regard homogeneity as being the *genetic* characteristic of space, of the plane, and of the line.

Throughout his writings on the foundations of geometry, Delboeuf places great emphasis on his view that *space*, the *plane* and the *straight* are not real things; on the contrary, these homogeneous forms are pure products of thought, purely ideal and immutable. Returning to the topic 33 years after his original publication of 1860, Delboeuf devotes his essay of 1893 entirely to distinguishing his ideal Euclidean space from the real space of experience (Delboeuf, 1893). Concerning Delboeuf's views on *real* space, Torretti (2012/1978) has brought attention to the following intriguing passage in which Delboeuf argues that the non-Euclidean geometries of constant curvature would in no ways help us to represent real space (Delboeuf, 1894a, p.372):[20]

> We can therefore say of Riemann and Lobachevsky's spaces that they are artificial spaces, like Euclidean space; and in this respect they are just as geometrical as Euclidean space. But they have no special quality to represent real space better than the latter. This [real space] certainly has a curvature, but this curvature is different at each of its points and varies there at each instant. The real figures, that is to say, the bodies, change with time and place. The constant curvatures of meta-Euclidian spaces are therefore as far from reality as is the homogeneity of Euclidean space.

From the present-day perspective, we are compelled to respond: *why not then ditch Euclidean space, and the "meta-Euclidean" spaces of constant curvature? Why not embrace the varying curvature of real space and apply Riemann's broader notion of differentiable manifolds?* Delboeuf seems on the point of anticipating the revolutions of the subsequent decades, but instead he passes by this and retreats to his aprioristic defence of Euclidean geometry. Torretti (2012/1978, p.300) takes this as evidence that Delboeuf had not read Riemann. However, it is more likely that Delboeuf avoided considering Riemann's geometries of changing curvature for the very same reason that most commentators did at the time; that, as Poincaré put it, these geometries are purely *analytic*, they do not permit the kinds of mathematical generalisations that are the essence of all *a priori synthetic* reasoning. In Delboeuf's terminology, they are not "geometrical". Delboeuf, as we have seen, sought to ground geometry in *intuition*, but our intuitions are powerless to predict how the curvature of real, empirical space might change from point to point. Space from the standpoint of experience would begin with diverse and heterogeneous phenomena, resisting all generalisations. But space, from the standpoint of mathematics, begins with our most basic intuition of pure homogeneity. As Delboeuf puts it (Delboeuf, 1860, p.73):[21]

[17] For more information concerning Leibniz's work on the foundations of geometry, see De Risi (2015).

[18] Translation of: "Le plan est une surface homogène; La droite est une ligne homogène; c'est-à-dire qu'une portion de plan , majorée , engendre le même plan; qu'une portion de droite , majorée , reproduit la droite. Nous pouvons donc regarder l'homogénéité comme étant le caractère génétique de l'espace , du plan , de la droite."

[19] It is this definition, that, as mentioned above, Delboeuf's mentor Ueberweg remarked had been given previously by Leibniz. The relevant passages can be found in (Leibniz, 1858, p.185, 188): "Recta est linea, cujus pars quaevis est similis toti", and "Ego quoque aliquas plani definitiones commentus sum. Una est, ut sit superficies, in qua pars similis toti", though they may also occur in other parts of Leibniz's works.

[20] Translation of: "On peut donc dire des espaces de Riemann et Lobatschewsky, que ce sont des espaces artificiels, comme l'espace euclidien; et sous ce rapport, ils sont tout aussi géométriques que l'espace euclidien. Mais ils n'ont pas qualité spéciale pour représenter mieux que lui l'espace réel. Celui-ci, comme je l'ai dit dans ma première étude, a certainement une courbure, mais cette courbure est différente en chacun de ses points et y varie à chaque instant. Les figures réelles, c'est-à-dire les corps, y changent avec le temps et avec le lieu. Les courbures constantes des espaces méteuclidiens sont donc aussi éloignées de la réalité que l'est l'homogénéité de l'espace euclidien."

[21] The full passage in French reads: "Il y a donc, comme nous l'avons dit dans le premier paragraphe de ce chapitre, une double géométrie. L'une s'efforce, une forme étant donnée, de la ramener à une forme idéale on en a des exemples dans la cristallographie, et dans les recherches de Kepler pour déterminer la na-

> There is therefore, [...] a double geometry. The first strives, given a natural form, to bring it back to an ideal form; [...] this experimental geometry studies bodies or phenomena independently of their substance [...]. The second, theoretical geometry, follows an inverse course, and, starting from ideal, absolute principles, creates an infinity of forms and seeks to make them coincide with natural forms.

Moreover, it is worth remarking that Riemann's own writings are not in contradiction with Delboeuf's position as Torretti seems to believe. Amid some of Riemann's most suggestive remarks, in which he proposes (as early as 1853) that the force of gravity be described along with inertia in terms of the dynamical geometry of a physical space, anticipating Einstein's equivalence principle (Riemann, 1876), we find the very same distinction between a *physical* (or real) and *geometrical* space that Delboeuf makes:[22]

> I seek the cause [of gravity] in the state of motion of the continuous substance spread throughout the entire infinite space. [...] this substance may be thought of as a physical space whose points move in geometrical space.

The only difference to Delboeuf is that Riemann ventures to propose some means by which this dynamical space of changing curvature might be described. Neither disagrees, however, on the distinctness of the concept of *geometrical* space.

3. Reflections and discussions

3.1. Mobility or Leibniz shifts?

Readers may have noticed the contrast between Delboeuf's arguments for scale-invariance and Helmholtz's requirements for the possibility of congruence. Helmholts's notion of congruence is empirically grounded, it depends upon the *physically realisable* motions of natural solids. Dilations of natural solids, on the other hand, are not physically realisable. On what grounds, then, do we assert the possibility of *similarities*?

In the physically realised motion of a natural solid, the given body is known to have been moved because it has changed its relation to other bodies. *Has it moved in geometrical space?* That is entirely a matter of convention since this space is a creation of our minds. The only space in which we know it to be moved is the *relative* space defined, and perhaps *conditioned*, by the surrounding bodies. By verifying the empirical possibility of congruence, we have only shown that this *physical* space—the relative space conditioned by surrounding bodies—is approximately isogenous. We have shown nothing of geometrical space.

The translations that are analogous to the dilatations imagined by Wallis, Laplace, Delboeuf, and others, are not translations of single bodies with respect to others, they are *Leibniz shifts*: motions of all bodies in the universe with respect to geometrical space itself. These motions lack any physical meaning, and take place only in our minds. We are not concerned with motions of some bodies with respect to others, since such motions could only tell us of the properties of bodies. We want to know about the properties of space itself, and, since we conceive this space as *passive*, we may assert that the relations among bodies should be invariant under Leibniz shifts. This condition tells us with certainty that the curvature of geometrical space is constant. Further, we also assert the invariance of the relations among bodies under *universal dilations*, which tells us that geometrical space must be Euclidean.

The impossibility of dilating natural solids with respect to one another informs us that the real, physical space they mutually inhabit possesses a definite scale, but tells us nothing about the properties of an ideal space. The converse is also true: the possibility of dilating all bodies with respect to an imagined space ensures that it must be Euclidean, but says nothing of the physical properties bodies possess with respect to one another. One class of motions is proper to the one, the other to the other, but the two are not interchangeable. In his *Metaphysical Foundations of Natural Science*, Kant explicitly distinguishes between absolute space, which is an ideal form, and empirical or relative space which are those spaces in which we perceive objects to be moved (Kant, 1970/1786, p.16-17). Therefore Helmholtz's claim that the possibility of congruence of natural solids implies anything about Kant's ideal forms, is simply mistaken.

If we admit that Helmholtz's empiricist method can only tell us of the geometry of a real or physical space, then history has vindicated Delboeuf's objection to this. As we saw in section 2.3.3, Delboeuf insisted that real space has a curvature which is "different at each of its points and varies there at each instant", therefore the non-Euclidean spaces of constant curvature "have no special quality to represent real space better than the [Euclidean]" (Delboeuf, 1894a, p.372). On the other hand, If we wish to determine the properties of a space conceived as a pure form in the Kantian sense, then, once again, it is Delboeuf's method which is more appropriate.

3.2. Russell's relative angles objection

The tendency to assert the relativity of position to the neglect of the relativity of magnitude is epitomised by Russell (1897, 1898). Towards the end of the 19th Century, like Helmholtz and Poincaré, Russell defended the notion that space should be of constant curvature, but that we had no criterion by means of which to favour Euclidean geometry. In his essay of 1898,

ture de l'orbite de Mars qui était toute tracée dans le ciel ; cette géométrie expérimentale étudie les corps ou les phénomènes indépendamment de leur substance (et de la substance transformée en force). L'autre, la géométrie théorique, suit une marche inverse, et, partant de principes idéaux, absolus, crée des formes à l'infini et cherche à les faire coïncider avec les formes naturelles."

[22] The contracted passage given above is from Peter Pesic's English translation (Riemann, 2007/1853). The full passage in German reads as follows: "Die nach Grosse und Richtung bestimmte Ursache (beschleunigende Schwerkraft); welche nach 3. in jedem Punkte des Raumes stattfindet, suche ich in der Bewegungsform eines durch den ganzen unendlichen Raum stetig verbreiteten Stoffes, und zwar nehme ich an, dass die Richtung der Bewegung der Richtung der aus ihr zu erldarenden Kraft gleih, und ihre Geschwindigkeit der Grosse der Kraft proportional sei. Dieser Stoff kann also vorgestellt werden als ein physischer Raum; dessen Punkte sich in dem geometrischen bewegen."

Russell addresses Delboeuf's argument for the apriority of Euclidean space directly (Russell, 1898):

> I come now to the principal argument in favour of the a priori character of Euclidean space, namely the argument which derives from the impossibility of an absolute magnitude. For this discussion, it will suit me better to adopt Delboeuf's terminology than to insist upon my own. [...] The question is: Can it be demonstrated that homogeneity is an a priori property of space?
>
> On this point, a strong argument in my favour is derived, I believe, from the absolute magnitude of angles. Those who affirm it to be evident a priori that the sides of a triangle can be lengthened in a given ratio without altering the angles, ought to hold, it seems to me, that it is equally possible to alter all the angles in a given ratio without altering the sides. But that, we know, is impossible in all Geometries. If the logically relative nature of all magnitude is admitted, I cannot see why the argument would apply only to linear dimensions, and not to angles, which are equally magnitudes.

This rather strange argument by Russell might be dismissed as an idiosyncrasy were it not that others have claimed the same. It is independently repeated by Torretti (2012/1978, p.297) for instance. Even Poincaré (1898) raises the same objection in his own essay of 1898:

> It is absurd, they say, to suppose a length can be equal to an abstract number. But why? Why is it absurd for a length and not absurd for an angle?

Both Russell and Poincaré's remarks were published in 1898. Delboeuf, who had died abruptly in 1896 at the age of 64, did not get a chance to address the objection. On his behalf, therefore, we will show that this argument is founded on a blatant misconception.

The rebuttal to this argument is contained in Russell's formulation of it; *the relativity of angles is impossible in all geometries.* Why is this?

One does not need to compare two angles in order to measure them. The angle of intersection of two lines is already a relation, a relation between the directions of these two lines. Angles are expressed by numbers, but these numbers are always ratios of the full *revolution*: 2π. This 2π, however, is not a property of space itself, it simply denotes the size chosen for the revolution. If we were to double the value attributed to all angles, we would necessarily have to double the value of the *revolution* as well, which would become 4π, but this would amount to a mere change of coordinates. Nothing of the *angles* themselves will have changed since the ratio of their new values to the *revolution* will have remained invariant. An angle denotes an objective relation between two directions, not relations of objects to space. While we may assert the relativity of directions (based on the isotropy of space), we cannot assert the relativity of angles. Similarly, while we may assert the relativity of magnitudes (based upon the homogeneity of space), we cannot assert the relativity of ratios between magnitudes. We cannot assert the relativity of angles therefore, for the very same reason that we cannot assert the relativity of ratios between magnitudes.

Angles differ from magnitudes, in that they do not live in an unbounded, *homogeneous* (in Delboeuf's sense) space. Rather, the set of possible angles between two lines traces an inherently *isogenous* "quantum": the circle. But, we might ask, what if space itself were isogenous too? Russell continues (Russell, 1898):

> We have an angular space-constant in every space, namely the four right angles. [...] angles are tied absolutely to their magnitude, and cannot be conceived as all expanded in a given ratio. We cannot therefore infer, from the fact that magnitude is relative, the impossibility of a space-constant.

The fallacy in Russell's argument is that he assumes that angles denote relations of things to space. In fact, they do not, and that is exactly the reason why they are "tied absolutely to their magnitude". They are already relations among things. But let us, for the sake of argument, entertain Russell's reasoning: We might imagine space as a hyperspherical surface possessing a definite scale defined by a *curvature-constant*, $\lambda \neq 0$, along with a definite *angular-constant*, 2π, at every point. In this case, scale too would occupy an isogenous space. But we are here witness to a sleight-of-hand. Delboeuf's assertion of the homogeneity of space is, above all, an assertion of its *ideality*. Real things do not have real relations to space, but only to one another. If space possessed a scale constant, then bodies would be endowed with real relations to space, and space would no longer be *ideal*. Russell asks: *what is the problem with that, after all, space already possesses an angular-constant?* But this angle constant is not a property of space, since angles are not relations of bodies to space, rather, as we have seen, angles are relations between the directions of lines.

As relations, angles are measurable in and of themselves, whereas the magnitudes of bodies must be compared to one another if they are to be measured. This is the source of their relativity. Epistemically, all measurement is a relation between two given things. As we have seen, Laplace (1835, p.472) in particular emphasised this point with regards to magnitude. By means of this principle, it is easy to rule out the possibility of using non-Euclidean geometries as forms for phenomena. If it is claimed that we live, or should represent ourselves to live, in a space of constant positive or negative curvature, we must ask upon what reason this claim is based. There are two options:

1. If this choice is grounded upon some empirical observations, suppose for instance that we live on the surface of a hypersphere; then according to the principle stated above, this hypersphere—whose curvature is measurable with respect to real objects—must be a real object itself.
2. If on the other hand, it is not grounded empirically, rather, this geometry is being used purely in its capacity as a form; then there is no reason to choose it over the Euclidean. In this case, Poincaré's simplicity criterion rules it out, moreover, a compensatory field would need to be invented to abrogate its needless effects.

In both cases we find therefore that the geometries of constant, nonzero curvature, cannot—despite what was asserted by Helmholtz and others—be used as forms for phenomena on account of the relativity of all measurement of sizes. This argument naturally carries over to the more general geometries of changing curvature as well, which are even less competent to be conceived as forms. We are forced to the conclusion that space, as pure form, must be Euclidean.

3.3. Synthetic knowledge and the passivity of space

Given the weakness of the *relative angles* objection, it is surprising that Poincaré approves it in his essay of 1898 (Poincaré, 1898). A decade later, however, Poincaré's views concerning the relativity of magnitude seem to have changed. In book II of his volume *Science and Method* (Poincaré, 1914/1908), amid comments concerning the relativity of space, Poincaré affirms the relativity of magnitude, citing Delboeuf as the principal proponent of this idea:[23]

> there is another [principle of the relativity of space], upon which Delbeuf [*sic*] has particularly insisted. Suppose that in the night all the dimensions of the universe become a thousand times greater: the world will have remained similar to itself, giving to the word similitude the same meaning as in Euclid, Book VI. Only what was a meter long will measure thenceforth a kilometer, what was a millimeter long will become a meter. [...] When I awake to-morrow morning, what sensation shall I feel in presence of such an astounding transformation ? Well, I shall perceive nothing at all. [...]

Poincaré uses this idea to deny that we can have knowledge of absolute magnitudes (Poincaré, 2015/1913, p.414), but does not go on to discuss Delboeuf's argument for the Euclidean nature of space.

Poincaré's deliberate avoidance of Delboeuf's thesis may be connected to his personal enthusiasm about non-Euclidean geometries as mathematical objects of study. In a well-known anecdote, Poincaré recounts how, out of the blue, upon stepping onto an omnibus in Coutances, it suddenly hit him with full clarity that "the transformations [he] had used to define the Fuchsian functions were identical with those of non-Euclidean geometry" (Poincaré, 2015/1913, p.417). This realisation would have taken place at some time before 1880 (Gray, 1997). It is plausible that Poincaré's personal involvement with the development of non-Euclidean geometries drove him away from the defenders of Euclidean apriorism, who, at the time, were largely considered to be a reactionary force, opposed to those that were creatively driving the progress of knowledge. This may have led him to seek out conventionalism as a midway compromise between empiricism and rationalism.

[23]This is a famous passage which has recently drawn attention due to its suggestion of the possibility of scale-invariant cosmological models. See for instance Gryb and Sloan (2021).

Elsewhere in his writing, however, Poincaré has based his *principle of relative motion* on an affirmation of the "passivity of space" (Poincaré, 2015/1913, p.83). Russell too, who, even more than Poincaré, defended non-Euclidean geometries of constant curvature, affirmed that space is *passive*. But what is the root of this intuition of space's passivity?

We have seen above that if we base some notion of the passivity of space on the invariance of bodies when they are moved with respect to one another, we will only have learned of the (approximate) passivity of a *physical* space (see section 3.1). This physical space does not permit the dilations of individual bodies, therefore it may be non-Euclidean. But *who are we to say that this physical space is passive?* Why should physical space allow for the possibility and mobility of rigid bodies? Surely, the question of whether a natural solid retains the same relations among its parts when it is moved with respect to other physical bodies is one that should be answered by empirical science. Indeed this is what was done, by the application of the equivalence principle—which, in the division between force and inertia, places gravity on the side of inertia and (chrono-)metricity—Einstein fulfilled Riemann's ideas and showed once and for all that the physical space which governs those motions which have traditionally been called "inertial" is *not* passive, but *dynamical*.

But what then of our intuition of the passivity of space, and the corollary *relativity of motion*? This law, and the intuition underlying it, can only be based in a truly Kantian conception of space, a space abstracted from all contingent phenomena. Notions of space's passivity, the relativity of magnitudes, and the relativity of motion are pervasive in Poincaré's works. Poincaré often justifies these ideas on the basis that contrary hypotheses would be "repugnant to the mind" (Poincaré, 2015/1913, p.107-109), but he does not discuss why we feel this repugnance. I submit that these intuitions we have of space, and of the relativity of motions and magnitudes, may be founded on what Delboeuf calls the intuition of homogeneity.

Quite apart from space and its geometry, we saw in section 1.1 that for Poincaré, the inductive method, which allows a formula to generalise over an infinity of cases, is the "veritable type of the synthetic a priori judgment" since it is "inaccessible to analytic demonstration and to experience" (Poincaré, 2015/1913, p.39). But what is it that makes this generalisation possible? Is it the mathematical concept of infinity? An infinite set which is not *ordered*, which is not in some respect homogeneous, does not permit of generalisations. It is not the notion of infinity that allows for reasoning by recurrence, rather, it is the intuition of *homogeneity*. It is because the number line is, in some respect, homogeneous, it is on account of its symmetry properties, that inductive generalisations are possible. The mathematical notion of infinite extent, of infinite iterability, is but a consequence of the idea of pure homogeneity, which, as Delboeuf has shown, contradicts boundedness. It is this homogeneity that allows one to generalise across an infinity of cases. Space is that which is homogeneous in every aspect. In this respect, we may regard it as the basic intuition upon which all synthetic a priori knowledge is based.

As Friedman (2009) describes, the mathematical sciences, for Poincaré, are organised hierarchically, with arithmetic at the top, followed by analysis, then geometry, then mechanics and the other physical sciences. With each step down we move further away from a priori reasoning and become more receptive to empirical results, so that geometry, while still a branch of mathematics, depends upon some conventions, the choice of which may be informed by experience. Arithmetic, for Poincaré, is therefore a more general science than geometry. However, the synthetic reasoning upon which it is based is not accounted for in detail by Poincaré beyond the observation that it permits the possibility of inductive generalisations. From a Delboeufian perspective, on the other hand, synthetic reasoning would consist of nothing other than the analytic decomposition of our pure intuitions of homogeneity; and from it the sciences of arithmetic and Euclidean geometry would be deduced. Arithmetic and Euclidean geometry would therefore stand on equal footing, while further conventions would be needed to specify non-Euclidean geometries. I propose, therefore, that we restructure this hierarchy, by placing at one pole Delboeuf's pure intuition of homogeneity, upon which all mathematical reasoning is based, while at the other pole, we place sense-experience, which is inherently heterogeneous, diverse and resistant to generalisations.

4. Morals for scientific methodology

As we have seen, unlike other neo-Kantian influences on contemporary philosophy of space and time, that of Cassirer and those of Helmholtz and Poincaré, Delboeuf's neo-Kantianism affirms the apriority of Euclidean geometry. It may be argued that Delboeuf's ideas, though perhaps interesting, have little relevance to present-day physics and philosophy of physics, since, after all, Delboeuf did not work directly in sphysics, his ideas had little or no influence on the development of Einstein's theories, and, unlike Cassirer, his philosophy was not developed in response to these paradigm-shifting ideas. In what respect, then, should we take this account seriously?

We have seen throughout this essay that the central insight upon which Delboeuf grounds his philosophy of geometry—that is, of the relativity of magnitude—was not unique to him. This insight dates back at least to Wallis in 1663, and was recognised by a variety of significant physicists and philosophers over the centuries. We even saw that Leibniz embarked upon a project very similar to Delboeuf's in his attempt to find stable foundations for geometry. Moreover, we have seen that Delboeuf's account is defensible in the context of the philosophies of geometry that were present at the time, it stands up to Russell and Poincaré's fallacious *relative angles* objection (sec.3.2), it is both more Kantian and more self-consistent than Hemlholtz's allegedly neo-Kantian approach (sec.3.1), and it even resolves certain problems in the foundations of Poincaré's philosophy of mathematics (sec.3.3).

Concerning the applications of these ideas to physics, it is clear that Delboeuf can only contribute on the *methodological* or *formal* side of things. We may propose a strict distinction between *geometrical* space, conceived as a form, and *physical* space (or space-time), conceived as part of the content of this form. This is, to a high degree, consistent with many of Einstein's own statements, for instance in his review of Meyerson's *La déduction relativiste*, Einstein writes (Lehmkuhl, 2014; Einstein and Metz, 1928):

> The fact that the metric tensor is denoted as "geometrical" is simply connected to the fact that this formal structure first appeared in the area of study denoted as "geometry". However, this is by no means a justification for denoting as "geometry" every area of study in which this formal structure plays a role, not even if for the sake of illustration one makes use of notions which one knows from geometry.

Indeed the distinction between geometrical space and physical space was even proposed by Riemann amid some of his most visionary remarks (see sec. 2.3.3). Since the advent of general relativity, a vast literature of flat space alternatives or subtle modifications has been proposed.[24] This literature raises a wide array of methodological advantages of working in flat space, including: (1) the recovery of a well-defined local gravitational energy and of global energy conservation laws (Rosen, 1940a,b; Logunov and Mestvirishvili, 1985),[25] (2) greater consistency with the methods in particle physics (Lasenby et al., 1998), (3) avenues towards unification with particle physics and prospects of developing a theory of quantum gravity (Dicke, 1957; Lasenby et al., 1998; Pitts and Schieve, 2001), (4) the possibility of implementing various interpretations of Mach's principle (Sciama, 1953; Dicke, 1957). Many of these models explicitly appeal to Poincaré's notion of the conventionality of geometry to justify their methods, however, given the findings of the present essay, we suggest that Delboeuf's forgotten arguments may help to provide a philosophical grounding for these flat space approaches.

The topics that Delboeuf's writings raise, however, are most relevant to certain recent developments in the physics and philosophy of cosmology. In recent years, Julian Barbour has been attempting to extend the Machian research program to encompass a requirement for the scale-invariance of cosmological models (Barbour, 2010). If we refuse to accept the existence of epistemically inaccessible absolutes, then the universe must consist only in the relative configuration of its parts—i.e., its *shape*. This way of thinking has led to the developement of the theory of *Shape Dynamics* (Barbour, 2012; Mercati, 2018). If we recognise that the shape of a body consists of the internal relations amongst its parts, while its size is an external relation to other bodies, then the universe as a whole, which has no external reference possesses only a shape. The central insight discussed in this essay—which was recognised by Wallis and Delboeuf—is that the reciprocal independence of shape and size implies the Euclidean nature of space. This essay may help provide grounds for Barbour *et al.*'s use of Euclidean space as a background for their models.

[24] A list of references was given in the introduction.

[25] In such theories, the principle of conservation of energy would appear, not as a contingent empirical fact, but as a guiding methodological principle.

It is only in the context of cosmological models, rather than in the study of subsystems of the cosmos, that transformations of all bodies with respect to space, i.e. *Leibniz shifts* or transformations by *similarity*, can be considered. Outside of shape dynamics, the requirement for the invariance of dynamics under similarity transformations in cosmology has been called "*dynamical similarity*", and it is a growing area of research in cosmology (Sloan, 2018; Gryb and Sloan, 2021; Bravetti et al., 2022). We hope that the ideas discussed in the present paper will help to provide some philosophical context and justification for these cosmological speculations.

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References

Arminjon, M., 2002. The scalar ether-theory of gravitation and its first test in celestial mechanics. International Journal of Modern Physics A 17, 4203–4208.

Barbour, J., 2010. The definition of mach's principle. Foundations of Physics 40, 1263–1284.

Barbour, J., 2012. Shape dynamics. an introduction, in: Quantum Field Theory and Gravity: Conceptual and Mathematical Advances in the Search for a Unified Framework. Springer, pp. 257–297.

Bitbol, M., Kerszberg, P., Petitot, J., 2009. Constituting objectivity: Transcendental perspectives on modern physics. volume 74. Springer Science & Business Media.

Bravetti, A., Jackman, C., Sloan, D., 2022. Scaling symmetries, contact reduction and poincaré's dream. arXiv preprint arXiv:2206.09911 .

Broekaert, J., 2005. A modified lorentz-transformation–based gravity model confirming basic grt experiments. Foundations of Physics 35, 839–864.

Carnot, L., 1803. Géométrie de position. JBM Duprat.

Cavalleri, G., Spinelli, G., 1980. Field-theoretic approach to gravity in the flat space-time. La Rivista del Nuovo Cimento (1978-1999) 3, 1–92.

Čulina, B., 2018. An elementary system of axioms for euclidean geometry based on symmetry principles. Axiomathes 28, 155–180.

Čulina, B., 2020. Euclidean geometry is a priori .

Čulina, B., 2023. Mathematics–an imagined tool for rational cognition. arXiv preprint arXiv:2306.03909 .

Davies, P., Falkowski, P., 1982. Quantum theory and the equivalence principle. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 381, 469–478.

De Risi, V., 2005. Leibniz on geometry: Two unpublished texts with translation and commentary. The Leibniz Review 15, 127–132.

De Risi, V., 2007. Geometry and Monadology: Leibniz's analysis situs and Philosophy of Space. Springer.

De Risi, V., 2015. Leibniz on the Parallel Postulate and the Foundations of Geometry. Springer.

Delboeuf, J., 1860. Prolégomènes philosophiques de la géométrie et solution des postulats. Muquardt.

Delboeuf, J., 1893. L'ancienne et les nouvelles géométries: Première Étude. l'espace réel est-il l'espace géométrique euclidien? Revue Philosophique de la France Et de l'Etranger 36, 449–484. URL: https://gallica.bnf.fr/ark:/12148/cb34349223n/date1894.r.

Delboeuf, J., 1894a. L'ancienne et les nouvelles géométries. ii les nouvelles géométries ont leur point d'attache dans la géométrie euclidienne. Revue Philosophique de la France Et de l'Etranger 37, 353–383. URL: https://gallica.bnf.fr/ark:/12148/cb34349223n/date1894.r.

Delboeuf, J., 1894b. L'ancienne et les nouvelles géométries: Troisième Étude: Les postulats réels de la géométrie euclidienne sont a la base des métagéométries. Revue Philosophique de la France Et de l'Etranger 38, 113–147. URL: https://gallica.bnf.fr/ark:/12148/cb34349223n/date1894.r.

Delboeuf, J., 1895. L'ancienne et les nouvelles géométries: Iv. – les axiomes et les postulats de la géométrie de l'espace homogène. Revue Philosophique de la France Et de l'Etranger 39, 345–371. URL: https://gallica.bnf.fr/ark:/12148/cb34349223n/date1894.r.

Deser, S., 1970. Self-interaction and gauge invariance. General Relativity and gravitation 1, 9–18.

Dewar, N., Eisenthal, J., 2020. A raum with a view: Hermann weyl and the problem of space. Thinking About Space and Time: 100 Years of Applying and Interpreting General Relativity , 111–132.

Dicke, R.H., 1957. Gravitation without a principle of equivalence. Reviews of Modern Physics 29, 363.

Einstein, A., Metz, A., 1928. A propos de la déduction relativiste de m. émile meyerson. Revue philosophique de la France et de l'étranger 105, 161–166.

Fang, J., Fronsdal, C., 1979. Deformations of gauge groups. gravitation. Journal of Mathematical Physics 20, 2264–2271.

Folina, J.M., 2016/1992. Poincaré and the Philosophy of Mathematics. Springer.

Friedman, M., 2001. Dynamics of reason. Csli Publications Stanford.

Friedman, M., 2009. Einstein, kant, and the relativized a priori, in: Bibtol, M., Kerzberg, P., Petitot, J. (Eds.), Constituting objectivity: transcendental perspectives in modern physics. Springer. chapter B1, pp. 253–268.

Friedman, M., 2014/1983. Foundations of space-time theories: Relativistic physics and philosophy of science. volume 113. Princeton university press.

Gray, J., 1997. Poincaré in the archives-two examples. Philosophia scientiae 2, 27–39.

Gryb, S., Sloan, D., 2021. When scale is surplus. arXiv preprint arXiv:2103.07384 .

Gupta, S.N., 1954. Gravitation and electromagnetism. Physical Review 96, 1683.

Heinzmann, G., 2001. The foundations of geometry and the concept of motion: Helmholtz and poincaré. Science in context 14, 457–470.

Helmholtz, H., 1866. On the factual foundations of geometry. Beyond Geometry: Classic Papers from Riemann to Einstein , 47–52.

Helmholtz, H., 1870. The origin and meaning of geometrical axioms (1870). Beyond Geometry: Classic Papers from Riemann to Einstein , 53–70.

Helmholtz, H.v., 1876. The origin and meaning of geometrical axioms. Mind 1, 301–321.

Hill, M., 1925. On the substitution of wallis's postulate of similarity for euclid's postulate of parallels, in: Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press. pp. 964–969.

Huggins, E.R., 1962. Quantum mechanics of the interaction of gravity with electrons: theory of a spin-two field coupled to energy. Ph.D. thesis. California Institute of Technology.

Jammer, M., 2013. Concepts of space: the history of theories of space in physics: third. Courier Corporation.

Kant, I., 1970/1786. Metaphysical foundations of natural science (j. ellington, trans.). Indianapolis, IN: Bobbs-Merrill.(Original work published 1786) .

Kant, I., 2004/1783. Immanuel Kant: Prolegomena to any future metaphysics: That will be able to come forward as science: With selections from the critique of pure reason. Cambridge University Press.

Kraichnan, R.H., 1955. Special-relativistic derivation of generally covariant gravitation theory. Physical Review 98, 1118.

Laplace, P.S., 1835. Oeuvres complètes de Laplace. volume 6. Gautier-Villars.

Lasenby, A., Doran, C., Gull, S., 1998. Gravity, gauge theories and geometric algebra. Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 356, 487–582.

Lehmkuhl, D., 2014. Why einstein did not believe that general relativity geometrizes gravity. Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 46, 316–326.

Leibniz, G.W., 1858. Leibnizens Mathematische Schriften, Herausgegeben Von C. I. Gerhardt. volume 1. Druck und Verlag von H. W. Schmidt.

Lie, S., 1893. Theorie der transformationsgruppen. volume 3. BG Teubner.

Lobachevsky, N., 1829a. A concise outline of the foundations of geometry. University of Kazan Messenger, Kazan .

Lobachevsky, N.I., 1829b. On the principles of geometry. Kazansky Vestnik 25, 178–187.

Logunov, A.A., Mestvirishvili, M.A., 1985. Relativistic theory of gravitation. Progress of theoretical physics 74, 31–50.

Mach, E., 1906. Space and geometry in the light of physiological, psychological and physical inquiry. Open court publishing Company.

Mercati, F., 2018. Shape dynamics: Relativity and relationalism. Oxford University Press.

Mittelstaedt, P., Barbour, J., 1967. On the geometrical interpretation of the theory of gravitation in flat space. Zeitschrift fur Physik 203, 82–90.

Nachtmann, O., Schmidle, H., Sexl, R., 1969. On the structure of field theories of gravitation(linear field theory of gravitation by considering gravitational field functions decomposition into spin components). Acta Physica Austriaca 29, 289–299.

Ogievetsky, V., Polubarinov, I., 1965. Interacting field of spin 2 and the einstein equations. Annals of Physics 35, 167–208.

Pitts, J.B., Schieve, W.C., 2001. Slightly bimetric gravitation. General Relativity and Gravitation 33, 1319–1350.

Poincaré, H., 1898. On the Foundations of Geometry... Open Court Publishing Company.

Poincaré, H., 1905/1902. Science and hypothesis. Science Press.

Poincaré, H., 1914/1908. Science and Method. Dover Publications.

Poincaré, Henri; Halsted, G.B., 2015/1913. The foundations of science. Cambridge library collection. History of science., Cambridge University Press.

Riemann, B., 1854. Über die hypothesen, welche der geometrie zugrunde liegen [1854]. H. Weyl, Das Kontinuum und andere Monographien. New York: Chelsea .

Riemann, B., 1876. Gesammelte mathematische Werke und wissenschaftlicher Nachlass. BG Teubner.

Riemann, B., 2007/1853. Two excerpts from riemann's nachlass (1853). Beyond Geometry: Classic Papers from Riemann to Einstein , 41.

Rosen, N., 1940a. General relativity and flat space. i. Physical Review 57, 147.

Rosen, N., 1940b. General relativity and flat space. ii. Physical Review 57, 150.

Russell, B., 1897. An Essay on the Foundations of Geometry, by Bertrand AW Russell... The University Press.

Russell, B., 1898. Les axiomes propres a euglide: Sont-ils empiriques? Revue de Métaphysique et de Morale 6, 759–776.

Ryckman, T., 2005. The reign of relativity: philosophy in physics 1915-1925. Oxford university press.

Ryckman, T.A., 2003. The philosophical roots of the gauge principle: Weyl and transcendental phenomenological idealism. Brading and Castellani (2003) , 61–88.

Sciama, D.W., 1953. On the origin of inertia. Ph.D. thesis. University of Cambridge.

Sloan, D., 2018. Dynamical similarity. Physical Review D 97, 123541.

Therrien, V.L., 2020. A diagram of choice: The curious case of wallis's attempted proof of the parallel postulate and the axiom of choice, in: Diagrammatic Representation and Inference: 11th International Conference, Diagrams 2020, Tallinn, Estonia, August 24–28, 2020, Proceedings 11, Springer. pp. 74–90.

Thirring, W.E., 1961. An alternative approach to the theory of gravitation. Annals of Physics 16, 96–117.

Torretti, R., 2012/1978. Philosophy of geometry from Riemann to Poincaré. volume 7. Springer Science & Business Media.

Wald, R.M., 2010. General relativity. University of Chicago press.

Wallis, J., 1696. De postulato quinto et definitione quinta. lib. 6. euclidis; disceptatio geometrica. De Algebra Tractatus; Historicus & Practicus. Cumvariis Appendicibus , 665–678.

Weinberg, S., 1964a. Derivation of gauge invariance and the equivalence principle from lorentz invariance of the s-matrix. Physics Letters 9, 357–359.

Weinberg, S., 1964b. Photons and gravitons in s-matrix theory: derivation of charge conservation and equality of gravitational and inertial mass. Physical Review 135, B1049.