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Mathematizing Metaphysics: The Case of the Principle of Least Action *

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Abstract

Standard narratives about physical teleology say its death was a *fait accompli* of the Scientific Revolution, but the principle of least action (PLA) has been taken to instantiate teleology's survival into Enlightenment physics. Other scholars claim this PLA-based teleological metaphysics fell to general philosophical attacks on final causes. None of these narratives fully captures the philosophical interest of its demise. It illustrates a metaphysics being refuted because it could not be coherently modeled in mathematics, hence directly through *mathematization* and not by philosophical argument or empirical test.

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1 Introduction

A venerable philosophical narrative says that teleological metaphysics died out in the 17th century: first banished by Bacon, Descartes, and Spinoza, it soon fell at the hands of mechanistic physics. The principle of least action (PLA), whose progenitors claimed to derive it from a metaphysical thesis about the operation of final causes in nature, renewed these philosophical controversies in the 1740s, drawing some of the Enlightenment's brightest luminaries into one of the liveliest *querelles* in the history of physics.

As originally stated, the PLA asserts that all physical processes realize the *end* of expending the smallest possible 'action of nature,' embodying the immanent final cause of *efficiency*, *economy*, or *simplicity*. The principle was *mathematized* as the assertion that Nature chooses particle trajectories that minimize $\int mvds$, the path integral of mass times velocity times distance, over the space of possible trajectories between two given points. From this infinite haystack of possible paths, Nature plucks the action-minimizing needle. This teleological interpretation played a role in the principle's early development, a fact which has been leveraged in arguing that teleology continued to enjoy a non-trivial place in Enlightenment physics, *pace* traditional narratives to the contrary (McDonough 2020).

Nonetheless, this metaphysics, created by Pierre Maupertuis with support from his colleague Leonhard Euler, soon encountered vehement opposition. Eventually — indeed, rather quickly — scientists rejected it. Why? This is the question for which I seek to develop a novel answer.

The core idea of this paper is that it was the act of *mathematizing* the original

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metaphysical principle that exposed it to the most serious and direct refutation. This was *not* simply because mathematization exposed it to precise measurement, empirical test, or other, more familiar sources of disconfirmation. Rather, the metaphysics underwriting the PLA was rejected by scientists because of obstacles emerging immanently within the process of mathematically representing or modelling 'the action of Nature' as a universally budgeted quantity. This entailed translating the implicit and explicit components of the metaphysical story into mathematical correlates. In particular, the metaphysics underwriting the principle became a set of formal *demands*. Purely formal labors of Maupertuis, Euler, and critics soon revealed they were jointly inconsistent. Mathematics and metaphysics clashed and, in the ensuing fray, the metaphysics lost. The PLA, newly deflated, remained an accepted principle of mathematical physics, and was even theoretically fruitful. But scientists decisively rejected its original metaphysical underpinning.

This idea has not, to my knowledge, received philosophical attention. Existing accounts of the rejection of the PLA's teleological metaphysics credit philosophical trends and arguments against final causes, or attacks from elsewhere in intellectual culture, such as Voltaire's widely-read satires, not mathematization (Pulte 1989; Schramm 1985). If anything, scholars have emphasized the role of mathematization in artificially *prolonging* the life of physical teleology by giving it a patina of rigor (Stöltzner 1994, p. 36).[1] And, as mentioned, some have proposed that it shows teleology not merely on artificial life support, but playing a significant role in 18th century science (McDonough 2020).

This is surprising, because the question of 'refutation via mathematics' goes back to

[1]I agree that mathematization may, at first, have had this effect; but the ultimate effect was, ironically, to deliver an even more decisive refutation.

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the dispute's origins. In a late attempt to defend Maupertuis and the PLA, Euler observed that their opponent Samuel König made a surprising claim: that the PLA and its metaphysical underpinning were not only false, but were "thoroughly overthrown by [König's] demonstrations, not Metaphysical, but Geometrical" (Euler 1753a). Euler objected that the PLA was both mathematically and metaphysically sound. I will argue that, while the PLA's teleological metaphysics may not have been overthrown by the Geometrical demonstrations of *König*, it was nonetheless refuted and rejected on grounds immanent to its mathematical elaboration.

This development is interesting because it shows mathematics playing a historically new role constraining metaphysics. Philosopher-scientists had previously adduced fundamental physical principles allegedly based on mathematizing metaphysical claims. But, when rejected, the constraints came from elsewhere. Descartes, for example, in Book II of his *Principia*, argued on metaphysical grounds that the totality of motion is conserved, since it is caused by an immutable God. He then *mathematized* the thesis by defining 'quantity of motion' as bulk times speed. But this theory is not plagued by mathematical inconsistencies. Indeed, Leibniz's famous 'brief demonstration' of its falsity relied on two key constraints: the (empirical) law of Galilean free fall and the (metaphysical) principle of equality of cause and effect. Conversely, prior clashes between metaphysics and mathematics could leave metaphysics unscathed. A plausible example, described by Harman (1983, p. 235), is Kant's 1747 *vis viva* essay, where he finds Leibniz's doctrine to be 'false in mathematics' and yet *true metaphysically*.

The PLA episode thus appears to instantiate a historically novel way of "accord[ing] ontological force to mathematical structure" (Mahoney 1998). Accordingly, philosophers interested in mathematization in science and its history or in the interaction between

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science and metaphysics will, hopefully, find this analysis of interest. It is partly inspired by, and hopefully provides additional perspective for, the debate over whether there can be non-causal scientific explanations of physical phenomena in which mathematics plays an indispensable explanatory role.[2]

Section 2 gives background and characterizes my objectives; section 3 gives my analysis of the immanent mathematical failure of the PLA's teleological metaphysics; and section 4 argues this led scientists to reject it. The PLA and the philosophical issues it raises are complicated, and have supported a rich scholarship through to the present.[3] It is impossible for me to adequately address the many existing perspectives on the abandonment of physical teleology linked to the PLA, even restricting my attention to the 18th century, but I try to address the readiest objections, especially in subsection 3.2.

[2]These are often called 'distinctively mathematical explanations', following Lange's (2013) terminology. The more recent account is his (2018). For an opposing, deflationary view, see Kuorikoski (2021).

[3]On the richness of the historical debate about the PLA and physical teleology, and its interwovenness with other philosophical and cultural questions, see (Stöltzner 1994), (Schramm 1985), (Lyssy 2022), and (McDonough 2020). On the PLA as an alternative foundation for physical theory, see (Sklar 2012, Ch. 11). On its connection to theoretical unification of physics, see (Stöltzner 2002, 2005). In relation to the metaphysics of modality, see (Butterfield 2005; Terekhovich 2017), and in relation to the metaphysics of causality, see (Ben-Menahem 2018, Ch. 6).

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2 Metaphysics and the Early PLA

2.1 The Maupertuis-Euler teleological metaphysics

Appeals to principles of nature's simplicity and efficiency date back to Aristotle or even the pre-Socratics, classically under the title *lex parsimoniae*.[4] After a long, relatively dormant period, these typically vague ideas of the simplicity of nature took on mathematical form in early modern physics. Fermat used his 'principle of least time' to derive and explain the classical laws of optics. In Maupertuis's later, more general vision, the Creator instituted a principle assigning Nature the end of carrying out its processes in the simplest, most efficient way possible (Maupertuis 1748a, p. 421). The content of this metaphysical principle is admittedly vague; I will use the label "principle of the simplicity or efficiency (or economy) of nature" (PSEN).

In Maupertuis' recounting, Fermat and Leibniz had appealed to the PSEN in particular applications, most notably optics. Maupertuis followed their lead, observing that the linear propagation and reflection of light, which exhibit how light always 'takes the simplest path,' seem to depend on the PSEN. For Maupertuis, that left the question of the refraction law. He thought his predecessors has gotten refraction wrong and that their success in mathematizing the PSEN was only partial. It was left to him to finally

[4]In Aristotle, related ideas ('nature does nothing in vain') appear in the *Politics* (Sections II and VIII), as well as in *Generation of Animals* (Section III) and other biological works. Yourgrau and Mandelstam (1968) stress the importance of *De Caelo*'s move from vague ideas of 'simplicity' to that of 'minimization,' a 'proto-mathematization' brought even closer to modernity by Hero of Alexandria's first-century treatment of the classical law of reflection.

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"reconcile [*accorder*] the law of refraction with the metaphysical principle" (Maupertuis 1748a, p. 423). Specifically, his initial project was to reconcile teleological explanation of the optical laws with what he (erroneously) believed to be the correct account of light, the Newtonian view that light consists of particles whose velocity increases with the density of the medium (Schramm 2005). The result was the PLA, which asserts that the *action*, or $\int mvds$, associated to the possible trajectories of a particle attains a minimum at the actual trajectory (Maupertuis 1748a, p. 423).[5] This definition of the 'action' was Leibniz's, who used it to prove a conservation principle. The thesis that it is *minimized* is Maupertuis'. Maupertuis claimed to derive all three optical laws from the PLA, and on that basis asserted that the quantity of action was "the true expense of Nature, and that which she economizes [*ménage*] as much as possible in the movement of light" (ibid., p. 423). By recovering the laws of optics from the PLA, he believed he had demonstrated the PSEN's deep truth, having brought the phenomena into accord with that "great principle" (ibid., p. 424).

Independently (and probably chronologically prior), Euler used an equivalent principle, writing down the action integral for a particle subject to a variety of conservative forces, and expressly asserting that it is *minimized* (1744, Addit. II, §2). Nonetheless, he always ceded priority to Maupertuis. For his part, Euler did not boast of *deriving* his physical principle from metaphysical ones. Nonetheless, he expressed hope that such a derivation would be forthcoming, the fruits of an inquiry "into the innermost laws of Nature and final causes" (ibid., Addit. II, §1). Maupertuis' paper likely appeared to him as a timely and welcome contribution (Schramm 2005).

[5]Note that in this and Euler's initial work, which treated a single particle, the mass was ignored.

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The thesis is that there is a fundamental physical quantity, the *action*, which can be regarded as "the action of Nature" in the context of the claim, stemming from the PSEN, that for all changes Nature chooses the trajectory requiring the least amount of it. The PLA was thereby adduced as the physical expression of the PSEN: it represents physically what it means for a natural process to be the simplest or most efficient, as demanded by the metaphysical principle, underwriting a picture of Nature as sublimely economical (or, if you like, 'lazy'). I will refer to this theory as the 'Maupertuis-Euler teleological metaphysics', hereafter METM.[6]

Unfortunately, it is hard to specify necessary *and sufficient* conditions for the success of the METM which do not involve God or design, and I am restricting my concern to its role in physics. But Maupertuis' comments suggest at least three *necessary* conditions for the PLA to express the operation of the immanent final cause of efficiency.[7]These conditions turn out to be inter-related, but are conceptually distinct:

1. (Budget) The action is a kind of 'resource' or 'budget'
2. (Minimization) The action is *minimized* in the processes covered by the law
3. (Universality) The principle covers all physical processes

[6]Euler's degree of commitment to the METM is a difficult question which I cannot go into deeply here. Though Maupertuis was its strongest devotee, it is equally clear that, in the period 1744-1753, Euler, across several works, endorsed, defended, and even premised arguments on the METM. That Euler's recognition of unavoidable mathematical difficulties *eventually* led him to discard the thesis is part of my argument.

[7] A fourth condition might be added: the PLA alone *uniquely determines* the actual

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2.2 Varieties of teleology in physics

Philosophers interested in the PLA and teleology have emphasized diverse senses of 'physical teleology,' which, though distinct, are complexly inter-related, making it a delicate matter to carve out a philosophical question about just one variety. To better distinguish the one concerning me here, I will mention other, often-discussed varieties.

Lyssy (2022) views Maupertuis and Euler as carrying on a Leibnizian heritage in the philosophy of science. Leibniz's teleological principles demanded that changes in Nature be the "most determined," *i.e.* they are characterized by either a minimum or maximum of some quantity. This teleology is 'formal' in that metaphysical significance attaches to the property of extremization, not the quantity extremized. It implies a directive for physics research as well as a unifying schema for physical laws.[8] A related set of ideas, deriving from William of Ockham, views simplicity or economy as (desirable) properties of our *theories*. The 'final cause' of simplicity characterizes the activity of theorizing, not nature.

By contrast with these 'formal' approaches to teleology, the METM was intended to express the PSEN, a metaphysical claim about Nature and not (merely) a criterion for our theories about it. Maupertuis implied that his success, where his predecessors fell short, was in reasoning to "the end of Nature" and finding "the quantity that we ought to regard as her expense [*dépens*] in the production of her effects" (Maupertuis 1748a,

path. This condition has the least direct textual support in Maupertuis and Euler. Lacking space, I omit it.

[8]Advocacy for using the PLA to unify physics reemerged in the late 19th and early 20th centuries, especially via Planck, though apparently only as a *formal* unification (Stöltzner 2002). For detailed analysis of varieties of 'formal teleology', see (Stöltzner 2005).

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p. 426). What is minimized is always the *action*, a 'substantive,' not merely 'formal,' final cause.

Maupertuis and Euler themselves attached *multiple* kinds of teleological significance to the PLA. Three aspects of the Maupertuis-Euler picture are reflected in the descriptions of (McDonough 2020, pp. 173–4, 176). First, Maupertuis took the simplicity and unifying power of the principle as the epistemic basis of an inference to *divine teleology*. Second, the PLA suggests that "teleology must operate within the order of nature," in the guise of efficiency, a kind of *immanent teleology* (ibid., p. 176). Finally, McDonough there gestures at an epistemic notion of "teleological reasoning," related to the notion of "optimality" explanations discussed in (McDonough 2022, §5.4).

Though there is much to say about each of these, I focus on the METM, a thesis about immanent physical teleology, not formal teleology, divine teleology, or teleological explanation. This picture starts with the PSEN, proceeds to formulate a quasi-physical idea of the 'action' of Nature as something it always minimizes, and then mathematizes this principle as 'the PLA': the claim that $\int mvds$ is minimized in actual natural processes, over the class of possible trajectories between fixed endpoints. The METM was arguably the most significant attempt in modern history to establish teleological metaphysics within physics.

2.3 Mathematization and the metaphysics-physics relation

The mechanism of refutation of the METM that I propose is, in Euler's phrase, by "demonstrations Geometrical, and not Metaphysical." Though its falsity is not exactly proved the way geometrical theorems are, it *is* refuted via formal mathematical

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considerations. Indeed, when we remain at the discursive level of metaphysical speculation, all three conditions from section 2.1 appear jointly possible, not susceptible to *a priori* philosophical disproof. Seemingly nothing prevents us from positing that Nature has a 'budget' of some universal 'resource' that she expends to produce her effects, and that she is always frugal in her purchases. A metaphysician could not hope to decisively *refute* it, except perhaps on the basis of a prior philosophical opposition to teleology, which may simply beg the question against the project of grounding physics in a teleological world-picture. Even scholars who credit philosophical analysis of epistemological bona fides (*e.g.* of the teleological 'derivation' of the PLA) acknowledge that this "may not by itself be a sufficient refutation" (Yourgrau and Mandelstam 1968, p. 174).

But everything changed when the principle was *mathematized*. Or so I argue. Once the action was specified as $\int mvds$, the principle was hoisted into the mathematical framework of the calculus (of variations), and the three features Budget, Minimization, and Universality had to be 'mathematized': modeled or represented in the mathematical theory of the PLA. As I reconstruct this history, complications for and criticism of the METM came on several levels, corresponding to these features.

At the bottom-most level, doubts arose about the mathematical definition of the action quantity itself. Critics questioned whether its mathematical properties justified giving it the sense of a 'budgeted resource' (Budget), and doubted whether, as defined, it could be physically or metaphysically fundamental. The middle level concerned the assertion that this (allegedly fundamental) quantity is economized (Minimization), underwriting a view of nature as "intending to be sparing." A thicket of problems soon sprang up for this claim, too.

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The only hope of rectifying these difficulties was to adjust the underlying metaphysical picture. Ultimately, however, this path was blocked at the highest level, Universality, which required the PLA to be a representational tool for solving physical problems. The PLA was meant to be a single, general principle, but, in application, Maupertuis was forced to divide it into three distinct forms which lacked a unifying metaphysical basis. Only one of them embodied, in any clear way, the original metaphysics which motivated the PLA, whereas the others appeared as arbitrary in relation to it. Euler attempted to address this difficulty mathematically, with Maupertuis' blessing, but this effort led him into self-contradiction, requiring him to both affirm and deny Minimization, or nature's "intention to be sparing." Critics came to complain that substantiating the PLA's generality required arbitrary reformulations and auxiliary assumptions that belied the absence of a truly universal metaphysical foundation. Collectively, these problems led to the METM's ruin.

The remarkable feature of this route to refutation is that it was entirely formal, based on immanent mathematical problems rather than on metaphysical or empirical argument. The tribunal of mathematics had determined that the METM's component features were not compossible, giving scientists *good reasons* to reject it. In section 4, I show how members of the scientific community highlighted these reasons, and the proponent of the METM most significant to our question, namely Euler, ceased to endorse it, after his valiant but ultimately doomed effort to save its mathematical viability.

I conclude that, around the mid-18th century, *mathematization* came to operate not only as a demand on physical theories to become more precise, predictively powerful, or capable of empirical confirmation, but also as a direct constraint on the metaphysical theses purported to ground them, arising as a consequence of the use of mathematics to

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represent or 'model' the metaphysics. Although commentators have occasionally remarked on broadly 'mathematical' reasons for the failure of the PLA's metaphysics, none has given the full picture of these reasons. Likely because of this, none has thematized the refutation of the METM the way I do here, or highlighted its novelty. By 'direct constraint' I just mean that complications and inconsistencies arose within the process of producing and working out the model *formally*, with no express attempt to 'compare the model with the empirical world'. Indeed, doubting the empirical accuracy of Maupertuis' or Euler's results was never in question. This episode thus appears to differ from otherwise analogous cases from the history of physics in which a metaphysical picture was mathematized, but was later rejected for more traditional reasons, such as the case of Descartes mentioned above. Implicitly, then, scientists had adopted a new constraint: a metaphysics that is mathematizable must have a consistent mathematical representation or model, and is otherwise discredited or disconfirmed. The presuppositions behind and full implications of adopting such a constraint merit further attention. Here, though, I begin by offering it as a philosophically interesting development in the relationship between metaphysics and physics, mediated by a historically new role for mathematics.

3 Mathematizing the PLA, Unraveling the METM

My analysis in this section is focused on problems emerging internally to the process of working out the PLA's theory, so my attention is largely restricted to the principal early theorists, Maupertuis and Euler. One can perceive their gradually growing awareness that mathematizing the theory of 'least action in nature' invites complications for the

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three conditions listed above. Mathematization puts significant pressure on each individually. Jointly, it makes them untenable. Or so, at any rate, I will now argue.

3.1 (Budget) Action as 'expense,' 'budget,' or 'resource'

In a paper read in 1744, Maupertuis announce that the action is Nature's budget: "this fund or budget [*fonds*], this quantity of action, which Nature saves up [*épargne*]," representing the amount of 'resources' available for purchasing physical effects (Maupertuis 1748a, pp. 424–5). The expression $\int mvds$ should support this interpretation as measuring an "expense of Nature." Maupertuis' contemporaries, too, wanted to know why that algebraic expression represented the "effort" or "expense" of nature (see section 4).

Maupertuis' discussion of the principle of least time (PLT) shows that he was sensitive to this question from the start (ibid.). The PLT was flawed, Maupertuis wrote, because it got the law of refraction wrong, incorrectly selecting time over space to be the minimized quantity. With no reason to privilege one over the other, we should expect both to appear in nature's budget (ibid., p. 16). Maupertuis further reasons, intuitively, that nature's action should be larger exactly when it moves a *greater mass*, moves it *faster*, or moves it *farther*, an explanation developed in his *Essai de cosmologie* (1750, p. 42). Evidently, Maupertuis understood he needed to justify why the algebraic form of his chosen quantity comported with its teleological interpretation.

Though his explanation has at least some intuitive appeal, an appeal to intuition is no demonstration. Further, Maupertuis' references to monetary expenditure imply that the 'action' should have further mathematical features, those of a currency. Giving

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content to the claim of 'economizing' need not require a natural or objective choice of unit. Yet the quantity of action must still support a meaningful zero point (compare: a difference between spending positive money and no money). Likewise, there must be a difference between an expenditure of action and its opposite (compare: paying out, versus being paid).

Mathematically, these conditions were not satisfied. Euler observed, in his (1753d), that the sign of the quantity of action (together with its scale) is insignificant, and also indicated that there may be no mathematically principled way to assign to the action a meaningful absolute level, since it contains an additive constant (ibid., §XIX). It was also evident that the action integral can be mathematically transformed at will into other quantities. Since $ds = vdt$, a substitution shows that $\int mvds = \int mv^2dt$, yielding a time-integral of *vis viva* (§§16-20). Indeed, he had already shown this in (1744), where he was evidently pleased to have shown the compatibility of Cartesian and Leibnizian concepts. What proved embarrassing to the metaphysics is that, in Leibniz's theory, *vis viva* is conserved. So this algebraic transformation immediately raised two difficult questions for the METM, both later pressed by d'Arcy (1756): (1) whether either quantity can be regarded as the *true* 'action of Nature,' and (2) whether nature minimizes action, or conserves it. Last, problems that could be formulated using path integrals of the purported 'budget' of $mvds$ include little more than the motion of point particles, whereas Maupertuis' ambitions were universal. As I will explain more fully in section 3.3, the quantity so defined is not mathematically, or even dimensionally, apt to characterize all mechanical effects, violating Universality. Collectively, these complications make it difficult to read the action as a quantity of some resource that Nature must economize.

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3.2 (Minimization) Action as minimized in natural processes

The action quantity must be *minimized* to do justice to the 'efficiency' aspect of the metaphysical *ur*-principle, the PSEN, which exposes the METM to the mathematical possibility of mechanical configurations that fail to minimize the action, or else its statical analogue, which Euler called the 'effort.'

In applying the PLA, one demands stationarity of the action integral, *i.e.* that its derivative (or variation) vanish. It is often observed that, besides minima, maxima and saddle points are also stationary, scotching a specifically minimizing interpretation. If one has not read the historical sources, one might think, implausibly, that Maupertuis failed to understand that stationarity does not imply minimization.

Maupertuis's thinking is better appreciated by comparing his (1740) with his (1748a). The first introduces the 'law of rest' – an important precursor to the PLA on which Euler also wrote several papers (1750a,b, 1753b,d). The law of rest is deduced from the stationarity of a certain expression (in modern terms, the *potential*; in Euler's, the 'effort') which is itself derived using other mechanical principles, through an analysis rooted in ideas of Daniel Bernoulli's. Maupertuis infers that its integral will attain a minimum or maximum, making it an *extremal*, not a minimal, principle. He proposes no teleological underpinning. Euler also attempted a 'metaphysical demonstration' of the law of rest, referencing fundamental facts about forces, *not* teleology (1753b). In Maupertuis' (1748a), by contrast, teleology expressly motivated a minimum condition, and *from* minimization, he inferred stationarity. Maupertuis did not posit a stationarity principle and then mistakenly infer that the action is minimized. Rather, he posited a metaphysics of least action, and mathematized it for use in mechanics, yielding the PLA.

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As mentioned, Euler also called for an inquiry into the final causes behind the PLA, although he did not undertake it himself.

But does the quantity of action, defined as $\int mvds$, always attain a minimum? And does this hold for other classes of problems supposed to fall under the auspices of the PLA, such as static equilibrium? Mathematical analyses of cases in which the action or effort attained a *maximum* soon posed a problem, the one usually emphasized in historical and philosophical commentary on the PLA, e.g. (Stöltzner 1994; Yourgrau and Mandelstam 1968). Maupertuis never acknowledged in print that maxima of the action are also possible. Euler, as we will see, was more equivocal.

In his (1753d), Euler described the case of a rigid bar fixed at its midpoint, free to rotate, with a weight hung from one end; the opposite end is pinned frictionlessly against a wall such that the weighted end is hefted up. The configuration, in unstable equilibrium, is at a maximum of the '*effort*,' an apparent counterexample obtained simply by toying around with a mathematical model.

To discount it, one might search for reasons to ignore the case as 'unphysical,' the way certain solutions to differential equations are thrown out as 'unphysical' (*e.g.*, if they blow up in finite time.) This would preserve the consonance between physical principle and underlying metaphysics. But the case involves nothing more outlandish than 'frictionless contact,' so no such considerations became available. A different gambit for safeguarding the teleological metaphysics is to weaken the thesis slightly. The *spirit* of the METM is that Nature "has it in view to make the sum of efforts as small as possible" (ibid., §XI). Perhaps the action and effort, though not always minimized in fact, always *tend to a minimum*. This amounts to replacing Minimization with:

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2. (Minimization*) The action (or effort) *tends to a minimum*; Nature *prefers* a minimum of the action.

The implication is that cases of effort-maximization should always be *unstable*. If perturbed, Nature will not return them to that configuration, because Nature tends toward, and strives to preserve, the effort-*minimizing* equilibria. We could then reasonably conclude that Nature's goal is revealed in its *tendency* towards minima.

This gambit seems to preserve the spirit of the METM, but it also raises further questions. Does Nature have ends that it can *fail* to attain? Worse, other action-maximizing cases emerged. The most important was reflection in a concave mirror, described by d'Alembert (1752; 1754) and d'Arcy (1756); on this, see also (Schramm 2005). Suppose light travels from point F to point f, reflecting at M in a plane mirror. Maupertuis' analysis (1748a) showed that the principle that the path of light minimizes action implies the law of reflection. Place, instead, a concave spherical mirror AMB whose tangent plane at M coincides with the plane mirror, as in Figure 1. The light's path will not differ, but the other parameters can be chosen so that this path locally *maximizes* the action. For instance, choose F and f equidistant from M and far enough apart that the ellipse through M with F and f as its foci, labeled oMp, passes outside the spherical mirror. By elementary properties of the ellipse, paths through points on AMB to the left or right of M have strictly lower action. (A second derivative test can be used, though the computation is somewhat laborious.) It appears neither Maupertuis nor Euler directly replied to this objection, which seems an obvious embarrassment to the view.

Indeed, it is so obvious that it may seem to imply an objection to my account. It is

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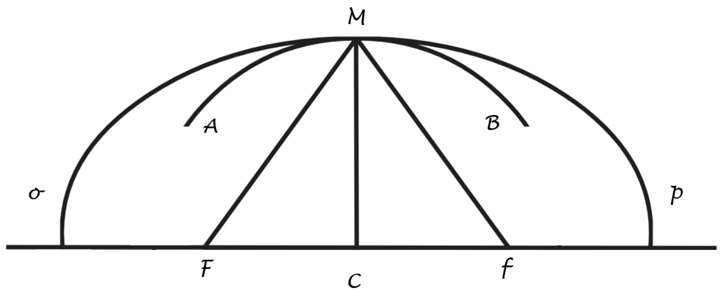

Figure 1: Reflection in concave mirrors can maximize action; based on (d'Arcy 1756)

trivial to see to see that the teleological reading fails, since it demands a minimum of the action. For example, Schramm (2005, p. 112) portrays the maximization of the action during reflection in a concave mirror as the single, decisive objection to the METM.[9]If this interpretation is correct, my account is largely redundant. Am I making much philosophical ado about nothing?

The failure of minimization with concave mirrors was, indeed, in some sense, "obvious." But this was not regarded as an automatic or direct refutation of the METM,

[9]In their classic book on variational principles, Yourgrau and Mandelstam draw the same conclusion from a different problem illustrating that particles take non-action-minimizing paths, namely orbital motion under a Coulomb force (1968, p. 175). They appear especially interested in having this objection, for the METM seems to postulate a "mysterious purposive agency," and is compared with scientific bogeys which are "dangerous because of their metaphysical insinuations" (ibid., p. 173). It is understandable that those who view physical teleology as a monster would want a silver bullet to kill it. My argument is merely that it was not that simple.

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and nor is it, logically, a sufficient refutation. One need not even retreat to weaker, Leibnizian versions of physical teleology requiring only extremization, as described, for example, in (McDonough 2022). Those wishing to retain the specific teleological metaphysics of 'efficiency' underlying the PLA could retouch the metaphysical picture without changing its spirit. As a start, this might involve adopting Minimization* to accommodate problematic cases involving unstable equilibria. If we ignore the other conditions on the METM, the concave mirror could also be accommodated. I cannot work out an auxiliary hypothesis in detail here. But here is how one possible response might go. Observe that, in the concave mirror case, no local minimum exists in the first place. Recast the PLA as the conjunction of two claims: (1)"*Whenever* the geometry of the constraints admits of a minimum of the action, *then* Nature will choose etc.. . . ," and (2) all geometries allow a minimum. Concave mirrors only refute the second claim, allowing defenders to hold onto the first as an expression of Nature's preference for minima. This way of avoiding the counterexample would of course lead to calls for further explanation. But the conditions required for minima had yet to be explained mathematically and so, especially in this vacuum, an enterprising metaphysician would surely be up to the task of suitably elaborating the account.

Thus, the failure of Minimization was not, by itself, a 'silver bullet.' Indeed, that view has trouble accommodating history: if correct, it would make a puzzle of the fact that scientists scrutinizing the PLA did not cite concave mirrors as an automatic and decisive refutation.[10] As exhibited in section 4, opponents nonetheless developed a rich,

[10]Euler himself understood that the mathematics he largely invented did not fundamentally distinguish between maxima and minima (nor would Maupertuis have been ignorant of this). Yet he persisted for years supporting the METM. Scholars have noted that Eu-

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formal critique based on a plurality of mathematical considerations falling into the categories I identify and thematize.[11] All three conditions were needed to block avenues for rescuing the overall metaphysical picture: they amount to sacrificing the METM's picture of action as a universally budgeted resource. Given all three conditions, the modification sketched above is no longer available, particularly violating Universality. It was collectively that these conditions allowed mathematics to firmly grab onto the METM and hold its metaphysical feet to the fire, leading scientists of the time to conclude that the PLA expresses nothing metaphysically deep — at any rate, nothing metaphysically deeper than the accepted laws of mechanics. With that said by way of forestalling an objection, I now turn to completing the analysis.

3.3 (Universality) The PLA as a universal mechanical principle

For both Maupertuis and Euler, the importance of the PLA, and the truth of its teleological interpretation, relies on its claim to generality. Its importance "consists in its Universality" (Euler 1753e, p. 201). This is what distinguishes it from the 'least time'

ler's 'preferential treatment' towards minima is something of a puzzle (Pulte 1989, 229n). Here is not the place to attempt to resolve it definitively, but note that the 'silver bullet' view requires us to regard him as foolish or self-deceiving, whereas my account exhibits a complexity to the case that makes reasonable an evolution of his attitudes over time.

[11]Existing scholarship, though rich, generally ignores the many other mathematical considerations contained in the responses of other prominent scientists like d'Arcy and d'Alembert. The works that do discuss these figures, like Pulte (ibid., pp. 225–30) or Schramm (1985, pp. 150–7), have aims that do not lead them to analyses of the kind I give here.

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and ‘least resistance’ principles of Fermat and Leibniz. As Lyssy puts it, Fermat’s optical law was allegedly “not a proper law by itself”, but at best derivative of the PLA (2022, p. 134). At worst, it was only “*par un pur hasard*” that their choices of the end of Nature yielded accurate formulas. By contrast, the PLA, in its generality, embodies “the reason for these phenomena,” the universal, immanent final cause of physical events (Euler 1753e, p. 208). Universality was also why the PLA was supposed to be superior to *non*-teleological principles, like conservation of *vis viva*, which, Maupertuis observed, is “true only for certain bodies” (Maupertuis 1748b).

That the end of Nature really is to minimize action is allegedly proved by the fact that the PLA can be used to derive the laws for *all* physical processes, which Maupertuis endeavors to demonstrate in his (1748a,b), also citing (Euler 1744).[12] So, setting aside considerations of mathematical tractability, the minimization of $\int mvds$ should be a criterion universally applicable to nature. By consequence, as a mathematical tool, it should be suitable for solving all mechanical problems. However, a suite of issues arose when the principle was translated into forms suitable for that task. In his (1748b), Maupertuis attempted to show that the PLA is “*si universel et si fécond*” that from it can be derived, not only the laws of optics, but also the law of the lever and the rules for elastic and inelastic collisions.

Unfortunately, the usual modeling procedure, in which one first assigns a quantity of action to each possible trajectory of the particle, is unsuited to these new problems. Take equilibrium problems. By hypothesis, the bodies are *not moving*. There is no nontrivial

[12] This is their *claim*. Lagrange later objected: “But it must be admitted that these applications are too particular to serve to establish the truth of a general Principle [of mechanics]” (Lagrange 1788, p. 188).

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'path', and the action is necessarily zero. There is simply no way to extract any information by imposing the PLA condition. As for collisions, the bodies are moving, and so their trajectories do possess a nonzero quantity of action. But here we are not interested in their trajectories. We seek their velocities post-collision. For that, the PLA as stated in 1744 is not a suitable representational tool.

The way Maupertuis addressed these problems looks a great deal like sleight-of-hand. He wrote two new "statements" of the principle, one each for collisions and for equilibrium: "In the Collision of Bodies, Motion is distributed in such a way that the quantity of action supposed by the change is the smallest possible. In Rest, the Bodies in equilibrium must be so situated, that if they were subject to a small Motion, the quantity of action would be least" (Maupertuis 1748b).

By what right does Maupertuis offer these as formulations *of* the PLA? It looks as though he simply asserted two *new* principles, banking on their resemblance to the original PLA to pass them off as equivalent assertions. To see this, let $\int mvds$ (a functional over trajectories) be the baseline definition of 'action.' Then the version of the PLA used to analyze optical laws, PLA_O, indeed says: nature chooses the trajectory which minimizes *action*. Call the version used to analyze collisions PLA_C. The application of PLA_C in Maupertuis' text shows that it comes to this: nature chooses the final velocity that minimizes *the instantaneous rate of change of action*. Finally, the version applied to static equilibrium, PLA_E, asserts: nature chooses, as the equilibrium state, that configuration for which the *action associated to an infinitesimal virtual movement away from that configuration* is a minimum. We are no longer even talking about 'possible paths', and the principle is now, logically, a conditional: *if* the system is in equilibrium, *then* it minimizes what might be called the 'virtual infinitesimal action'.

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This last concept, moreover, has dubious mathematical bona fides, insofar as it is defined by integration over an infinitesimally short 'path'.

The present point is not to deny that PLA_O, PLA_C, and PLA_E are related. But they are different assertions, with different content. (Indeed, as mentioned, PLA_E, or the 'law of rest,' was not even teleological when it originally appeared in (1740).) They each assert the minimization of a different quantity relative to a different definition of the space of possibilities. And they are suitable for treating non-overlapping classes of problems.[13]Translating the metaphysical principle into a tool in the mathematical discipline of mechanics was supposed to exhibit that it is the unique teleological law governing the course of nature. But in actuality it gave rise to a problem for that metaphysics. For, we no longer have a single principle governing physical processes at all, but an ensemble of principles which at best bear a loose conceptual association to each other. The formal properties of the original PLA that were supposed to express the METM are not even shared among its new 'versions.'

As for Euler, initially, he was delighted to see the PLA find wider application. But he also noticed these difficulties. He tried to address them formally, by demonstrating that these principles are mathematically equivalent. A proof of the equivalence of the law of rest (PLA_E) and the 'proper' PLA of 1744 (PLA_O) was first sketched at the end of (1750b), then treated at greater length in (1753d). Maupertuis himself, in concluding his *Réponse* to d'Arcy's first attack, referred, one imagines with relief, to Euler's proof of the

[13]The principle PLA_E has a further peculiarity. When it is applied to the equilibrium of the lever, for example, it gives the position x of the fulcrum that *would* leave the lever balanced. This will be the 'actual' position of the fulcrum, not by *Nature's* decree, but only should the experimenter choose to place it there.

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equivalence of the law of rest and the principle of least action (1754). Both of them appear to have understood that the significance of the principle and its metaphysical purport was at stake.

What about this proof? The only direction treated in detail proceeds from the law of rest (PLA_E) to the 'proper' PLA (PLA_O). Euler's proof relies on a number of auxiliary assumptions, one being the principle of conservation of *vis viva*. Indeed, Euler already understood the need to restrict the principle to conservative systems in his (1744). Others had remarked that the need to import this independent energy constraint weakens the PLA's claim to universality. It is, indeed, difficult to see how the resources of the METM could motivate this condition. The most striking assumption in the proof, however, is a premise to the effect that action minimization *locally* implies action minimization over the whole path. As it turns out, there is no *mathematical* justification for this assumption, since it is not a mathematical theorem. Where, then, does it come from? It is a *metaphysical premise* based on the METM. Euler reasoned: "For if the intention of Nature is to be as sparing as possible with the sum of efforts [in equilibrium], this [intention] must extend also to movement, provided we take the efforts, not only as they subsist in an instant, but in all the instants through which the movement lasts, taken together [*i.e.*, the action]" (Euler 1753d, §XII).

Euler deployed a (metaphysical) premise about the intentions of Nature in an otherwise mathematical proof of the equivalence of these two principles. Even so, what he eventually proved is not that $\int mvds$ is minimized, but that $Const. - \int mvds$ is minimized. The constant may be determined by the particle's initial velocity, but even setting that aside, the sign is the reverse of what was intended, an awkward result. Of course, the mathematical procedure for applying the principle to find solutions to

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problems in mechanics is unaffected by a change of sign. Yet his excuses go further. He wrote: "But though the difference between a maximum and a minimum may seem large, *it is, however, of no consequence in Nature itself*" (Euler 1753d, §XIX, my emphasis).

This is striking. Euler had just leaned on the METM to provide a bridge from local to global minimization. This was to *justify* his equivalence proof, allowing him to rescue Universality. If, as he wrote, "the intention of Nature is to be as sparing as possible," one would naturally expect that the difference between maximizing and minimizing must be of some consequence (recall 3.1). But he was forced to deny that very claim and, in so doing, to sacrifice a core component of the teleological metaphysics. He tried to purchase Universality by sacrificing Minimization, while at the same time using 'Nature's intention to minimize' in his argument for Universality.

By now, I submit, the features of the mathematized concept of action, as prescribed by the underlying metaphysics, have given rise to severe contradictions, providing excellent reasons to reject that metaphysics. Maupertuis may have sought to give teleology new life by placing it on the solid foundation of mathematics. But in doing so, he appears, ironically, to have exposed himself to a novel and distinctively mathematical way of undermining his metaphysical project.

4 Rejection of the Teleological Metaphysics of the PLA

I conclude that, within about ten years, the advocates of the PLA faced contradictions between the metaphysical picture, motivated by the PSEN, and the mathematical

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resources used to model it. But did these reasons in fact operate as an effective constraint on metaphysics? I argue that they did. This is best appreciated by seeing how such considerations informed the scientific community's rejection of this metaphysics. I have already hinted at the reactions of the likes of d'Alembert and d'Arcy, and a closer look would seem to confirm that considerations of the kind I identify motivated their own critiques of the METM.

D'Alembert was no friend of teleology, and might easily have dismissed the METM for general philosophical reasons of obscurity or lack of rigor, as he did with other teleological ideas. This is *not* what he did. Indeed, in his article 'Final causes' (1752), d'Alembert discussed Maupertuis only to approve of his attacks on fallacious uses of final causes, staying silent on the PLA. Elsewhere, in the article 'Action,' he expressed skepticism about the algebraic from of the action quantity. The most extensive discussion of the PLA resides in the article 'Cosmologie' (1754), which contains the most measured and detailed critical analysis I have come across.[14]

D'Alembert was prepared to take Maupertuis' ideas seriously: the article praises Maupertuis for turning the vague PSEN into a precise principle of mathematical physics. Nonetheless, d'Alembert brought a sharp critical eye to the PLA and the METM. His analysis agrees with my own on many matters of detail, but more importantly, in its overall character: the METM falls victim to a suite of problems immanent to its formal elaboration, falling into the categories I identify. For reasons of space, I provide only a synopsis. Against the METM, d'Alembert cites: the non-significance and "arbitrariness" of the algebraic combination $mvds$ ("we can make as many mathematical combinations

[14]Though co-authored with Samuel Formey, d'Alembert wrote the second part, concerning the PLA.

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as we like of these two things [space and time], and can call all of this *action*; but the primitive, metaphysical idea behind the word *action* will not be made any more clear" (p. 297); that certain problems Maupertuis tackles with action are more appropriately formulated via *vis viva* ("but when we substitute here the word *vis viva* for that of the *action*..." p. 296); the failure of minimization with concave mirrors (p. 295); the need to formulate mathematically distinct versions of the principle for different problems ("the principle applies to many other cases, with some modifications that are more or less arbitrary," p. 297); and the difficulty in making sense of the action associated to changes taking place in zero time ("in the case of hard bodies, the change happening in an indivisible instant, the time is zero, and in consequence the action is null" p. 296). These considerations parallel those I raised regarding Budget, Minimization, and Universality. Evidently, he was concerned with much more than just the failure of Minimization. Though he mentions the problem of concave mirrors briefly, the majority of his analysis concerns other difficulties. By the end of the article, d'Alembert both circumscribes the domain of application of the PLA and withholds support for the METM.

D'Arcy is more polemical, but echoes many of these points (his articles and the *Encyclopédie* cite each other). Worth particular mention is his attack on the algebraic expression $mvds$ as representing the 'action of Nature', the issue I called Budget. This is in part because he wishes to defend his own theory, arguing that nature's action both has a different algebraic form and that it is actually conserved. He also forcefully presses the failure of Minimization, and a potential fourth problem, also formal, which I did not discuss: that the PLA alone does not always determine a unique trajectory (see note 7).

As for Maupertuis, he did not give up on the teleological interpretation of the PLA that he invented. But the significance of his position should not be overestimated. Aside

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from the critics just mentioned, Samuel König dragged him into an ugly priority dispute, claiming that Maupertuis's PLA was both false and plagiarized. Voltaire penned a vicious satire of him. Clergymen attacked him for trying to supplant their preferred teleological arguments for God. Embattled, his reputation and his priority claim at risk, and his health degenerating, it was an inopportune time to publicly reconsider the significance of his discovery. Maupertuis was also most committed to the theological implications of his principle, giving him additional stake in the viability of its metaphysics. This indicates apportioning less significance to Maupertuis' attitudes on the scientific merits.

In my view, Euler's attitude shift is more significant, not only because of his preeminence in the Enlightenment, but because he was the earliest and most vocal and consistent ally of Maupertuis. Although he never, as far as I am aware, repudiated the METM in print, the evidence here is telling. As we saw, Euler had to internalize serious contradictions between the METM and the implied features of the PLA's mathematical theory. Given that, as Maupertuis' ally, it would have been awkward for him to acknowledge these tensions publicly, I regard it as indicative of Euler's attitudes that his advocacy on behalf of the PLA effectively ceases after 1753. It was around then that he made a last, valiant effort, publishing a defense of the PLA (1753e), a reply to König (1753c), and an octavo book collecting his defenses along with the original attack by König (Euler 1753a). But after this, he ceased to discuss teleology in connection to action, despite enjoying many more active research years. Nor was he without opportunities to reflect on foundational and philosophical issues in physics. Touching on the topic of action in Letter 78 (dated November 22, 1760) of the enormously successful *Lettres à une Princesse d'Allemagne* (1768), Euler remained deafeningly silent on the subject of teleology, once so intimately associated with the PLA. Happy to credit

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Maupertuis as the PLA's discoverer, and noting that it once elicited both extraordinary praise and extraordinary criticism, he wrote that the principle is nothing more than a consequence of the impenetrability of bodies. Somewhat slyly equivocating on the word 'action', he had more or less returned to the view pre-dating his association with Maupertuis: the PLA is merely a consequence of the properties of bodies, no longer attributed to Nature's "intention to be sparing."

Although the mathematicians who continued work on variational principles did not provide detailed excursions into metaphysical interpretation, what they do say seems to further confirm the role of mathematical considerations in warding off physical teleology. As Fraser (1983, p. 233) notes, Lagrange was comfortable speaking of 'least action' in his early correspondence with Euler, but by 1760 the phrase disappears from his official lexicon in his memoir on the delta-calculus. It reappears in some historical remarks in the *Méchanique analitique* of 1788, where Lagrange recounts the controversy over whether the action is minimized or conserved (à la d'Arcy). Dismissing the whole debate, he wrote: "as if these vague and arbitrary denominations constituted the essence of the laws of nature, and could, by some secret virtue, erect the simple results of the known laws of mechanics on top of [a foundation of] final causes" (quoted in (ibid., translation modified)).

Lagrange also complained about Maupertuis' *applications* of the PLA to diverse problems: "[T]hey have, besides, something vague and arbitrary about them, that could not but render uncertain the consequences that we might draw regarding the exactitude of this Principle" (Lagrange 1788, p. 188). The failure of Universality and what Lagrange here calls "vagueness and arbitrariness" are sides of a coin. The allegedly universal PLA had a form unsuited to static equilibrium and collisions. Maupertuis addressed this by a

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fudge: (vague) constructions involving 'infinitesimal virtual paths' and (arbitrary) decisions to instead minimize the *rate of change* of the action. Lagrange clarified that the true, general principle, which he himself generalized out of the work of Maupertuis and Euler, is an extremal, not a minimal, principle. He concluded: "Such is the principle I, however improperly, here give the name 'least action', and which I regard not as a metaphysical principle, but as a simple result of the laws of Mechanics."

The fourth of Jacobi's *Vorlesungen über Dynamik*, delivered in 1842-3, also invoke formal considerations for rejecting the METM. The PLA is formulated in the modern notation of variational calculus:

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum_i m_i ds_i^2} = 0 \tag{1}$$

(where U is the potential and h an arbitrary constant). Jacobi then comments: "It is hard to find a metaphysical cause for the principle of least action when it is expressed, as it must be, in this true form" (Jacobi 1866, p. 45). The descendent variational principles of mechanics had thus quickly ceased to be interpreted teleologically by those developing the theory. And its development continued in the absence of a proof that the action is universally minimized. Indeed, Jacobi's tools (the theory of conjugate points) for determining conditions for minima likely only further undermined the proposal that minima possess fundamental (meta-)physical significance. The force of these formal reasons has been borne out by discussion of the PLA into subsequent centuries. In reference works for physicists, for example, one finds lists of considerations which, from the perspective of contemporary mathematical physics, defeat the teleological metaphysics which originally motivated the PLA (Gray 2009). Many of these notably

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diverse reasons rely on later theoretical developments, but they are all best understood as formal, or as immanent to the mathematics.

5 Conclusion

In the 1740s, a metaphysical thesis asserting the efficiency of Nature came to underwrite a new, general principle of physics. But, as I have argued, the process of working out these ideas soon led to contradictions between the metaphysics of 'least action in Nature' and the mathematical theory which was supposed to provide a rigorous foundation for it. As a result, this teleological metaphysics could no longer find footing in its mathematical representation. Formal, mathematical considerations — not *a priori* philosophical reasoning or empirical test — resulted in the rejection of a metaphysical thesis. Mathematization was supposed to finally place physical teleology on a rigorous foundation. Ironically, it provided novel, scientific reasons to decisively reject it.

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