The prince and the pauper
A quantum paradox of Hilbert-space fundamentalism

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I prove the existence of infinitely many physically distinct worlds represented by the same state vector and evolving according to the same law. This gives a constructive refutation of "Hilbert-space fundamentalism", the hypothesis that from the abstract state vector and the Hamiltonian all features of the physical world emerge uniquely, including space, all physical objects and their properties, and the decomposition into subsystems (Carroll 2021). This thesis was previously refuted in (Stoica 2021) in full generality, but the proof was mathematically very abstract, while the present constructive proof is, hopefully, easily accessible to the down-to-earth intuition of the working physicists and philosophers of physics.

Keywords: Hilbert-space fundamentalism; quantum-first approaches; preferred basis problem; no-go theorem

I. INTRODUCTION

A quantum system, which may be the entire world, is represented by a unit vector $|\psi(t)\rangle$ called *state vector*. $|\psi(t)\rangle$ belongs to a *state space* $\mathcal{H}$, a complex vector space endowed with a scalar product $\langle\psi|\xi\rangle = \langle\xi|\psi\rangle^*$, having some continuity properties that make it a *Hilbert space*.

A system in the state $|\psi(0)\rangle$ changes, after a time interval t, according to the *evolution equation*:

$$|\psi(t)\rangle = \widehat{\mathbf{U}}_t\,|\psi(0)\rangle\,, \tag{1}$$

where $\widehat{\mathbf{U}}_t = e^{-i/\hbar\widehat{\mathbf{H}}t}$, $\hbar$ is the *reduced Planck constant*, and $\widehat{\mathbf{H}}$ is the *Hamiltonian operator*. $\widehat{\mathbf{H}}$ is time independent for closed systems and even for the entire universe. The *evolution operators* $\widehat{\mathbf{U}}_t$ preserve the complex vector space structure and the scalar product of $\mathcal{H}$, so they are *unitary*.

Definition BQS. We call the triple $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi(t)\rangle)$ *basic quantum structure*.

This quantum formalism raises the following problem:

Question 1. *Does the basic quantum structure $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi(t)\rangle)$ give a complete description of reality?*

It is sometimes claimed that the answer to Question 1 is yes (Carroll 2021, Carroll and Singh 2019). This thesis, coined "Hilbert-space fundamentalism" in (Carroll 2021), is assumed in numerous research programs, especially Quantum Gravity programs in which spacetime emerges from the quantum structure. A discussion and more references are given in (Stoica 2021), where it was shown that the answer is negative by giving an admittedly very abstract but fully general proof. Also numerous counterexamples can be found in (Stoica 2022b).

In Section §II I show that the same state vector and the same Hamiltonian describe physically distinct histories corresponding to different measurement outcomes. This provides a very simple, intuitive and constructive proof that the answer to Question 1 is negative. This supplements the proof from (Stoica 2021) and the counterexamples from (Stoica 2022b) with more concrete situations that are more accessible to the readers that are too busy or less patient with mathematical abstractions, but have a good physical intuition of quantum theory. In Section §III we will see that this ambiguity extends to the classical level of reality, even in the absence of explicitly quantum measurements. Section §IV concludes with a brief discussion of the implications of this result. The more technical parts are exiled in Appendix §A.

II. THE PRINCE AND THE PAUPER

Edward is a young dreamer with bold ideas, who wants to make positive contributions to the world. He wants to invest, thinking that money would help him achieve his goals. Since he has a risk-embracing attitude, he decides to let quantum measurements make financial decisions for him. Or maybe he is just practical, not wanting to waste too much time making decisions based on incomplete information.

So whenever he thinks of choosing between two possible investments, or between buying or selling stocks, he lets quantum chance decide for him, and he faithfully bids accordingly. He can do this by making quantum measurement on qubits, or by using Vaidman's *Quantum World Splitter* (Vaidman 2022).

Suppose that if the qubit turns out to be in the state $|+\rangle$ Edward becomes very wealthy. Let's represent the world's state in which Edward is rich like a prince by

$|\psi_+\rangle = \Big|$  $\Big\rangle$. (2)

But if the opposite result $|-\rangle$ is obtained, he becomes poor like a pauper, and the world's state becomes:

$|\psi_-\rangle = \Big|$  $\Big\rangle$. (3)

This scenario is inspired by Mark Twain (Twain 1882). The images are A.I. generated (Zendesk 2023). Based on this scenario, I prove the following result:

Theorem 1. *The same basic quantum structure can represent an unlimited number of physically distinct realities.*

Proof. Let the $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle)$ be the basic quantum structure, assumed to describe a world in which Edward is rich like a prince.

Consider an alternative universe, with the same degrees of freedom and the same evolution law. There is no need for an alternative universe to exist, later we will see that this can be dropped and we can talk about the same universe. But for now it's pedagogically easier to introduce our problem by assuming an alternative universe. Let $(\mathcal{H}', \widehat{\mathbf{H}}', |\phi(t)\rangle')$ be its basic quantum structure.

The two basic quantum structures $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle)$ and $(\mathcal{H}', \widehat{\mathbf{H}}', |\phi(t)\rangle')$ are *isomorphic* if there is a unitary operator $\widehat{\mathcal{T}} : \mathcal{H} \to \mathcal{H}'$ so that

$$\begin{cases} \widehat{\mathcal{T}}\,|\psi_+(t)\rangle = |\phi(t)\rangle' \\ \widehat{\mathcal{T}}\widehat{\mathbf{H}}\widehat{\mathcal{T}}^{-1} = \widehat{\mathbf{H}}'. \end{cases} \tag{4}$$

If they are isomorphic, for any structure $\mathcal{S}$ supposed to emerge from the basic quantum structure $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle)$, there is an equivalent structure $\mathcal{S}'$ emerging from $(\mathcal{H}', \widehat{\mathbf{H}}', |\phi(t)\rangle')$, induced by an isomorphism as in equation (4) from the structure $\mathcal{S}$.

I will show that the basic quantum structure $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle)$ is isomorphic to both $(\mathcal{H}', \widehat{\mathbf{H}}', |\psi_+(t)\rangle')$ and $(\mathcal{H}', \widehat{\mathbf{H}}', |\psi_-(t)\rangle')$, where $|\psi_+(t)\rangle'$ and $|\psi_-(t)\rangle'$ are two physically distinct worlds resulting from the same qubit measurement. This will entail that there are physically distinct structures, so they don't emerge uniquely.

For this, let's return to young Edward and his usage of quantum measurements for decision making. We will consider a copy of Edward and his world represented in $\mathcal{H}'$. Let the universe, prior to the qubit measurement, be in the state

$$|Q\rangle\,|\text{ready}\rangle\,. \tag{5}$$

If $|Q\rangle = |+\rangle$, the universe evolves unitarily like

$$|+\rangle\,|\text{ready}\rangle \mapsto |\psi_+\rangle = \Bigg|$$  $$\Bigg\rangle\,, \tag{6}$$

while if $|Q\rangle = |-\rangle$, its unitary evolution is

$$|-\rangle\,|\text{ready}\rangle \mapsto |\psi_-\rangle = \Bigg|$$  $$\Bigg\rangle\,. \tag{7}$$

From Lemma 1 (see Appendix §A), there is a unitary transformation $\widehat{\mathbf{S}}$ of the total Hilbert space $\mathcal{H}'$ that preserves the Hamiltonian $\widehat{\mathbf{H}}'$ so that

$$\widehat{\mathbf{S}}\,\Bigg|$$  $$\Bigg\rangle = \Bigg|$$  $$\Bigg\rangle\,. \tag{8}$$

We found two basic quantum structure isomorphisms,

$$\begin{cases} \widehat{\mathcal{T}} & : (\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle) \to (\mathcal{H}', \widehat{\mathbf{H}}', |\psi_+(t)\rangle') \\ \widehat{\mathbf{S}}\widehat{\mathcal{T}} & : (\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle) \to (\mathcal{H}', \widehat{\mathbf{H}}', |\psi_-(t)\rangle'). \end{cases} \tag{9}$$

Therefore, $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi_+(t)\rangle)$ represents both a world in which Edward is rich, and a world in which he is poor.

Moreover, since Edward makes many financial decisions by using qubit measurements, the number of alternative worlds represented by the same basic quantum structure has an exponential, unlimited growth.

All these worlds may be physically very different, from containing a bankrupt Edward living on the street, to versions of Edward that built various financial empires, depending on the investment he made as advised by the results of the quantum measurements. □

Therefore, the basic quantum structure $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi(t)\rangle)$ doesn't give a complete description of reality.

III. AMBIGUITY AT THE CLASSICAL LEVEL

While the world is described by quantum theory, it appears to us classical, at least as long as we don't appeal to quantum measurements. But quantum measurements are ubiquitous, for example sight is a quantum measurement. When we observe visually the positions and shapes of objects, sight works like a position measurement. When we observe color, it works like a momentum measurement, since the wavelength of light is proportional to the momentum.

Both the emergence of classicality at the macro level and the quantum measurements work in the same way. A quantum measurement leads to a superposition of states in which the pointer state has observably distinct states. Whatever resolves this superposition, whether it is the wavefunction collapse or decoherence or another mechanism, it also ensures that at the macro level superposition is gone, and the world appears to us classical.

The observables that represent positions and momenta have continuous spectra, the full set of real numbers $\mathbb{R}$. This is in contrast with the qubit observables, which have only two eigenvalues. But Lemma 2 shows that the resulting states corresponding to two distinct eigenvalues are related by a unitary transformation that preserves the evolution law.

Therefore, the possible states of the macro level of reality that result from the same initial state vector can be described by the same basic quantum structure $(\mathcal{H}, \widehat{\mathbf{H}}, |\psi(t)\rangle)$. This provides, again, an unlimited number of concrete counterexamples to the Hilbert-space fundamentalism thesis. These counterexamples don't even require explicitly quantum measurements, only naked eye observations of the world.

IV. DISCUSSION

If Hilbert-space fundamentalism were true, space, fields on space, the decomposition into subsystems, a preferred basis, and every other physical feature of the world would emerge uniquely from the state vector and the Hamiltonian.

In (Stoica 2021) it was shown that whenever such a structure emerges from the state vector and the Hamiltonian, it is not unique. The only structures that can emerge uniquely are those that can't exhibit physical differences, even in relation with the state vector. But the actual structures exhibit such differences. For example the wavefunction changes with respect to space and to the tensor product decomposition of the Hilbert space that corresponds to subsystems. Numerous counterexamples to Hilbert-space fundamentalism were provided in (Stoica 2022b). One may object that the particular Hamiltonian of our world is not among those examples, but in (Stoica 2022a) it was shown that if it is not, the world can return to a past state.

The results from (Stoica 2021) were shown to affect all theories that assume an affirmative answer to Question 1, whether they rely on state vector reduction or branching (*e.g.* the version of Everett's Interpretation coined by Carroll and Singh "Mad-dog Everettianism"), proposals based on decoherence, and proposals that spacetime emerges from a purely quantum theory of gravity. This doesn't mean that such approaches are useless, just that they can't give a complete description of reality.

The proof given in (Stoica 2021) is fully general, but it was largely ignored, maybe because it's quite abstract, using tensors on the Hilbert space and invariants, and because it's an existence proof without many constructive counterexamples. The most intuitive and constructive counterexample was obtained using transformations of the form $\widehat{\mathbf{S}} = \widehat{\mathbf{U}}_t$, which preserve the Hamiltonian and its relation with the state vector. This implies that the present time state vector also describes the past and future states of the world. But the constructions present here give concrete examples of alternative realities represented by the same state vector and Hamiltonian.

If we assume Hilbert-space fundamentalism, Theorem 1 implies that Edward from a world in which he is poor also lives in a world in which he is rich, and he can realize this just by passively changing the basis of the Hilbert space. Section §III implies that he can do this without even making explicitly quantum measurements. By changing the basis using $\widehat{\mathbf{S}} = \widehat{\mathbf{U}}_t$, he can also passively travel in time (Stoica 2021). Such paradoxes show that the abstract state vector and the Hamiltonian provide an incomplete description of reality.

There are infinitely many continuous families of unitary transformations that commute with the Hamiltonian and preserve the state vector (Stoica 2021). Those from this article are just more intuitive, physically.

Appendix A: Proof of Lemma 1

To prove Lemma 1, needed in the proof of Theorem 1, we recall the *standard model of quantum measurements* (see *e.g.* Mittelstaedt 2004, §2.2(b), and Busch et al. 1995, §II.3.4). Realistic examples of such measurements are described in (Busch et al. 1995, §VII), including spin measurements using the Stern-Gerlach device, photon polarization measurements, various photon counters and measurements, and various beam splitter experiments.

We will apply the model to the measurement of a qubit observable $\widehat{\mathbf{A}}$ whose eigenvalues are ± 1.

In the case of spin measurements using the Stern-Gerlach apparatus, the possible outcomes of the measurement are distinguished by the region of a photographic plate hit by the observed particle. Therefore, the pointer observable corresponds to position. Similarly, we choose a pointer operator $\widehat{\mathbf{Z}}$ with the spectrum equal to $\mathbb{R}$. By working in the interaction picture, we can take the free Hamiltonians of the two systems to be zero, without loss of generality. This allows us to focus only on the interaction Hamiltonian. The Hamiltonian is

$$\widehat{\mathbf{H}} = \widehat{\mathbf{H}}_{\text{int}} = -g\widehat{\mathbf{A}} \otimes \widehat{\mathbf{p}}_{\mathbf{Z}}, \tag{A1}$$

where $\widehat{\mathbf{p}}_{\mathbf{Z}}$ is the canonical conjugate of the pointer operator $\widehat{\mathbf{Z}}$. The coupling g is constant in the interval $[0,T]$ and negligible outside this interval.

Let $\{|\lambda, a\rangle\}_{a\in\mathscr{A}}$ be a set of orthonormal eigenvectors of $\widehat{\mathbf{A}}$ corresponding to the eigenvalue λ. To account for the possible degeneracy of the eigenvalues, they are indexed by a label $a \in \mathscr{A}$. Since all eigenspaces of $\widehat{\mathbf{A}}$ have the same dimension, we can choose the same set of labels for all λ. They can be absent if the eigenvalues are unique.

Let $|\zeta\rangle$ be the pointer eigenvector corresponding to the eigenvalue $\zeta \in \mathbb{R}$. Since $\widehat{\mathbf{p}}_{\mathbf{Z}}$ is the canonical conjugate of the pointer operator $\widehat{\mathbf{Z}}$, they satisfy the *canonical commutation relation*

$$[\widehat{\mathbf{Z}}, \widehat{\mathbf{p}}_{\mathbf{Z}}] = i\hbar\widehat{\mathbf{I}}. \tag{A2}$$

The operator $\widehat{\mathbf{p}}_{\mathbf{Z}}$ generates the translations on the set of eigenvectors of $\widehat{\mathbf{Z}}$, *i.e.* for $\tau \in \mathbb{R}$,

$$e^{-i\tau\widehat{\mathbf{p}}_{\mathbf{Z}}} |\zeta\rangle = |\zeta + \tau\rangle. \tag{A3}$$

Then, for any eigenvector $|\lambda, a\rangle$ of $\widehat{\mathbf{A}}$ and any time interval $t \in [0, T]$, we obtain (Mittelstaedt 2004, §2.2(b)),

$$\begin{aligned} \widehat{\mathbf{U}}_t |\lambda, a\rangle |\zeta\rangle &= e^{-\frac{i}{\hbar}\widehat{\mathbf{H}}t} |\lambda, a\rangle |\zeta\rangle \\ &= e^{\frac{i}{\hbar}g\widehat{\mathbf{A}}\otimes\widehat{\mathbf{p}}_{\mathbf{Z}}t} |\lambda, a\rangle |\zeta\rangle \\ &= |\lambda, a\rangle e^{igt\lambda\widehat{\mathbf{p}}_{\mathbf{Z}}} |\zeta\rangle \\ &= |\lambda, a\rangle |\zeta - gt\lambda\rangle. \end{aligned} \tag{A4}$$

If the ready pointer state is calibrated to be $|0\rangle$ and the resulting pointer state after the time interval T is

$|-gT\lambda\rangle$, the corresponding eigenvalue of $\widehat{\mathbf{A}}$ for the observed system is read to be λ. Therefore,

$$\widehat{\mathbf{U}}_t |\lambda, a\rangle |\text{ready}\rangle = |\lambda, a\rangle |\text{result} = \lambda\rangle . \quad \text{(A5)}$$

Lemma 1. For the qubit measurement there is a unitary transformation $\widehat{\mathbf{S}}$ of the total Hilbert space so that

$$[\widehat{\mathbf{S}}, \widehat{\mathbf{H}}] = \widehat{\mathbf{0}} \quad \text{(A6)}$$

and

$$\widehat{\mathbf{S}} |+\rangle |\text{result} = +1\rangle = |-\rangle |\text{result} = -1\rangle . \quad \text{(A7)}$$

Proof. In the standard measurement scheme, we choose the unitary transformation defined on the basis vectors of the Hilbert space by

$$\widehat{\mathbf{S}} |\lambda, a\rangle |\zeta\rangle = |-\lambda, a\rangle |-\zeta\rangle . \quad \text{(A8)}$$

Therefore condition (A7) is satisfied.

For the condition (A6), we notice that for all $t \in [0, T]$

$$\begin{aligned} \widehat{\mathbf{U}}_t\widehat{\mathbf{S}} |\lambda, a\rangle |\zeta\rangle &\overset{(A8)}{=} \widehat{\mathbf{U}}_t |-\lambda, a\rangle |-\zeta\rangle \\ &\overset{(A4)}{=} |-\lambda, a\rangle |-\zeta + gt\lambda\rangle \\ &\overset{(A8)}{=} \widehat{\mathbf{S}} |\lambda, a\rangle |\zeta - gt\lambda\rangle \\ &\overset{(A4)}{=} \widehat{\mathbf{S}}\widehat{\mathbf{U}}_t |\lambda, a\rangle |\zeta\rangle . \end{aligned} \quad \text{(A9)}$$

Therefore, for all $t \in [0, T]$,

$$\widehat{\mathbf{U}}_t\widehat{\mathbf{S}} = \widehat{\mathbf{S}}\widehat{\mathbf{U}}_t. \quad \text{(A10)}$$

By taking the limit $t \searrow 0$, it follows that $\widehat{\mathbf{H}}\widehat{\mathbf{S}} = \widehat{\mathbf{S}}\widehat{\mathbf{H}}$, so condition (A6) is satisfied too. □

Remark 1. The result from Lemma 1 extends easily to any observable $\widehat{\mathbf{A}}$ that has $-\lambda$ as an eigenvalue whenever λ is an eigenvalue, and whose eigenspaces have equal dimension. This includes the cases when the spectrum of $\widehat{\mathbf{A}}$ is a continuous interval $(-\lambda_{\max}, +\lambda_{\max})$ or $[-\lambda_{\max}, +\lambda_{\max}]$. □

In addition, in the case when the spectrum of the observable $\widehat{\mathbf{A}}$ is $\mathbb{R}$, we have the following result.

Lemma 2. If the spectrum of $\widehat{\mathbf{A}}$ is $\mathbb{R}$ and the eigenspaces have equal dimension, for any pair of non-null eigenvalues $\lambda_1 \neq \lambda_2$ there is a unitary transformation $\widehat{\mathbf{S}}$ of the total Hilbert space so that $[\widehat{\mathbf{S}}, \widehat{\mathbf{H}}] = \widehat{\mathbf{0}}$ and

$$\widehat{\mathbf{S}} |\lambda_1\rangle |\text{result} = \lambda_1\rangle = |\lambda_2\rangle |\text{result} = \lambda_2\rangle . \quad \text{(A11)}$$

Proof. This is achieved by the unitary transformation

$$\widehat{\mathbf{S}} |\lambda, a\rangle |\zeta\rangle = \left|\frac{\lambda_2}{\lambda_1}\lambda, a\right\rangle \left|\frac{\lambda_2}{\lambda_1}\zeta\right\rangle . \quad \text{(A12)}$$

Then,

$$\begin{aligned} \widehat{\mathbf{U}}_t\widehat{\mathbf{S}} |\lambda, a\rangle |\zeta\rangle &\overset{(A12)}{=} \widehat{\mathbf{U}}_t \left|\tfrac{\lambda_2}{\lambda_1}\lambda, a\right\rangle \left|\tfrac{\lambda_2}{\lambda_1}\zeta\right\rangle \\ &\overset{(A4)}{=} \left|\tfrac{\lambda_2}{\lambda_1}\lambda, a\right\rangle \left|\tfrac{\lambda_2}{\lambda_1}\zeta - gt\tfrac{\lambda_2}{\lambda_1}\lambda\right\rangle \\ &\overset{(A12)}{=} \widehat{\mathbf{S}} |\lambda, a\rangle |\zeta - gt\lambda\rangle \\ &\overset{(A4)}{=} \widehat{\mathbf{S}}\widehat{\mathbf{U}}_t |\lambda, a\rangle |\zeta\rangle . \end{aligned} \quad \text{(A13)}$$

Therefore, $\widehat{\mathbf{U}}_t\widehat{\mathbf{S}} = \widehat{\mathbf{S}}\widehat{\mathbf{U}}_t$ for all $t \in [0, T]$ and the limit $t \searrow 0$ gives $\widehat{\mathbf{H}}\widehat{\mathbf{S}} = \widehat{\mathbf{S}}\widehat{\mathbf{H}}$. □

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