# Perturbative Expansions and the Foundations of Quantum Field Theory

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**Abstract**

Perturbative expansions have played a peculiarly central role in quantum field theory, not only in extracting empirical predictions but also in investigations of the theory's mathematical and conceptual foundations. This paper brings the special status of QFT perturbative expansions into focus by tracing the history of mathematical physics work on perturbative QFT and situating a contemporary approach, perturbative algebraic QFT, within this historical context. Highlighting the role that perturbative expansions have played in foundational investigations helps to clarify the relationships between the formulations of QFT developed in mathematical physics and high-energy phenomenology.

## Contents

# 1 Introduction

In applied mathematics one is often in a situation where exact solutions can be obtained in an idealised limit $\alpha \to 0$, but empirically relevant $\alpha > 0$ models resist analytic treatment. The perturbative strategy is to set up a series expansion for quantities of interest, of the form

$$F(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + ... \tag{1}$$

in order to try to extract information about the $\alpha > 0$ regime. In some very special cases, it may be possible to determine the form of the series to all orders and associate a sum with it (using methods like Borel summation in the case of an asymptotic expansion). In the more usual case where the large-order behaviour is not under control, however, it is still possible to use truncations of the series as a numerical approximation. In the context of quantum field theory, truncations of a series expansion in the interaction coupling—a parameter which describes the strength of interactions between quantum fields—are the most important source of the empirical predictions tested in collider experiments.

Approximation methods of this kind are sometimes viewed as pragmatically necessary for applications but irrelevant to investigations of the philosophical and mathematical foundations of a theory. This dismissive view of approximation methods is problematic on general grounds (de Olano et al., 2022), but in the case of QFT, where perturbative expansions have played a peculiarly central role in the development of the theory, it is especially implausible. There is no denying that weak coupling expansions shape the interpretative glosses found in standard QFT pedagogy, with the concepts of the virtual particle and the force mediating boson, to name a few prominent examples, being spelt out in Feynman diagram language. However, even in foundationally orientated literature on QFT, perturbative expansions loom larger than is often appreciated. While, there is a long history of attempts to develop non-perturbative characterisations of QFT in mathematical physics, exemplified by the axiomatic QFT and constructive QFT traditions, the development of these frameworks has been strongly influenced by the perturbative approach. Furthermore, there is also a long history of direct engagement with the mathematical structure and conceptual foundations of QFT perturbative expansions themselves. This second strand of mathematical work on QFT has been largely neglected in the recent philosophical and historical literature on QFT and is also not widely known in mainstream theoretical physics. Our goal in this paper, therefore, is to bring this current into clearer focus by surveying early foundational engagement with perturbative QFT and situating a prominent contemporary approach, perturbative algebraic QFT (pAQFT), within this broader historical narrative.

The first part of this paper (section 2) sketches the development of mathematical QFT from the 1950s, structuring the story around the reception of various divergence difficulties that arose within the perturbative formalism: ultraviolet divergences, the large-order divergence of the renormalised series, infrared divergences, and the more obscure "surface divergences". The main claim is that we can see foundational work on QFT as dividing into two strands in the 1950s, in response to the problems facing the perturbative formalism: the axiomatic QFT road, which attempted to develop non-perturbative formulations of the theory, and the causal perturbation theory road, which focused on clarifying the mathematical underpinnings of the series expansion itself. The second part of the paper (section 3) presents pAQFT as a continuation of this story which brings these two strands of development together again and vindicates the continuing significance of perturbation expansions in contemporary work on the foundations of QFT. Highlighting the role that perturbative expansions have played in investigations of the foundations of QFT helps to clarify the relationships between the formulations of the theory found in mathematical physics and high-energy physics phenomenology, and we conclude with some brief reflections on this point.

# 2 Early Foundational Engagement with the Perturbative Expansion

## 2.1 Ultraviolet Divergences

From the very beginning, attempts to quantise field theories were plagued by ultraviolet divergences—divergent integrals which, when written in momentum-space, blow up in the region of large momentum. Such integrals, in fact, originally cropped up in the quantisation of exactly solvable free (i.e. non-interacting) models. Ultraviolet divergent expressions for the eigenstates of a quantum string were already obtained in Born, Heisenberg and Jordon's famous "*Dreimännerarbeit*" (Born et al., 1926), with the same behaviour being found in the quantisation of the free electromagnetic field (Jordan and Pauli, 1928). These original ultraviolet divergences were relatively calmly dealt with, however, as it was argued they could be eliminated by subtracting an infinite constant from the ground state energy, a procedure which was eventually understood as following from the normal ordering of the field operators (discussed further in section 3.1). Given this successful evasion, when ultraviolet divergent integrals appeared in the second-order expansion coefficients of the perturbation series in the interaction coupling established in Heisenberg and Pauli's founding paper on quantum electrodynamics (QED), it was initially unclear how bad the problem really was. These perturbative ultraviolet divergences proved to be of a more disturbing kind, however. Oppenheimer (1930) argued that the divergences in Heisenberg and Pauli's perturbative scheme implied not only that the absolute values but the differences between energy eigenstates were infinitely large, concluding that second-order approximants obtained from their perturbation series were empirically incoherent.[1]

It was this problem with the second-order perturbative coefficients which sparked serious worries about the basic consistency of interacting QFTs in the 1930s and 1940s, already demonstrating the central role of the perturbative expansion in the foundational appraisal of the theory. Bohr and Rosenfeld (1933)'s discussion of field measurements seems to have convinced many that the source of perturbative infinities lay in the local point-like nature of the QED interaction term. Accordingly, before the war, many of the key theorists had been attempting to move away from a local field theoretic approach to relativistic quantum theory, in one way or another. Heisenberg came to associate the ultraviolet divergences with the need for a fundamental length which would cut-off arbitrarily high momentum modes and eventually put forward the S-matrix as an alternative starting point for the dynamics of relativistic quantum theories (Blum, 2017). Meanwhile, Dirac, Kramer, and Feynman in his early work with Wheeler, connected the ultraviolet divergences in QFT perturbation theory to unresolved problems with the infinite self-energy of point charges in classical electrodynamics, looking for new classical theories, and new methods of quantisation, which would lead to an infinity-free quantum theory of the electromagnetic interaction (Rueger, 1992).

After the war, a rather more conservative response to the ultraviolet divergences problem was developed, however. The work of Feynman, Schwinger and Tomonaga introduced new, more perspicuous, formulations of the QED weak coupling expansion and demonstrated that the ultraviolet divergences appearing in the coefficients could be systematically removed via a procedure which came to be known as renormalisation (Schweber, 1994). Freeman Dyson's classic pair of papers Dyson (1949a,b) synthesised this work into a coherent and predictively powerful formalism that forms the basis of the approach to perturbative QFT one still finds in textbooks today.

Dyson set up the series expansion for the S-matrix, an operator mapping asymptotic states at $t \to -\infty$ to $t \to +\infty$ (that had previously been introduced by Heisenberg, as noted above). Separating the Hamiltonian into a free and interacting part, $H = H_0 + V$, Dyson made use of the interaction picture (more on this below) to derive a series expansion of the form:

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} dt_1 ... \int_{-\infty}^{\infty} dt_n \mathcal{T}[V(t_1)...V(t_n)], \tag{2}$$

where $\mathcal{T}$ indicates that operator products appearing inside the brackets are to be time-

[1] The reception of the ultraviolet divergences problem in the 30s and 40s is a complex matter which is treated only very briefly here. See Rueger (1992) and Schweber (1994), chapters 1 and 2, for more detailed accounts.

ordered, i.e. operators with earlier time arguments are to the right of operators with later time arguments. We discuss the problem of providing a mathematically rigorous definition of this time-ordered product in what follows, but naively implementing time-ordering by multiplying by Heaviside functions in the time arguments leads to ultraviolet divergent integrals of the same type found in the 1930s. This is where renormalisation comes in. The key idea behind this procedure was that the ultraviolet divergent part of the coefficients could be "subtracted" via the introduction of infinite counterterms, leading to a redefinition of the dynamical parameters of the model. In the case of QED, it proved to be sufficient to introduce renormalised versions of the finite number of dynamical parameters appearing in the original Lagrangian (the theory is said to be renormalisable).

Once the ultraviolet divergences had been cured in this way it was possible to obtain extremely accurate empirical predictions from truncations of the Dyson series. Indeed, some of the most accurate empirical predictions in the history of physics have been obtained via renormalised perturbation theory. What was less clear was what all this meant for the foundational standing of QFT. Had the worries about the pathological short-distance behaviour of QED, and QFT more generally, been solved or merely skirted around for calculational purposes? We can see the 1950s as a period of critical reflection on what had been achieved by the invention of perturbative renormalisation. It was in this moment that some theorists came to see increased standards of mathematical rigour as an important precondition for bringing resolutions to the persistent foundational questions surrounding QFT. We thus see a new sub-community, which we anachronistically refer to as mathematical QFT, gradually emerge in this period.

Renormalisation had not, in fact, solved all of the problems associated with QFT perturbative expansions, never mind QFT as a whole, and in the following four sub-sections we use the continuing struggles with perturbative divergences problems of various kinds to frame the early development of mathematical QFT. Sections 2.2 and 2.3 discuss the large-order behaviour of the series expansion and the infrared divergent integrals which also appear in the coefficients. It is argued that these issues played a role in pushing many foundationally minded theorists away from perturbation theory and in search of a coherent non-perturbative formulation of QFT. This ultimately gave rise to the operator algebra approach to QFT and the Haag-Kastler axioms. The next two sections discuss another less well-known strand of foundationally orientated work on QFT which focused on the perturbative formalism itself. This tradition, which would come to be called causal perturbation theory, originated in a critique of the dynamical assumptions underlying Dyson's derivation of the perturbative expansion, manifesting in so-called surface divergences (section 2.4). It would go on to produce a new understanding of the ultraviolet divergence problem and a mathematically rigorous reconstruction of the perturbative renormalisation procedure (section 2.5).

The picture of early mathematical/foundational work on QFT which emerges from our

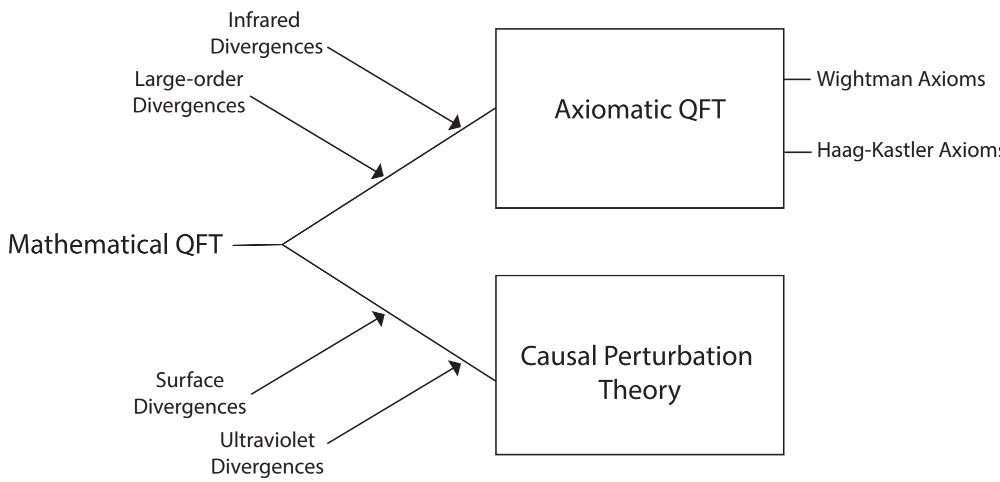


Figure 1: A representation of the development of mathematical QFT in the 1950s–1970s, emphasising that the axiomatic QFT and causal perturbation theory strands primarily responded to different issues with Dyson's renormalised perturbative expansion for the S-matrix.

discussion then, is one of two parallel research programs - one non-perturbative (yet still responding to problems that manifest within perturbation theory) and one perturbative. This is illustrated in figure 1.

## 2.2 Large-Order Divergences

Perhaps the most optimistic take on the foundational potential of the new renormalised perturbation series was held by Freeman Dyson. Dyson saw his 1949 papers as the first step in a programme which would use the renormalised expansion to put QFT on a firm mathematical footing.[2] A crucial component of this programme was the hope that after the ultraviolet divergences had been handled, the expansion converged so that the sum

[2] To give some more details of Dyson's programme in this period, the goal was to use the renormalised expansion to construct Heisenberg field operators, rather than just the asymptotic S-matrix (Dyson, 1951a). In order to achieve this he introduced the so-called intermediate representation, a representation of the time-evolution of the theory which separated the high and low momentum field modes, with the hope being that the high momentum part of the theory could be constructed via a convergent renormalised series expansion Dyson (1951b,c). Interestingly, while Dyson's intermediate representation was not taken up in the 50s, Kenneth Wilson mentions it as an inspiration for his approach to the renormalisation group (Wilson, 2002).

could be used to construct exact solutions to the QED field equations. While Dyson was initially hopeful about the convergence of the perturbative expansion circa 1951 it soon became clear that this rosy scenario was not realised; even after renormalisation the perturbation series of QED, and probably any other realistic interacting field theory, was divergent (for a more detailed account of these developments see Blum (2023)).

Dyson himself published a heuristic argument to that effect in 1952 (Dyson, 1952). If the perturbation expansion of some quantity $F(e^2)$ in QED was to converge it would have to be analytic in some finite radius of convergence in the complex plane. Thus, if it was to converge for small positive values of $e^2$, it must also converge for small negative values as well. But, Dyson reasoned, QED with a negative interaction coupling would have a potential which was unbounded from below. Since electrons and positrons would attract in this model, the system would be unstable to the creation of arbitrarily many electron-positron pairs. Dyson thus found it impossible to believe that $F(e^2)$ could be analytic on the negative axis. In the language of complex analysis, there would thus be a cut on the negative axis, meaning that the radius of convergence of the expansion was zero. This verdict was soon bolstered by diagram counting arguments published by Hurst and Thirring (Hurst, 1952; Thirring, 1953). At each order of perturbation theory the number of Feynman diagrams which contribute to the series coefficients at each order greatly increases, as loops can be inserted in more and more positions on the diagrams' internal legs. Using the $\phi^3$ scalar theory as a toy model, these authors argued that the number of diagrams at each order increases factorially, meaning that the growth of the coefficients would eventually overpower the suppression effect from repeatedly multiplying by a small expansion parameter.

While it had not been conclusively demonstrated that QFT perturbation series generically diverge, all indications seemed to point in this direction. Compared to the ultraviolet divergences problem, the large-order divergence of the perturbative expansion did not spark the same atmosphere of crisis, and indeed we find a very quiet response to these results in the succeeding literature. This is partially explained by the fact that, unlike the ultraviolet divergences in the coefficients, the large-order behaviour of the series did not threaten the predictivity of the perturbative approximation strategy. QED perturbation theory was clearly at least an asymptotic series which displayed apparent convergence in the leading terms of the series. Indeed, both the Dyson and the Hirst-Thirring arguments suggested that the number of terms before the series started diverging was roughly the inverse of the fine structure constant, so the QED expansion could be expected to produce a converging approximation for the first 137 terms or so—far beyond what could ever feasibly be calculated.[3]

[3]When it came to models of the strong nuclear interaction the expansion parameter was expected to be much larger than QED's, and consequently the fact that the expansion was likely only asymptotic had a more direct phenomenological relevance—see de Olano et al. (2022) for some discussion. The large-order

The large-order divergence of the expansion arguably did have a crucial impact on the development of foundational and mathematical work on QFT, however. In a retrospective review of the development of axiomatic QFT Arthur Wightman wrote of this period:

> "[A]re the renormalized series convergent or divergent? Although there is even yet no proof in quantum electrodynamics, it is generally believed on the basis of a study of simpler cases that they are divergent... Thus at this stage in the development, which occured in the early 1950's, the key problem was to find some non-perturbative approach to the solutions of quantum field theory." (Wightman, 1976, 195).

Similarly, when Arthur Jaffe, a key architect of the constructive field theory programme, gave a more rigorous demonstration of the divergence perturbative expansion of scalar QFT models in the 1960s, the conclusion he drew was that "some approximation scheme more sophisticated than perturbation theory must be used to investigate solutions of the field equations" (Jaffe, 1965, 9). It seems clear, therefore, that the large-order divergence problem put an end to Dyson's dream of resolving the foundational problems of QFT using perturbation theory alone, and provided a powerful impetus for early axiomatic field theorists to develop a non-perturbative characterisation of interacting QFTs.[4]

It might be tempting to identify this turn away from perturbation theory in the early axiomatic QFT movement as the origin of the divide between the mathematical and phenomenological wings of high energy theory. This would be misleading for a number of reasons, however. Firstly, there were actually good phenomenological reasons to attempt to escape the shackles of the perturbative expansion in the 1950s and 1960s, as most theorists became increasingly sceptical about the possibility of usefully applying weak coupling approximations to the strong nuclear interaction. For a time at least, foundational and empirical motivations aligned on this point, leading to perhaps surprising interplay between axiomatic QFT and the heuristic dispersion relations based approach to non-perturbative calculations which flourished in this period.[5] Secondly, as we have already intimated,

behaviour of the series would also become important once more with the rise of perturbative quantum chromodynamics (QCD) in the 70s and 80s, leading to a new wave of results on the large-order behaviour of QFT perturbative expansions.

[4]The impression that the perturbative expansion was not powerful enough to resolve the foundational problems of QFT was likely further bolstered by new putatively non-perturbative results which emerged in the 1950s. Källén and Landau put forward new arguments for thinking that QED broke down in the ultraviolet regime which seemed to be non-perturbative in nature and potentially more severe than the (Källén, 1953; Landau and Pomeranchuk, 1955). These developments, and the broader context of debates about the inconsistency of QFT in the 1950s, are discussed in Blum (2023).

[5]As is evidenced by the curious 1961 volume "Dispersion Relations and the Abstract Approach to Field Theory" which gathered together early axiomatic QFT papers with more phenomenologically orientated works by Mandelstram, Pomeranchuk and others as if they were all contributions to a common programme (Klein, 1961). Fleshing out these connections is a thus far unexplored historical problem.

axiomatic QFT was not the only programme that can be placed under the heading of mathematical QFT and there was in fact a substantial amount of mathematical work on the perturbative expansion, as we will discuss further in sections 2.4 and 2.5. Finally, however, moving beyond perturbation theory does not mean ignoring perturbative results entirely, and indeed we see substantial interplay between axiomatic QFT and perturbation theory, as we touch on in the following section.

## 2.3 Infrared Divergences

Ultraviolet divergences are not, in fact, the only type of divergent integral expressions that appear in the coefficients of QFT perturbation series. Theories featuring massless fields, and especially gauge theories like QED, display so-called infrared divergences as well—integrals in the series coefficients which blow up in the region of arbitrarily small momentum (again, when Fourier transformed into momentum-space). Where the ultraviolet divergences are associated with the short-length scale structure of the theory, these infrared divergences are associated with long-distance structure and the limit of infinite volume. While infrared divergences did not spark the same level of angst about the consistency of QED they too received serious attention in the 1930s (see Blum (2015) for a historical analysis of this early period). Most importantly, Bloch and Nordsieck (1937) argued that the infinitely many "soft", i.e. low momentum, photons in asymptotic scattering states which gave rise to the infrared divergences could not be detected experimentally and consequently the infrared divergent parts of the coefficients must cancel in the series expansions of properly defined asymptotic observables—see Miller (2021) for a philosophical discussion. Various questions about how to deal with infrared divergences in practice remained, however, as well as the foundational question of why this second class of divergences occurs in the first place. One might argue that the problem of understanding the mathematical origins of infrared divergences, rather than the infamous ultraviolet divergences, actually had a more direct impact on the formation of axiomatic QFT—and especially the algebraic approach. A brief discussion of this theme will allow us to highlight the interplay between early axiomatic QFT and perturbation theory.

One of the earliest mathematical studies of QFT, that of Friedrichs (1953), already manifested this focus on infrared divergences. Friedrichs was among the first to highlight a crucial feature which differentiates QFT from non-relativistic quantum mechanics: the existence of unitarily inequivalent Hilbert space representations of the commutation relations.[6] The presence of infinitely many degrees of freedom circumvents the Stone-

[6]The fact that infinite degrees of freedom led to inequivalent Hilbert space quantisations was noted already by Von Neumann (1939). Besides Friedrichs, another early proponent of the significance of this fact was Irvin Segal, who advocated an algebraic approach to quantum theory long before Haag and Kastler; see Segal (1947, 1959).

von Neumann theorem, a result which ensures that the Hilbert space representation of the canonical commutation relations of a typical finite system is unique up to a unitary transformation. Friedrichs explicitly linked the perturbative infrared divergences to the presence of inequivalent Hilbert spaces arising from infinitely many outgoing soft photons. Around the same time Van Hove (1952) and Haag (1955) also put forward results which shed light on the infrared divergence issue. What came to be known as Haag's theorem established that the vacuum state associated with the free theory and the interacting theory live in unitarily inequivalent Hilbert spaces. All this suggested that the Fock space representation employed in the perturbative formalism was the culprit and that the infrared divergences arose from integrating over states in the "wrong representation of the Weyl relations" (Wightman, 1976, 205). Many of the founding papers of the axiomatic field theory tradition were thus concerned with developing a mathematically precise description of asymptotic scattering states without employing perturbative machinery. The LSZ formalism was a crucial starting point here (Lehmann et al., 1955), with later works by Haag (1958) and Ruelle (1962) framed in terms of shoring up the rigour of the LSZ scattering theory. Already in these papers, one finds a focus on algebras of Hilbert space operators associated with local space-time regions, hinting at an algebraic approach to the theory.

This line of development starting from the infrared divergences problem was thus a key precursor to what became known as algebraic QFT (AQFT). The axioms for relativistic QFT put forward by Haag and Kastler (1964) were based on the notion of a C*-algebra, a natural abstraction from concrete algebras of bounded self-adjoint operators on Hilbert space. C*-algebras of observables $\mathfrak{A}(\mathcal{O})$ are associated with closed regions $\mathcal{O}$ on Minkowski space-time, with states identified with linear maps from elements of this algebra to the numbers. One could then write down a series of axioms that this algebra net could reasonably be expected to satisfy thereby obtaining a fully non-perturbative statement of the physical content of a QFT. One axiom which will be important for later discussion is known as micro-causality:

> **Micro-causality**. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two spacelike separated regions then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$,

inforcing the independence of spacelike separated observables.[7] Another assumption of the Haag-Kaster axioms was the existence of a unique Poincare invariant vacuum state associated with each Hilbert space representation of the algebra. This allowed contact to be made with the Wightman axioms, an alternative system of axioms developed in the

[7]The idea that a causality condition of some kind needed to be added to Poincare covariance was prevalent across a wide range of theoretical approaches to relativistic quantum theory in the 50s and 60s, as evidenced by Bogoliubov's causality condition discussed in section 2.4. This broader context is discussed in Blum and Fraser (2023).

same period which had been used to prove spin-statistic and PCT theorems (Streater and Wightman, 1964). A key advantage of the AQFT approach, however, was that the unitarily inequivalent Hilbert space quantisations of an infinite system could be understood as representations of a common algebraic structure, thus providing an ideal framework for describing infrared phenomena like superselection (Doplicher et al., 1971, 1974), unsurprisingly given the roots of this approach in the infrared divergence problem.

These brief remarks only scratch the surface of the early development of axiomatic QFT, which currently lacks a dedicated historical treatment. These preliminary points about the relevance of the infrared divergences problem are sufficient to establish that a complete historical account of the axiomatic approach will certainly have to take interactions with the perturbative formalism into account. The goal of the axiomatic programme was to go beyond perturbation theory rather than throw it away entirely, and early axiomatisers were often quite explicit in their attempts to abstract general properties from the features of the perturbative expansion—Haag (1955) checks his earliest set of axioms are satisfied in the perturbative setting, and Wightman is also clear that the singularity properties of the interacting field operators are conjectured on the basis of the behaviour of n-point functions in perturbation theory (Wightman, 1976).

In the end, axiomatic QFTs mission to go beyond the perturbative expansion was at best a rather partial success, however. While some general structural results and qualitative accounts of infrared phenomena could be extracted from the new axiomatic systems, difficulties remained with connecting them to empirically successful models. In the late 60s a sub-programme known as constructive field theory emerged, dedicated to verifying the existence of interacting models of the Wightman and Haag-Kastler axioms. While there were some impressive achievements constructing 2-dimensional interacting models, interacting theories on 4-dimensional Minkowski space-time, and specifically the gauge theories making up the standard model, remain beyond our grasp even today. Furthermore, while there was widespread hope that perturbation theory might be circumvented as the primary predictive approach to QFT in the 1950s and 1960s (as mentioned in section 2.2) the asymptotic freedom of non-abelian gauge theories, and the acceptance of QCD as the theory of the strong nuclear force triggered a new renaissance of perturbation theory in the 1970s and 1980s. Contrary to some theorist's expectations the perturbative expansion continued to be the primary tool for analysing empirically relevant models and a robust non-perturbative approach to interactions did not emerge.

## 2.4 Surface Divergences

We have seen that the axiomatic QFT tradition strove to move beyond the perturbative expansion and develop a new non-perturbative language for the theory. There was another parallel strand of mathematically orientated work on QFT, that also goes back to the

1950s, which instead attempted to clarify the mathematical structure of the perturbative expansion itself. One prominent programme in this second branch of mathematical QFT is what came to be called causal perturbation theory. The following two sections sketch the development of this approach (a more detailed account is found in Blum and Fraser (2023)).

While the large-order divergence of the perturbation series suggested that it could not provide a complete characterisation of the mathematical structure of QFT, the perturbative formalism still had its own internal foundational issues which remained unresolved. Furthermore, since perturbation theory had so far been the only successful calculation scheme in QFT, these problems were, from a certain point of view, more natural targets for mathematical clarification. As N. N. Bogoliubov, one of the founding figures of the causal perturbation theory approach remarked, focusing on the perturbative expansion,

> "best corresponds to the actual modern state of field theory, where up to the present various formal expansions in powers of the smallness of the interaction cannot be removed and where all fundamental results were obtained with the help of these expansions" (Bogoliubov, 1955).[8]

Where the axiomatic QFT program was, in a way, quite revolutionary in its ambitions to develop a new non-perturbative language for QFT, causal perturbation theory can be thought of as a more reformist operation, attempting to shore up the conceptual underpinnings and mathematical rigour of the tried and tested perturbative treatment of interactions.

One of the key ideas driving the causal perturbation theory approach was a critique of the dynamical assumptions Dyson had adopted in his derivation of the series expansion of the S-matrix. The post-war formulation of perturbative QED had been based on the so-called interaction picture. In this representation of the time evolution, the Hamiltonian of a field theory is split into a free and interacting part, $H = H_0 + V$, and states are taken to evolve under the interaction Hamiltonian alone with the remaining free time evolution shifted into the operators. Schwinger and Tomonaga had independently pointed out that one could formulate a fully covariant evolution equation in the interaction picture by considering variations with respect to a space-like hypersurface $\sigma$:

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)} = V\Psi(\sigma). \tag{3}$$

Dyson (1949a)'s derivation of the perturbative expansion for the S-matrix had been based on iteratively integrating this equation.

[8]Thanks to Kseniia Mohelsky for translating this Russian language source.

We have already mentioned one reason to worry about the cogency of this scheme. Haag's theorem implies that a unitary transformation linking the free and interacting theory could not in fact exist (as Haag (1955) pointed out explicitly). Here, we focus on a lesser-known problem with the interaction picture dynamical framework which manifests within the perturbative expansion coefficients themselves. It turns out that generalizing from the asymptotic S-matrix, and working with a perturbative expansion of Dyson's time-evolution operator at finite times leads to yet another sort of divergence in the coefficients, variously called boundary, surface and Stueckelberg divergences (in the admittedly scant literature which discusses this issue at all).

The statement of this issue goes back to Stueckelberg (1951), Ernst Stueckelberg being the original architect of the causal perturbation theory approach. Stueckelberg (1951) posed the question of how one could extend the description of scattering afforded by Dyson's expansion of the S-matrix to finite times—essentially relaxing the idealisation of treating incoming and outgoing scattering states at temporal infinity. He proposed modifying the S-matrix framework by introducing a "switching function", $g(x)$, that takes a value of 1 in some finite space-time region and is zero elsewhere, thus "turning on the interaction" in a finite region only. One could, therefore use a generalised S-matrix, $S(g(x))$, to represent a scattering event that takes place in a finite space-time region, with the usual asymptotic S-matrix being recovered in the limit $g \to 1$. Stueckelberg pointed out that implementing the Schwinger-Tomonaga interaction picture equation at a finite time was equivalent to turning on the interaction discontinuously, i.e. taking the switching function to be a Heaviside function of the space-time arguments. This, he pointed out, led to a new type of divergence in the perturbative coefficients which were not removed by renormalisation—the so-called surface divergences.[9] His solution was to use a smooth switching function to turn on the interaction, but Stückelberg reasoned, this would mean moving away from the Schwinger-Tomonaga differential equation as the basis for deriving the perturbative expansion.

Stueckelberg and his collaborators proposed setting up the weak coupling expansion starting from a causality condition (Stueckelberg and Rivier, 1950a,b). This alternative to Dysonian perturbation theory was largely ignored in the west but was taken up by the Soviet mathematician and physicist, Nicolay Bogoliubov. Whereas Stueckelberg had been a contemporary of Feynman, Schwinger and Tomonaga, and often styled his approach as a rival theory, Bogoliubov saw in it the potential for a rationale reconstruction of the conventional perturbative formalism motivated primarily by concerns about mathematical rigour. Where Stueckelberg had struggled to provide a mathematically precise formulation of his causality condition, Bogoliubov made use of the smooth switching function, $g(x)$, to do so. Suppose we have two switching functions $g_1$ and $g_2$, such that all of the spacetime

[9] For a detailed reconstruction of this problem see Blum and Fraser (2023).

points in the support of $g_1$ are in the past with respect to some reference frame, and all of the points in the support of $g_2$ are in the future. Bogoliubov's causality condition is then:

$$S(g_1 + g_2) = S(g_2)S(g_1), \tag{4}$$

This condition can be read as ruling out the possibility of retro-causality, i.e. the possibility of the results of a latter period of scattering affecting the results of an earlier period of scattering. It is thus, at least on the surface, a conceptually distinct notion of relativistic causality from the micro-causality condition of AQFT, which is naturally read as ruling out superluminal influence (we will see in section 3.2 pAQFT brings some clarity to the relationship between these two causality conditions).

The method of constructing the series expansion advanced by Bogoliubov was to start with an abstract expansion of the form:

$$S(g) = \sum_{n=0} \frac{1}{n!} \int S_n(x_1, ..., x_n) g(x_1)...g(x_n) dx_1...dx_n, \tag{5}$$

where the $S_n$ coefficients are so far unspecified. By imposing his new causality condition, along with the unitarity and Lorentz invariance of the S-matrix, one obtained a relationship between the coefficients at each order:

$$S_n(x_1, ..., x_n) = S_m(x_1, ..., x_m) S_{n-m}(x_{m+1}, ..., x_n), \tag{6}$$

where the points $x_1 - x_m$ are in the past with respect to the points $x_{m+1} - x_n$. Starting from a stipulation of the form of the first-order term:

$$S_1(x) = iV(x), \tag{7}$$

which essentially amounts to selecting an interaction potential for the model, one could iteratively construct the higher-order terms in the series to all orders and reproduce the basic form of the Dyson series: i.e. one ends up with series coefficients consisting of time-ordered products of $V$.

There is one important difference between the series expansion derived from Bogoliubov's causality condition and the conventional Dyson series, however. Ultraviolet divergences arise in the expansion coefficients due to singularities associated with coincident spacetime arguments—i.e. $x_1 = x_2 = ... = x_n$. In the conventional approach, this can be seen as flowing from the implementation of the time-ordered product via a discontinuous Heaviside function. Starting from Bogoliubov's causality condition, on the other hand, left the form of the series coefficients at these problematic coincident points unspecified since it only makes a statement about the ordering of temporally separated events. Consequently, Stueckelberg, Bogoliubov, and later on Epstein and Glaser, took the causal derivation of the perturbative expansion to provide a superior starting point for a more rigorous treatment of the ultraviolet divergences problem, as we discuss in the following section.

## 2.5 Ultraviolet Divergences, Again

Another major point of contention with the conventional formulation of perturbative QFT was the mathematical rigour of the renormalisation procedure. While it had clearly been predictively successful, even its inventors acknowledged that the way it was implemented was mathematically dubious. In the conventional approach, one initially writes down a divergent integral expression and then introduces a regulator (such as an ultraviolet cut-off) which induces convergences. "Bare" parameters were introduced, ultimately being identified with a series with infinite coefficients to be subtracted by equally problematic infinite counterterms when the regulator was removed. There was a pressing question about how to make mathematical sense of this procedure.

It was within the causal perturbation theory program that these worries about the mathematical foundations of perturbative renormalisation were most explicitly addressed, and a precise formulation of the ultraviolet divergence problem was finally achieved. Gaining a more thorough understanding of the problem of ultraviolet divergences would hinge on utilising ideas from the theory of distributions. Independently of Arthur Wightman, who employed distribution theory in his non-perturbative axiomatisation of QFT (Streater and Wightman, 1964) mentioned in section 2.3, Stueckelberg and Bogoliubov used distribution theory to explain why ultraviolet divergences occur in the perturbative coefficients, and to recast the renormalization procedure as a process of providing a proper definition for products of singular distributions appearing in the expansion.

The notion of a distribution, or generalised function, was relatively new in mathematics at this time, having been formalised in its modern form by Laurent Schwartz (Schwartz, 1951). It is possible to uniquely associate an ordinary locally integrable function, $f(x)$, with a functional that takes test functions, $\psi(x)$ (which are required to satisfy certain conditions) to the numbers:

$$T_f : \psi(x) \to \int_{-\infty}^{\infty} f(x)\psi(x)dx. \tag{8}$$

The idea is to consider a more general class of functionals on the space of test functions $\mathcal{D}(\mathbb{R}^n)$.

$$\{T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C} \mid T \text{ is linear and continuous}\}. \tag{9}$$

This includes all of the functionals associated with integrable functions, the regular distributions, but also contains additional objects, the singular distributions. The archetypical example of the latter is the dirac delta 'function' which can be identified with the distribution:

$$T_\delta : \psi(x) \to \int_{-\infty}^{\infty} \delta(x)\psi(x)dx = \psi(0), \tag{10}$$

A key difference between distributions and ordinary integrable functions is that their products are not generally well-defined. The square of the Dirac delta distribution, for instance, is ill-defined, and in general, giving meaning to products of distributions with overlapping singularities requires care.

In a remarkable, but cryptic article, Stueckelberg and Petermann (1953) explicitly advanced the claim that the ultraviolet divergences in perturbative quantum field theory arise due to the naive multiplication of singular distributions since the Feynman propagator has a singularity at the origin (i.e. for coincident spacetime arguments). They say that the central mathematical problem of perturbative QFT is therefore to provide a proper definition for the products of distributions which appear in the series coefficients. Bogoluibov and his school also advanced this distribution theoretic recasting of the perturbative renormalisation procedure. Bogoliubov and Parasiuk (1957) set out a way of thinking about the problem which is very similar in spirit to the approach of contemporary practitioners of perturbative algebraic QFT. They say that one can start by constructing a product of propagators on a restricted space of test function which vanish at the origin, thus temporarily sidestepping the problem posed by the singularities. This is the information which can be extracted from Bogoliubov's causality condition, since, as we noted above, it does not fix the form of the perturbative coefficients at coincident points. The problem is then one of defining the *extension* of this product to the full space of test functions—we will see that Bogoliubov and Parasiuk's framing of the renormalisation procedure is essentially the same as that adopted in pAQFT (section 3.2).

Both Stueckelberg and Petermann (1953) and Bogoliubov and Shirkov (1959) claim that the products of distributions appearing in perturbative QFT can be given a proper mathematical definition, but it is not unique; the solution is instead a class of distributions which differ via the addition of Dirac delta distributions and their derivatives at the origin coincident points. This ambiguity is equivalent to the freedom to subtract an arbitrary finite part and define different renormalisation schemes in the conventional approach to perturbative renormalisation. The causal perturbation theory approach thus provided a mathematical reconstruction of the perturbative renormalisation procedure, based not on the subtraction of infinities but on the fixing of ambiguities flowing from the multiplication of singular distributions in the time-ordered product. It was actually Stueckelberg and Petermann (1953) who first posited a group of transformations between these alternative ways of fixing the renormalization ambiguity and Bogoliubov and Shirkov who coined and popularised the term renormalisation group. While this is little appreciated today, the renormalisation group concept has its origins in the causal perturbation theory programme; see Fraser (2021) for a detailed historical analysis.

While Stueckelberg and Bogoliubov had all of the basic ingredients of the causal perturbation theory formalism set out in the 1950s it would be some decades before the details were filled in with full mathematical rigour. It was the papers of Epstein and

Glaser in the 70s which really brought the programme to fruition, forming the central reference point for later mathematical physics work on perturbative QFT (Epstein and Glaser, 1973). Adopting Bogoliubov's causality condition, Epstein and Glaser carried out the project of constructing distributional products occurring at each order iteratively without ever dealing with an ultraviolet divergent expression. In later decades the causal perturbation theory approach was further developed and refined. Gunter Sharf extended the Epstein and Glaser's approach to gauge theories, showing that it was possible to fully reconstruct the perturbative expansions of QED and QCD in the causal approach (Sharf, 1989). Where Epstein and Glaser had used a method of distribution splitting to provide a proper definition for the time-ordered products appearing in the expansion, Steinmann (1971) and Stora (1993) laid the groundwork for an alternative approach based on Bogoliubov's original suggestion of defining the distributional products via a method of extension (more on this in section 3.2). Stueckelberg and Bogoliubov's ideas about the perturbative renormalisation group were also made more rigorous, with the existence of a map between different ways of fixing the ambiguity in the distributional products coming to be known as *the main theorem of renormalisation* (Popineau and Stora, 2016).

The proceeding discussion of the history of foundational work on perturbative QFT is rather preliminary and many gaps in the story remain to be filled in. One important development we have not mentioned, for instance, is the BPHZ formalism. This framework was designed to rigorously prove that the ultraviolet divergences could be systematically removed to all orders of perturbation theory, and while it has both formal and historical overlap with the causal perturbation theory approach it also has its own story that is yet to receive a proper historical treatment. Still, we have said enough to establish the existence of a well-developed tradition of mathematically orientated work on QFT perturbation theory. For several decades, causal perturbation theory and axiomatic QFT developed more or less independently from each other. This can partially be explained by the fact that they adopted different responses to the problems of conventional renormalised perturbation theory: causal perturbation theory carrying out a repair operation on the conventional formalism, with axiomatic QFT attempting to transcend the limitations of the perturbative approach. Still, connections between these two wings have been made in contemporary work in mathematical physics, as will be discussed in detail in the following section.

# 3 Perturbative Algebraic Quantum Field Theory: A Contemporary Perspective

## 3.1 Origins: Generalising the Haag-Kastler Axioms

At the turn of the 21st century causal perturbation theory and axiomatic QFT were brought together in an approach called *perturbative algebraic quantum field theory* (pAQFT). This framework offers a view of the role of perturbation theory in contemporary work on the foundations of QFT. The key move of this approach is to consider a generalization of the standard algebraic axiomatization of QFT. Where the Haag-Kastler axioms had taken C*-algebras to be the basic objects, in pAQFT one considers algebras of formal power series (i.e. series where one does not care about convergence), which form a weaker *-algebra structure that does not satisfy all of the properties of a C*-algebra. This more permissive construal of what a QFT is allows one to use the rigorous versions of the perturbative expansion developed in the causal perturbation theory approach to construct interacting models.

This marrying of the algebraic and causal perturbation theory traditions was, in fact, preceded and facilitated by another line of generalization of the Haag-Kastler axioms: their extension to QFT on curved space-times.[10] By the 1990s, understanding QFT on curved space-time had become a pressing problem, spurred on by groundbreaking work on black hole thermodynamics. It turned out that the algebraic approach to QFT is very well suited for the generalization to curved space-times, since the construction of the algebras of observables is local, i.e. the observables themselves behave, to a large extent, as if they were in flat Minkowski spacetime. Global aspects of the theory are included in the specification of a *state*. In QFT on Minkowski spacetime, the natural choice of a state is the *vacuum*. On generic curved spacetimes, however, a distinguished state with all the good properties of the Minkowski vacuum might not exist. The main advantage of starting from the Haag-Kastler axioms here was that, where the Minkowski vacuum state plays a seemingly inextricable role in the Wightman axioms for relativistic QFT, AQFT separates the construction of the local algebra from the construction of a state making the generalization to a curved space-time setting much easier to achieve.

The basic concepts of AQFT on curved spacetime, later referred to as *locally covariant quantum field theory (LCQFT)*, were laid during an Oberwholfach workshop in September 2000, where the main discussants included Romeo Brunetti, Klaus Fredenhagen, Stefan Hollands, Bernard Kay and Bob Wald, and shortly afterwards appeared in a series of

[10]Interestingly, the idea of connecting the causal perturbation theory and algebraic approach to QFT was suggested much earlier by Il'in and Slavnov (1978). This does not seem to have been developed further at the time, but it does suggest that the pieces were in place to connect causal perturbation theory and AQFT by that time, raising the historical question of why it took so long for pAQFT to be developed.

seminal papers: (Brunetti et al., 2003; Hollands and Wald, 2001, 2002). Looking at these papers, it is clear that from the outset the framework was also suited to deal with ultraviolet divergence issues since locally covariant quantum fields were in fact *normally ordered*. Normal ordering can be thought of as the first step in renormalization, where one tries to define local field operators. On Minkowski spacetime normal-ordering amounts to putting all the creation operators on one side and all the annihilation operators on the other. This procedure can also be formally thought of as subtracting "an infinite constant" from what would be the "naive" product of those field operators. Another way to think about this procedure is to use the *point-splitting* prescription. For example, to define $\Phi^2(x)$ one considers the limit:

$$\lim_{y \to x} (\Phi(x)\Phi(y) - w(x,y)1) \,,$$

where $\Phi(x)$, $\Phi(y)$ are field operators (or more precisely operator-valued distributions) $w$ is an appropriately chosen distribution that cancels out the singularity of $\Phi(x)\Phi(y)$ at the coinciding points limit. The resulting normally-ordered field operators are also called *Wick polynomials*. One way to describe these objects abstractly, using algebras, is to use the construction known as the *Borchers-Uhlman algebra* (see Hack (2015)). Here we will follow the approach of Dütsch and Fredenhagen (2001) and treat Wick polynomials as smooth functionals on the space of field configurations.

An important conceptual step to make, in order to include objects like Wick polynomials into the AQFT framework was to weaken a key assumption of the Haag-Kastler approach, namely that the local algebras of observables must form $C^*$-algebras. Instead one considers a more general class of *involutive topological algebras*; Involution, denoted by $*$, allowing one to define *conjugates* of operators and the topology allowing one to talk about *convergence*. The main concepts of AQFT, though originally formulated for $C^*$-algebras, can actually apply in this more general setting, allowing much of the machinery of AQFT to be extended to the curved space-time setting and, as we shall see, to perturbative models. It is worth pointing out that there are also costs associated with this generalization, however, since some of the structural results achieved by traditional AQFT rely on the stronger properties of C*-algebras and therefore cannot be recovered in this more general setting (more on this point in section 3.3)

Let us state three axioms of AQFT that carry over to both curved spacetimes and perturbative models. As in the original Haag-Kastler approach, we associate algebras of observables $\mathfrak{A}(\mathcal{O})$ with spacetime regions $\mathcal{O}$, however, these algebras are now only required to be topological *-algebras and $\mathcal{O}$ are *relatively compact* regions of a general spacetime manifold $M$. We require the following three axioms to hold:

A 1 **Isotony**. If we have a smaller region $\mathcal{O}_1$ contained in a larger region $\mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.

A 2 **Micro-causality**. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two spacelike separated regions, i.e. no causal (timelike or null) curve can connect them. Then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$, i.e. operators localised in spacelike separated regions commute.

A 3 **Time-slice**. Let $\mathcal{N}$ be a region containing a Cauchy surface for the region $\mathcal{O}$ (i.e. a hyperspace such that every causal curve intersects it exactly once). Then $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{N})$, i.e. the algebra of the given region can be reconstructed from the algebra of a time-slice within that region. This is the quantum version of the well-posedness of the initial value problem.

Assumptions about the vacuum state, often included in the list of Haag-Kastler axioms, are simply omitted.

Locally covariant QFT greatly clarified how to make sense of free QFTs on curved space-times but as with traditional AQFT there was a problem with incorporating interactions. pAQFT was developed essentially in parallel with the LCQFT framework and always intended to work on curved backgrounds as well as the more familiar Minkowski space setting (as can be seen from a careful study of (Brunetti et al., 2009)). The additional step we need to make to embrace perturbation theory is to further weaken the assumption of what $\mathfrak{A}(\mathcal{O})$ should be. To make sense of perturbation theory in a mathematically rigorous way, we allow $\mathfrak{A}(\mathcal{O})$ to be the algebra of *formal power series* (i.e. as series considered independently of any notion of convergence) with coefficients in topological involutive algebras. It is then possible to construct interacting models satisfying axioms A 1 - A 3 by adapting the rigorous version of the perturbative expansion developed in the causal perturbation theory programme. Crucially, where early axiomatic QFT had rejected perturbative methods at least partly because of the likely non-convergence of the series the pAQFT approach shows that it is possible to do a surprising amount by simply bracketing the convergence question and working with formal power series. pAQFT can thus be understood as adopting a different, more deflationary, response to the large-order divergence problem discussed in section 2.2.

## 3.2 The Construction of Perturbative Models

This section describes how models of the weakened algebraic axioms can be constructed using the perturbative expansion. More detailed and comprehensive accounts can be found in Fredenhagen and Rejzner (2015) and Rejzner (2016).

To give a qualitative outline, the model construction method described here uses two mathematical principles to define two types of product:

- **Deformation**. We start with a Poisson algebra of classical observables (specifically, functionals of classical field configurations). Quantization is achieved by a deformation of this algebra's product. We will use this approach to define a non-commutative

***-product** appropriate for a space of formal power series. This product acts as the involution operator for the algebra and defines the relevant commutation relations.

- **Extension**. Following causal perturbation theory, the problem of ultraviolet divergences in the series expansion is understood as one of providing a proper mathematical definition for the **time-ordered product**. Consequently, this product also needs to be defined on our space of observables. This is achieved via a process of *extension*: one first defines the product in a domain of less singular objects, before extending the product to the full physically relevant space.

The first step is to specify the spacetime $M$ that we want to work on. It is assumed that this spacetime is globally hyperbolic, i.e. of the form $\Sigma \times \mathbb{R}$, where $\Sigma$ is a Cauchy surface. Such spacetimes do not have *closed causal curves*, so, intuitively speaking, it is impossible to travel or to send information back in time. Next, we fix the space of field configurations $\mathcal{E}$ that we wish to consider. This could be, for example, $\mathcal{C}^\infty(M, \mathbb{R})$ for the free scalar field or more generally a space of sections of some vector bundle over $M$. For simplicity of exposition, we will restrict ourselves here to the case of a scalar field $\varphi(x)$.

Classical observables of the theory can be described as real-valued functions on $\mathcal{E}$, i.e. a classical observation assigns a number (the result of the measurement) to a given field configuration. For mathematical convenience we assume our observables to be smooth functionals on $\mathcal{E}$ and with the quantization in mind, we want them to be complex-valued, so elements of $\mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$. Among all functionals, we distinguish the ones that are *local*. This means that they can be written as

$$F(\varphi) = \int_M \alpha(\varphi(x), \partial\varphi(x), \ldots, \partial^k\varphi) d\mu(x)\,,$$

where $\alpha$ is a smooth function, $d\mu$ is a distinguished invariant measure induced by the metric of our spacetime and $\partial^k$ denotes the $k$-th order derivative of $\varphi$, where we suppressed all the spacetime components of such derivatives. We see that a local functional essentially depends only on values of the field $\varphi$ and its derivatives up to order $k$ at the same point. We can multiply local functionals together to obtain functionals called *multilocal*.

When it comes to dynamics we work with a modified version of the Lagrangian formalism, in anticipation of utilising ideas from causal perturbation theory later on. Section 2.4 introduced a generalised version of the S-matrix $S(f)$, but one can also introduce switching functions $f(x)$ into the Lagrangian formalism itself (as was in fact originally done by Bogoliubov (Bogoliubov and Shirkov, 1959)). Let $\mathcal{D} \equiv \mathcal{C}_c^\infty(M, \mathbb{R})$ be the space of smooth compactly-supported functions on $M$ (our test functions). A *generalised Lagrangian* is a map $\mathcal{D} \ni f \mapsto L(f)$, where $L(f)$ is a local functional. In practice, one takes the usual Lagrangian density and smears it with $f(x)$. For example, the Lagrangian of the free scalar field is:

$$L_0(f) = \frac{1}{2}\int_M \left(\nabla^\nu\varphi\nabla_\nu\varphi - m^2\varphi^2\right) f d\mu_f$$

Switching functions play a number of roles in the pAQFT approach. As in causal perturbation theory, we will use the support of these functions to formulate statements about the causal properties of the generalised S-matrix (though we will ultimately use the support of local functionals for this purpose rather than referring to the switching function directly). The switching functions also get rid of both the surface and infrared divergences that would appear in the perturbative coefficients. Perturbative infrared divergences occur only in the limit $f(x) \to 1$, and as Dütsch and Fredenhagen (1999) point out, this limit is not actually required for constructing local algebra nets, however.[11] pAQFT can thus also take a deflationary view of the infrared divergence problem, though this does mean losing contact with results in asymptotic scattering theory developed in the traditional axiomatic approach.

Some properties of the free classical theory are important to discuss. One defines the *Euler-Lagrange derivative* of $L_0$ as follows:

$$\langle dL_0(\varphi), \psi\rangle \doteq \left\langle L_0(f)^{(1)}(\varphi), \psi\right\rangle ,$$

where $f \equiv 1$ on supp$\psi$ and $\psi \in \mathcal{D}$ and the pairing $\langle .,.\rangle$ is given by pointwise multiplying and integrating over all $M$. Since $L_0$ is local, $dL_0(\varphi)$ can be expressed as a differential operator, i.e.

$$dL_0(\varphi) = P\varphi\,.$$

For the free scalar field $P = -(\Box + m^2)$, where $\Box$ is the *d'Alembertian*, i.e. the wave operator. Operators of this type have some very nice properties, in particular, the existence of unique retarded and advanced Green functions $\Delta^{\mathrm{R/A}}$. This means that we can specify the initial data in a small time slice, in the neighbourhood of a Cauchy surface and this data can either be propagated to the future (by applying the retarded Green function) or to the past (by applying the advanced Green function). These causal properties of distinguished Green functions play a crucial role in what follows.

One of the difficulties with the formulation of classical field theories on curved spacetimes is the desire to keep covariance. In classical mechanics, the dynamics can be formulated by specifying the Hamiltonian of the theory and introducing the canonical bracket on the space of functions on the phase space, which is a *Poisson bracket*. When generalising to field theory, we note that in order to obtain a Hamiltonian, one needs to use a particular foliation into Cauchy surfaces. This is not very covariant, so we prefer to remain in

[11] Furthermore, the $f(x) \to 1$ limit may not even make sense in a curved spacetime setting so there may actually be physical reasons not to take this limit.

the Lagrangian framework. The remaining problem of introducing an appropriate Poisson structure has been solved by Peierls in Peierls (1952). It is defined as follows:

$$\{F, G\}(\varphi) \doteq \left\langle F^{(1)}(\varphi), \Delta G^{(1)}(\varphi) \right\rangle ,$$

where $F, G$ are local functionals on $\mathcal{E}$ and $\Delta \doteq \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan function. The Peierls bracket can be understood as the difference between the retarded and advanced response of observable $G$ to perturbing the action $L_0$ by adding $F$ to it. As $(\Delta^{\mathrm{A}})^T = \Delta^{\mathrm{R}}$, $\Delta$ is antisymmetric. Another important feature of $\{.,.\}$ is that it is well defined an *all* local functionals, not just on local functionals on the space of solutions to the equations of motion. We say that it is defined *off-shell* (as opposed to making sense only *on-shell*). The Peierls bracket involves taking derivatives of functionals, so in general $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ are *distributions*, i.e. could be singular. Since $\Delta$ is also a distribution, the product of $F^{(1)}(\varphi)$, $G^{(1)}(\varphi)$ and $\Delta$ is not necessarily well-defined on arbitrary smooth functionals. It is in fact well-defined on multilocal ones, but this class is not closed under taking the bracket, i.e. $\{F, G\}$ is not necessarily multilocal. There exists, however, a slightly larger space of functionals $\mathcal{F}$, which is closed under the bracket Dütsch and Fredenhagen (2001).

Now that the classic system is characterised we can move on to quantisation. As previously announced, this will be achieved by means of a deformation of the Poisson algebra of functionals that we have just introduced (the bracket is the Peierls bracket and the product is the pointwise product $\cdot$ of functionals). The goal here is to introduce a non-commutative product $\star$ on the space of formal power series $\mathcal{F}[[\hbar]]$, such that:

$$F \star G \xrightarrow{\hbar \to 0} F \cdot G\,, \qquad \frac{1}{i\hbar}(F \star G - G \star F) \xrightarrow{\hbar \to 0} \{F, G\}\,.$$

A first attempt at defining such a product is the Moyal exponential formula

$$e^{\frac{i}{2}\hbar \left\langle \Delta, \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} \right\rangle} F(\varphi_1) G(\varphi_2) \big|_{\varphi_1 = \varphi_2 = \varphi} .$$

This formula would satisfy the two properties listed above, but it does not work for general $F$ and $G$, since $\Delta$ is too singular. The solution is to add to $\frac{i}{2}\Delta$ a symmetric distribution (so that it does not affect the commutator), which would make the singularity structure better. Fortunately, such modification is possible for the spacetimes of interest (though it is not unique). We introduce

$$\Delta^+ = \frac{i}{2}\Delta + H\,,$$

where $H$ is chosen so that the singularity structure of $\Delta^+$ allows for applying the Moyal formula to local functionals. We define the star product as

$$F \star G := e^{\hbar \left\langle \Delta^+, \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} \right\rangle} F(\varphi_1) G(\varphi_2) \big|_{\varphi_1 = \varphi_2 = \varphi} .$$

The resulting product plays the role of the operator product of quantum theory. It is the product that defines the relevant commutation relations and will generate our *-algebra.

In order to construct interacting models we need to define another product on our space of functionals: the *time-ordered product*. Recall that causal perturbation theory introduced the generalised S-matrix $S(f(x))$. We now generalise one step further and consider S-matrices $S(F)$ associated with the local functional $F$. The support of the functional $F$ will play the same role as the support of the switching function in implementing a causal factorisation property of the sort introduced by Bogoliubov. We want to identify these generalised S-matrices with a series of time-ordered products $\mathcal{T}_n$:

$$S(F) = \sum_{n=0} \frac{1}{n!} \mathcal{T}_n(F, \ldots, F) \,. \tag{11}$$

Note that we do not require the functional $F$ to live on the space of solutions, so this is essentially the off-shell analogue of equation (5).

As we have seen, the problem of ultraviolet divergences arises from a naive treatment of the singular distributions that occur in these time-ordered products. Accordingly, we first start by defining these time-ordered products on a smaller class of less singular functionals than the local functionals. The *regular functionals* are smooth compactly-supported functionals whose derivatives are smooth, i.e. $F^{(n)}(\varphi)$ is a smooth function on $M^n$ for all $n \in \mathbb{N}$ and all $\varphi \in \mathcal{E}$. For such functionals, we can introduce the time-ordered product:

$$F \cdot_{\mathcal{T}} G := e^{\hbar \left\langle \Delta^F, \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} \right\rangle} F(\varphi_1) G(\varphi_2) \big|_{\varphi_1 = \varphi_2 = \varphi} \,, \tag{12}$$

where $\Delta^F := \frac{i}{2}(\Delta^R + \Delta^A) + H$ is the Feynman propagator. One can also write the time-ordered product as

$$F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1} F \cdot \mathcal{T}^{-1} G) \,,$$

where

$$\mathcal{T} := e^{\hbar \left\langle \Delta^F, \frac{\delta^2}{\delta\varphi^2} \right\rangle}$$

is the *time-ordering operator*.[12]

Here is where the singularity structure of the Feynman propagator starts playing a crucial role. Because the Feynman propagator $\Delta^F$ is much more singular than $\Delta^+$, formula

[12]This operator is in fact closely related to the path integral. Formally, we have that

$$(\mathcal{T} F)(0) = \int F(\varphi) d\mu_{\Delta^F}(\varphi) \,,$$

where $d\mu_{\Delta^F}(\varphi)$ is the Gaussian measure with covariance $i\Delta^F$ and the integral is the standard path integral. This connection to more conventional path integral approaches to QFT provides a powerful starting point for formulating general conjectures starting from pAQFT.

(12) cannot be directly applied. However, it does allow one to construct the $n$-fold product of $n$ local functionals $F_1, \ldots F_n$, as long as their supports are pairwise disjoint, so for such functionals, we can define

$$\mathcal{T}_n(F_1, \ldots, F_n)\,.$$

This is where the concept of *extension* comes in. Causal perturbation theory is exactly the right tool to extend such $\mathcal{T}_n$s to their *renormalised* counterparts that are defined on arbitrary local functionals. Originally, such extensions were done by using *causal splitting* of distributions (Epstein and Glaser, 1973; Scharf, 1995), however, a more modern approach involves distributional extensions using the *Steinmann scaling degree*, following (Steinmann, 1971). For the more modern rendering of that idea see for example Brunetti and Fredenhagen (2000). As was discussed in section 2.5 a time-ordered product can be defined but it is not unique, with the ambiguity in defining the extension to local functionals being governed by the Stueckelberg-Petermann renormalization group. The notion of the renormalization group has in fact been clarified within the pAQFT approach, with the relationship between the Stueckelberg-Petermann and Wilsonian renormalization groups spelt out in mathematically precise terms (Brunetti et al., 2009).[13]

To take stock, we now have two products to work with: the non-commutative $\star$ and the commutative $\cdot_{\mathcal{T}}$ (time-ordered product). So far, however, our construction has proceeded using the free Lagrangian $L_0$ and has proceeded off-shell. We now want to construct interacting models, i.e. observables of the theory with Lagrangian $L_0 - V$. This, of course, is where the perturbative expansion comes in. The generalised S-matrices of the interacting theory are identified with time-ordered exponential series with respect to $V$, which are not required to converge:

$$S(V) = e_{\mathcal{T}}^{iV/\hbar}\,.$$

Interacting observables are obtained using the Bogoliubov forumula:

$$R_V(F) = S(V)^{-1} \star (S(V) \cdot_{\mathcal{T}} F)\,,$$

where the inverse is with respect to $\star$ and $F$ is a given classical functional. They are generated by the *relative S-matrix*:

$$S_V(F) := S(V)^{-1} \star S(V + F)\,.$$

[13] There remains the question whether the renormalised $\mathcal{T}_n$s can be seen as $n$-fold products coming from some binary product. This question was answered affirmatively by Fredenhagen and Rejzner (2013). We will keep denoting this, now renormalized, time-ordered product by $\cdot_{\mathcal{T}}$, with the corresponding $\mathcal{T}$. The idea to treat the renormalised time-ordered product as a binary product opened up many new possibilities since one can drastically simplify the algebraic structure of the theory.

We now have all the pieces in place to demonstrate that models satisfying our weakened axioms A1-A3 have been constructed. One can define the algebra associated to a region $\mathcal{O} \subset M$ as the algebra generated by $S(F)$, where $F \in \mathcal{F}_{\mathrm{loc}}$, with respect to the product $\star$. This assignment automatically satisfies the isotony axiom A1. Causal factorisation holds in this algebras, so for $F_1$ with the support not later than $F_2$ (i.e. such that the support of $F_1$ does not intersect the future of the support of $F_2$), we have (independently of the choice of $F$):

$$S(F_2 + F + F_1) = S(F_2 + F) \star S(F)^{-1} \star S(F + F_1) .$$

Using this, we can now prove that the causality axiom A2 holds for our net of algebras. Indeed, for $F_1$ and $F_2$ with spacelike supports, we have both $F_1$ not later than $F_2$ and $F_2$ not later than $F_1$. Hence, in this case (setting $F = 0$):

$$S(F_2) \star S(F_1) = S(F_2 + F_1) = S(F_1) \star S(F_2) ,$$

so $S(F_1)$ and $S(F_2)$ commute. This establishes the connection between the Bogoliubov causality condition and the algebraic "microcausality" principle adopted by Haag and Kastler: in the pAQFT setting the former implies the latter. The time-slice axiom is related to the dynamics of the theory and thus requires us to go on-shell by requiring our functionals to be in the space of solutions, or, equivalently, after taking an appropriate quotient of our algebra of S-matrices. After this is done it can be shown to hold (Chilian and Fredenhagen, 2008).

## 3.3 Beyond the Perturbative Expansion

The proceeding section shows that once the Haag-Kastler axioms have been sufficiently weakened a plethora of perturbative interacting models can be shown to exist, including the 4-dimensional gauge theories on Minkowski space-time that make up the standard model of particle physics. This addresses a central problem facing the axiomatic QFT tradition—the difficulty of constructing realistic interacting models of the proposed systems of axioms. We should stress, however, that pAQFT also has important limitations and does not simply render the older non-perturbative approach to axiomatic QFT redundant.

For one thing, pAQFT inherits the phenomenological limitations of the perturbative approach to QFT. While the rise of non-abelian gauge theories sparked a renaissance of perturbation theory, it also became clear that there were aspects of these models that could not be described perturbatively. Key phenomena like quark confinement are not expected to have a perturbative description and the strong coupling regime of QCD is generally expected to be dominated by contributions to observable quantities which are non-analytic in the coupling and thus invisible to the perturbation approximation. The perturbative models constructed in pAQFT thus do not capture all of the physical content of realistic gauge theories. Furthermore, from a mathematical physics perspective, motivations remain for pursuing the stricter notion of a QFT as a net of C*-algebras. One reason for this is that many of the successes of traditional AQFT, especially general structural results and qualitative accounts of infrared phenomena mentioned in section 2.3, depend on the stronger properties of C*-algebras and therefore cannot be recovered in the pAQFT setting. Under this heading, the Doplicher-Haag-Roberts theory of superselection (Doplicher et al., 1971, 1974), and modular theory (Takesaki, 1970), loom large.

There is thus a continued motivation to pursue constructive field theory in the traditional sense; that is to try to construct interacting models in the non-perturbative C*-algebra setting. Recently, Buchholz and Fredenhagen (2020) have proposed a way to use the intuitions gained from pAQFT to make this further step. The idea is relatively simple. One considers first the $C^*$-algebra generated by a family of unitaries $S(F)$ labelled by local functionals $F$. These unitaries can again be identified with generalised S-matrices. Next, one quotients this algebra by relations that correspond to relations appearing in pAQFT. Working by analogy, one requires that:

$$S(F_2 + F + F_1) = S(F_2 + F)S(F)^{-1}S(F + F_1)\,,$$

if the support of $G$ is not later than the support of $F$. One also requires a relation that can be seen as the unitary version of the Schwinger-Dyson equation, namely that:

$$S(F) = S(F^{\psi} + \delta L_0(\psi))\ , \quad \psi \in \mathcal{D}$$

where $L_0$ is the generalised Lagrangian of the free theory,

$$F^{\psi}[\phi] \doteq F[\phi + \psi]\ , \quad \delta L_0(\psi) := L_0(f)^{\psi} - L_0(f)\ ,$$

for any $f \in \mathcal{D}$ satisfying $f \equiv 1$ on $\operatorname{supp}\psi$. Note that $\delta L_0$ is the difference quotient version of $dL_0$, so this equation tells us something about the dynamics of the theory. Another important relation, dubbed the *unitary anomalous master Ward identity* in Brunetti et al. (2023), allows one to study how classical global symmetries of the theory can be broken upon quantisation. This is quantified by means of the *anomalous Noether theorem*. The same paper also introduces the non-perturbative version of the Stückelberg-Petermann renormalisation group, which acts on the $C^*$-algebra of generalised S-matrices.

This recent work can be understood as another instance of a methodological strategy we saw in the historical part of the paper: working results out concretely using the perturbative expansion before exporting them to a non-perturbative context. This, once again, reinforces the significance of the perturbative expansion even for the most intractable foundational questions about QFT.

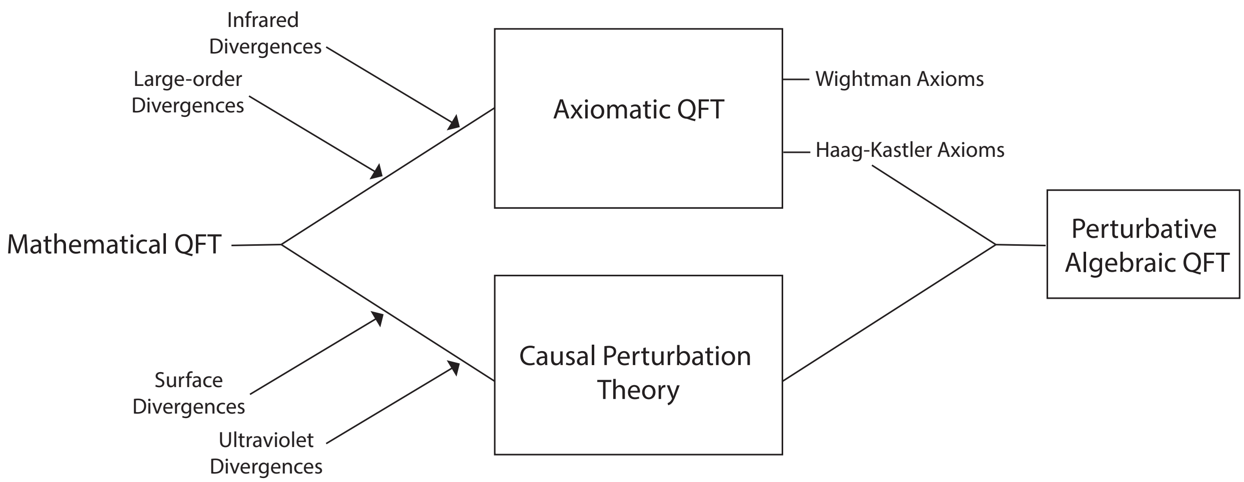


Figure 2: Continuation of Figure 1, depicting perturbative algebraic QFT (qAQFT) as bringing the earlier strands of axiomatic QFT and causal perturbation theory together.

# 4 Conclusion

Let us summarise the historical picture which emerges from the preceding discussion. During the 1950s and 1960s, a distinctively mathematical approach to QFT emerged, based on following a higher standard of mathematical rigour and using tools from pure mathematics to illuminate foundational questions about the theory. Mathematical QFT split into (at least) two largely independent strands of development, which can be thought of as responding to different issues with the formalism of renormalised perturbation theory: axiomatic QFT, which tried to develop a non-perturbative language for QFT, and causal perturbation theory, which instead clarified the mathematical structure of the perturbative approach. In recent decades, these two arms of mathematical QFT have been brought together by pAQFT; this is illustrated in Figure 2.

In addition to mediating between these two previously disconnected strands of mathematical QFT, pAQFT plays another important role in relating the results of mathematical QFT to particle physics phenomenology, which is still largely based on the perturbative approximation scheme. We conclude with some comments on this aspect.

One of the apparent difficulties with approaching QFT from a foundational point of view is the existence of different formulations of the theory employed in mainstream theoretical physics and mathematical physics. This has been seen as posing a dilemma: which version of the theory ought we to start with when asking questions about what QFT is telling us about the physical world? In the philosophy of science literature this framing of the situation has been dramatised in a debate between Wallace (2011) and Doreen Fraser (2011), with Fraser arguing that axiomatic QFT is the correct starting point for the purposes of philosophical interpretation on grounds of their superior mathematical rigour and unificatory power, while Wallace arguing that the conventional empirically successful approach to the theory which ought to be our focus. The tradition of mathematical work on the perturbative expansion we have highlighted in this article arguably tells against a sharp distinction between the physicist's and mathematician's version of QFT, since in some sense it lies in an intermediate position between axiomatic QFT and conventional textbook treatment of interactions. It allows us to formulate the predictively powerful perturbative approximation methods driving particle physics phenomenology in a mathematically rigorous way and situates perturbatively characterised gauge theories within the algebraic QFT framework.

In other words, the gap between the physicist's and mathematician's versions of QFT is not actually as wide as has sometimes been thought. Relationships between the various formulations of the theory are now quite well understood. This suggests that the choice between axiomatic and conventional QFT may be something of a false dilemma. What this means for how philosophers can most fruitfully engage with QFT is a discussion for another time.

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