The Non-Relativistic Geometric Trinity of Gravity

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Abstract

The geometric trinity of gravity comprises three distinct formulations of general relativity: (i) the standard formulation which interprets gravity in terms of spacetime curvature, (ii) the teleparallel equivalent of general relativity which interprets gravity in terms of spacetime torsion, and (iii) the symmetric teleparallel equivalent of general relativity (STEGR) which interprets gravity in terms of spacetime non-metricity. In this article, we complete a non-relativistic geometric trinity of gravity, by (a) taking the non-relativistic limit of STEGR to determine its non-relativistic analogue, and (b) demonstrating that this non-metric theory is equivalent to Newton–Cartan theory and its teleparallel equivalent, i.e., the standard curvature and torsion based theories in the non-relativistic regime that are both geometrised versions of classical Newtonian gravity.

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1 Introduction

It has become increasingly well-known that general relativity (GR) constitutes but one vertex in a 'geometric trinity' of gravitational theories [1]. The other two vertices of this trinity are the 'teleparallel equivalent general relativity' (TEGR), in which the curvature degrees of freedom of GR are traded for spacetime torsion, and the 'symmetric teleparallel equivalent general relativity' (STEGR), in which the curvature degrees of freedom of GR (and torsion degrees of freedom of TEGR) are traded for spacetime non-metricity. The actions of all three theories are equivalent up to a total divergence term—in this sense, all three theories are dynamically equivalent.

In a parallel vein, it has been known since Trautman in the 1960s [2] that standard Newtonian gravity can be formulated similarly to GR in the sense that (non-relativistic) gravitational effects become a manifestation of spacetime curvature: this theory is known as Newton–Cartan (NC) theory, and was first developed in the 1920s by Cartan and Friedrichs: see [3–5] for the original sources, and [6] for a recent review of non-relativistic gravity. (To be clear: in this article, by 'non-relativistic', we always mean 'Newtonian', rather than 'ultra-relativistic'; we will leave the construction of an ultra-relativistic geometric trinity to future work.) In [7], it was shown that there is a precise sense in which classical Newtonian gravitation can be understood as the teleparallelised version of NC, that we name TENC, in which the gravitational potential can be understood as a manifestation of the 'mass torsion' which arises once one gauges the Bargmann algebra (e.g., [8]). Adding to this, it was shown recently in [9] that TENC can be secured as the non-relativistic limit of TEGR using a $1/c$ expansion of the TEGR field equation (in [7] the same result was shown using null reduction), just as NC is by now well-known to be the non-relativistic limit of GR (on which see [6] and references therein).

These results invite the following question: *can one complete a non-relativistic geometric trinity*, by constructing a non-metric theory equivalent to both NC (understood as a theory with spacetime curvature) and TENC (understood as a torsionful theory)? In this article, we answer this question in the affirmative—indeed, we triangulate a non-relativistic version of Newtonian gravity (which we dub 'symmetric teleparallel equivalent Newton–Cartan' in analogy with its relativistic parent, shortened to STENC) in two ways: (a) by taking the non-relativistic limit of STEGR (using the same $1/c$ expansion developed in [10]), and (b) by proving that it includes classical Newtonian gravity, thereby obtaining the analogues of the Trautman recovery theorems (see [11, Ch. 4]) for STENC.

The structure of this article is as follows. In Sec. 2, we review the essential details of the geometric trinity of gravity; in Sec. 3, we construct STENC by taking the non-relativistic limit of STEGR and show that it is equivalent to the NC formulation; in Sec. 4, we discuss general properties of the non-relativistic trinity and of STENC specifically. We close in Sec. 5 with some discussions of the upshots of this work.

2 Background: The Geometric Trinity

The bulk of this section constitutes a review of the relativistic geometric trinity of gravity (Sec. 2.2). In addition, we review briefly the state-of-play regarding geometric reformulations of non-relativistic gravity (Sec. 2.3).

2.1 Notation

Since we will deal with four different connections, introduced at the relativistic and the non-relativistic levels, we here define the following notation:

(i) The Lorentzian Levi-Civita connection ${\Gamma^\mu}_{\alpha\beta}$ relative to the Lorentzian metric $g_{\mu\nu}$, with covariant derivative ∇, and Riemann tensor ${R^\mu}_{\alpha\beta\nu}$.

(ii) The general affine connection introduced at the relativistic level ${\bar{\Gamma}^\mu}_{\alpha\beta}$, with covariant derivative $\bar{\nabla}$, torsion ${\bar{T}^\mu}_{\alpha\beta}$, and Riemann tensor ${\bar{R}^\mu}_{\alpha\beta\nu}$.

(iii) The (symmetric) Galilean connection ${\hat{\Gamma}^\mu}_{\alpha\beta}$ (i.e., non-relativistic connection) relative to $(\tau_\mu, h^{\mu\nu})$, with covariant derivative $\hat{\nabla}$, and Riemann tensor ${\hat{R}^\mu}_{\alpha\beta\nu}$.

(iv) The general affine connection introduced at the non-relativistic level ${\tilde{\Gamma}^\mu}_{\alpha\beta}$, with covariant derivative $\tilde{\nabla}$, torsion ${\tilde{T}^\mu}_{\alpha\beta}$, and Riemann tensor ${\tilde{R}^\mu}_{\alpha\beta\nu}$.

When considering the non-relativistic limit, both the Lorentzian connection ${\Gamma^\mu}_{\alpha\beta}$ (and related variables) and the general affine connection ${\bar{\Gamma}^\mu}_{\alpha\beta}$ (and related variables) are written as Taylor series of the speed of light. The full series will be denoted by an upper "λ" (e.g., ${\overset{\lambda}{\Gamma}{}^\mu}_{\alpha\beta}$), and the n-th order will be denoted by an upper "(n)" (e.g., ${\overset{(n)}{\Gamma}{}^\mu}_{\alpha\beta}$).

Finally, as will be detailed in Sec. 2.3, a non-relativistic (Galilean) structure does not possess a metric allowing for raising and lowering indices. Therefore, when introducing tensors at the non-relativistic level (denominated with a hat), the position of the indices will be fixed.

2.2 Relativistic gravity

Spacetime theories are typically formulated in terms of a metric tensor $g_{\mu\nu}$ and an affine connection ${\bar{\Gamma}^\alpha}_{\mu\nu}$. General relativity (GR) is of course the paradigmatic theory of gravity and makes use of the Levi-Civita connection ${\Gamma^\alpha}_{\mu\nu}$, setting ${\bar{\Gamma}^\mu}_{\alpha\beta} = {\Gamma^\mu}_{\alpha\beta}$, with components

$$ {\Gamma^\alpha}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \right), \tag{1} $$

which is the unique connection that is compatible with the metric and torsion-free. The metric-compatibility condition is given by $\nabla_\alpha g_{\mu\nu} = 0$ and the torsion-free condition is given by ${\Gamma^\alpha}_{[\mu\nu]} = 0$ [12, Ch. 3]. Famously, GR describes gravity as a manifestation of spacetime curvature, as encoded in the Riemann tensor, which has components

$$ {R^\alpha}_{\beta\mu\nu}(\Gamma) := \partial_\mu {\Gamma^\alpha}_{\nu\beta} - \partial_\nu {\Gamma^\alpha}_{\mu\beta} + {\Gamma^\alpha}_{\mu\lambda} {\Gamma^\lambda}_{\nu\beta} - {\Gamma^\alpha}_{\nu\lambda} {\Gamma^\lambda}_{\mu\beta}. \tag{2} $$

Spacetime curvature measures the rotation of a vector when it is parallel transported along a closed curve.

One can alter or otherwise relax the above assumptions in order to construct spacetime theories that manifest torsion and/or non-metricity. Torsion is given by the antisymmetric part of the connection

$$T^{\alpha}{}_{\mu\nu}\left(\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) := 2\bar{\Gamma}^{\alpha}{}_{[\mu\nu]}, \tag{3}$$

and can be thought of as a measure of the non-closure of the infinitesimal parallelogram formed by two vectors being parallel transported along each other. Non-metricity is given by the non-vanishing of the covariant derivative of the metric tensor

$$Q_{\alpha\mu\nu}\left(\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) := \bar{\nabla}_{\alpha} g_{\mu\nu}, \tag{4}$$

and can be thought of as a measure of how the length of a vector changes when parallel transported.

We can thus categorize spacetimes as

(i) metric (i.e., $Q_{\alpha\mu\nu}\left(\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) = 0$),

(ii) torsionless (i.e., $T^{\alpha}{}_{\mu\nu}\left(\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) = 0$),

(iii) flat (i.e., $R^{\alpha}{}_{\beta\mu\nu}\left(\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) = 0$).

Curvature, torsion, and non-metricity are all possible geometric properties of an affine connection in relation with a Lorentzian metric. A completely general affine connection $\bar{\Gamma}^{\alpha}{}_{\mu\nu}$ can be decomposed in the following way [13]:

$$\bar{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{5}$$

where

$$K^{\alpha}{}_{\mu\nu} := \frac{1}{2}\bar{T}^{\alpha}{}_{\mu\nu} + g^{\alpha\sigma}\bar{T}^{\gamma}{}_{\sigma(\mu}\, g_{\nu)\gamma} \tag{6}$$

is referred to as the 'contortion tensor', and

$$L^{\alpha}{}_{\mu\nu} := \frac{1}{2} g^{\alpha\sigma} Q_{\sigma\mu\nu} - g^{\alpha\sigma} Q_{(\mu\nu)\sigma} \tag{7}$$

is referred to as the 'distortion tensor'. Consequently, the difference tensor $\Gamma^{\mu}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\beta}$ between the Levi-Civita connection and a general affine connection have the form

$$\Gamma^{\mu}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\beta} = g^{\mu\sigma}\left(\bar{\nabla}_{(\alpha} g_{\beta)\sigma} - \frac{1}{2}\bar{\nabla}_{\sigma} g_{\alpha\beta}\right) - \frac{1}{2}\bar{T}^{\mu}{}_{\alpha\beta} - g^{\mu\sigma}\bar{T}^{\nu}{}_{\sigma(\alpha}\, g_{\beta)\nu}. \tag{8}$$

We can use Eq. (5) to facilitate translations between different spacetime theories with different connections (and associated different geometrical properties). As we have seen, GR is a spacetime theory that is metric and torsionless, but non-flat as the Levi-Civita connection in general possesses curvature. In this article, we will also be concerned with two other spacetime theories: the 'teleparallel equivalent general relativity' (TEGR) and the 'symmetric teleparallel equivalent general relativity' (STEGR). TEGR spacetimes are metric and flat but in general possess torsion; STEGR spacetimes are torsionless and flat but in general possess non-metricity. Both TEGR and STEGR are dynamically equivalent to GR, in the sense that the actions of all three theories are equivalent up to total divergence terms [see Eqs (12)–(13)]; thereby, these theories are capable of modelling the same empirical phenomena, and constitute a 'geometric trinity' of gravity—see [1, 14–16] for recent discussions.

For example, we can find GR's torsionful and non-metric equivalents by taking the expressions for the Riemann curvature and Ricci scalar in GR in terms of the Levi-Civita connection, and re-expressing these in terms of the 'Weitzenböck' connection of TEGR or the non-metricity connection of STEGR. Consider that we can express a generic Riemann curvature tensor $\bar{R}^{\alpha}{}_{\beta\mu\nu}$ as [17]:

$$\bar{R}^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} M^{\alpha}{}_{\nu\beta} - \nabla_{\nu} M^{\alpha}{}_{\mu\beta} + M^{\gamma}{}_{\nu\beta} M^{\alpha}{}_{\mu\gamma} - M^{\gamma}{}_{\mu\beta} M^{\alpha}{}_{\nu\gamma} + \bar{T}^{\alpha}{}_{\beta\mu} M^{\beta}{}_{\alpha\nu}\,, \tag{9}$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the standard Riemann tensor from the Levi-Civita connection ∇_{μ} and $M^{\alpha}{}_{\mu\nu} := K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$. This formula is the heart of the trinity of GR.

One can choose to work with TEGR and the contortion tensor (i.e., $K^{\alpha}{}_{\mu\nu} \neq 0$ and $L^{\alpha}{}_{\mu\nu} = 0$) or with STEGR and the distortion tensor (i.e., $K^{\alpha}{}_{\mu\nu} = 0$ and $L^{\alpha}{}_{\mu\nu} \neq 0$). Upon index contraction, one constructs the curvature scalar and finds:

$$-R = \frac{1}{4}\bar{T}_{\alpha\mu\nu}\bar{T}^{\alpha\mu\nu} + \frac{1}{2}\bar{T}_{\alpha\mu\nu}\bar{T}^{\mu\alpha\nu} - \bar{T}^{\alpha}{}_{\alpha\mu}\bar{T}^{\beta}{}_{\beta\nu}g^{\mu\nu} + 2\nabla_{\alpha}\bar{T}^{\lambda\alpha}{}_{\lambda} \quad \text{with} \quad Q_{\mu\alpha\beta} = 0, \tag{10}$$

and

$$-R = g^{\mu\nu}\left(L^{\alpha}{}_{\mu\beta}L^{\beta}{}_{\nu\alpha} - L^{\alpha}{}_{\alpha\beta}L^{\beta}{}_{\mu\nu}\right) + \nabla_{\alpha}\left(Q^{\alpha}{}_{\lambda}{}^{\lambda} - Q^{\lambda}{}_{\lambda}{}^{\alpha}\right) \quad \text{with} \quad \bar{T}^{\alpha}{}_{\mu\nu} = 0. \tag{11}$$

Importantly, this shows that the scalar expressions of curvature, torsion, and non-metricity are equivalent up to a boundary term.[1] This justifies the above claim that GR, TEGR, and STEGR can be formulated in terms of dynamically equivalent Lagrangian expressions, respectively as

$$\mathcal{L}_R := g^{\mu\nu}R_{\mu\nu}, \tag{12}$$

$$\mathcal{L}_T := \frac{1}{4}T_{\alpha\mu\nu}T^{\alpha\mu\nu} + \frac{1}{2}T_{\alpha\mu\nu}T^{\mu\alpha\nu} - T^{\alpha}{}_{\alpha\mu}T^{\beta}{}_{\beta\nu}g^{\mu\nu}, \tag{13}$$

$$\mathcal{L}_Q := g^{\mu\nu}\left(L^{\alpha}{}_{\mu\beta}L^{\beta}{}_{\nu\alpha} - L^{\alpha}{}_{\alpha\beta}L^{\beta}{}_{\mu\nu}\right). \tag{14}$$

From these Lagrangians, the same equation of motion is found, but written as a function

[1] See [18–20] for some discussions concerning the role and significance of these boundary terms.

of either the Levi-Civita Ricci curvature $R_{\mu\nu}$, the contorsion $K^{\mu}{}_{\alpha\beta}$, or the distorsion $L^{\mu}{}_{\alpha\beta}$:

$$\text{GR}: \begin{cases} R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T^{\alpha}{}_{\alpha}}{2} g_{\mu\nu}\right), \\ \text{with} \quad \bar{T}^{\alpha}{}_{\mu\nu} = 0 \quad \text{and} \quad Q_{\mu\alpha\beta} = 0, \end{cases} \tag{15}$$

$$\Updownarrow$$

$$\text{TEGR}: \begin{cases} -\bar{\nabla}_{\alpha} K^{\alpha}{}_{\mu\nu} + \bar{\nabla}_{\mu} K^{\alpha}{}_{\nu\alpha} - K^{\alpha}{}_{\mu\beta} K^{\beta}{}_{\alpha\nu} + K^{\alpha}{}_{\alpha\beta} K^{\beta}{}_{\mu\nu} \\ \qquad\qquad + \bar{T}^{\alpha}{}_{\beta\mu} K^{\beta}{}_{\alpha\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T^{\alpha}{}_{\alpha}}{2} g_{\mu\nu}\right), \\ \text{with} \quad \bar{R}^{\mu}{}_{\alpha\beta\sigma} = 0 \quad \text{and} \quad Q_{\mu\alpha\beta} = 0, \end{cases} \tag{16}$$

$$\Updownarrow$$

$$\text{STEGR}: \begin{cases} -\bar{\nabla}_{\alpha} L^{\alpha}{}_{\mu\nu} + \bar{\nabla}_{\mu} L^{\alpha}{}_{\nu\alpha} - L^{\alpha}{}_{\mu\beta} L^{\beta}{}_{\alpha\nu} + L^{\alpha}{}_{\alpha\beta} L^{\beta}{}_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T^{\alpha}{}_{\alpha}}{2} g_{\mu\nu}\right), \\ \text{with} \quad \bar{R}^{\mu}{}_{\alpha\beta\sigma} = 0 \quad \text{and} \quad \bar{T}^{\alpha}{}_{\mu\nu} = 0, \end{cases} \tag{17}$$

where $T_{\mu\nu}$ is the energy momentum tensor. These equivalences are a direct consequence of relation (9), relating the Riemann tensors of two different connections.

While these particular theories are empirically equivalent to each other, there are a number of reasons why physicists are interested in investigating such alternative geometric representations. One reason has to do with the fact that these theories possess different gauge structure. In particular, TEGR and STEGR can be understood as gauge theories of translations [21, 22], which allows one to formulate the theories in a language more closely resembling other fundamental interactions and potentially suggests different routes towards quantisation. Another reason can be found in resolving cosmological puzzles. Despite the incredible successes of the current ΛCDM model, there are a number of unresolved issues that are the subject of heated debate, including our modeling of both early and late time expansion of the universe [23–27]. While the theories within the trinity are indeed equivalent, their geometric structures based on curvature, torsion, and non-metricity suggest different routes to modifying gravity. Indeed, the equivalence is broken when we move to modifications that consist in higher order scalar invariants of the relevant geometric quantities. That is, e.g., $f(R)$, $f(T)$, and $f(Q)$ theories are not equivalent to each other, and this has motivated exploring this theory space as possible novel realisations of dark energy, inflation, the astrophysics of black holes, and bouncing cosmologies [28–32]

2.3 Non-relativistic gravity

So much by way of background on the relativistic geometric trinity of gravity; what is the current state-of-the-art with respect to non-relativistic physics? It has been known since the 1960s that standard, flat-space classical Newtonian gravity can be formulated as a curved spacetime theory known as 'Newton–Cartan theory' (NC), in a way closely reassembling GR. The geometric structure involved in the NC formulation is known as a

'Galilean structure', defined by degenerate spatial and temporal metrics $h^{\mu\nu}$ and τ_μ that are orthogonal, i.e., $\tau_\mu h^{\mu\nu} := 0$, and equipped with a Galilean connection $\hat{\Gamma}^\alpha{}_{\mu\nu}$ that is separately compatible with both spatial and temporal metrics such that $\hat{\nabla}_\alpha h^{\mu\nu} := 0$ and $\hat{\nabla}_\alpha \tau_\mu := 0$. Similarly to GR, the dynamical degrees of freedom of NC theory are captured by its curvature tensor $\hat{R}^\alpha{}_{\beta\mu\nu}$, with the field equations

$$\begin{aligned} &\hat{R}_{\mu\nu} = 4\pi\rho\tau_\mu\tau_\nu, \\ &\hat{R}^\alpha{}_\nu{}^\mu{}_\beta = \hat{R}^\mu{}_\beta{}^\alpha{}_\nu, \\ &\hat{\nabla}_\mu T^{\mu\nu} = 0, \end{aligned} \tag{18}$$

where ρ is the mass density. The first equation is known as the Newton–Cartan equation, and encodes the Newton-Poisson equation (see e.g. [11, Ch. 4]). The second equation is a condition (due to Trautman) that the curvature tensor must satisfy such that inertial frames, a defining feature of Newtonian theory, exist. The third equation is just the conservation of the energy-momentum tensor. Contrary to the GR case, the first equation does not implies the third one, which, therefore, is an additional independent equation.

The equivalence of this spacetime theory with classical Newtonian gravity is most easily obtained by projecting along $h^{\mu\nu}$ and τ_μ the above system of equations [33], i.e., performing a 3+1-projection. Recovering classical Newtonian theory is also codified in the Trautman geometrisation and recovery theorems [11, Ch. 4], where one can define another connection ∇' such that $\hat{\nabla}_\beta v^\alpha = \nabla'_\beta v^\alpha - \tau_\beta \tau_\nu v^\nu h^{\alpha\mu} \nabla'_\mu \Phi$ where Φ is the familiar Newtonian gravitational potential. Upon assuming an isolated system, i.e., no cosmic expansion, one can then show that this connection is also compatible with the metrics and leads to a flat spacetime where one recovers the familiar Poisson equation $\nabla'^\alpha \nabla'_\alpha \Phi = 4\pi\rho$.

Newton–Cartan theory, and non-relativistic gravity more generally, is still a very active field of research as it has found important applications in non-relativistic holography [34], quantum gravity [35, 36], and condensed matter systems [8, 37–39]. What constitutes much more recent knowledge is that it is possible to obtain a teleparallel equivalent Newton–Cartan theory (TENC) [7, 9], similarly as TEGR for relativistic gravity. In this TENC formulation, the gravitational field present in the classical Newtonian formulation can be understood as the torsion of the mass gauge field m_μ obtained by gauging the Bargmann algebra [7]. In other words, the torsion of the flat-torsionful-metric connection $\tilde{\nabla}$ present in TENC is sourced by the mass gauge field. Moreover, TENC can be obtained by taking a $1/c^2$ expansion of TEGR [9].

In view of the relativistic trinity presented in Sec. 2.2, this invites the following questions: (a) can one construct a non-metric non-relativistic theory of gravity by taking a $1/c^2$ expansion of STEGR, and (b) is that theory equivalent to the NC theory, thereby retrieving the classical Newtonian theory? In the remainder of this article, we answer in the affirmative both (a) and (b), obtaining, as a result, a symmetric teleparallel equivalent Newton–Cartan theory (STENC). Thereby, we fill in the dotted lines in the Figure 1, and so complete for the fist time a non-relativistic geometric trinity of gravity, which we summarise in Sec. 3.5.

$$
\begin{tikzcd}
\mathrm{GR} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] \arrow[dd, "c\to\infty"'] & & \mathrm{TEGR} \arrow[dd, "c\to\infty"] \\
& \mathrm{STEGR} \arrow[ur, Leftrightarrow] \arrow[dd, dashed, "c\to\infty"] & \\
\mathrm{NC} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow, dashed] & & \mathrm{TENC} \\
& \mathrm{STENC} \arrow[ur, Leftrightarrow, dashed] &
\end{tikzcd}
$$

Figure 1: The geometric trinity and its (conjectured) non-relativistic limit.

3 The Non-Relativistic Limit of the Trinity

The goal of this section is to perform the non-relativistic limit of the STEGR field equations. For this, we will first define what is, in general, a contorsion $\hat{K}^{\mu}{}_{\alpha\beta}$ and a distorsion $\hat{L}^{\mu}{}_{\alpha\beta}$ relative to a Galilean connection $\hat{\nabla}$ (Sec. 3.1). We will then define the non-relativistic limit using Lorentzian and Galilean structures (Sec. 3.2), and apply this limit to a relativistic contorsion $K^{\mu}{}_{\alpha\beta}$, distorsion $L^{\mu}{}_{\alpha\beta}$ and difference tensor $\Gamma^{\mu}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\beta}$ (Sec. 3.3). Finally, in Sec. 3.4, we perform the limit of the relativistic trinity, hence rederiving the TENC formulation, but most importantly, obtaining the STENC formulation.

3.1 Distorsion and contorsion tensors with respect to a Galilean connection

In this section, we define the notion of distorsion $\hat{L}^{\mu}{}_{\alpha\beta}$ and contorsion $\hat{K}^{\mu}{}_{\alpha\beta}$ relative to a Galilean connection $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$ and a general affine connection $\tilde{\Gamma}^{\mu}{}_{\alpha\beta}$.

Contrary to the Levi-Civita connection of a Lorentzian metric, a (symmetric) Galilean connection defined with respect to a spatial metric $h^{\mu\nu}$ and a temporal metric τ_μ is not unique. Two freedoms exist: (i) in the choice of a timelike vector B^μ defined such that $B^\mu \tau_\mu = 1$; (ii) in the choice of a 2-form $\kappa_{\mu\nu}$ called the Coriolis field. The connection coefficients $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$ have the form

$$
\hat{\Gamma}^{\mu}{}_{\alpha\beta} = h^{\mu\sigma}\left(\partial_{(\alpha} b_{\beta)\sigma} - \frac{1}{2}\partial_\sigma b_{\alpha\beta}\right) + B^\mu \partial_{(\alpha}\tau_{\beta)} + 2\tau_{(\alpha}\kappa_{\beta)\nu} h^{\mu\nu}, \tag{19}
$$

where $b_{\mu\nu}$ is the spatial projector orthogonal to B^μ defined such that $b_{\mu\nu}B^\mu := 0$ and $h^{\alpha\mu} b_{\beta\mu} := \delta^\alpha_\beta - \tau_\beta B^\alpha$.

We now consider an (additional) general affine connection $\tilde{\Gamma}^{\alpha}{}_{\mu\nu}$ whose torsion is denoted $\tilde{T}^{\alpha}{}_{\mu\nu} := 2\tilde{\Gamma}^{\alpha}{}_{[\mu\nu]}$. Using the general formula (19) for a symmetric Galilean connection, the difference tensor $\hat{\Gamma}^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta}$ takes the form

$$\hat{\Gamma}^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = h^{\mu\sigma}\left(\tilde{\nabla}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\tilde{\nabla}_{\sigma}b_{\alpha\beta}\right) + B^{\mu}\tilde{\nabla}_{(\alpha}\tau_{\beta)} + 2\tau_{(\alpha}\kappa_{\beta)\nu}h^{\mu\nu} - \frac{1}{2}\tilde{T}^{\mu}{}_{\alpha\beta} - h^{\mu\sigma}\tilde{T}^{\nu}{}_{\sigma(\alpha}b_{\beta)\nu}. \tag{20}$$

From this formula, we can define what is a Galilean contorsion tensor $\hat{K}^{\mu}{}_{\alpha\beta}$, by assuming metricity, i.e., assuming $\tilde{\nabla}_{\alpha}h^{\mu\nu} = 0$ and $\tilde{\nabla}_{\mu}\tau_{\nu} = 0$:

$$\begin{aligned} -\hat{K}^{\mu}{}_{\alpha\beta} &:= \hat{\Gamma}^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta} \quad \left(\text{with} \quad \tilde{\nabla}_{\alpha}h^{\mu\nu} = 0,\ \tilde{\nabla}_{\mu}\tau_{\nu} = 0\right) \qquad (21) \\ &= h^{\mu\sigma}\left(\tilde{\nabla}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\tilde{\nabla}_{\sigma}b_{\alpha\beta}\right) + 2\tau_{(\alpha}\kappa_{\beta)\nu}h^{\mu\nu} - \frac{1}{2}\tilde{T}^{\mu}{}_{\alpha\beta} - h^{\mu\sigma}\tilde{T}^{\nu}{}_{\sigma(\alpha}b_{\beta)\nu}. \end{aligned}$$

We can also define the Galilean distorsion tensor $\hat{L}^{\mu}{}_{\alpha\beta}$, by assuming $\tilde{\nabla}$ to be torsionless, i.e., $\tilde{T}^{\mu}{}_{\alpha\beta} = 0$:

$$\begin{aligned} -\hat{L}^{\mu}{}_{\alpha\beta} &:= \hat{\Gamma}^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta} \qquad \left(\text{with} \quad \tilde{T}^{\mu}{}_{\alpha\beta} = 0\right) \qquad (22) \\ &= h^{\mu\sigma}\left(\tilde{\nabla}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\tilde{\nabla}_{\sigma}b_{\alpha\beta}\right) + B^{\mu}\tilde{\nabla}_{(\alpha}\tau_{\beta)} + 2\tau_{(\alpha}\kappa_{\beta)\nu}h^{\mu\nu}. \end{aligned}$$

Note that, contrary to the Lorentzian case, the relation $\hat{\Gamma}^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = -\hat{L}^{\mu}{}_{\alpha\beta} - \hat{K}^{\mu}{}_{\alpha\beta}$ does not hold, as the two definitions given above are only valid if the other one is set to zero. This means that the notion of contorsion and distorsion in the Galilean case is ill-defined for a general affine connection.

3.2 The non-relativistic limit

In this section, we define the non-relativistic limit we will use to obtain STENC from the relativistic trinity. The limit begins with an expansion of the relativistic objects in terms of powers of c, the speed of light. As there is no absolute velocity in non-relativistic physics, one takes $c \to \infty$, which can be thought of as 'flattening' the null cones at all spacetime points. Essentially, "in the limit the cones are all tangent to a family of hypersurfaces, each of which represents "space" at a given "time", which corresponds to the standard Newtonian picture of spacetime" [40]. More precisely, the fundamental ansatz of the non-relativistic limit is to consider a Lorentzian metric admitting of a Taylor series in terms of $\lambda := 1/c^2$ given by the following [10]:

$$\overset{\lambda}{g}{}^{\mu\nu} := h^{\mu\nu} + \lambda \overset{(1)}{g}{}^{\mu\nu} + \lambda^2 \overset{(2)}{g}{}^{\mu\nu} + \mathcal{O}\left(\lambda^3\right), \tag{23}$$

$$\overset{\lambda}{g}_{\mu\nu} := -\frac{1}{\lambda}\tau_{\mu}\tau_{\nu} + \overset{(0)}{g}_{\mu\nu} + \lambda \overset{(1)}{g}_{\mu\nu} + \mathcal{O}\left(\lambda^2\right), \tag{24}$$

where $h^{\mu\nu}$ is a tensor whose kernel is 3-dimensional and τ_μ is a 1-form. The leading orders of this Taylor series are degenerate, contrary to the full Lorentzian metric.[2]

The defining relation $\overset{\lambda}{g}{}^{\mu\alpha}\overset{\lambda}{g}_{\mu\beta} = \delta^\alpha_\beta$ for the full Lorentzian metric implies the orthogonality relation $\tau_\mu h^{\mu\nu} = 0$, along with the following formulae for $\overset{(1)}{g}{}^{\mu\nu}$ and $\overset{(0)}{g}_{\mu\nu}$:

$$\overset{(1)}{g}{}^{\mu\nu} = -B^\mu B^\nu + k^{\mu\nu}, \tag{25}$$

$$\overset{(0)}{g}_{\mu\nu} = b_{\mu\nu} - 2\tau_\mu \tau_\nu \phi, \tag{26}$$

where $k^{\mu\nu}\tau_\mu \coloneqq 0$, and B^μ and $b_{\mu\nu}$ are defined after Eq. (19). Finally, ϕ is an arbitrary scalar (see e.g. [10, 33] for further discussion related to these objects). Both B^μ and ϕ have some gauge freedom coming from the infinitesimal gauge freedom in the definition of the Taylor series (see Appendix F. in [42] for a detailed discussion).

The non-relativistic expansion defined in Eqs. (23)–(24) in orders of λ implies that the standard Levi-Civita connection expands as $\overset{\lambda}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{(-1)}{\Gamma}{}^\alpha{}_{\mu\nu} + \overset{(0)}{\Gamma}{}^\alpha{}_{\mu\nu} + \mathcal{O}(\lambda)$. However, it is only the zeroth order of the expansion of this connection that transforms as a connection, so it is only this order that can properly serve as a connection for the theories that emerge in this limit [43]. Consequently, $\overset{(-1)}{\Gamma}{}^\alpha{}_{\mu\nu}$ must vanish, which happens when we impose that $d\tau = 0$ [6, 10]; or in other words, when τ is closed, which gives us a notion of absolute time inherent in standard Newtonian spacetime theories. The zeroth order of the expansion of the Levi-Civita connection then defines a (symmetric) Galilean connection $\hat{\nabla}_\alpha$ compatible with $h^{\mu\nu}$ and τ_μ, as introduced in Sec. 2.3.

It should be noted that this is not the only way one can proceed. As detailed in [44, 45], one can take the non-relativistic limit of GR using a more general connection that does not force us to impose any conditions on τ. When we do impose $d\tau = 0$, we recover familiar NC, which has been dubbed 'Type I' Newton–Cartan theory in these papers, whereas relaxing this condition to $\tau \wedge \mathrm{d}\tau = 0$ leads to 'Type II' Newton–Cartan theories. Here we will adopt the condition that τ_μ is closed, even exact, thereby staying within the realm of traditional Newtonian spacetime theories with a notion of absolute time. We proceed now to take the non-relativistic limit of STEGR, which will involve taking the non-relativistic limit of the distortion tensor $L^\alpha{}_{\mu\nu}$ that distinguishes this theory from GR.

3.3 Distorsion and contorsion in the non-relativistic limit

3.3.1 General formulae

If the Galilean connection $\hat{\Gamma}^\mu{}_{\alpha\beta}$ derives from the non-relativistic limit (as defined in the previous section) of a Lorentzian connection $\overset{\lambda}{\Gamma}{}^\mu{}_{\alpha\beta}$, assuming τ_μ to be closed, then it has

[2] Rendall [41] showed that there is not much other choice than this ansatz for the leading orders, but also that odd orders in c should be considered. However, for the first few orders that we consider, these odd orders are purely gauge. Therefore, not considering them should not change the result of the paper.

the form:

$$\hat{\Gamma}^{\mu}{}_{\alpha\beta} \coloneqq \overset{(0)}{\Gamma}{}^{\mu}{}_{\alpha\beta} = h^{\mu\sigma}\left(\partial_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\partial_\sigma b_{\alpha\beta}\right) + B^\mu \partial_{(\alpha}\tau_{\beta)} + \tau_\alpha\tau_\beta h^{\mu\nu}\partial_\nu\phi. \tag{27}$$

Compared to a general Galilean connection (19), the connection deriving from a non-relativistic limit has an exact Coriolis field as $\kappa_{\mu\nu} = \tau_{[\mu}\partial_{\nu]}\phi$.

Now the goal is to take the limit of the contorsion $\overset{\lambda}{K}{}^{\mu}{}_{\alpha\beta}$ [defined by (6)], the distorsion $\overset{\lambda}{L}{}^{\mu}{}_{\alpha\beta}$ [defined by (7)], and the difference tensor $\Gamma^{\mu}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\beta}$. As for the Levi-Civita connection, the general affine connection $\bar{\Gamma}^{\mu}{}_{\alpha\beta}$ should, *a priori*, have a Taylor series as a function of λ. Let us assume that the leading order of this series is the zeroth order, i.e. $\overset{\lambda}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta} = \overset{(0)}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta} + \mathcal{O}(\lambda)$. We will discuss this hypothesis in Sec. 3.3.4. We obtain for the contorsion

$$\begin{aligned}
-\overset{\lambda}{K}{}^{\mu}{}_{\alpha\beta} = \frac{1}{\lambda}&\left[h^{\sigma\mu}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\tau_{\beta)}\tau_\gamma\right] \\
&- \frac{1}{2}\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} - h^{\mu\sigma}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\overset{(0)}{g}_{\beta)\gamma} + \overset{(1)}{g}{}^{\sigma\mu}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\tau_{\beta)}\tau_\gamma + h^{\sigma\mu}\overset{(1)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\tau_{\beta)}\tau_\gamma + \mathcal{O}(\lambda),
\end{aligned} \tag{28}$$

and for the distorsion

$$\begin{aligned}
-\overset{\lambda}{L}{}^{\mu}{}_{\alpha\beta} = \frac{1}{\lambda}&\left[h^{\sigma\mu}\left(\tau_\alpha\overset{(0)}{\bar{\nabla}}_{[\sigma}\tau_{\beta]} + \tau_\beta\overset{(0)}{\bar{\nabla}}_{[\sigma}\tau_{\alpha]}\right)\right] \\
&- h^{\mu\sigma}\overset{(1)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\tau_{\beta)}\tau_\gamma + \left(2\phi h^{\sigma\mu} + \overset{(1)}{g}{}^{\sigma\mu}\right)\left(\tau_\alpha\overset{(0)}{\bar{\nabla}}_{[\sigma}\tau_{\beta]} + \tau_\beta\overset{(0)}{\bar{\nabla}}_{[\sigma}\tau_{\alpha]}\right) \\
&+ h^{\sigma\mu}\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_\sigma b_{\alpha\beta}\right) + B^\gamma\overset{(0)}{\bar{\nabla}}_{(\alpha}\tau_{\beta)} + \tau_\alpha\tau_\beta h^{\gamma\sigma}\partial_\sigma\phi + \mathcal{O}(\lambda),
\end{aligned} \tag{29}$$

where $\overset{(0)}{\bar{\nabla}} \coloneqq \partial + \overset{(0)}{\bar{\Gamma}}$. These two formulae imply that the limit of the relativistic formula (8) is

$$\begin{aligned}
\overset{\lambda}{\Gamma}{}^{\mu}{}_{\alpha\beta} - \overset{\lambda}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta} = h^{\sigma\mu}&\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_\sigma b_{\alpha\beta}\right) + B^\gamma\overset{(0)}{\bar{\nabla}}_{(\alpha}\tau_{\beta)} + \tau_\alpha\tau_\beta h^{\gamma\sigma}\partial_\sigma\phi \\
&- \frac{1}{2}\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} - h^{\mu\sigma}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}b_{\beta)\gamma} + \mathcal{O}(\lambda),
\end{aligned} \tag{30}$$

where we used the fact that τ_μ is closed, implying $\overset{(0)}{\bar{\nabla}}_{[\sigma}\tau_{\beta]} = -\frac{1}{2}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\alpha\beta}\,\tau_\gamma$.

We see that, in general, the contorsion and the distorsion tensors have negative orders in the limit. However, the difference tensor (30) only has positive orders. Furthermore, this tensor does not depend on first orders of the torsion tensor contrary to the contorsion and the distorsion. The formula (30) is very similar to the general Galilean formula (20) when associating the zeroth order $\overset{(0)}{\bar{\nabla}}$ with $\tilde{\nabla}$, the only difference being the exactness of the Coriolis field.

3.3.2 Non-relativistic limit with torsion and metricity

We assume that $\overset{\lambda}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta}$ is metric. In the limit, this implies

$$\overset{(0)}{\bar{\nabla}}_{\mu}h^{\alpha\beta} = 0 \quad ; \quad \overset{(0)}{\bar{\nabla}}_{\mu}\tau_{\nu} = 0, \tag{31}$$

implying $\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\alpha\beta}\,\tau_{\gamma} = 0$. Using $\overset{\lambda}{\hat{L}}{}^{\mu}{}_{\alpha\beta} = 0$, we also have

$$h^{\mu\sigma}\overset{(1)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}\tau_{\beta)}\tau_{\gamma} = h^{\sigma\mu}\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_{\sigma}b_{\alpha\beta}\right) + \tau_{\alpha}\tau_{\beta}h^{\gamma\sigma}\partial_{\sigma}\phi. \tag{32}$$

Finally, the contorsion tensor in the limit is

$$\begin{aligned} -\overset{\lambda}{\hat{K}}{}^{\mu}{}_{\alpha\beta} = {} & h^{\sigma\mu}\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_{\sigma}b_{\alpha\beta}\right) + \tau_{\alpha}\tau_{\beta}h^{\gamma\sigma}\partial_{\sigma}\phi \\ & - \frac{1}{2}\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} - h^{\mu\sigma}\overset{(0)}{\bar{T}}{}^{\gamma}{}_{\sigma(\alpha}b_{\beta)\gamma} + \mathcal{O}\left(\lambda\right). \end{aligned} \tag{33}$$

Compared to the general formula (21) for Galilean contorsion, the one obtained in the limit features an exact Coriolis field. That form of the contorsion tensor at zeroth order of the non-relativistic limit is in agreement with the one derived by Schwartz [9, Eq. (2.33a)]. His "$\left(\tau_{(\alpha}f_{\beta)}{}^{\mu}\right)$" is replaced by "$\frac{1}{2}\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} - h^{\sigma\mu}\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_{\sigma}b_{\alpha\beta}\right) - \tau_{\alpha}\tau_{\beta}h^{\gamma\sigma}\partial_{\sigma}\phi$" in our case, which has the same properties when projected along $h^{\mu\nu}$ and τ_{μ}.

From the metricity relations (31), we can also show

$$\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta}\tau_{\mu} = 0 \quad ; \quad \overset{(0)}{\bar{T}}{}^{(\alpha}{}_{\mu\nu}\,h^{\beta)\mu}h^{\nu\sigma} = 0. \tag{34}$$

3.3.3 Non-relativistic limit without torsion and with non-metricity

We assume that $\overset{\lambda}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta}$ is torsionless, which implies $\overset{(0)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} = 0$ and $\overset{(1)}{\bar{T}}{}^{\mu}{}_{\alpha\beta} = 0$. Then the distorsion tensor becomes in the limit

$$-\overset{\lambda}{\bar{L}}{}^{\mu}{}_{\alpha\beta} = h^{\sigma\mu}\left(\overset{(0)}{\bar{\nabla}}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\overset{(0)}{\bar{\nabla}}_{\sigma}b_{\alpha\beta}\right) + B^{\mu}\overset{(0)}{\bar{\nabla}}_{(\alpha}\tau_{\beta)} + \tau_{\alpha}\tau_{\beta}h^{\mu\sigma}\partial_{\sigma}\phi + \mathcal{O}(\lambda). \tag{35}$$

Compared to the general formula (22) for a general Galilean distorsion tensor, the one obtained in the limit features an exact Coriolis field.

3.3.4 Limit of a flat (reference) connection

The specificity of taking the non-relativistic limit of TEGR or STEGR compared to GR is that there is an additional field, namely $\bar{\nabla}$, whose behaviour in the limit needs to be understood. In the previous section, the formulae derived and depending on $\bar{\nabla}$ are valid regardless of the Riemann curvature of that connection. However, in the case of interest for this paper, the connection $\bar{\Gamma}$ is flat. Therefore, each order of its Riemann tensor must be zero. Since we assumed that the leading order of the connection is the zeroth order, then that order is a connection whose Riemann tensor $\overset{(0)}{\bar{R}}{}^{\mu}{}_{\alpha\beta\nu}$ is zero. In other words, this implies that the zeroth order of the connection $\overset{\lambda}{\bar{\Gamma}}{}^{\mu}{}_{\alpha\beta}$ is also flat.

Therefore, the connection $\overset{(0)}{\bar{\nabla}}$ appearing in the distorsion and contorsion tensors in the non-relativistic limit is flat. This was expected of course, but not that trivial considering the fact the reference connection has, in general, a Taylor series. This zeroth order will correspond to the flat affine connection present in the non-relativistic trinity, and will be denoted by $\tilde{\nabla}$.

Remark 1. That results obtained in the limit hold only if $\overset{\lambda}{\bar{\Gamma}}$ has positive orders. If we allow for negative orders, then the flatness condition $\overset{\lambda}{\bar{R}}{}^{\mu}{}_{\alpha\beta\nu} = 0$ implies the constraint

$$\overset{(N)}{\bar{\Gamma}}{}^{\alpha}{}_{\mu\lambda}\overset{(N)}{\bar{\Gamma}}{}^{\lambda}{}_{\nu\beta} = \overset{(N)}{\bar{\Gamma}}{}^{\alpha}{}_{\nu\lambda}\overset{(N)}{\bar{\Gamma}}{}^{\lambda}{}_{\mu\beta}, \tag{36}$$

where $N < 0$ is the leading order, which is by definition strictly negative. As for the Levi-Civita connection $\overset{\lambda}{\Gamma}{}^{\mu}{}_{\alpha\beta}$, because only the zeroth order of the connection is not a tensor, all the other orders are tensors, then $\overset{(N)}{\bar{\Gamma}}$ is a tensor. Therefore, the above condition is a tensor equation. We suspect that this constraint implies $\overset{(N)}{\bar{\Gamma}} = 0$, which would mean that the reference connection necessarily has positive orders once it is considered flat, hence justifying the hypothesis. But we have not been able to prove such a result.

3.4 Non-relativistic limit of STEGR

In the case of taking the non-relativistic limit of TEGR, the prescription followed in [9] is the following one: (a) write the Einstein equation of GR in terms of the TEGR contortion (and associated torsion); (b) take the non-relativistic limit—constructed in the above way—of that equation. Following the same prescription for the non-relativistic limit of STEGR, we first recall the expression in Eq. (17), and take the non-relativistic limit. Taking the non-relativistic limit of the rhs gives [10]:

$$\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T^{\alpha}{}_{\alpha}}{2}g_{\mu\nu}\right) \to 4\pi G\rho\tau_{\mu}\tau_{\nu}. \tag{37}$$

The validity of this formula requires the leading orders of the energy-momentum tensor $\overset{\lambda}{T}{}^{\mu\nu}$ to be positive [42]. Cases where these orders are negative have been studied in [45], dubbed “strong gravity”. In the present paper, we will not consider this possibility.

We proceed now to take the limit of the lhs of the field equation (17), which is the Ricci tensor of the Levi-Civita connection expressed in terms of the distortion and the non-metric connection. As mentioned earlier, when taking the non-relativistic limit of the connection, we will be seeking the zeroth order term in the expansion. This means that we will be searching for $\hat{L}^{\alpha}{}_{\mu\nu} := \overset{(0)}{L}{}^{\alpha}{}_{\mu\nu} = \overset{(0)}{\bar{\Gamma}}{}^{\alpha}{}_{\mu\nu} - \overset{(0)}{\Gamma}{}^{\alpha}{}_{\mu\nu} = \tilde{\Gamma}^{\alpha}{}_{\mu\nu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu}$, which is given by the leading order term in Eq. (35). Consequently, the non-relativistic limit of the field Eq. (17) of STEGR is:

$$-\tilde{\nabla}_{\alpha}\hat{L}^{\alpha}{}_{\mu\nu} + \tilde{\nabla}_{\mu}\hat{L}^{\alpha}{}_{\nu\alpha} - \hat{L}^{\alpha}{}_{\mu\beta}\hat{L}^{\beta}{}_{\alpha\nu} + \hat{L}^{\alpha}{}_{\alpha\beta}\hat{L}^{\beta}{}_{\mu\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu}, \tag{38}$$

where

$$-\hat{L}^{\alpha}{}_{\mu\nu} = h^{\gamma\alpha}\left(\tilde{\nabla}_{(\mu}b_{\nu)\gamma} - \frac{1}{2}\tilde{\nabla}_{\gamma}b_{\mu\nu}\right) + B^{\alpha}\tilde{\nabla}_{(\mu}\tau_{\nu)} + \tau_{\mu}\tau_{\nu}h^{\alpha\gamma}\partial_{\gamma}\phi. \tag{39}$$

This is the field equation of STENC.

As it is formulated in Eq. (39), the non-relativistic distorsion tensor is not written explicitly as a function of a single non-metricity tensor as in the relativistic case (7). There are two reasons for this: first, we do not have a single metric like in GR, and second, the distorsion tensor does not depend only on these two metrics, but also on B^{μ} and $b_{\mu\nu}$. This means that there are four different non-metricities that can naturally be defined in the STENC formulation:

$$\hat{Q}_{\mu}{}^{\alpha\beta} := \tilde{\nabla}_{\mu}h^{\alpha\beta} \quad ; \quad \hat{Q}_{\mu\nu} := \tilde{\nabla}_{\mu}\tau_{\nu} \quad ; \quad \hat{Q}_{\alpha\mu\nu} := \tilde{\nabla}_{\alpha}b_{\mu\nu} \quad ; \quad \hat{Q}_{\mu}{}^{\nu} := \tilde{\nabla}_{\mu}B^{\nu}. \tag{40}$$

We recall that since there is no duality between forms and vectors with Galilean structures, i.e. no way of raising and lowering indices, then $\hat{Q}_{\mu}{}^{\nu}$ and $\hat{Q}_{\mu\nu}$ (as well as $\hat{Q}_{\mu\alpha\beta}$ and $\hat{Q}_{\mu}{}^{\alpha\beta}$) have to be understood as different tensors. Consequently, the distorsion tensor of STENC can be written in the form

$$-\hat{L}^{\alpha}{}_{\mu\nu} = h^{\gamma\alpha}\left(\hat{Q}_{(\mu\nu)\gamma} - \frac{1}{2}\hat{Q}_{\gamma\mu\nu}\right) + B^{\alpha}\hat{Q}_{(\mu\nu)} + \tau_{\mu}\tau_{\nu}h^{\alpha\gamma}\partial_{\gamma}\phi. \tag{41}$$

Writing the non-relativistic distorsion in this way, one can clearly see the analogy with the relativistic version given in Eq. (7). It should be noted; however, that this non-relativistic distortion tensor that emerges in the limit is not uniquely defined, as it is expressed in terms of the non-metricity of the spatial projector $b_{\mu\nu}$. Here, the fixed spacetime structure is given by τ_{μ} and $h^{\mu\nu}$. Furthermore, one can compute the non-metricity of the fixed spatial metric $h^{\mu\nu}$ and find that it is given by $\tilde{\nabla}_{\alpha}h^{\mu\nu} = -2h^{(\mu\sigma}h^{\nu)\gamma}\left(\hat{Q}_{(\alpha\sigma\gamma)} - \frac{1}{2}\hat{Q}_{\sigma\alpha\gamma}\right) - 2B^{(\mu}h^{\nu)\sigma}\hat{Q}_{(\alpha\sigma)}$. However, it is not possible to write the distorsion uniquely as a function of $\hat{Q}_{\mu}{}^{\alpha\beta}$ and $\hat{Q}_{\mu\nu}$, showing, again, that some gauge freedom remain in the definition of this tensor.

3.5 The non-relativistic trinity

In conclusion, the non-relativistic limit of the trinity of general relativity gives the three following equivalent systems of equation for non-relativistic gravitation:
(i) the curved–torsionless–metric formulation (NC) is

$$\begin{cases} \hat{R}_{\mu\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu}, \\ \hat{\nabla}_{\mu}T^{\mu\alpha} = 0, \\ \hat{\Gamma}^{\mu}{}_{\alpha\beta} = h^{\gamma\mu}\left(\partial_{(\alpha}b_{\beta)\mu} - \frac{1}{2}\partial_{\mu}b_{\alpha\beta}\right) + B^{\gamma}\partial_{(\alpha}\tau_{\beta)} + \tau_{\alpha}\tau_{\beta}h^{\gamma\mu}\partial_{\mu}\phi, \end{cases} \tag{42}$$

(ii) the flat–torsionful–metric formulation (TENC) is

$$\begin{cases} -\tilde{\nabla}_\alpha \hat{K}^\alpha{}_{\mu\nu} + \tilde{\nabla}_\mu \hat{K}^\alpha{}_{\nu\alpha} - \hat{K}^\alpha{}_{\mu\beta}\hat{K}^\beta{}_{\alpha\nu} + \hat{K}^\alpha{}_{\alpha\beta}\hat{K}^\beta{}_{\mu\nu} + \tilde{T}^\alpha{}_{\beta\mu}\hat{K}^\beta{}_{\alpha\nu} = 4\pi G\rho\tau_\mu\tau_\nu, \\ \tilde{\nabla}_\mu T^{\mu\alpha} - 2T^{\sigma(\alpha}\hat{K}^{\mu)}{}_{\mu\sigma} = 0, \\ -\hat{K}^\mu{}_{\alpha\beta} = h^{\sigma\mu}\left(\tilde{\nabla}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\tilde{\nabla}_\sigma b_{\alpha\beta}\right) + \tau_\alpha\tau_\beta h^{\gamma\sigma}\partial_\sigma\phi - \frac{1}{2}\tilde{T}^\mu{}_{\alpha\beta} - h^{\mu\sigma}\tilde{T}^\gamma{}_{\sigma(\alpha}b_{\beta)\gamma}, \\ \text{with} \quad \tilde{R}^\mu{}_{\alpha\beta\nu} = 0 \quad ; \quad \tilde{\nabla}_\mu h^{\alpha\beta} = 0 \quad ; \quad \tilde{\nabla}_\mu \tau_\nu = 0. \end{cases} \tag{43}$$

(iii) the flat–torsionless–non-metric formulation (STENC) is

$$\begin{cases} -\tilde{\nabla}_\alpha \hat{L}^\alpha{}_{\mu\nu} + \tilde{\nabla}_\mu \hat{L}^\alpha{}_{\nu\alpha} - \hat{L}^\alpha{}_{\mu\beta}\hat{L}^\beta{}_{\alpha\nu} + \hat{L}^\alpha{}_{\alpha\beta}\hat{L}^\beta{}_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu, \\ \tilde{\nabla}_\mu T^{\mu\alpha} - 2T^{\sigma(\alpha}\hat{L}^{\mu)}{}_{\mu\sigma} = 0, \\ -\hat{L}^\mu{}_{\alpha\beta} = h^{\sigma\mu}\left(\tilde{\nabla}_{(\alpha}b_{\beta)\sigma} - \frac{1}{2}\tilde{\nabla}_\sigma b_{\alpha\beta}\right) + B^\gamma\tilde{\nabla}_{(\alpha}\tau_{\beta)} + \tau_\alpha\tau_\beta h^{\gamma\sigma}\partial_\sigma\phi, \\ \text{with} \quad \tilde{R}^\mu{}_{\alpha\beta\nu} = 0 \quad ; \quad \tilde{T}^\mu{}_{\alpha\beta} = 0. \end{cases} \tag{44}$$

We recall that $T^{\mu\nu}$ is the energy-momentum tensor.

The Galilean connection $\hat{\Gamma}^\mu{}_{\alpha\beta}$, contorsion tensor $\hat{K}^\mu{}_{\alpha\beta}$ and distorsion tensor $\hat{L}^\mu{}_{\alpha\beta}$ are not uniquely defined since there is a gauge freedom in the choice of the observer 4-velocity B^μ with respect to which $b_{\mu\nu}$ is defined, and in ϕ. The origin of this freedom can be traced back to the gauge freedom in the Taylor series of the Lorentzian metric when considering the non-relativistic limit [42, Appendix F.].

In each case, the Coriolis field is exact, which implies the existence of vorticity-free observers and the absence of gravitomagnetism, two necessary features for the theory to be considered "Newtonian". Furthermore, since the first equations in each case are equivalent (considering the definition of $\tilde{\nabla}$ along with the formulae for $\hat{K}^\mu{}_{\alpha\beta}$ and $\hat{L}^\mu{}_{\alpha\beta}$), and correspond to the Newton–Cartan equation, then each of these systems of equations are equivalent and lead to the classical Newtonian theory (for an isolated system) [46] and to the cosmological Newtonian theory (if closed boundary conditions are considered) [33], as explained in Sec. 3.6. The first system has been known since Trautman [46], while the second was derived by Read and Teh [7], Schwartz [9]. What is new in the present paper is the third system, defining the STENC formulation. Hence, this completes the construction of the non-relativistic trinity of gravity, as summarised in Figure 2.

3.6 Recovering the (classical) Newtonian theory

Classical Newtonian theory is described by a system of equations defined on a 3-manifold. Therefore, recovering that theory from the Newton-Cartan formulation requires obtaining spatial equations from one of the Newton-Cartan systems of the non-relativistic trinity. The most general way of performing such a recovery is through a 3+1–projection where the 4-dimensional spacetime equations are projected along the time metric τ_μ and the spatial metric $h^{\mu\nu}$. In the relativistic case, such a projection leads to the so-called 3+1–GR formulation, a way a describing the Einstein equation with equations defined on a 3-manifold, allowing for, in particular, numerical resolution of that equation [47].

When performing the 3+1–projection on the NC system, one does not directly obtain the classical Newtonian equations (in which, in particular, the Poisson equation is $\Delta_h \phi = 4\pi G\rho$, with Δ_h the spatial Laplacian), but similar equations featuring additional fields that depend on the choice of boundary conditions [33].[3] Only when the physical fields are considered integrable and the space infinite, in other words choosing an isolated system, then the 3+1-projection of the NC system leads to the classical Newtonian theory. However, if closed boundary conditions are chosen, which is more representative of a cosmological setup, then, instead, the cosmological Newtonian theory is retrieved, in which expansion is present. This was first studied by [48] in the homogeneous case, and fully derived in the general (inhomogeneous) case by [33]; see [49] for related foundational discussions.

The fact that the NC formulation naturally features expansion is usually missed in references studying the recovery of (classical) Newtonian theory. The main reason is that the spatial metric is, in general, assumed to be δ_{ij} in some coordinate systems, i.e. independent of time, which imposes the vanishing of expansion. In other references [e.g., 9], a covariant condition is taken by assuming the existence of observers whose 4-velocity u^μ satisfies $h^{\mu(\alpha}\hat{\nabla}_\mu u^{\beta)} = 0$. This is exactly the covariant condition for vanishing expansion in NC, as shown in [33] (see also Sec. 4.1).

Now, with regard to the recovery process from STENC, since that system of equations is equivalent to the NC formulation, the same recovery is therefore possible. This means that STENC leads to either the classical or the cosmological Newtonian theory, depending on the choice of boundary conditions, answering by the positive to the second question raised at the end of the introduction.

4 Discussion

Having now constructed the non-relativistic geometric trinity of gravity, in this section we discuss (i) observers in the non-relativistic trinity (Sec. 4.1), (ii) the special case of Weylian (i.e., pure trace) non-relativistic non-metricity (Sec. 4.2), and (iii) different possible choices for non-relativistic non-metricity in relation to the Poisson equation (Sec. 4.3).

4.1 Observers in the non-relativistic trinity

In non-relativistic theories, a (timelike) observer is defined by a 4-velocity u^μ unit with respect to the time metric, i.e. $u^\mu \tau_\mu = 1$. Two types of observers are of major importance: geodesic observers and inertial observers. The former are related to the equivalence prin-

[3] This paper refers to '1+3' instead of '3+1' for the projection. In general relativity these two procedures differ depending on whether the time vector is vorticity free or not. However, in non-relativistic theories with an exact τ_μ, 3+1 and 1+3 projections become degenerate. We choose to use the '3+1' denomination in the present paper.

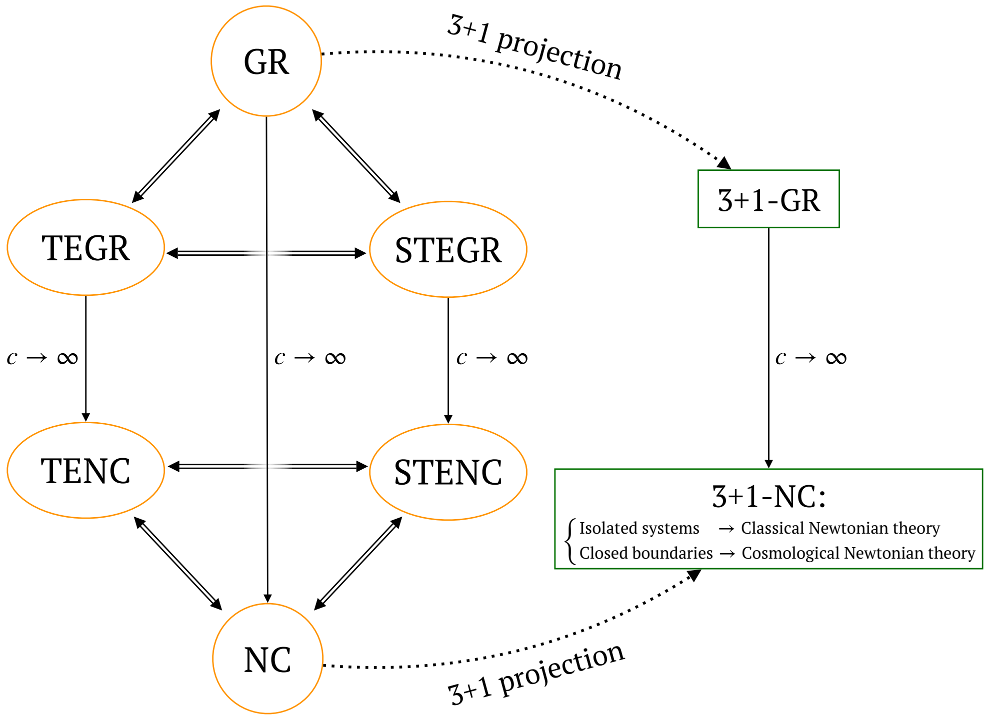

Figure 2: The geometric trinity of gravity, its now-constructed non-relativistic counterpart, and the relations of both to the $3+1$ formulations of GR and NC, respectively.

ciple while the latter to the concept of inertial frames of Newtonian theory as well as the gravitational field:

(i) *Geodesic observers:* As for GR, in non-relativistic gravitation an observer subject only to gravity follows the geodesics (related to the Galilean connection) of spacetime. In the trinity, this translate into the following three equations:

$$\mathrm{NC}: \quad u^\mu \hat{\nabla}_\mu u^\alpha = 0, \tag{45}$$

$$\mathrm{TENC}: \quad u^\mu \tilde{\nabla}_\mu u^\alpha = u^\mu u^\nu \hat{K}^\alpha{}_{\mu\nu}, \tag{46}$$

$$\mathrm{STENC}: \quad u^\mu \tilde{\nabla}_\mu u^\alpha = u^\mu u^\nu \hat{L}^\alpha{}_{\mu\nu}. \tag{47}$$

(ii) *Inertial observers:* As shown in [33], the NC equation implies the existence of an inertial observer (called Galilean observer in this paper) which is defined by a 4-velocity G^μ with the following constraints:[4]

$$h^{\mu(\alpha}\hat{\nabla}_\mu G^{\beta)} = \chi h^{\alpha\beta} + \Xi^{\alpha\beta}, \tag{48}$$

$$h^{\mu[\alpha}\hat{\nabla}_\mu G^{\beta]} = 0, \tag{49}$$

[4] In terms of a (unique) scalar-vector-tensor decomposition of symmetric tensors, the first condition means that the expansion tensor $h^{\mu(\alpha}\hat{\nabla}_\mu G^{\beta)}$ of G^μ has no gradient part [33].

where χ is a scalar field (imposed to be a spatial constant by the Newton–Cartan equation), representing global expansion, and $\Xi^{\mu\nu}$ is a traceless-harmonic spatial tensor (with respect to the spatial metric $h^{\mu\nu}$) representing anisotropic expansion. The value of the expansion fields χ and $\Xi^{\mu\nu}$ depend on boundary conditions. For an isolated system, where $\chi = 0$ and $\Xi^{\mu\nu} = 0$, i.e. no expansion, the first condition implies that in a coordinate system where $\partial_t^\mu = G^\mu$, the spatial metric components are independent of time. If expansion is present with $\chi \neq 0$ or $\Xi^{\mu\nu} \neq 0$, one or several scale factors appear. The second condition means that G^μ is vorticity free.

The gravitational field g^μ in the Newton–Cartan formulation is elegantly defined as the opposite of the 4-acceleration of an inertial observer:

$$g^\alpha \coloneqq -G^\mu \hat{\nabla}_\mu G^\alpha. \tag{50}$$

From the Newton–Cartan equation and the exactness of the Coriolis field, we can show that g^μ is vorticity free (i.e. no gravitomagnetism), and that its potential is the scalar field Φ entering into the Poisson equation [33]. These properties also follow from the definition of an inertial observer. In the language of the non-relativistic trinity, the gravitational field can be written in the following forms:

$$\text{TENC}: \quad g^\alpha = -G^\mu \tilde{\nabla}_\mu G^\alpha + G^\mu G^\nu \hat{K}^\alpha{}_{\mu\nu}, \tag{51}$$

$$\text{STENC}: \quad g^\alpha = -G^\mu \tilde{\nabla}_\mu G^\alpha + G^\mu G^\nu \hat{L}^\alpha{}_{\mu\nu}. \tag{52}$$

Remark 2. We suspect that in both cases the gauge freedom in the definition of $\tilde{\nabla}$ allows us to assume $G^\mu \tilde{\nabla}_\mu G^\alpha = 0$, with the elegant interpretation that inertial observers are not accelerating with respect to the flat connection. However, we were not able to prove this property.

4.2 The Weylian special case

As we have been concerned with theories that exhibit general non-metricity, we consider now the special case of 'Weylian' non-metricity, of the form

$$Q_{\alpha\mu\nu} = \sigma_\alpha g_{\mu\nu}, \tag{53}$$

where σ is a 1-form. In other words, we consider now the case in which the non-metricity is 'pure trace'. This form of non-metricity is of both historical and modern interest. Historically, Weyl famously generalized Riemannian geometry by relaxing the metric compatibility condition in a failed attempt to unify gravity and electromagnetism [50]. While this particular attempt was unsuccessful, Weyl geometries and Weyl-inspired gravitational theories themselves are still an active field of research as these theories manifest scale-invariance as they are invariant under the well-known 'Weyl transformations' [51, 52]:

$$\begin{aligned} g^{\mu\nu} &\mapsto e^{-f} g^{\mu\nu}, \\ \sigma_\alpha &\mapsto \sigma_\alpha - \bar{\nabla}_\alpha f. \end{aligned} \tag{54}$$

Most relevantly to us, [51] shows how to formulate GR in the language of Weyl geometry by adopting the condition in Eq. (53). This leads to a distortion tensor given by:

$$L^{\alpha}{}_{\mu\nu} = -\sigma_{(\mu}\delta^{\alpha}_{\nu)} + \frac{1}{2}\sigma^{\alpha}g_{\mu\nu}. \tag{55}$$

In the following, we proceed to take the non-relativistic limit of this ansatz in the same way as before, using the Taylor expansion defined in Sec. 3. First, we must consider the Taylor expansion of σ:

$$\overset{\lambda}{\sigma}_{\mu} = \overset{(0)}{\sigma}_{\mu} + \lambda\overset{(1)}{\sigma}_{\mu} + \mathcal{O}\left(\lambda^2\right). \tag{56}$$

The non-metricity in this Weylian case then becomes

$$\tilde{\nabla}_{\alpha}h^{\mu\nu} = -\overset{(0)}{\sigma}_{\alpha}h^{\mu\nu} \quad ; \quad \tilde{\nabla}_{\mu}\tau_{\nu} = \frac{1}{2}\overset{(0)}{\sigma}_{\mu}\tau_{\nu}, \tag{57}$$

where, again, we denote $\overset{(0)}{\bar{\nabla}}$ by $\tilde{\nabla}$ which is flat and torsionless. (Note that this is consistent with [53].) We see that in the limit the Weylian form in the ansatz (54) remains present for $\tilde{\nabla}_{\mu}h^{\alpha\beta}$ and $\tilde{\nabla}_{\mu}\tau_{\nu}$ as these two derivatives are both proportional to the 1-form $\overset{(0)}{\sigma}_{\mu}$ obtained in the limit. This suggests that conformal gravity theories should have a non-relativistic limit preserving the conformal symmetry, but transferring it to the spatial and the temporal metrics.

Regarding the distorsion tensor, as before, when taking the non-relativistic limit, we are searching for $\hat{L}^{\alpha}{}_{\mu\nu} := \overset{(0)}{L}{}^{\alpha}{}_{\mu\nu} = \overset{(0)}{\bar{\Gamma}}{}^{\alpha}{}_{\mu\nu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu} = \tilde{\Gamma}^{\alpha}{}_{\mu\nu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu}$ while assuming $\bar{\nabla}$ to be torsionless, but with the 'Weylian' versions of the non-metric objects. One obtains a (-1)-order term $\overset{(-1)}{L}{}^{\alpha}{}_{\mu\nu} = -\frac{1}{2}h^{\alpha\beta}\overset{(0)}{\sigma}_{\beta}\tau_{\mu}\tau_{\nu}$ which has to vanish to obtain a Galilean connection in the limit. This gives us $h^{\alpha\beta}\overset{(0)}{\sigma}_{\beta} = 0$. In other words, σ_{μ} is proportional to the time metric, i.e., $\sigma_{\mu} = \psi\,\tau_{\mu}$, where ψ is an arbitrary scalar. This implies

$$\begin{aligned}\overset{(0)}{L}{}^{\alpha}{}_{\mu\nu} &= -\overset{(0)}{\sigma}_{(\mu}\delta^{\alpha}_{\nu)} - \frac{1}{2}\left(\overset{(1)}{g}{}^{\alpha\beta}\overset{(0)}{\sigma}_{\beta} + h^{\alpha\beta}\overset{(1)}{\sigma}_{\beta}\right)\tau_{\mu}\tau_{\nu} \\ &= -\psi\,\tau_{(\mu}\delta^{\alpha}_{\nu)} + \frac{1}{2}\left(\psi B^{\alpha} - h^{\alpha\beta}\overset{(1)}{\sigma}_{\beta}\right)\tau_{\mu}\tau_{\nu}.\end{aligned} \tag{58}$$

Using this result and $\tilde{\nabla}_{[\mu}\overset{(0)}{\sigma}_{\nu]} = 0$ (that σ is closed is implied by the non-metricity condition), the field equations under the Weylian assumption for non-metricity are given by

$$-\tilde{\nabla}_{\mu}\overset{(0)}{\sigma}_{\nu} - \frac{1}{2}\overset{(0)}{\sigma}_{\mu}\overset{(0)}{\sigma}_{\nu} + \frac{1}{2}\tilde{\nabla}_{\alpha}\left[\overset{(1)}{g}{}^{\alpha\beta}\overset{(0)}{\sigma}_{\beta} + h^{\alpha\beta}\overset{(1)}{\sigma}_{\beta}\right]\tau_{\mu}\tau_{\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu}. \tag{59}$$

4.3 Choice of non-metricity and the Poisson equation

In this section, we show that there are two mirror ansatzes that can be imposed on the non-relativistic non-metricity tensors defined in Eq. (40), such that the field equation of STENC becomes explicitly the Poisson equation written in terms of a spacetime connection. The two conditions are:

(i) $\hat{Q}_{\mu\alpha\beta} = 0$ and $\hat{Q}_{\mu}{}^{\nu} = 0$.

(ii) $\hat{Q}_{\mu}{}^{\alpha\beta} = 0$ and $\hat{Q}_{\mu\nu} = 0$.

Both of these approaches restrict the possible physical solutions as spatial expansion is set to zero with $\hat{Q}_{\mu}{}^{\alpha\beta} = 0$ or $\hat{Q}_{\mu\alpha\beta} = 0$, meaning that only an isolated system can be considered. The goal of these ansatzes is mainly to illustrate how the physical degrees of freedom can be shifted between the different non-metricities.

With the first choice, the distorsion tensor becomes

$$\hat{L}^{\alpha}{}_{\mu\nu} = -B^{\alpha}\tilde{\nabla}_{(\mu}\tau_{\nu)} - \tau_{\mu}\tau_{\nu}h^{\alpha\gamma}\partial_{\gamma}\phi. \tag{60}$$

Furthermore, this choice implies that $\hat{L}^{\alpha}{}_{\alpha\beta} = \hat{L}^{\alpha}{}_{\nu\alpha} = B^{\alpha}\hat{Q}_{(\alpha\beta)} + \tau_{\alpha}\tau_{\beta}h^{\alpha\gamma}\partial_{\gamma}\phi = 0$. The second term vanishes by the orthogonality of the spatial and temporal metrics, while the first term vanishes as setting $\hat{Q}_{\mu}{}^{\nu} = 0$ implies that $B^{\alpha}\hat{Q}_{\alpha\mu} = 0$ because $\tilde{\nabla}_{\mu}\left(B^{\alpha}\tau_{\alpha}\right) = \hat{Q}_{\mu}{}^{\alpha}\tau_{\alpha} + B^{\alpha}\hat{Q}_{\mu\alpha} = 0$. Consequently, the field equation simplifies to:

$$-\tilde{\nabla}_{\alpha}\hat{L}^{\alpha}{}_{\mu\nu} - \hat{L}^{\alpha}{}_{\mu\beta}\hat{L}^{\beta}{}_{\alpha\nu} = 4\pi G\rho\,\tau_{\mu}\tau_{\nu}, \tag{61}$$

Carrying through the calculations, we have that

$$\begin{aligned}
-\tilde{\nabla}_{\alpha}\hat{L}^{\alpha}{}_{\mu\nu} &= 2\hat{Q}_{\alpha\mu}\tau_{\nu}h^{\alpha\gamma}\tilde{\nabla}_{\gamma}\phi + \tau_{\mu}\tau_{\nu}\tilde{\nabla}_{\alpha}\left(h^{\alpha\gamma}\tilde{\nabla}_{\gamma}\phi\right), \\
-\hat{L}^{\alpha}{}_{\mu\beta}\hat{L}^{\beta}{}_{\alpha\nu} &= -2\hat{Q}_{\alpha\mu}\tau_{\nu}h^{\alpha\gamma}\tilde{\nabla}_{\gamma}\phi.
\end{aligned} \tag{62}$$

The field equation of STENC then takes the form

$$\tau_{\mu}\tau_{\nu}\tilde{\nabla}_{\alpha}\left(h^{\alpha\gamma}\tilde{\nabla}_{\gamma}\phi\right) = 4\pi G\rho\tau_{\mu}\tau_{\nu}, \tag{63}$$

which is the Poisson equation expressed in terms of a curvature-free, torsion-free connection that also possesses non-metricity. For this version of Newtonian gravity, the compatibility conditions of the metrics are given by:

$$\tilde{\nabla}_{\alpha}h^{\mu\nu} = 2B^{(\mu}\hat{Q}_{\alpha\gamma}h^{\nu)\gamma} \quad ; \quad \tilde{\nabla}_{\mu}\tau_{\nu} = \hat{Q}_{\mu\nu}. \tag{64}$$

A similar equation was obtained in [7, Eq. 31] and [9, Eq. 4.20] in the torsional case. These papers fixed both the expansion and spatial torsion to vanish, and obtained the 'standard' formulation of Newtonian gravity in terms of a torsional connection rather that the non-metric connection here. See e.g. [54] for further discussion of these versions of Newtonian gravity that result from such geometric considerations.

In the second choice, the distorsion tensor takes the form

$$\hat{L}^{\alpha}{}_{\mu\nu} = -\tau_{\mu}\tau_{\nu}h^{\alpha\gamma}\partial_{\gamma}\phi, \tag{65}$$

leading the the field equation

$$\tau_{\mu}\tau_{\nu}h^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\phi = 4\pi G\rho\tau_{\mu}\tau_{\nu}, \tag{66}$$

which is formally equivalent to Eq. (63).

At this stage, in either ansatz, one gauge freedom remains in the definition of B^μ. A natural gauge fixing is to assume $B^\mu = G^\mu$, i.e., the gauge dependent 4-velocity refers to an inertial observer, which is a gauge choice that can always be taken [33, 42]. This implies that the gauge dependent scalar ϕ becomes the gravitational potential Φ present in the gravitational field $g^\mu = h^{\mu\nu}\partial_\nu \Phi$ that we defined in Sec. 4.1. Then, in both of the above ansatzes, the gravitational field is directly encoded in the distorsion tensor.

There is nevertheless a major conceptual difference between the two approaches, as with the second one, the connection $\tilde{\nabla}$ is a Galilean connection, i.e. it is compatible with the space and time metrics. In this sense, arguably we lose to some degree the philosophy of the STENC formulation, where the goal was to have two symmetric connections: one Galilean and one non-Galilean.

5 Conclusion

In this article, we have taken the non-relativistic limit of the symmetric teleparallel equivalent formulation of general relativity (STEGR), and have obtained the symmetric teleparallel equivalent formulation of Newton–Cartan theory (STENC). Just as in the relativistic case, the gravitational degrees of freedom in STENC are in the non-metricity; thereby, we have triangulated a non-metric alternative theory to Newton–Cartan theory (NC) and its equivalent teleparallel version (TENC). We have therefore completed a non-relativistic geometric trinity for gravity—this also makes good on a question raised in [9] as to what one would obtain on taking the non-relativistic limit of STEGR.

In completing this work, we cast new light and understanding upon the relationship between the non-relativistic limit and geometrical reformulations of spacetime theories, as well as come to understand better the geography of the 'space of spacetime theories' more generally (cf. [55, 56]). Moreover, the existence of the non-relativistic geometric trinity need not be a mere theoretical or philosophical curiosity: it is already known in the case of the relativistic geometric trinity that different nodes of the trinity are more or less apt to represent different physical scenarios (e.g., black hole boundaries—see [18]); in principle, we expect the same to be true in the non-relativistic case, although we will leave such explorations for future work.

There are many future prospects to the present work. To name three:

(i) Recently, in [44], a novel version of NC (so-called 'Type II NC') has been constructed by taking a more careful and systematic $1/c$ expansion of the GR dynamics. Type II NC has revealed several novel features of non-relativistic gravitational theories, including that non-relativistic theories can account for many of the strong gravitational effects previously believed to belong to relativistic theories and can also reproduce much of the solution space of GR [45, 57]. This raises the question: what would be the 'Type II' equivalents of TENC and STENC? Indeed, finding such theories would have conceptual payoff, for in this article we have demonstrated equivalence of the three vertices of the non-relativistic geometric trinity only at the level of equations of motion, whereas the equivalence of the relativistic geometric trinity can—as we have seen—be demonstrated at the level of the action. However, action principles for the 'Type I' theories provably do not exist [45]; not so for 'Type II' theories (and, indeed, an action for Type II NC is explicitly constructed in [45]); therefore, to construct a non-relativistic trinity using action principles (as in the relativistic case), one would have to construct and work with the 'Type II' theories.

(ii) It has very recently been shown that there exists an 'extended' geometric trinity between $f(R)$, $f(T, B)$ and $f(Q, \tilde{B})$ theories (for boundary terms B and $\tilde{B}$) [15]—does a similar extension of the non-relativistic geometric trinity exist? (For further discussion on this issue in the relativistic case, see [58, 59].).

(iii) It is possible to understand different nodes of the relativistic geometric trinity as different gauge theories of gravity: GR can be understood as a gauge theory of the Lorentz transformations; TEGR as a gauge theory of the translations; and STEGR as a gauge theory of shear/scale (and—as mentioned above—also the translations) [60]. This invites the question: can the nodes of the non-relativistic trinity also be understood as gauge theories of gravity in a similar manner? We are optimistic about the prospects for an affirmative answer here but, again, we will leave a full study of this question for future work.

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