Non-relativistic twistor theory: Newtonian limits and gravitational collapse

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Abstract

Recently, Dunajski and Gundry (2016) have developed an extension of twistor theory to the non-relativistic domain. Unlike relativistic twistor theory, their approach is able to reproduce the entire space of models of Newton-Cartan theory. I critically assess the significance of non-relativistic twistors, in particular with respect to proposals by Dunajski and Penrose (2023) that using non-relativistic twistors to describe gravitationally induced collapse could play a part in solving the quantum measurement problem.

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1 Introduction

Twistor theory, originally developed by Roger Penrose (1967), has been pursued for many decades now both as a reformulation of general relativity (GR), and for its applications outside of that field e.g. in Yang-Mills theories and string theory (see Atiyah et al. (2017) for a recent review article on the subject). The twistor programme has precipitated a number of technical advances, for example, in the classification of manifolds with exotic holonomy groups (see e.g. Merkulov and Schwachhofer (1999) and Merkulov (1995)), and in the computation of scattering amplitudes in particle physics (see e.g. Adamo et al. (2011), Cachazo et al. (2015), and Mason and Skinner (2014)). Despite this by now well-established history, twistor theory remains relatively underexplored in the philosophy literature (Bain (2006), Gajic et al. (2023) are notable exceptions).

Meanwhile, recent years have seen a resurgence of interest in Newton-Cartan theory (NCT) from both physicists and philosophers. On the physics side, NCT has found *inter alia* applications to condensed matter phenomena (especially the fractional quantum Hall effect, see Geracie et al. (2016), Son (2013), and Wolf, Read, and Teh (2022)), Hořava-Lifshitz gravity (Hartong and Obers 2015), non-relativistic holography (Christensen et al. 2014), and non-relativistic string theory (Harmark et al. 2017); moreover, work on NCT has motivated exploring other non-relativistic theories of gravity such as the off-shell ('type-II NCT') non-relativistic limit presented in Hansen et al. (2019a,b, 2020) which are able to reproduce many of the strong-field gravitational effects previously held to be the purview of relativistic physics. On the philosophy side, NCT has been explored in connection with the question of the 'correct' spacetime setting for Newtonian gravitation (see e.g. Dewar (2018), Knox (2014), Saunders (2013), and Weatherall (2016)), and most recently has been understood as just one node of a non-relativistic geometric trinity of gravity (March et al. 2023; Read and Teh 2018; Wolf and Read 2023), in which gravitational effects can be equivalently understood as a manifestation of either curvature, torsion, or non-metricity.

In a recent paper, Dunajski and Gundry (2016) have brought these two physics traditions into contact with one another by developing a non-relativistic twistor theory. Even more recently, Dunajski and Penrose (2023) have proposed to use non-relativistic twistor theory in the description of gravitationally-induced 'collapse' of the quantum state, thereby making contact with another well-established philosophy literature—namely, on the quantum measurement problem. As with previous work on twistor theory (and given the recentness of the Dunajski-Penrose proposal) , these developments have thus far attracted little-to-no attention within the philosophical literature.

The aim of this paper is to discuss these proposals, and thereby contribute somewhat to filling this gap. Accordingly, the paper will be structured as follows. In §2, we briefly review the details of non-relativistic twistor theory, before in §3 clarifying some of the conceptual background specific to non-relativistic twistor theory. We then turn, in §4, to Dunajski and Penrose's recent arguments about non-relativistic twistors and gravitational collapse, and discuss some of the issues facing their proposal. We close in §5.

2 Newton-Cartan twistor theory

In this section, we present the basic mathematics of NCT, as well as a brief introduction to twistors and the non-relativistic twistor theory of Dunajski and Gundry (2016). Further details on relativistic twistor theory can be found in Huggett and Tod (1994), Penrose and Rindler (1984, 1986), and Ward and Wells (1990) as well as the review article Atiyah et al. (2017); see also Bain (2006) and Gajic et al. (2023) for a philosophically-oriented presentation. Appendix A contains some details on the construction of line bundles; see also appendix B for details on Kodaira deformation theory and cohomology, and we refer the reader to these where appropriate.

Beginning with Newton-Cartan theory (NCT), this theory has kinematical possibilities of the form $\langle M, t_a, h^{ab}, \nabla, \Phi\rangle$, where M is a differentiable four-manifold,[1] t_a and h^{ab} are orthogonal temporal and spatial metrics i.e. such that $t_a h^{ab} = 0$, Φ represents material fields, and ∇ is a torsion-free and compatible (in the sense that $\nabla_a t_b = \nabla_a h^{bc} = 0$) derivative operator on M. Note in particular that the metrics t_a and h^{ab} are degenerate: see e.g. Malament (2012, §4.1). The dynamical possibilities for NCT are encoded in the geometrised Poisson equation:

$$R_{ab} = 4\pi\rho t_a t_b, \tag{1}$$

where R_{ab} is the Ricci curvature of the NCT connection ∇, along with the curvature conditions:[2]

$$R^{a\ c}_{\ b\ d} = R^{c\ a}_{\ d\ b}, \tag{2}$$

$$R^{ab}_{\ \ cd} = 0. \tag{3}$$

Turning now to twistor theory, whilst there are several routes to defining twistors (see, e.g. Penrose and Rindler (1986) and Ward and Wells (1990)), here we will discuss just two. The basic idea behind twistor theory is that conformally invariant dynamics on some spacetime M can be equivalently represented via putatively geometrical statements about some (complex) twistor space. With this in mind, let M be a (pseudo)-Riemannian four-manifold. Locally, there exists a canonical bundle isomorphism (the Klein correspondence, see e.g. Ward and Wells (1990)) $\mathbb{C} \otimes TM \cong \mathbb{S} \otimes \mathbb{S}'$, where $\mathbb{S}$, $\mathbb{S}'$ are complex rank-2 vector bundles over M with symplectic structures ε, ε'. Sections of $\mathbb{S}$, $\mathbb{S}'$ are spinors λ^A, $\mu^{A'}$. If $x^{AA'}$ is the coordinate of a point in M, then points in the complexified spacetime $M_{\mathbb{C}}$[3] can be identified with pairs of spinors under this isomorphism via the 'incidence relation'

$$\lambda^A = ix^{AA'}\mu'_A. \tag{4}$$

This takes us immediately to our two definitions of twistors. For the first, we can fix a pair of spinors $(\lambda^A, \mu^{A'})$ and ask: what is the locus of points $x^{AA'}$

[1] Assumed connected, Hausdorff, and paracompact.

[2] See e.g. Malament (2012, §4.3) for further discussion of these conditions.

[3] That is, a (four complex dimensional) spacetime whose tangent space $TM_{\mathbb{C}}$ is locally isomorphic to the product bundle $\mathbb{C} \otimes TM$.

(in some such region $O_{\mathbb{C}}$ of $M_{\mathbb{C}}$) satisfying the incidence relation? This locus is a totally null two-plane with self-dual tangent bivector, known as an α-plane. Secondly, given a pair of spinors $(\lambda^A, \mu^{A'})$ satisfying the incidence relation for some $x^{AA'}$, one can show that these spinors arise as the solutions of the twistor equation[4]

$$\nabla_{A'}{}^{(A}\omega^{B)} = 0. \tag{5}$$

Since the incidence relation is invariant under rescalings of the pair $(\lambda^A, \mu^{A'})$, this motivates the definition of projective twistor space: we take it to be the space $PT = \mathbb{CP}^3 - \mathbb{CP}^1$ of pairs $(\lambda^A, \mu^{A'})$, $\mu^{A'} \neq 0$ satisfying the incidence relation coordinatised by homogeneous coordinates $\lambda^1 : \lambda^2 : \mu^1 : \mu^2$.[5] Equivalently, we can take PT to be the space of α-planes in $O_{\mathbb{C}}$—thus, the projective twistor space encodes the conformal structure of O.

Thus far, our construction has been entirely local. If we wish to tackle the problem of representing the entire spacetime M twistorially, we now have a distribution of α-planes on $M_{\mathbb{C}}$. When this distribution is integrable, we can proceed as before, now defining PT as the three-parameter space of integral two-surfaces of this distribution (known as α-surfaces). The crucial question is then: under what conditions on the original spacetime M is the distribution of α-planes integrable? In relativistic twistor theory, one central result in this direction is the Penrose (1976) non-linear graviton theorem, which establishes an equivalence between self-dual (SD) vacuum solutions of the Einstein field equations,[6] and twistor spaces PT with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of $PT \to \mathbb{CP}^1$.[7]

This forms the basis for Dunajski and Gundry's (2016) construction of the non-relativistic twistor correspondence. Dunajski and Gundry show, by considering the patching relation on the relativistic twistor space (on the preimage of $U_\lambda \cap U_\mu$ in PT for patches U_λ, U_μ on $\mathbb{CP}^1$), that in the non-relativistic limit the normal bundle of rational curves in the twistor space 'jumps' from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ $\mathcal{O} \oplus \mathcal{O}(2)$. Their central result is then as follows:

Proposition 2.1 (Dunajski and Gundry (2016))**.** *There is a one-to-one correspondence between line bundles over $\mathcal{O} \oplus \mathcal{O}(2)$ which are trivial on real twistor lines and vacuum Newton-Cartan connections on M.*

Dunajski and Gundry then show that arbitrary Newton-Cartan connections can be constructed from non-trivial rank-two vector bundles over PT which restrict to $\mathcal{O} \oplus \mathcal{O}(2)$ on twistor lines.

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[4] See, e.g. Penrose and Rindler (1986, ch. 6) for details.

[5] For further discussion on this point, see, e.g. Gajic et al. (2023) and Penrose (1967).

[6] Self-duality is a property of the Weyl tensor on the (complexified) spacetime: $C^a{}_{bcd}$ is SD iff $C^a{}_{bcd} = \star C^a{}_{bcd}$, where $\star C^a{}_{bcd} := 1/2\varepsilon^{nm}{}_{cd} C^a{}_{bnm}$ is the Hodge dual of $C^a{}_{bcd}$.

[7] Here, $\mathcal{O}(n)$ is the holomorphic line bundle over $\mathbb{CP}^1$ with (first) Chern class n, see appendix A. The normal bundle is defined as follows: if $i : N \to M$ is an immersion, then the normal bundle of N in M is the bundle $T_{M/N} \to N$, where $T_{M/N} := i^*TM/TN$ is the quotient of TM by TN.

3 Relativistic vs. non-relativistic twistors

3.1 The non-relativistic twistor correspondence

One striking difference between relativistic and non-relativistic twistor theory is the absence of restrictions in proposition 2.1 on which Newton-Cartan spacetimes may be represented twistorially. By contrast, in the relativistic case there are a number of *partial* results linking twistor space structures with (classes of) relativistic spacetimes (the non-linear graviton theorem being one such example) but as yet no recipe for constructing twistor spaces from generic relativistic spacetimes. One crucial question about the non-relativistic twistor correspondence is therefore as follows: why does this work in the non-relativistic, but not the relativistic case?

Dunajski and Gundry (2016) do not discuss this issue. However, it turns out that the key to understanding this is to recall that what gets encoded in the holomorphic structure of the twistor space is the conformal structure of the corresponding (non-)relativistic spacetime. For this, we need to say something about the relevant notion of conformal structure under consideration. For spacetimes with a non-degenerate (pseudo-)Riemannian metric g_{ab}, conformal structure is standardly defined as follows: it is an equivalence class $[g_{ab}]$ of conformally equivalent metrics, where g_{ab}, g'_{ab} are conformally equivalent just in case $g_{ab} = \Omega^2 g'_{ab}$ for some Ω. In non-relativistic spacetimes, however, the metrics are degenerate, which motivates an alternative definition of conformal structure apposite for non-relativistic spacetime theories. Within the literature on this topic, two such definitions have been proposed. The first is due to Ewen and Schmidt (1989):

Conformal structure (ES): Let $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla \rangle$ be a non-relativistic spacetime. The Ewen-Schmidt conformal structure $\mathcal{C}_{ES}$ of $\mathfrak{M}$ is the set $\{\sigma\}$ of unparametrised spacelike geodesics of ∇.

The second definition is due to Curiel (2015):[8]

Conformal structure (C): Let M be a differentiable four-manifold. A Curiel conformal structure $\mathcal{C}_C$ on M is an integrable rank-3 distribution on M with leaves S, and an equivalence class of conformally equivalent spatial metrics $[h^{ab}]$ on M such that the following both hold: (a) there exists a one-form t_a on M, orthogonal to all the $h^{ab} \in [h^{ab}]$ such that for all points p, all S, and all tangent vectors σ^a to S at p, $t_n \sigma^n = 0$, and (b) at least one representative of $[h^{ab}]$ is flat.

It follows that if h^{ab} is conformally equivalent to some flat spatial metric, then any non-relativistic spacetime $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla \rangle$ determines a unique $\mathcal{C}_C$. Note, however, that condition (b) means that there is no well-defined notion

[8] Here I am departing somewhat from Curiel's presentation, which defines separately notions of spatial and temporal conformal structure for a non-relativistic spacetime $\langle M, t_a, h^{ab}, \nabla \rangle$ and then demands compatibility of the two. This is harmless, since in spatially flat spacetimes (which is the case Curiel considers, see below for further discussion) the two are equivalent.

of Curiel conformal structure for spacetimes where h^{ab} does not satisfy this condition.[9]

Let us compare these two definitions. Given the Ewen-Schmidt conformal structure of a non-relativistic spacetime $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla\rangle$, we can define a foliation of M into spacelike hypersurfaces (since, up to conformal factors, there will be a unique one-form which annihilates all tangent vectors to all the $\sigma \in \mathcal{C}_{ES}$). But $\mathcal{C}_{ES}$ gives us more than this: we can also recover a notion of spatial *projective structure*—i.e. the class of spatial derivative operators D induced on each spacelike hypersurface for which the $\sigma \in \mathcal{C}_{ES}$ can all be reparameterised as geodesics.

By way of contrast, the Curiel conformal structure $\mathcal{C}_C$ of $\mathfrak{M}$ also gives us a foliation of M into spacelike hypersurfaces; unlike $\mathcal{C}_{ES}$, however, $\mathcal{C}_C$ does not fix the projective structure of the leaves of the foliation, instead, it determines a conformal structure on each leaf. So whilst $\mathcal{C}_{ES}$ and $\mathcal{C}_C$ are not orthogonal, neither is strictly weaker than the other. Moreover, the fact that $\mathcal{C}_{ES}$ partially determines the projective geometry of $\mathfrak{M}$ suggests that it is $\mathcal{C}_C$, rather than $\mathcal{C}_{ES}$, which is the closer analogue of relativistic conformal structure for the non-relativistic context.

There is one feature of Curiel's definition which we have not yet dealt with. As it stands, $\mathcal{C}_C$ cannot be applied to spacetimes where h^{ab} is not conformally equivalent to some flat spatial metric. Whilst this condition holds trivially on-shell in Newton-Cartan theory—if $R^{ab}{}_{cd} = 0$ then h^{ab} is flat (Malament 2012, proposition 4.2.4)—we will see later that assuming conformal flatness of the spatial leaves from the outset obscures the special role which the on-shell Newton-Cartan spatial geometry has to play in explaining why it is that a full twistor-correspondence exists in the non-relativistic case. Given this, I will adopt the following generalisation of Curiel's definition, which drops the condition (b):

Conformal structure: Let M be a differentiable four-manifold. A non-relativistic conformal structure $\mathcal{C}$ on M is an integrable rank-3 distribution on M with leaves S, and an equivalence class of conformally equivalent spatial metrics $[h^{ab}]$ on M such that there exists a one-form t_a on M with the following properties: t_a is orthogonal to all the $h^{ab} \in [h^{ab}]$, and for all points p, all S, and all tangent vectors σ^a to S at p, $t_n \sigma^n = 0$,

I will say that a Newtonian spacetime is spatially conformally flat iff at least one representative of $[h^{ab}]$ is flat. It follows that a Curiel conformal structure on M is a spatially conformally flat non-relativistic conformal structure on M.

To see how the non-relativistic conformal structure can be encoded in the twistor space structure, it is helpful to begin by considering the complexified spacetime $M_{\mathbb{C}}$. We can define (via holomorphic extension of t_a) a canonical closed one-form T_a on $M_{\mathbb{C}}$, which induces a fibration $\pi : M_{\mathbb{C}} \to \mathbb{C}$. We can also define (via holomorphic extension of h^{ab}) a spatial metric H^{ab} on $M_{\mathbb{C}}$, which

[9] As far as Curiel's motivation for introducing condition (b) goes, it seems to be just that (b) holds on-shell in NCT.

induces a holomorphic Riemannian metric on the fibres $\pi^{-1}(T)$. Now say that a vector $\sigma^a \in T_p M_{\mathbb{C}}$ at a point $p \in M_{\mathbb{C}}$ is *totally null* iff σ^a is null with respect to both T_a and H^{ab}. Thus totally null vectors lie in the fibres $\pi^{-1}(T)$, and are null with respect to the induced metric on those fibres.[10] It follows that the conformal structure $\mathcal{C}$ of $\langle M, t_a, h^{ab}, \nabla \rangle$ can be completely characterised by the totally null vector sub-bundle of $TM_{\mathbb{C}}$.[11]

Note that this gives us something close to Ewen and Schmidt's definition, but crucially, M must first be complexified for this to work; specification of the spacelike vectors (or even the spacelike geodesics) at each point does not carry enough information to fix orthogonality relations between pairs of spacelike vectors in the foliation. It also gives us something close to the relativistic case, where the conformal structure of a Lorentzian manifold $\langle M, g_{ab} \rangle$ can equivalently be defined as the null vector sub-bundle of TM.[12] So whilst it is less straightforward to define conformal structure for *real* non-relativistic spacetimes, there is a natural analogue of relativistic conformal structure available for *complex* non-relativistic spacetimes.

This suggests an alternative, 'horizontal' path to constructing the non-relativistic twistor space. Recall that in the relativistic case, the twistor space could be defined as the space of α-surfaces in $M_{\mathbb{C}}$—2-surfaces all of whose tangent planes are null with respect to G_{ab} and whose tangent bivectors are self-dual. So rather than taking the $c \to \infty$ limit of the relativistic twistor space, we might instead try to define the non-relativistic twistor space, in analogy to the relativistic case, as the space of totally null 2-surfaces with self-dual tangent bivector in $M_{\mathbb{C}}$—where the relevant notion of totally null is now as defined above.[13] The relationship between these two approaches is summarised in Figure 1. Constructing the bottom edge of Figure 1 will be what is needed to illuminate the fact that there exists a full non-relativistic, but not a full relativistic twistor correspondence.

For this, note first that since non-relativistic α-surfaces are null with respect to the closed one-form T_a, they can be parameterised by the (complex) time coordinate T. Moreover, since the conformal geometry is the same on each of the spatial leaves (since H^{ab} is flat),[14] this furnishes the space of non-relativistic

[10] Recall that Riemannian metrics on a complex manifold admit non-vanishing null vectors.

[11] Why? Because the conformal structure associated to a holomorphic Riemannian metric is completely characterised by a specification of its (possibly complex) null vectors at each point; moreover, up to conformal factors there will be a unique one-form which annihilates all totally null vectors on $M_{\mathbb{C}}$. The latter fixes the fibration $\pi : M_{\mathbb{C}} \to \mathbb{C}$, the former fixes the conformal geometry on each of the fibres $\pi^{-1}(T)$.

[12] See e.g. Linnemann and Read (2021) for discussion of the relationship between this approach and the definition of conformal structure as an equivalence class of conformally equivalent metrics.

[13] In fact, one might prefer this approach over that of Dunajski and Gundry (2016) since the equations involved remain finite throughout, whereas the relativistic incidence relation blows up in the non-relativistic limit. At the very least, the horizontal construction should provide reassurance that this is not a pathological feature of Dunajski and Gundry's approach.

[14] This is connected with the fact that the dynamics of NCT are time-translation invariant. In particular, for $M \cong \mathbb{R}^4$ we can equivalently obtain the spatial leaves S of $M_{\mathbb{C}}$ by quotienting the space $M_{\mathbb{C}}$ under the action of $\mathbb{C}$.

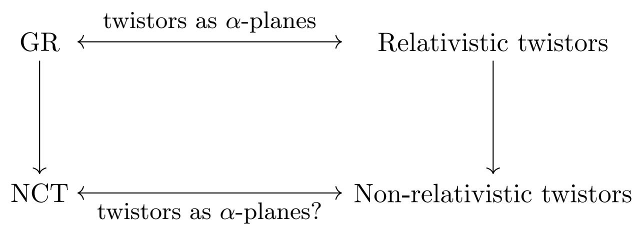

Figure 1: 'Horizontal' vs. 'vertical' approaches to the non-relativistic twistor correspondence. Vertical arrows represent the non-relativistic limit.

α-surfaces with a Cartesian product structure: we take it to be $\mathbb{C} \times \mathcal{N}_S$, where $\mathcal{N}_S$ is the space of null two-surfaces associated to the spatial leaves S of $M_{\mathbb{C}}$ (which, recall, are complex Riemannian three-spaces).

Understanding the space of null two-surfaces associated to complex Riemannian three-spaces is therefore our remaining task. Under what conditions does there exist a (two-parameter) family of null two-surfaces in such a space? This problem has been studied extensively in the twistor literature (originally in relation to the twistorial description of non-Abelian monopoles, see e.g. Hitchin (1982b, 1983)); the central result here is due to Hitchin (1982a):

Proposition 3.1 (Hitchin 1982)**.** *Let $\langle M, g_{ab} \rangle$ be a Riemannian three-manifold. Then there exists an integrable distribution of null two-planes on $M_{\mathbb{C}}$ with respect to the holomorphic extension of g_{ab} iff $R_{ab} = \Lambda g_{ab}$ for some constant Λ. The two-parameter space of integral surfaces of this distibution is a family of rational curves with normal bundle $\mathcal{O}(2)$.*

Such a distribution of null two-planes is called a *mini-twistor distribution*, and the associated space of integral surfaces of the distribution a *mini-twistor space*.

This completes our inventory of tools needed to construct the bottom edge of Figure 1. On-shell in Newton-Cartan theory, $R^{ab} = 0$. It follows from the foregoing discussion and proposition 3.1 that the Newtonian twistor space has normal bundle $\mathbb{C} \times \mathcal{O}(2) = \mathcal{O} \oplus \mathcal{O}(2)$. This brings out the reason why a full twistor correspondence exists in the non-relativistic case: it is a consequence of the fact that (a) the non-relativistic totally null vectors lie in the spatial leaves of $M_{\mathbb{C}}$, and (b) the on-shell Newton-Cartan geometry is spatially conformally flat. This guarantees that the distribution of non-relativistic α-planes in $M_{\mathbb{C}}$ is integrable, regardless of which particular model of Newton-Cartan theory we are interested in. By way of contrast, the conditions coming from Fröbenius' theorem for the distribution of null two-planes of a Lorentzian metric to be integrable are not in general satisfied on-shell in GR. One can understand the non-linear graviton theorem of Penrose (1976) as saying that precisely in the case of SD vacuum spacetimes, these integrability conditions *are* in fact satisfied.

This is in agreement with the analysis of Dunajski and Gundry (2016), who show, by considering the behaviour of the twistor equation in the $c \to \infty$ limit, that the relativistic α-surfaces become spacelike and are the integral surfaces of the mini-twistor distribution on the fibres of $M_{\mathbb{C}} \to \mathbb{C}$. However, our current path has given us a much deeper understanding of this result: it is precisely this distribution of α-planes which encodes the non-relativistic conformal structure. The fact that this conformal structure is invariant on-shell, and moreover is spatially conformally flat, is what explains the existence of a full non-relativistic twistor correspondence.

3.2 Kodaira instability and jumping lines

The other striking difference between the non-relativistic and relativistic twistor correspondence is that, unlike in the relativistic case, the non-relativistic twistor space is unstable under general Kodaira deformations (Dunajski and Gundry 2016). Whilst the details of Kodaira deformation theory are somewhat technically involved (and are contained in appendix B), the basic idea is that we can deform the complex structure of the twistor space by replacing the patching relations F (defined on the preimage of $U_\lambda \cap U_\mu$ for patches U_λ, U_μ on $\mathbb{CP}^1$) with parameterised patching relations $F(b_1, ..., b_i)$ with the parameters $(b_1, ..., b_i)$ taking values in some base space B (possibly a complex manifold).

Dunajski and Gundry (2016) consider a particular class of Kodaira deformations which deform the patching relations in the T-direction which parametrises the $\mathcal{O}$ factor of the non-relativistic twistor space. Since for a trivial line bundle with total space $\mathbb{C} \times \mathcal{O}(2) \to \mathcal{O}(2)$ over $\mathbb{CP}^1$ we can always take $T = \tilde{T}$, this amounts to replacing the non-relativistic twistor space $\mathbb{C} \times \mathcal{O}(2)$ with a non-trivial line bundle $L \to \mathcal{O}(2)$. Under such deformations, the normal bundle of rational curves in twistor space 'jumps' discontinuously from $\mathcal{O} \oplus \mathcal{O}(2)$ to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (Dunajski and Gundry 2016). This corresponds to a singularity in the conformal structure of the associated spacetime (Jones and Tod 1985); the deformed metric structure is non-degenerate and corresponds to an SD vacuum solution of GR (Hitchin 1982a; Penrose 1976).

Our previous discussion of Newtonian conformal structure (§3.1) gives us the resources to understand this result in greater detail. As noted above, the Kodaira deformations under which the normal bundle of the non-relativistic twistor space is unstable can be realised by replacing the space $\mathbb{C} \times \mathcal{O}(2)$ with a non-trivial line bundle $L \to \mathcal{O}(2)$. This construction manifestly does not preserve the Cartesian product structure $\mathbb{C} \times \mathcal{N}_S$ of the normal bundle of rational curves in L when $L \to \mathcal{O}(2)$ is non-trivial. However, we saw in §3.1 that it is precisely this Cartesian product structure which encodes the fibration $M \to \mathbb{R}$ induced by t_a. It is for this reason that Kodaira deformations in the T-direction of the non-relativistic twistor space do not preserve the non-relativistic conformal structure of the associated spacetime, resulting in the 'jump' from degenerate Riemannian to Lorentzian geometry.

This means that the interpretation of general Kodaira deformations of the non-relativistic and relativistic twistor spaces are importantly different. First, consider the case of relativistic vacuum SD spacetimes. Under the twistor correspondence, points in such spacetimes map to rational curves in twistor space with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. It follows from a theorem due to Kodaira (1963) (see appendix B) that under 'small' Kodaira deformations, the twistor space will still admit a four-parameter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This allows us to interpret the Kodaira deformation as inducing a map between different relativistic SD spacetimes, which in turn can be interpreted as a change in the curvature (with respect to the unique Levi-Civita connection induced by the metric) on the associated spacetime manifold.

But now consider what happens in the non-relativistic case. We have seen that general Kodaira deformations of the non-relativistic twistor space do not preserve the holomorphic structure of the normal bundle, which 'jumps' from $\mathcal{O} \oplus \mathcal{O}(2)$ to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. As discussed above, this means that the deformations induce a map from non-relativistic to relativistic SD spacetimes.

Now, I claim, this map cannot be interpreted as (just) a change in curvature. To see why, recall that curvature is strictly speaking a property of the connection. The same connection may be compatible with multiple different metric structures—for example, a flat connection on $M \cong \mathbb{R}^4$ is trivially compatible with Lorentzian, Riemannian, and degenerate non-relativistic metric structures. The basic point is that facts about metric signature do not supervene on facts about the Riemann curvature of a compatible connection. But we have just seen that metric signature is not invariant under general Kodaira deformations of the non-relativistic twistor space. This means that Kodaira deformations which do not preserve the patching relation for the T-direction on the non-relativistic twistor space cannot be interpreted as just changing the curvature, since the deformed spacetime manifestly depends on facts which are not specifiable from the undeformed metric structure and deformed Riemann curvature.

3.3 Motivating non-relativistic twistor theory

Our final point concerns the motivation for exploring non-relativistic twistor theory, particularly as a route to solving the quantum measurement problem. To be completely clear, we have no qualms *per se* with the idea that non-relativistic twistor theory should be a field of interest to either physicists or philosophers. On the one hand, if twistor theory does end up playing a role in some eventual theory of quantum gravity, and in light of the technical achievements of the relativistic twistor programme (see section 1), it might be of practical interest to explore similar constructions in the non-relativistic case. On the other hand, the existence of a non-relativistic twistor correspondence which parallels the relativistic one might be thought to have independent theoretical interest, insofar as this sheds light on and increases our understanding of the relationships between relativistic and non-relativistic theories of gravity.

Whilst at present speculative, these reasons for exploring non-relativistic twistor theory are interesting and legitimate. Here, however, we wish to focus on two further motivations for non-relativistic twistor theory which are put forward by Dunajski and Penrose (2023):

> [W]hatever it is that actually goes on physically when the wave-

> function collapses, i.e. the reduction of the quantum state, or R-process—this being taken to be an objective physical process—must have a curious 'retro-active' aspect to it if taken to be a 'classically real' physical process [...]. Such ontological puzzles do not present difficulties in the Newtonian limit, so it makes sense, at our present level of understandings (sic) to concentrate on the Newtonian situation, where such issues of the precise timing of the R-process can be evaded.
>
> Quantum theory is inherently non–local, and this non–locality is likely to prevail in any modifications resulting from incorporating gravitational fields. Newtonian twistor theory is also non–local: space–time points correspond to extended and global objects (rational curves) in the twistor space. It is hoped [...] that combining quantum non–locality with Newtonian twistor non–locality can shed light on the role of gravity in quantum state reduction. (Dunajski and Penrose 2023, p. 1)

In the above passage, Dunajski and Penrose make the following two suggestions:

- It is helpful to focus on non-relativistic twistor theory in discussions of the quantum measurement problem because non-relativistic theories render unproblematic questions of 'when' collapse of the wavefunction occurs (by virtue of being non-relativistic).
- Twistor theory (whether relativistic or non-relativistic) is a promising route to understanding quantum gravity's role in the measurement problem because (a) quantum theory is non-local, and (b) twistor theory is non-local, insofar as it represents spacetime points by extended objects.

We are sceptical of both. On the first, consider the 'retro-active' aspect of 'collapse' to which Dunajski and Penrose refer. Although Dunajski and Penrose (2023) are not explicit about what this is supposed to involve, Penrose's earlier (Penrose 1988) writings on the topic suggest that they have in mind the disagreement between non-comoving observers about the time-ordering of 'collapse' for measurements made on spacelike separated subsystems of a joint system.[15] But note that this *only* presents a puzzle if collapse of the wavefunction is taken to be a 'real physical process'. This interpretation of the von Neumann-style collapse postulate is not compulsory—as witness e.g. Everettian approaches to quantum theory, in which the dynamical evolution of the wavefunction is exclusively unitary and 'collapse' is recovered as an approximation to the unitary evolution of the quantum state in the presence of environment-induced decoherence. From this perspective, Dunajski and Penrose's argument for moving to the non-relativistic case does not even get going.

[15]Compare e.g. "Perhaps most baffling is the non-local and seemingly relativity-conflicting behaviour in EPR-type (Clausen-Aspect) experiments. Spacelike separated measurements take place. There is a conflict between the apparent time-ordering of the 'reductions' due to these two measurements. *'When'* do these reductions *'actually'* take place?" (Penrose 1988, emphasis in original)

Even setting this aside there are worries. For one, focussing on theories with non-relativistic spacetime structure does not by itself obviate the need to address questions of when—if at all—'collapse' of the wavefunction occurs; it merely ensures that non-comoving observers will agree on the time-ordering of spacelike-separated collapse events, *if* we first fix a time for those events to take place. And whilst it is of course legitimate to attempt solutions to the measurement problem within the non-relativistic regime, at some point one has to reckon with the problem of extending these solutions to relativistic quantum theory.[16] The worry that focussing too much on the non-relativistic case obscures the difficulties with finding such an extension is especially pressing in the case of twistor theory, where a full relativistic twistor correspondence has yet to be worked out.

Turning now to bullet point two, whilst twistor theory may have a role to play in discussions of quantum gravity, the fact that spacetime points are represented by extended objects in twistor space (henceforth 'twistor non-locality') has nothing *per se* to do with quantum non-locality—i.e. the failure of factorisability of the joint probability distributions for measurements conducted on (possibly spacelike separated) subsystems of a joint system when conditionalised on the ontic state of the joint system.[17] To show that twistor non-locality and quantum non-locality can be related in some meaningful way would be a substantive achievement; terminology aside, there is no *a priori* connection between the two.

4 The Dunajski-Penrose proposal

This concludes our discussion of the conceptual background to non-relativistic twistor theory. With this in hand, we now turn to Dunajski and Penrose's discussion of non-relativistic twistor theory and gravitationally-induced collapse of the quantum wavefunction. Our aim here is to (a) reconstruct their proposal to use non-relativistic theory in the description of gravitational collapse, and thereby (b) assess and clarify some of the issues surrounding these proposals.

Accordingly, we begin by reviewing the central ideas of the gravitational collapse programme put forward by Penrose (1996, 2014). Penrose's argument—as I understand it—is as follows. Consider a massive particle in a superposition of two spatial locations in a spacetime which admits global timelike Killing fields.

[16] There is some subtlety here, since the recently-discovered 'type II' Newton-Cartan theory of Hansen et al. (2019a,b, 2020) is non-relativistic, yet is able to pass most current empirical 'tests' of GR (Wolf, Sanchioni, et al. 2023). Note, however, that it is the standard 'type I' Newton-Cartan theory with which Dunajski and Penrose (2023) are concerned; moreover, as is clear from §3.1, there is no guarantee of a non-relativistic twistor correspondence once one drops the condition of spatial conformal flatness, nor for (generically torsionful) type II NCT connections. For further discussion of the prospects for relativistic extensions of different solutions to the quantum measurement problem, see e.g. Myrvold (2022) and Wallace (2022).

[17] Here we are eliding to some extent the distinction between quantum non-locality and non-separability, and between different notions of quantum non-locality; for discussion of these differences see e.g. Cheng (2023).

In order to describe such a superposition in a theory where spacetime is dynamical (in the sense that spacetime structure varies across the dynamically possible models of the theory *a la* Curiel (2016)), we must first settle on a standard for identifying spacetime points across different terms in the superposition. Penrose's strategy here is to invoke two principles (see e.g. Adlam et al. (2022) for further discussion on this). First, the principle of diffeomorphism invariance is invoked to argue that the procedure for identifying points of different spacetimes must proceed on the basis of coordinate-independent features of the spacetimes in question. Secondly, the Einstein equivalence principle is invoked to argue that we can nevertheless use coordinates to make such an identification, since by using local inertial coordinates in (some region of) both spacetimes, we can (locally) identify the geodesics of the two. In general, however, no *global* identification of the geodesics of spacetimes in the superposition will be possible.

The next step is to recall that we are assuming that the spacetimes under consideration all admit global timelike Killing fields,[18] which define time-translation operators for each term in the superposition. But the fact that there is no global identification of the geodesics of these spacetimes blocks the construction of global time-translation operators for the superposed state. Penrose claims that this will make the superposition unstable, since without global time-translation operators, there will be an inherent energy uncertainty in the superposed state. Here, Penrose draws an analogy to particle decay: the energy uncertainty ΔE will result in a spontaneous collapse of the superposition with some finite lifetime given by $\hbar/\Delta E$. He also shows (Dunajski and Penrose 2023; Penrose 2014) that the phase difference between the superposed Newton-Cartan spacetimes will in general have a cubic time-dependence, which he claims means that without collapse, we will be unable to extrapolate this construction to quantum field theory (QFT), since the spacetimes will be associated with different QFT vacua.

Now, there are a number of criticisms one might have of this proposal. (For example, one might question Penrose's claim that the correct identification of spacetime points across superposed states involves identifying their geodesics; one might question whether the resulting ill-definedness of *global* time-translation operators is a problem, given that we can still construct well-defined time-translation operators for each spacetime in the superposition (as in Giacomini and Brukner (2022)); one might question the strength of Penrose's analogy between particle decay and gravitational collapse (see Gao (2013)); one might further question Penrose's view of cubic time-dependence in the phase difference leading to different QFT vacua as motivation for collapse *specifically*, rather than e.g. the move to a richer Hilbert space structure, or simply as a manifestation of the fact that we are working within a non-quantum-field-theoretic framework.) However, we will bracket these issues for the moment, since our focus here is specifically on the relationship between Penrose's proposal for gravitational collapse and non-relativistic twistor theory. On this, Dunajski and Penrose (2023) write:

[18] Note that this holds trivially in NCT.

> It is reasonable to assume that [the reduction of the quantum state] happens in between two times t_0, and t_1, and introduces enough curvature that the $\mathcal{O}\oplus\mathcal{O}(2)$ Newtonian twistor space is deformed. While this may correspond to a discontinuous jump in the space–time structure we propose that the twistor space survives the reduction [...], but the 4-parameter family of curves with $\mathcal{O}\oplus\mathcal{O}(2)$ normal bundle disappear, and needs to be replaced by a new family. (Dunajski and Penrose 2023, p. 12)

Their proposal is then as follows. Parameterise the $\mathcal{O}$ factor of the non-relativistic twistor space by T, and let T_0, T_1 correspond to t_0, t_1 respectively. At T_0, introduce a Kodaira deformation of the patching relation in the T-direction. The result is a relativistic twistor space with $\mathcal{O}(1)\oplus\mathcal{O}(1)$ normal bundle. At T_1, take the $c\to\infty$ limit of this twistor space as outlined in (Dunajski and Gundry 2016) to obtain a new non-relativistic twistor space, whose rational curves correspond to the new spacetime points. Thus, Dunajski and Penrose (2023, p. 13) claim, "although the space time seems to bifurcate and collapse in the R-process, the twistor space is one complex three-fold. The curves in the R-process change their holomorphic type."

This construction raises (at least) two immediate questions:

1. What are the reasons for thinking that the twistor space survives the quantum state reduction?
2. How should we interpret the Kodaira deformation and non-relativistic limit in the Dunajski-Penrose proposal; what, if anything, is their relation to spacetime superpositions; and how, if at all, do they correspond to the glosses given in terms of 'introducing spacetime curvature' and 'spacetime bifurcating and collapsing'?

On (1), Dunajski and Penrose do not make their reasoning explicit. However, the same question is addressed in detail in earlier work by Penrose (1988), who writes on quantum state reduction:

> One might have thought that any such violently discontinuous change in the state of the world—if it were a *real* effect—ought to be more noticeable as to *when* it actually takes place! Also, since the Schrödinger equation is such a nice smooth analytic thing, it seems odd that Nature should choose to execute such violently discontinuous jumps from time to time. [...]
>
> It is conceivable that a twistor-type viewpoint could provide some sort of resolution of this puzzle. Suppose that reduction is a gravitational effect [...] and that space-time is described twistorially. [...] Now suppose that, with some measurement, the twistor space becomes sufficiently 'curved' that the original family of holomorphic lines, representing spacetime points, peters out—and we must switch to a *new* family in order to keep going. Somehow the geometry of

space-time seems to jump—yet in the 'actuality' of twistor space there is *no* jump—just a (necessary) shift in viewpoint. (Penrose 1988, emphasis in original)

Whilst the details of the argument in this passage are admittedly sketchy, Penrose's underlying idea seems clear: the fact the twistor space survives the state reduction process is supposed to help make sense of the discontinuous changes in spacetime geometry associated with state reduction, since we are to take the twistor space as fundamental. To what extent does the Dunajski-Penrose proposal make good on this aim?

To begin with, it is not obvious that the Dunajski-Penrose proposal helps explain why we do not observe "when" state reduction takes place. If anything, it makes this fact even more mysterious: if the Dunajski-Penrose proposal is to be taken seriously, state reduction is associated with a discontinuous shift between non-relativistic and relativistic metric structure, which *prima facie* one would expect to have empirically detectable consequences. Pending a detailed analysis of why we should *not* in fact expect this shift to make itself manifest to experiment, the Dunajski-Penrose proposal does little to recommend itself on this front.

This takes us to our second point. *If* one has antecedent reasons for thinking that twistors are more fundamental than spacetime points, and *if* a detailed twistor analysis of why we should not expect to detect the discontinuous changes in spacetime geometry associated with state reduction can be given, then the Dunajski-Penrose proposal might make these shifts more palatable. But these are two very big 'ifs'! Moreover, given our present understanding of the twistor correspondence, there seem to be good reasons for *not* taking twistors to be more fundamental than spacetime points. For example, as Gajic et al. (2023) discuss, the fact that the twistor correspondence is bidirectional means that twistor theory fails at least one plausible criterion for fundamentality—namely non-derivability.[19]

Moving now to (2), we should note that the explicit form of the dynamics which might underwrite the Dunajski-Penrose proposal have yet to be fleshed out,[20] so the analysis here will only be partial. However, there are still several points worth making. Our first has to do with Dunakski and Penrose's description of what goes on in their model of the state-reduction process. On this, one can distinguish three separate issues:

- By construction, the Dunajski-Penrose proposal involves only a single non-relativistic twistor space before the state-reduction process, and only a single relativistic twistor space during the state-reduction process. It is

[19]This worry is particularly pressing in the relativistic case. Here, not only is there a bidirectional correspondence between twistor spaces and spacetimes in those sectors of GR where a twistor correspondence does exist; outside of those sectors, we have spacetimes but as yet no known associated twistor spaces.

[20]Though note in particular that these cannot be the Newton-Cartan dynamics, which leave the metric signature invariant; c.f. also discussions of signature change in the physics literature (see e.g. Ellis (1992), Ellis et al. (1992), Gibbons and Hartle (1990), and Kossowski and Kriele (1993)), initially inspired by the 'no-boundary' proposal of Hartle and Hawking (1983).

therefore unclear how spacetime could be said to "bifurcate" during the Dunajski-Penrose state-reduction process.[21]

- As argued in §3.2, the Kodaira deformation considered by Dunajski and Penrose cannot be interpreted as (just) a change in spacetime curvature. Therefore, on the Dunajski-Penrose proposal, it is not strictly speaking correct to say that the state-reduction process "introduces enough curvature that the [...] twistor space is deformed."

- On the Dunajski-Penrose proposal, the "collapse" of spacetime at the end of the state-reduction process is modelled by taking the non-relativistic limit of the relativistic twistor space. It is not clear in what sense this limit corresponds to "collapse", unless "collapse" just means a discontinuous change in conformal structure.

The next feature of the Dunajski-Penrose proposal we will consider is the non-relativistic limit which is taken at the end of the state-reduction process. Dunajski and Penrose do not say anything to motivate physically the taking of this limit, which can be understood as essentially a formal mathematical device for recovering a non-relativistic twistor space from the (deformed) relativistic one.[22] However, this raises an obvious worry. The Dunajski-Penrose proposal makes essential use of the fact that the non-relativistic twistor space is unstable under a particular class of Kodaira deformations. We have already seen that this is not the case for relativistic twistor spaces. Some outline of how the Dunajski-Penrose proposal is supposed to generalise to the relativistic case is therefore needed if the proposal is to remain viable as a description of gravitationally induced collapse outside of the non-relativistic regime.

Our final point has to do with the relationship between the Dunajski-Penrose proposal and the quantum measurement problem. In particular, there is nothing in the Dunajski-Penrose proposal which has to do with quantum theory or spacetime superpositions *per se*. Rather, what the proposal gives is a recipe for constructing non-relativistic twistor spaces with 'collapse events' from ones without such events. Whilst the motivation behind this may be quantum-mechanical, the construction itself is not. It is therefore not clear that the Dunajski-Penrose proposal substantially illuminates the connections between gravity, quantum mechanics, and 'collapse' of the wavefunction.

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[21] Dunajski and Penrose are not explicit about what they mean by 'bifurcation' here, though their terminology suggests that they might have in mind something like non-Hausdorff spacetime structure. Whilst non-Hausdorff spacetimes have been explored in the GR literature (see also Luc and Placek (2020) for philosophical discussion), there have not yet been any attempts to extend this to non-Hausdorff NCT; moreover, there is as yet no worked-out twistor correspondence for non-Hausdorff spacetimes. So although one might charitably read Dunajski and Penrose's talk of 'bifurcation' as directed at some future non-Hausdorff extension of their proposal, the prospects for such an extension are at best unclear.

[22] Recall that one cannot use another (small) Kodaira deformation here, since small Kodaira deformations of the $\mathcal{O}(1) \oplus \mathcal{O}(1)$ relativistic twistor space preserve the holomorphic structure of the normal bundle; see §3.2, appendix B, and references therein.

5 Close

To continue with the issues raised at the end of the previous section, whilst currently highly programmatic in nature, the Dunajski-Penrose proposal is certainly an interesting development in the gravitational collapse literature which is worthy of study. Moreover, as noted in §3.3, there are a number of reasons for taking an interest in non-relativistic twistor theory more generally, independently of the Dunajski-Penrose proposal. Given the relative lack of philosophical literature on twistor theory, and the recentness of the Dunajski-Penrose proposal, our focus in this article has been to clarify the conceptual background to non-relativistic twistor theory, as well as to lay out in detail the issues facing the Dunajski-Penrose proposal.

Accordingly, our conclusions here may seem to be somewhat negative. On a more positive note, getting clear on what these issues are paves the way both for future developments of the Dunajski-Penrose proposal to address these issues, and for future philosophical analysis of these developments, should they be forthcoming.

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A Line bundles

In this appendix, we briefly review the basic construction of line bundles over projectivised vector spaces.

Definition A.1 (Projectivisation of a vector space)**.** *Let V be a vector space over a field F. The projectivisation PV of V is the quotient space $PV := (V \setminus \{0\})/F^\times$, $F^\times := F \setminus \{0\}$.*

$P\mathbb{C}^n$ is denoted $\mathbb{CP}^n$. The complex projective line $\mathbb{CP}^1$ is also called the Riemann sphere, and considered as a submanifold of some complex manifold is a rational curve. If PV is a projectivised vector space, then each point in PV carries the action of the multiplicative group $F^\times$. This provides a natural way of constructing a fibre bundle over PV with fibres F:

Definition A.2 (Tautological line bundle on a projective space)**.** *Let v, $[v]$ denote points in V, PV respectively. The tautological line bundle on PV is the sub-bundle $\{(v, [v]) \in V \times PV | v \in [v]\} \to PV$ of $V \times PV \to PV$, and is denoted $\mathcal{O}_V(-1)$.*

From the tautological line bundle, we can then construct further line bundles on PV:

Definition A.3 ($\mathcal{O}_V(n)$)**.** *Let $\mathcal{O}_V(-1)$ be the tautological line bundle on PV. The construction of $\mathcal{O}_V(n)$ for arbitrary integers n proceeds via two steps:*

- $\mathcal{O}_V(1) := \mathcal{O}_V(-1)^*$*;*
- $\mathcal{O}_V(n) := \mathcal{O}_V(1)^{\otimes n}$*.*

If $F = \mathbb{C}$, then n is the (first) Chern class of $\mathcal{O}_V(n)$. Chern classes are characteristic classes for complex vector bundles which arise naturally in algebraic topology (see e.g. Milnor and Stasheff (1974)), algebraic geometry (see e.g. Hartshorne (1977)), and differential geometry (see e.g. Chern (1995)). When $V = \mathbb{C}$, we write $\mathcal{O}_{\mathbb{C}}(n) := \mathcal{O}(n)$. It is a result due to Birkhoff and Grothendiek that any holomorphic vector bundle on $\mathbb{CP}^1$ can be constructed from the $\mathcal{O}(n)$:

Proposition A.1 (Birkhoff-Grothendiek theorem)**.** *Any rank-k vector bundle on $\mathbb{CP}^1$ is isomorphic to a direct sum of line bundles $\mathcal{O}(n_1) \oplus ... \oplus \mathcal{O}(n_k)$ for integers $n_1, ..., n_k$.*

B Kodaira deformation theory

In this appendix, we give a brief introduction to Kodaira deformation theory. We also present details on some of the basic ideas from sheaf cohomology needed for the central results in Kodaira deformation theory.

B.1 Complex analytic families

We begin with two definitions:

Definition B.1 (Complex analytic family)**.** *A complex analytic family of complex manifolds is a complex fibre bundle $\pi : \mathcal{M} \to B$ such that π is holomorphic and has Jacobian whose rank is equal to the dimension of B. $\mathcal{M}$ is a family of compact complex manifolds iff the fibres $\pi^{-1}(b)$ at each point $b \in B$ are compact.*

Definition B.2 (Stable submanifold of a complex manifold)**.** *Let N be a compact complex submanifold of M. N is a stable submanifold of M iff for any complex analytic family $\pi : \mathcal{M} \to B$ such that $\pi^{-1}(b) = M$ for a point $b \in B$, there exist a neighborhood U of b in B and a fibre submanifold $\pi : \mathcal{N} \to U$ with compact fibres of the complex fibre manifold $\mathcal{M}_{|U}$ such that $\mathcal{N} \cap M = N$.*

Given a complex analytic family of compact complex manifolds $\mathcal{M}$, and a submanifold N of $M = \pi^{-1}(p)$, there are two natural questions which one might be interested in:

- How does the complex structure of the fibres of $\mathcal{M}$ vary with respect to points in the base space?
- Under what conditions is N a stable submanifold of M?

B.2 Sheaf cohomology

It turns out that useful answers to the questions raised at the end of §B.1 can be given in terms of sheaf cohomology classes associated to the topological spaces of interest (the construction can also be done using Čech cohomology, since Čech cohomology classes are isomorphic to sheaf cohomology classes for sheaves on paracompact Hausdorff spaces; see Huggett and Tod (1994) for a presentation along those lines). We begin with some preliminaries from cohomology theory, and then go on to consider their generalisations to sheaf cohomology:

B.2.1 Cohomology

Definition B.3 ($\mathbb{Z}$-graded abelian group)**.** *A $\mathbb{Z}$-graded abelian group is an abelian group C which admits a direct sum decomposition*

$$C = \bigoplus_{n\in\mathbb{Z}} C_n$$

where the C_n are abelian groups.

All graded abelian groups will be assumed $\mathbb{Z}$-graded.

Definition B.4 (Homomorphism of graded abelian groups)**.** *A homomorphism $h : C \to D$ of graded abelian groups is a group homomorphism such that $h(C_n) \subset D_n$.*

Definition B.5 (Homomorphism of degree k)**.** *A homomorphism $h : C \to D$ of degree k of graded abelian groups is a group homomorphism such that $h(C_n) \subset D_{n+k}$.*

This allows us to introduce the notion of a *chain complex*:

Definition B.6 (Chain complex)**.** *A chain complex (C, ϕ) is a graded abelian group together with a homomorphism ϕ of degree -1 (called a boundary homomorphism) such that $\phi^2 = 0$.*

A chain complex gives us a sequence of abelian groups

$$... \to C_{n+1} \overset{\phi_{n+1}}{\to} C_n \overset{\phi_n}{\to} C_{n-1} \to ...$$

where $\phi_{n+1} \circ \phi_n = 0$. Note in particular that $\mathrm{im}(\phi_{n+1}) \subseteq \ker(\phi_n)$. We define:

Definition B.7 (Boundary)**.** *An n-boundary is the image* $\mathrm{im}(\phi_{n+1})$.

Definition B.8 (Cycle)**.** *An n-cycle is the kernel* $\ker(\phi_n)$.

We then have

Definition B.9 (Exact sequence)**.** *A sequence of abelian groups is exact iff all n-boundaries and n-cycles are equal, i.e.* $\mathrm{im}(\phi_{n+1}) = \ker(\phi_n)$ *for all n. An exact sequence is short exact iff it has the form* $0 \to A \to B \to C \to 0$.

Given a chain complex (C, ϕ), it follows that the quotient groups $\text{im}(\phi_{n+1})/\ker(\phi_n)$ encode information about the extent to which the sequence

$$... \to C_{n+1} \stackrel{\phi_{n+1}}{\to} C_n \stackrel{\phi_n}{\to} C_{n-1} \to ...$$

fails to be exact at each n. This is made precise in the notion of *homology*:

Definition B.10 (Homology)**.** *The nth homology class $H_n(C)$ of (C, ϕ) is the quotient $H_n(C) := \text{im}(\phi_{n+1})/\ker(\phi_n)$.*

We have the following useful result about homology:

Proposition B.1. *A short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups induces a long exact sequence*

$$H_0(A) \to H_0(B) \to H_0(C) \to H_1(A) \to ...$$

on homology groups.

The idea of cohomology is then just that when the action of $\mathbb{Z}$ on C is free we can make an analogous construction for the dual groups and dual maps of the chain complex (C, ϕ).

Definition B.11 (Cochain complex)**.** *Let (C, ϕ) be a chain complex such that the action of $\mathbb{Z}$ on C is free. The n cochain C^n of (C, ϕ) is the dual group $C^n := \text{hom}(C_n, \mathbb{Z})$*

A cochain complex gives us a sequence of abelian groups

$$... \to C^{n-1} \stackrel{\phi^{n-1}}{\to} C^n \stackrel{\phi^n}{\to} C^{n+1} \to ...$$

where ϕ^n is the dual map to ϕ_n and $\phi^n \circ \phi^{n-1} = 0$. So this time, we have $\text{im}(\phi^n) \subseteq \ker(\phi^{n-1})$. We therefore define

Definition B.12 (Coboundary)**.** *An n-coboundary is the image $\text{im}(\phi^n)$.*

Definition B.13 (Cocycle)**.** *An n-cocycle is the kernel $\ker(\phi^{n-1})$.*

Definition B.14 (Cohomology)**.** *The nth homology class $H^n(C)$ of (C, ϕ) is the quotient $H^n(C) := \text{im}(\phi^n)/\ker(\phi^{n-1})$.*

Proposition B.2. *A short exact sequence $0 \to A \to B \to C \to 0$ of abelian (dual) groups induces a long exact sequence*

$$H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to ...$$

on cohomology groups.

B.2.2 Sheaves

Definition B.15 (Abelian sheaf)**.** *Let X be a topological space. An abelian sheaf $\mathcal{S}$ over X is a topological space and a map $\pi : \mathcal{S} \to X$ such that*

- *π is a local homomorphism;*
- *The stalks $\mathcal{S}_x := \pi^{-1}(x)$ are topological abelian groups (i.e. abelian groups whose group operations are continuous).*

We record for future use the following fact about abelian sheaves:

Proposition B.3. *The sections $\mathcal{S}(U)$ of $\mathcal{S}$ over any open set $U \subset X$ form an abelian group.*

In fact, it is standard to use proposition B.3 as part of the *definition* of abelian sheaves, but for our purposes here we can make do with the (rather simpler) definition B.15, although it obscures somewhat the relationship to sheaf theory more generally. In any case, all sheaves will henceforth be assumed abelian. Our next step is to define the notion of a structure-preserving map between sheaves:

Definition B.16 (Sheaf homomorphism)**.** *Let $\mathcal{S}$, $\mathcal{T}$ be sheaves over X. A continuous map $\phi : \mathcal{S} \to \mathcal{T}$ is a sheaf homomorphism iff for all $x \in X$ $\phi : \mathcal{S}_x \to \mathcal{T}_x$.*

One application of this idea is to define the notion of an *injective resolution* of a sheaf:

Definition B.17 (Injective sheaf)**.** *An injective sheaf I is a sheaf such that for any homomorphism $f : \mathcal{S} \to I$ and any monomorphism $g : \mathcal{S} \to \mathcal{T}$ there exists a homomorphism $h : \mathcal{T} \to I$ such that $h \circ g = f$.*

Definition B.18 (Injective resolution of a sheaf)**.** *An injective resolution of a sheaf $\mathcal{S}$ is an exact sequence $0 \to \mathcal{S} \to I_0 \to I_1 \to I_2 \to ...$ where the I_i are injective sheaves.*

Given an injective resolution $0 \to \mathcal{S} \to I_0 \to I_1 \to I_2 \to ...$ of $\mathcal{S}$, it follows from proposition B.3 that we have the cochain complex

$$0 \to \mathcal{S}(X) \to I_0(X) \to I_1(X) \to I_2(X) \to ...$$

which immediately allows us to make contact with our discussion of cohomology in §B.2.1. However, we first need to address the question: under what conditions do injective resolutions of a sheaf exist? The answer is provided by the following proposition:

Proposition B.4. *Let $\mathcal{S}$ be an abelian sheaf. Then there exists an injective sheaf I and a monomorphism $\mathcal{S} \to I$.*

(This is a statement of the fact that the category of abelian sheaves has enough injections, see e.g. Hartshorne (1977).) Thus for any sheaf we can define

Definition B.19 (Sheaf cohomology)**.** *Consider the cochain complex*

$$0 \to \mathcal{S}(X) \to I_0(X) \xrightarrow{\phi_0^*} I_1(X) \xrightarrow{\phi_1^*} I_2(X) \xrightarrow{\phi_2^*} ...$$

where $0 \to \mathcal{S} \to I_0 \to I_1 \to I_2 \to ...$ *is an injective resolution of* $\mathcal{S}$*. The ith cohomology group* $H^i(X, \mathcal{S}) := \ker(\phi_i^*)/\mathrm{im}(\phi_{i-1}^*)$*.*

This definition of sheaf cohomology classes only makes sense if the $H^n(X, \mathcal{S})$ are independent of the choice of injective resolution. Standard arguments in the theory of abelian categories (see e.g. Hartshorne (1977)) imply that this is indeed the case:

Proposition B.5. *The cohomology classes* $H^n(X, \mathcal{S})$ *are independent of the choice of injective resolution* $0 \to \mathcal{S} \to I_0 \to I_1 \to I_2 \to ...$ *of* $\mathcal{S}$*.*

We have from proposition B.2 the following useful result about sheaf cohomology:

Proposition B.6. *A short exact sequence* $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{R} \to 0$ *of sheaves over* X *induces a long exact sequence*

$$H^0(X, \mathcal{S}) \to H^0(X, \mathcal{T}) \to H^0(X, \mathcal{R}) \to H^1(X, \mathcal{S}) \to ...$$

on cohomology groups.

B.3 The Kodaira theorems

We now have the tools in place to address our two questions from §B.1. To motivate our answer to the first (see Zykoski (2018) for a detailed discussion of this), let $\pi : \mathcal{M} \to B$ be a complex analytic family, and consider some open $U \subset B$. Fix a covering of $\pi^{-1}(U)$ by coordinate charts φ_λ defined on patches U_λ of $\pi^{-1}(U)$ which all agree on their restrictions to $U \cap U_\lambda$, and let $F_{\lambda\mu}$ be patching relations on $U_\lambda \cap U_\mu$. We are interested in how the complex structure of the fibres $\pi^{-1}(b)$ changes as we move between 'neighbouring' points in the base space. Since this complex structure is encoded in the patching relations on each fibre $\pi^{-1}(b)$, it makes sense to say that to first order, this is completely characterised by the first derivatives of the $F_{\lambda\mu}$ with respect to the coordinates on U. This suggests that to answer our question we should consider the relationship between $T\mathcal{M}_{|M}$ and T_bB. Since $\pi : \mathcal{M} \to B$ is a complex analytic family, we have the map

$$d\pi : T\mathcal{M}_{|M} \to T_bB$$

which induces a short exact sequence of sheaves

$$0 \to TM \to T\mathcal{M}_{|M} \to T_bB \otimes \mathcal{O}_M \to 0.$$

(Here we are abusing notation by eliding the distinction between the sheaf of sections of a vector bundle and the bundle itself.) Since $H^0(M, T_bB \otimes \mathcal{O}_M) \cong T_bB$, by proposition B.6 this gives us the following map:

Definition B.20 (The Kodaira-Spencer map)**.** *The map* $T_bB \to H^1(M, TM)$ *obtained by taking the long exact cohomology sequence of* $0 \to TM \to T\mathcal{M}_{|M} \to T_bB \otimes \mathcal{O}_M \to 0$. *is called the Kodaira-Spencer map.*

which makes precise our earlier gloss of Kodaira deformations from §3.2. Turning now to the second question raised in §B.1, a partial answer to this question is given by a theorem due to Kodaira:

Proposition B.7 (Kodaira 1963)**.** *Let* N *be a compact complex submanifold of* M*, and let* $\mathcal{N}_{N/M}$ *be the normal sheaf of* N *in* M.[23] *Then* N *is a stable submanifold of* M *if* $H^1(V, \mathcal{N}_{N/M}) = 0$.

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[23] That is, the sheaf of sections of the normal bundle of N in M.

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