Lorentz invariance and quantum mechanics

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Abstract

Bohmian mechanics and spontaneous collapse models are theories that overcome the quantum measurement problem. While they are naturally formulated for non-relativistic systems, it has proven difficult to formulate Lorentz invariant extensions, primarily due to the inherent non-locality, which is unavoidable due to Bell's theorem. There are trivial ways to make space-time theories Lorentz invariant, but the challenge is to achieve what Bell dubbed "serious Lorentz invariance". However, this notion is hard to make precise. This is reminiscent of the debate on the meaning of general invariance in Einstein's theory of general relativity. The issue there is whether the requirement of general invariance is physically vacuous (in the sense that any space-time theory can be made generally invariant) or whether it is a fundamental physical principle. Here, we want to consider two of the more promising avenues that have emerged from that debate in order to explore what serious Lorentz invariance could mean. First, we will consider Anderson's approach based on the identification of absolute objects. Second, we will consider a relativity principle for isolated subsystems. Using these criteria, we will evaluate a number of Lorentz invariant Bohmian models and a spontaneous collapse model, finding that the latter satisfies both criteria, while there are some Bohmian models that violate the criteria. However, some Bohmian models that satisfy both criteria still do not seem seriously Lorentz invariant. While these notions may hence still not capture exactly what serious Lorentz invariance ought to be, they clarify what aspects of relativity theory (in addition to locality) may need to be given up in passing from classical to quantum theory.

1 Introduction

In the non-relativistic domain, there exist precise versions of quantum mechanics that do not suffer from the measurement problem. Examples are Bohmian mechanics [1–3] and spontaneous collapse theories [4]. The extension of these theories to the relativistic domain has proven challenging. One aspect is to deal with the phenomenon of particle creation and annihilation. Bohmian [5–8] as well as spontaneous collapse models [9]

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have been developed to account for this. However, the simplest formulation of those models employs a preferred reference frame. Even though this frame may not be detectable and the corresponding statistical predictions may be Lorentz invariant, like in the Bohmian theories, the dynamical dependence on the frame implies a violation of Lorentz invariance at the fundamental level.

Lorentz invariance at the fundamental level is hard to achieve. Only in recent years, some (toy) models have been proposed [10–26]. It is hard because these theories must necessarily be non-local, i.e., they must entail influences over space-like distances, as a result of Bell's theorem [27–29]. The non-locality does not exclude Lorentz invariance. Lorentz invariance concerns a dynamical symmetry and can hold for a non-local theory, examples being tachyonic theories or the Wheeler–Feynman theory of electrodynamics. Nevertheless, in the context of quantum mechanics, it seems difficult to marry the non-locality with Lorentz invariance.

Prima facie, it might sound strange that a Lorentz invariant formulation is difficult. After all, every textbook on relativistic quantum mechanics or quantum field theory is concerned with such formulations. However, these textbooks tend to ignore the collapse postulate. In standard quantum mechanics, the wave function is supposed to collapse upon measurement, but if the collapse is instantaneous, it requires the specification of a particular time coordinate, thereby violating the Lorentz invariance.

When we say that a Lorentz invariant quantum theory is hard to devise, we mean Lorentz invariance in what Bell called a "serious" way [30]. After all, there are trivial ways to make space-time theories Lorentz invariant, even theories that are not regarded as such, like Newtonian mechanics [10, 26, 31].

To see this, let us first recall what is meant by Lorentz invariance. A physical theory concerns the specification of a set $\mathcal{K}$ of kinematically allowed histories (e.g., particle world lines or space-time fields) and a subset $\mathcal{D} \subseteq \mathcal{K}$ of dynamically allowed histories that satisfy the laws of motion. For a theory in Minkowski space-time, a Lorentz transformation will map possible histories to histories, yielding a permutation of $\mathcal{K}$. A theory is Lorentz invariant iff each Lorentz transformation Λ maps dynamically allowed histories to dynamically allowed histories, i.e., $\Lambda\mathcal{D} = \mathcal{D}$.

Consider now Newtonian mechanics. Newtonian mechanics is naturally formulated in Galilean space-time and the natural space-time symmetry group is that of the Galilean transformations [32,33]. However, the theory can trivially be recast in Minkowski space-time in a Lorentz invariant way. To do this, consider a Lorentz frame F and postulate the Newtonian dynamics to hold in this frame. As such, the theory is of course not Lorentz invariant. Namely, denoting the set of dynamically allowed histories by $\mathcal{D}^F$ (i.e., the set of trajectories that form solutions of Newton's equations in the frame F), then for a Lorentz transformation Λ, we have that $\Lambda\mathcal{D}^F \neq \mathcal{D}^F$. That is, the Lorentz transformation of a solution to Newton's equations is generically not a solution. However, consider a reformulation where the frame is part of the degrees of freedom, so that a kinematically allowed history is now (F, X), with a law such that $(F', X') \in \mathcal{D}$ iff there is a Lorentz transformation Λ, such that $(F', X') = \Lambda(F, X)$, with $X \in \mathcal{D}^F$. The theory is Lorentz invariant since $\Lambda\mathcal{D} = \mathcal{D}$ for all Lorentz transformations Λ.

The example can perhaps be made less abstract as follows. A frame determines a future causal unit-normal vector field n^μ which is normal to the constant-time surfaces and is constant over space-time, i.e., it satisfies

$$\partial_\nu n^\mu = 0. \tag{1}$$

Using this vector field n^μ, Newton's equations can be written explicitly in Lorentz invariant form, with the familiar form of the equations holding in the frame where $n^\mu = \delta^\mu_0$. The vector field n^μ is introduced as an extra dynamical quantity, but clearly it satisfies a Lorentz invariant law.

So, the trick is to introduce the frame (or the vector field n^μ) as extra space-time structure. Another possibility would be to let the frame be determined by the matter distribution itself. In that case, there would be *no* extra space-time structure. For example, Bell [30] (crediting Baumann) suggested that Newtonian mechanics can be made Lorentz invariant by letting the frame be the center-of-mass frame determined by the matter distribution (assuming the latter is finite). Indeed, if this frame could be determined in a Lorentz invariant way from the trajectories, then the theory would be Lorentz invariant. However, it is not clear how this proposal is supposed to work. In the first place, it is problematic to have a relativistic generalization of the notion of center-of-mass frame that shares the simple properties of the Newtonian one [34]. Furthermore, the Newtonian dynamics determines a *Newtonian* center-of-mass frame (where the total Newtonian momentum is zero), but does not necessarily admit a relativistic center-of-mass frame, which is independent of the Lorentz frame in which it is defined, and which is what is required for a Lorentz invariant formulation. Conversely, a relativistic center-of-mass frame does necessarily determine a Newtonian one. (For example, the relativistic center-of-mass frame could be defined as the frame where *relativistic* total 3-momentum, i.e., the spatial part of the total 4-momentum, of the particles vanishes. But the relativistic 3-momentum differs from the Newtonian 3-momentum and therefore the latter generically does not vanish in such a frame.) So, it is unclear how exactly to realize Bell's proposal. But perhaps there are other ways of letting the frame be determined by the matter distribution.

In any case, it is clear that Newtonian mechanics, as well as other space-time theories, can be reformulated in a Lorentz invariant way. However, such formulations cannot be considered as seriously Lorentz invariant. But what exactly does "serious Lorentz invariance" mean? Bell found this hard to make precise [30]. Since Bell, little has been said about this in the context of quantum mechanics. On the other hand, there has been an extensive debate on this for general relativity [35]. While Einstein motivated his field equations by requiring invariance under general coordinate transformations, it was argued by Kretschmann [36], very early on, that this motivation was physically vacuous, since non-invariant theories could trivially be made invariant. There are some attempts to make precise what serious general invariance could mean. Here, we want to consider these in the context of quantum theory.

One of the best developed attempts is that of Anderson [37]. Anderson distinguishes between the *covariance group* and the *symmetry group* of a theory. A covariance group is a group of transformations that maps solutions of the dynamical equations to solutions (i.e., leaves $\mathcal{D}$ invariant). (This is how we characterized Lorentz invariance before.) The group action defines an equivalence relation, with an equivalence class containing solutions that are related by elements of the covariance group. The symmetry group is defined as the largest subgroup of the covariance group that leaves invariant the *absolute objects* of the theory. The absolute objects are distinguished from the *dynamical objects* and are the geometric objects (on some manifold) that appear, together with all their transformations under the covariance group, in every equivalence class. Anderson's analysis, presented here in a nutshell, could serve as a possible candidate to characterize serious Lorentz invariance. If the Lorentz group is a covariance group, then it will also be a symmetry group if it leaves invariant all the absolute objects of the theory. An object will be absolute if it is the same for each solution, up to a Lorentz transformation. (One of these absolute objects is of course the Minkowski metric.)

What does Anderson's analysis entail for the Lorentz invariant formulation of Newtonian mechanics? The covariance group is clearly the group of Lorentz transformations. But n^μ (or F) is an absolute object, since for any two possible solutions (n^μ, X) and (n'^μ, X'), there is a Lorentz transformation that maps n^μ to n'^μ. Since the Lorentz group does not leave n^μ invariant, the symmetry group is not the Lorentz group; the Lorentz invariance is not serious.

As has been observed before, Anderson's approach is not without problems [35, 38], and similar problems will resurface in the present context.

Another possible characterization of serious Lorentz invariance is through a relativity principle for (approximately) isolated subsystems [39–42]. This is inspired by the following observation. Empirical evidence for dynamical symmetries is usually obtained by considering transformations of dynamically isolated subsystems within the universe (i.e., subsystems whose motion does not depend on the environment), rather than of the universe itself. Symmetry transformations of the whole universe will not lead to observable differences, but a transformation of just an isolated subsystem — leaving the environment invariant — may lead to a physically distinct situation, albeit unobservable from within the subsystem itself. The prime example is Galileo's ship experiment which can be taken (when suitably qualified) as empirical evidence for Galilean boost symmetries. Namely, the situations where the ship is at rest or moving with respect to the shore are physically distinct, but unobservable from within the ship.

A theory can now be said to satisfy a relativity principle for isolated subsystems when solutions of the equations of motion can be transformed into other (approximate) solutions by applying the transformations (Lorentz transformations in the present case) to subsystems that are (approximately) dynamically isolated. For a stochastic theory, also the probabilities for the subsystem should be transformed and match the probabilities set by the theory for the whole universe.

Note that we do not require that the subsystem be completely isolated. In reality, this will usually not be the case, with some interaction with the environment remaining. But it suffices that the interaction can be arbitrarily small in principle. In order not to trivialize this principle, it should also be understood that the theory should allow for approximately isolated subsystems in the first place and that these can be of arbitrary complexity, in the sense that if the dynamical laws allow for a particular system, then they should also allow for multiple such systems in the universe, perhaps separated far away from each other, that evolve approximately independently.

This relativity principle is admittedly somewhat imprecise, but this will suffice for our purposes. Note that in our characterization of it, we did not refer to empirical aspects, as is sometimes done (see, e.g., [40]). Our formulation seems more up to the task of investigating serious Lorentz invariance in the context of quantum mechanics since the latter's statistical character might entail that objects violating Lorentz invariance like a preferred frame are unobservable.

Now let us apply this to the Lorentz invariant formulation of Newtonian mechanics. An approximately isolated system will consist of particles that do not (or only very weakly) interact with their environment, e.g., because of substantial spatial separation. The history (n^μ, X) of the universe can then be decomposed as $(n^\mu_s, X_s; n^\mu_e, X_e)$, where X_s and X_e respectively denote the trajectories of the subsystem and environment, and n^μ_s and n^μ_e are respectively the unit-normal vector at the space-time region of the subsystem and environment. A Lorentz boost of the subsystem (n^μ_s, X_s), to (n'^μ_s, X'_s), will lead to the history $(n'^\mu_s, X'_s; n^\mu_e, X_e)$ of the universe. This is however not a solution of the dynamical equations, since the law (1) for n^μ implies that it is constant over space-time, which is not the case since $n'^\mu_s \neq n^\mu_e$. Hence, the Lorentz invariant formulation of Newtonian mechanics does not satisfy the relativity principle for isolated subsystems.

This should be contrasted with for example the Wheeler–Feynman theory, which counts as Lorentz invariant in Anderson's sense and satisfies the relativity principle for subsystems (with a subsystem being dynamically isolated when it is sufficiently distant from its environment).

Here, we will consider some Lorentz invariant Bohmian and spontaneous collapse models. This will be done in a setting of a fixed number of non-interacting Dirac particles, which makes the analysis relatively simple, avoiding the complications of for example quantum field theory. While the particles are non-interacting, the theories are still non-local because the wave function may be entangled.

The many worlds theory is one of the other main approaches to solve the measurement problem. However, in this theory, at least in the way usually understood, and unlike in Bohmian mechanics or spontaneous collapse theories, space-time and stuff in space-time are not considered fundamental, but emergent. This requires a different discussion of space-time symmetries, which we will not embark on here. We will only briefly consider some versions of this theory that do have an ontology in space-time on the fundamental level.

We focus here on the possible meaning of serious Lorentz invariance. But this discussion should be viewed in the broader context of what it could mean for a theory to be fundamentally relativistic. We will return to this question in the conclusion.

The outline of the paper is as follows. In the next section, we will first consider Anderson's analysis in the context of the Klein–Gordon theory, rehearsing difficulties with this approach that will also resurface in the context of quantum mechanics. In section 3, we turn to the Galilean invariance of non-relativistic Bohmian mechanics, for which both Anderson's criterion and the relativity principle are satisfied. In sections 4 and 5, respectively, Lorentz invariant Bohmian and spontaneous collapse models are discussed, followed by a brief consideration of the many worlds theory in section 6. We conclude in section 7.

2 The Klein–Gordon theory

A prime example of Anderson's analysis, as well as its shortcomings, is that of the Klein–Gordon equation in Minkowski space-time:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0. \tag{2}$$

This equation, with $\eta_{\mu\nu}$ the Minkowski metric, is Poincaré invariant (i.e., invariant under Lorentz transformations as well as translations in space and time). It can be cast in a generally invariant way by replacing the Minkowski metric by a Lorentzian metric $g_{\mu\nu}$ and demanding that the corresponding Riemann curvature tensor $R_{\mu\nu\kappa\sigma}$ vanishes. The resulting equations are

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0, \tag{3}$$

$$R_{\mu\nu\kappa\sigma} = 0. \tag{4}$$

The latter entails that the metric is the Minkowski one up to a space-time diffeomorphism. The solutions of (3) and (4) correspond to diffeomorphisms of $(\eta_{\mu\nu}, \phi)$ where ϕ is a solution of (3).[1] As such, this theory has the same empirical content as (3). But since the metric is always the Minkowski one up to a diffeomorphism, it is an absolute object according to Anderson and hence the symmetry group is not that of general coordinate transformations, but of the Poincaré group.

As suggested by Pitts [38], consider now replacing (4) by

$$R_{\mu\nu\kappa\sigma} = \frac{k}{12}(g_{\mu\sigma}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\sigma}), \tag{5}$$

with some constant k. This entails that the space-time is maximally symmetric with constant curvature k [43, p. 141]. This theory is not equivalent to (2), since the space-time is not necessarily flat. Nevertheless the extension could be made as minimal as possible, e.g., by restricting the possible values of k, and leaving it as a contingency whether the world is flat or not. In any case, the upshot is that the metric no longer qualifies as an absolute object, since space-time diffeomorphisms do not change the curvature.[2]

[1] Basically the same trick is played here as in making Newtonian mechanics Lorentz invariant. Namely, the metric is considered to be part of the degrees of freedom, in addition to the field, and then a new law is formulated that has as solutions all the space-time diffeomorphisms of the Minkowski metric and field solutions of (3).

[2] Other examples of this sort are given by Torretti [44] and Norton [35, p. 848]. Toretti considers Newtonian gravity where space has a constant curvature. For a law that singles out a particular constant

So, an absolute object in the theory can trivially be turned into a non-absolute object by enlarging the space of solutions in such a way that it contains solutions for which the object under consideration is not related by an element of the covariance group. The new theory is no longer equivalent to the original theory, but nevertheless if the actual world is described well with the original theory, it will also be by the new theory.

Interestingly, Anderson's analysis does not stop here. While the metric itself is not an absolute object, part of it *is* (just like in general relativity), as argued in [38], which means that the theory described by (5) is still not generally invariant in Anderson's sense.

3 Non-relativistic Bohmian mechanics

Before turning to the possibility of Lorentz invariant formulations of Bohmian mechanics, it is instructive to recall the Galilean invariance of the non-relativistic theory [45], and see how it satisfies Anderson's criterion and the relativity principle.

Bohmian mechanics describes point-particles whose motion depends on the wave function. The wave function ψ satisfies the usual Schrödinger equation[3]

$$\mathrm{i}\partial_t \psi_t(x) = \left(-\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V(x) \right) \psi_t(x) \,, \qquad x = (\mathbf{x}_1, \dots, \mathbf{x}_N), \tag{6}$$

while the motion of the particles, with positions $\mathbf{X}_k$, $k = 1, \dots, N$, is determined by

$$\frac{d\mathbf{X}_k(t)}{dt} = \mathbf{v}_k^{\psi_t}(\mathbf{X}_1(t), \dots, \mathbf{X}_N(t)), \tag{7}$$

where

$$\mathbf{v}_k^{\psi} = \frac{\hbar}{m_k} \mathrm{Im} \frac{\boldsymbol{\nabla}_k \psi}{\psi} = \frac{1}{m_k} \boldsymbol{\nabla}_k S, \qquad \psi = |\psi| \mathrm{e}^{\mathrm{i}S/\hbar}. \tag{8}$$

This theory is Galilean invariant in that a Galilean transformation will map solutions $(\mathbf{X}_1(t), \dots, \mathbf{X}_N(t), \psi_t(\mathbf{x}_1, \dots, \mathbf{x}_N))$ of (6) and (7) to solutions. For example, for a boost with velocity $\mathbf{v}$, the transformation reads

$$\mathbf{X}_k(t) \to \mathbf{X}'_k(t) = \mathbf{X}_k(t) + \mathbf{v}t, \tag{9}$$

$$\psi_t(x) \to \psi'_t(x) = \psi_t(\mathbf{x}_1 - \mathbf{v}t, \dots, \mathbf{x}_N - \mathbf{v}t) \exp\left[\mathrm{i} \sum_{k=1}^{N} m_k \left(\mathbf{v} \cdot \mathbf{x}_k - \frac{1}{2} v^2 t \right) \right]. \tag{10}$$

Since there are no absolute objects (apart from those of Galilean space-time), the theory is Galilean invariant according to Anderon's analysis. The theory also satisfies

curvature, the spatial geometry is an absolute object. But by allowing different spatial geometries of constant curvature, it is no longer absolute. Norton extended this example to Friedmann–Lemaître–Robertson–Walker space-times.

[3]Throughout, we assume units such that $\hbar = c = 1$.

the relativity principle. To see this, consider a system described by the coordinates $\mathbf{x}_1, \dots, \mathbf{x}_M$ and an environment described by the coordinates $\mathbf{x}_{M+1}, \dots, \mathbf{x}_N$. Assume further that the potential $V(x)$ contains no interaction terms between the system and environment. Then a wave function which is initially a product of the form

$$\psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi_s(\mathbf{x}_1, \dots, \mathbf{x}_M)\psi_e(\mathbf{x}_{M+1}, \dots, \mathbf{x}_N) \tag{11}$$

will remain a product at all times. As a consequence, the velocity field is

$$\mathbf{v}_k^{\psi_t}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \begin{cases} \mathbf{v}_k^{\psi_{s,t}}(\mathbf{x}_1, \dots, \mathbf{x}_M) & \text{if} \quad k \leqslant M, \\ \mathbf{v}_k^{\psi_{e,t}}(\mathbf{x}_{M+1}, \dots, \mathbf{x}_N) & \text{if} \quad k > M. \end{cases} \tag{12}$$

Hence, it is clear that the relativity principle holds. One can apply a Galilean transformation of just the subsystem and obtain another solution of the dynamical equations. The assumption that the potential contains no interaction between the system and environment can be relaxed. It is sufficient that the interaction be negligible, which will be the case when the interaction potential falls off with distance and ψ_s and ψ_e have the bulk of their support on configurations corresponding to spatial regions that are well separated. (Furthermore, since the velocity of a configuration is only determined by the value of the wave function in the neighbourhood of that configuration, it is actually sufficient that the wave function obtains the form (11) in a sufficiently large region near the configuration. One could use the notion of conditional wave function, which is the natural definition of the wave function of a subsystem, to make this more precise [45].)

4 Lorentz invariant Bohmian models

4.1 Dynamics

We will consider a few representative examples of Lorentz invariant Bohmian models [10–20] in the simplified setting of a fixed number of free Dirac particles [11, 18]. While there are no interactions, there can still be entanglement, which is the source of the non-locality.

In the multi-time picture, the wave function $\psi = \psi(x_1, ..., x_N)$ is a function on $\mathcal{M}^N$, with $\mathcal{M}$ Minkowski space-time, and takes values in the N-particle spin space $(\mathbb{C}^4)^{\otimes N}$. To simplify later discussions, we will not require complete anti-symmetry of the wave function. The wave function satisfies the N Dirac equations

$$\mathrm{i}\gamma_k^\mu \partial_{k,\mu}\psi - m_k\psi = 0, \tag{13}$$

with $k = 1, ..., N$. Here, $\gamma_k^\mu = I \otimes \cdots \otimes I \otimes \gamma^\mu \otimes I \otimes \cdots \otimes I$ with the Dirac matrix γ^μ at the k-th of the N places. There are also N particles with worldlines $X_k(s)$, $k = 1, \dots, N$, parameterized by s. In the models we will consider, the equations of motion for the particles are of the form

$$\dot{X}_k(x) \propto J_k^{\Sigma_{k,x},\psi}\left(X_1^{\Sigma_{k,x}}, \dots, X_N^{\Sigma_{k,x}}\right). \tag{14}$$

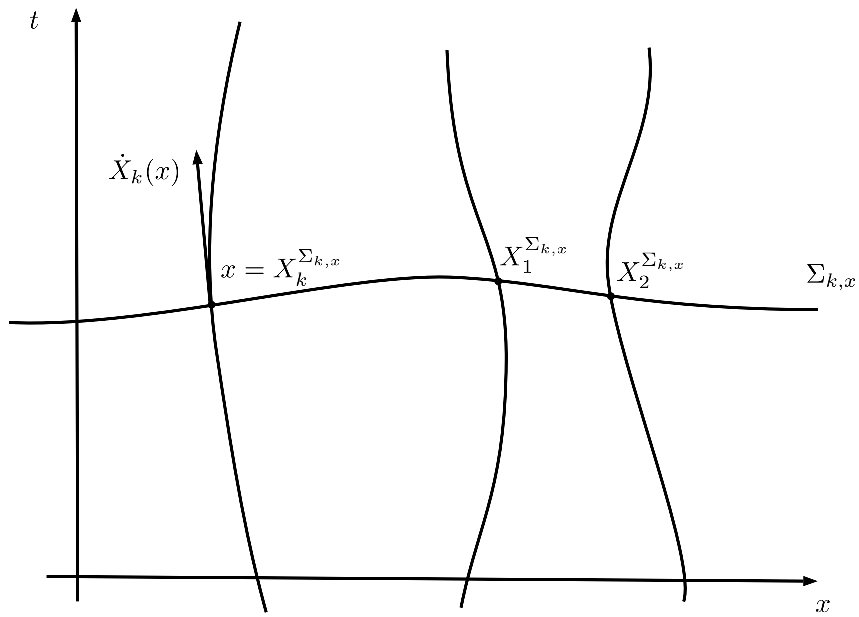

Figure 1: The tangent vector $\dot{X}_k(x)$ of the k-th worldline at the point x depends on the surface $\Sigma_{k,x}$ and the crossings of this surface with the other worldlines, cf. Eq. (14). (Only one spatial direction is shown.)

As illustrated in Fig. 1, $\dot{X}_k(x)$ is the unit tangent vector to the k-th worldline at the space-time point x, $\Sigma_{k,x}$ is a space-like hypersurface (to be specified below) for the kth particle that contains x and $X_i^{\Sigma_{k,x}}$ is the crossing of the i-th worldline with the surface $\Sigma_{k,x}$. The vector field $J_k^{\Sigma,\psi}$ is

$$\left[J_k^{\Sigma,\psi}\right]^{\mu_k}(x_1,\dots,x_N) = J^{\mu_1\dots\mu_N}(x_1,\dots,x_N)n_{\mu_1}(x_1)\dots\widehat{n_{\mu_k}(x_k)}\dots n_{\mu_N}(x_N), \qquad (15)$$

where n^μ is the unit vector field normal to Σ,

$$J^{\mu_1\dots\mu_N}(x_1,\dots,x_N) = \overline{\psi}(x_1,\dots,x_N)\gamma_1^{\mu_1}\dots\gamma_N^{\mu_N}\psi(x_1,\dots,x_N), \qquad (16)$$

and the hat on $n_{\mu_k}(x_k)$ denotes that this factor should be omitted.

So, the theory involves the particles, the wave and the surfaces $\Sigma_{k,x}$. It remains to specify these surfaces. There are different possibilities. The surfaces may be determined by the wave function or the worldlines, or may satisfy some law, possibly also depending on the wave function or particle positions. The surfaces may or may not determine a foliation of space-time (see Fig. 2).

A foliation $\mathscr{F}$ can be completely characterized by its unit normal vector field n^μ. A unit vector field n^μ will determine a foliation iff $n \wedge \mathrm{d}n = 0$, where $n = n_\mu \mathrm{d}x^\mu$ is the one-form associated to the vector field.

The following six choices for the surfaces $\Sigma_{k,x}$ are considered:

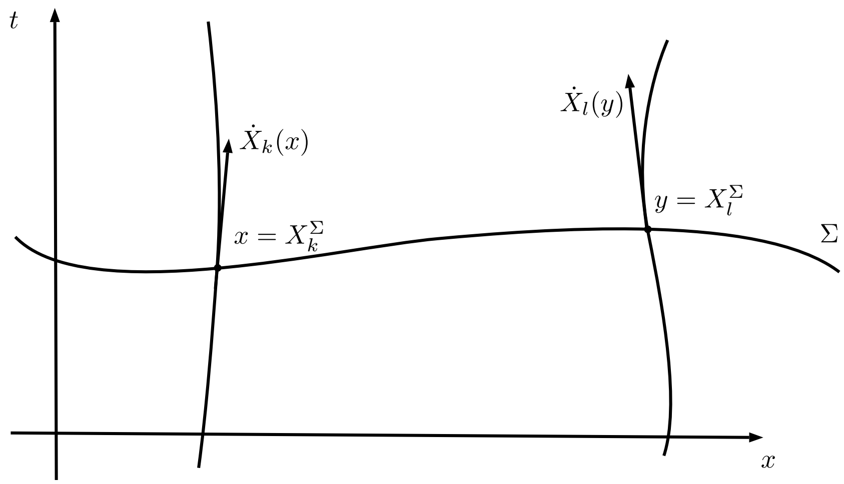

Figure 2: Trajectories and tangent vectors in the case of a foliation, with a leaf Σ. In this case, $\Sigma_{k,x} = \Sigma_{l,y} = \Sigma$.

(i) The $\Sigma_{k,x}$ are the leaves of a foliation $\mathscr{F}$, with unit normal vector field n^μ, that satisfies [11, p. 2736]:

$$\partial_\mu n^\nu = 0. \tag{17}$$

This law determines a foliation in terms of hyperplanes. (Given a particular space-like hyperplane as an "initial" leaf of the foliation, the law determines the whole foliation.)

(ii) The $\Sigma_{k,x}$ are the leaves of a foliation $\mathscr{F}$, with unit normal vector field n^μ, that satisfies [17]:

$$\partial_\mu n_\nu - \partial_\nu n_\mu = 0. \tag{18}$$

This law implies that $n^\mu \partial_\mu n_\nu = 0$. Hence, the vector field n^μ does not change in the direction normal to the surfaces. (Again, given a particular space-like Cauchy[4] hypersurface as an initial leaf of the foliation, the law determines the whole foliation.[5] The distance between two surfaces, in the normal direction, will be constant along the surface.)

(iii) The $\Sigma_{k,x}$ are the leaves of a foliation $\mathscr{F}$, with unit normal vector field n^μ, that satisfies [20]:

$$n \wedge \mathrm{d}n = 0. \tag{19}$$

This implies that there is a foliation, which is not further restricted. (Given an initial space-like Cauchy surface, there are many foliations that include it as a leaf.)

[4] A Cauchy surface crosses every inextensible curve just once. A space-like hypersurface in not necessarily a Cauchy surface.

[5] Specified as such, the foliation may have kinks, i.e., points where the leaves are not smooth. The Bohmian dynamics can still be defined for such foliations [46], but such details do not concern us here.

(iv) The $\Sigma_{k,x}$ are the leaves of a foliation that is covariantly determined by the wave function [11, 18]. That is, the law specifies a map $\psi \to \mathscr{F}$, such that the diagram

$$\begin{tikzcd} \psi \arrow[r] \arrow[d, "U_\Lambda"'] & \mathscr{F}^{\psi} \arrow[d, "\Lambda"] \\ \psi' \arrow[r] & \mathscr{F}^{\psi'} \end{tikzcd} \tag{20}$$

is commutative. Λ is a Lorentz transformation, with $\mathscr{F}' = \Lambda \mathscr{F}^{\psi}$ the corresponding natural transformation of the foliation and $\psi' = U_\Lambda \psi$ is the corresponding transformation of the wave function, i.e.,

$$\psi'(x_1, \dots, x_N) = (U_\Lambda \psi)(x_1, \dots, x_N) = D_\Lambda \otimes \cdots \otimes D_\Lambda \psi(\Lambda^{-1} x_1, \dots, \Lambda^{-1} x_N), \tag{21}$$

where D_Λ is the representative of Λ in the representation of the Lorentz group for the Dirac theory [47].

An example is the map $\psi \to P^\mu$, with P^μ the total energy-momentum 4-vector determined by ψ [18]. This vector satisfies $\partial_\nu P^\mu = 0$, so that it determines a foliation in terms of hyperplanes normal to it.[6]

Explicitly, this vector is given by

$$P^\mu = \sum_{k=1}^{N} \int_\Sigma d\sigma_\nu(x) T_k^{\mu\nu}(x), \tag{22}$$

where

$$T_k^{\mu_k \nu_k}(x_k) = \int_\Sigma d\sigma_{\nu_1}(x_1) \dots \widehat{\int_\Sigma d\sigma_{\nu_k}(x_k)} \cdots \int_\Sigma d\sigma_{\nu_N}(x_N)$$
$$\overline{\psi}(x_1, \dots, x_N) \gamma_1^{\nu_1} \dots \gamma_{k-1}^{\nu_{k-1}} \frac{\mathrm{i}}{2} \left(\overleftrightarrow{\partial}^{\mu_k} \gamma_k^{\nu_k} + \gamma_k^{\mu_k} \overleftrightarrow{\partial}^{\nu_k} \right) \gamma_{k+1}^{\nu_{k+1}} \dots \gamma_N^{\nu_N} \psi(x_1, \dots, x_N), \tag{23}$$

with Σ some arbitrary space-like hypersurface and $\overleftrightarrow{\partial}^\mu = \frac{1}{2}(\overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu)$.[7] This expression does not depend on the choice of Σ, since the divergence of the integrand with some ∂_{ν_i} gives zero [47, p. 58]. It also implies that $T_k^{\mu_k \nu_k}(x_k)$ does not depend on the x_i, $i \neq k$.

(v) The $\Sigma_{k,x}$ are the surfaces orthogonal to

$$P_k^\mu = \int_\Sigma d\sigma_\nu(x) T_k^{\mu\nu}(x). \tag{24}$$

[6] So, this foliation determines a center-of-mass frame for the wave function. As such, this proposal is akin to Bell's attempt to formulate a Lorentz invariant version of Newtonian mechanics using the center-of-mass frame determined by the particles.

[7] For a single particle, $T^{\mu\nu}(x)$ is the symmetrized energy-momentum tensor.

For each k, the constancy of the P_k^μ determines a foliation in terms of hyperplanes. But this foliation might be different for different k.[8] So, it is not necessarily the case that the velocity fields are determined by a single foliation. It might be the case, as illustrated in Fig. 3, that $\Sigma_{k,x} \neq \Sigma_{l,y}$, where y is the crossing of $\Sigma_{k,x}$ with the l-th worldline.

(vi) The $\Sigma_{k,x}$ are the hyperplanes orthogonal to $\dot{X}_k(x)$. Generically, these surfaces do not determine a foliation (not even for a fixed k).

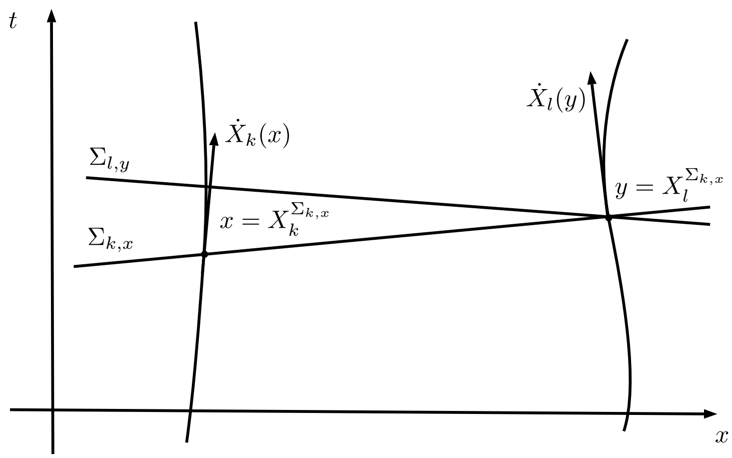

Figure 3: Trajectories and tangent vectors in the case of theories (v) and (vi). Situation where the surfaces do not determine a foliation, since $\Sigma_{k,x} \neq \Sigma_{l,y}$, even though $y \in \Sigma_{k,x}$.

Let us first mention some properties of these theories. In each case, the single-particle law is

$$\dot{X}^\mu(x) \propto J^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x). \tag{25}$$

So, the possible trajectories are the integral curves of the 4-current J^μ, which does not depend on any surface $\Sigma_{k,x}$.

The dynamics is non-local, since for an entangled wave function the velocity of a particle will depend on the positions of the other particles along the relevant space-like hypersurface Σ. This means that the velocity does not only depend on what happened in the past light cone, but also by what happens outside it. Such non-locality is unavoidable by Bell's theorem.

While there are influences over space-like distances, the trajectories themselves are never space-like. That is because the tangent vector $\dot{X}_k(x)$ is always time-like or light-like, i.e., $\left[J_k^{\Sigma,\psi}\right]^{\mu_k}\left[J_k^{\Sigma,\psi}\right]_{\mu_k} \geqslant 0$ for all k and choices of space-like hypersurface Σ [11]. (At least in theory (i), the particles typically do not reach the light-speed [48].)

[8] For a completely anti-symmetric wave function, all the P_k^μ are equal and the law reduces to the one considered in (iv).

All these theories are also Lorentz invariant (or, in Anderson's terminology, the Lorentz group forms the covariance group); a Lorentz transformation Λ maps solutions $(X_1, \dots, X_N, \psi)$ of the dynamical equations to solutions $(\Lambda X_1, \dots, \Lambda X_N, U_\Lambda \psi)$, with $U_\Lambda \psi$ defined in (21).

The theories (i)-(iii) contain extra space-time structure, namely the foliation. The trajectories depend on this foliation, but the foliation itself is independent of the particles. In the theories (iv)-(vi), there is no extra space-time structure. The surfaces $\Sigma_{k,x}$ are completely determined by the wave function in theories (iv) and (v) and by the worldlines in theory (vi).

In the theories (i)-(iv), the surfaces $\Sigma_{k,x}$ are part of a single foliation of space-time. For these theories, there is a notion of quantum equilibrium and it can be argued that these theories agree with the usual quantum predictions [11,26,49]. Namely, in quantum equilibrium, the density of crossings of the trajectories with a leaf Σ of the foliation with normal field n^μ is given by

$$\rho^{\Sigma,\psi}(x_1, \dots, x_N) = J^{\mu_1 \dots \mu_N}(x_1, \dots, x_N) n_{\mu_1}(x_1) \dots n_{\mu_N}(x_N), \tag{26}$$

relative to the volume defined by the induced metric on Σ, and evaluated on Σ^N. (In the case Σ is a hyperplane, $\rho^{\Sigma,\psi}$ equals the familiar expression $\psi^\dagger \psi$, with ψ the wave function on that hyperplane.) The crossings with a space-like Cauchy surface Σ' that is not a leaf in the foliation will generically *not* be distributed according to $\rho^{\Sigma',\psi}$ [10]. Nevertheless, it can be shown that actual measurements along an arbitrary space-like Cauchy surface will give results in accordance with the Born rule. This also means that the foliation itself is unobservable. So, we are in this prima facie awkward situation where the foliation plays a dynamical role for the trajectories, but is absent in the statistical predictions of the theory. (In quantum non-equilibrium, the foliation *would* be observable.)

In the theories (v) and (vi), on the other hand, the surfaces are (generically) not part of a single foliation.[9] This not only implies that it is hard to establish existence of solutions (especially in the case of theory (vi), where the stipulation of the surfaces is not even independent of (14), which might make the theory overconstrained), it also implies that it is hard to perform a statistical analysis and to compare the theory with the usual quantum predictions. Hence, it is not at all clear to what extent the predictions of these theories agree with those of quantum theory. (In the non-relativistic limit, assuming that there is a frame where the velocities of all the particles are small compared to the light speed, the surfaces $\Sigma_{k,x}$ nearly agree and presumably the predictions of non-relativistic quantum mechanics are recovered.)

[9] In [12–15, 19] other laws are considered that are not based on a foliation and which hence have the problem of statistical transparency. For example, in [14, 19], the surfaces are given by the future light-cones and the contraction in (15) is with the tangent vectors of the particles rather than the normals to the surface (which do not exist in this case).

4.2 Anderson's criterion in terms of absolute objects

According to Anderson's definition, the vector field n^μ and hence the corresponding foliation are absolute objects in the theory (i), but not in the theories (ii) and (iii). Namely, in the theory (i), just as in the Lorentz invariant formulation of Newtonian mechanics, the constancy over space-time of the vector field n^μ entails that any two such fields are related by a Lorentz transformation. This is not the case in theories (ii) and (iii), because the law for n^μ allows for vector fields (and hence foliations) that are not connected by a Lorentz transformation. As such, the group of Lorentz transformations counts as the symmetry group in the theories (ii) and (iii), but not in the theory (i).

However, this conclusion seems counterintuitive and the concern is the same as the one voiced in section 2 in relation to the Klein–Gordon theory. The theories (ii) and (iii) extend the set of solutions of theory (i). So, if the actual world is described by theory (i), it will also be by theories (ii) and (iii). That the actual world is described by a foliation in terms of hyperplanes, rather than curved surfaces, is then just a contingent fact. It then seems odd that the vector field is absolute in theory (i), but not in theories (ii) and (iii).

The original motivation of Anderson was actually to capture the idea that absolute objects act on the other objects but are not acted back on [37, p. 73 and 83]. In this case, Anderson's definition fails to capture that idea. In each of the theories (i)-(iii), the foliation acts on the particles, without back-reaction. However, as already observed by Norton [35], the notion of action without back-reaction is hard to make precise in a sensible way. Here too, in the context of Bohmian mechanics, this idea seems immediately problematic. The wave function acts on the particles but without back-reaction. But it does not seem to make sense to regard it as an absolute object.

The theories (iv)-(vi) contain no extra space-time structure like a foliation; there is just the wave and the particles. As such, there are no absolute objects (other than the Minkowski metric). However, there still is a dynamically distinguished foliation in the case of (iv). It is along the leaves of this foliation that the non-locality will be instantaneous.

So, in summary, adopting Anderson's principle as a criterion for serious Lorentz invariance would mean that only theory (i) would not count as seriously Lorentz invariant. But this seems at odds with the intuition that theories (i)-(iii) seem on par concerning possible serious Lorentz invariance.

4.3 Relativity principle for subsystems

To describe an isolated subsystem, consider a product wave function[10]

$$\psi(x_1,\dots,x_N) = \psi_s(x_1,\dots,x_M)\psi_e(x_{M+1},\dots,x_N). \tag{27}$$

[10] If the form (27) holds on a surface $\Sigma_1\times\cdots\times\Sigma_N$, with the Σ_i Cauchy surfaces (e.g., Σ_i constant-time surfaces in some Lorentz frame), then it will hold on $\mathcal{M}^N$. Note that this form cannot obtain if the wave function is completely anti-symmetric. However, such a form could hold locally on $\Sigma_1\times\cdots\times\Sigma_N$, which would be sufficient to carry out a similar analysis.

The wave functions ψ_s and ψ_e will then separately satisfy the many particle Dirac equations (13). Since

$$J_k^{\Sigma,\psi}(x_1,\dots,x_N) = \begin{cases} J_k^{\Sigma,\psi_s}(x_1,\dots,x_M)\rho^{\Sigma,\psi_e}(x_{M+1},\dots,x_N) & \text{if } k \leqslant M, \\ \rho^{\Sigma,\psi_s}(x_1,\dots,x_M)J_k^{\Sigma,\psi_e}(x_{M+1},\dots,x_N) & \text{if } k > M, \end{cases} \tag{28}$$

the corresponding particle equations (14) can be written as

$$\dot{X}_k(x) \propto \begin{cases} J_k^{\Sigma_{k,x},\psi_s}(X_1^{\Sigma_{k,x}},\dots,X_M^{\Sigma_{k,x}}) & \text{if } k \leqslant M, \\ J_k^{\Sigma_{k,x},\psi_e}(X_{M+1}^{\Sigma_{k,x}},\dots,X_N^{\Sigma_{k,x}}) & \text{if } k > M. \end{cases} \tag{29}$$

For the subsystem consisting of the particles $1,\dots,M$ to be isolated, also the particle dynamics should decouple. That is, the evolution of the particles $1,\dots,M$ should not depend on the wave function ψ_e or the particle positions $X_{M+1},\dots,X_N$. Because of the form (29), there can only be such a dependence if the surface $\Sigma_{k,x}$ has it. For the theories we consider, this only happens in the case (iv), where the surface $\Sigma_{k,x}$ depends on the universal wave function ψ. (In theory (v), the vectors P_k^μ, $k = 1,\dots,M$, and hence the corresponding surfaces $\Sigma_{k,x}$, only depend on ψ_s.) So, in the case of theory (iv), there are no isolated subsystems, except in the special case where the wave function ψ_s is completely separable, i.e.,

$$\psi_s(x_1,\dots,x_M) = \psi_1(x_1)\dots\psi_M(x_M), \tag{30}$$

because then the velocity of each particle is of the form (25) and does not depend on a surface $\Sigma_{k,x}$. So, if the wave function of the subsystem is of the form (30), then the dynamics of the particles $1,\dots,M$ does not depend on the other particles. If it is *not* of that form, then the dynamics of (some of) the particles $1,\dots,M$ will depend on ψ_e. Namely, if the wave function ψ_s is entangled, then the non-local influences between the particles $1,\dots,M$ will be instantaneous along the leaves of the foliation determined by the total wave function ψ. As a result, the relativity principle will hold for subsystems for which the wave function is of the form (30), but not for those not of that form. Because if ψ_s is of the form (30), then for a given solution of the dynamics one can always apply any Lorentz transformation on just this subsystem to obtain a new possible solution, because the particle trajectories are the integral curves of a 4-vector of the form (25). (These solutions are even exact in this case because of the simplified setup of non-interacting particles.) As such, the relativity principle only holds for a very limited set of subsystems. It does not hold for systems of arbitrary complexity (as characterized above) needed for a proper application of the principle.

So, apart from theory (iv), there exist isolated subsystems and Lorentz transformations can be considered of these subsystems that leave the environment invariant. As is easy to see, applying such a transformation leads to a new solution only in the theories (ii), (iii), (v) and (vi), but not in theory (i). In the latter case, the problem is exactly the same as in the Lorentz invariant formulation of Newtonian mechanics. As with the theory (iv), the relativity principle holds merely for subsystems whose wave function is completely separable.

In summary, the theories (ii), (iii), (v) and (vi) satisfy the relativity principle for isolated subsystems. The difference with Anderson's principle is that now also theory (iv) fails to qualify as seriously Lorentz invariant. As with Anderson's principle, it seems counterintuitive that theory (i) does not qualify as seriously Lorentz invariant, while theories (ii) and (iii), which enlarge the set of solutions of theory (i), do. (One could actually enlarge the solution space of (iv) too, to have a theory that satisfies the relativity principle, by stipulating that in the case of a separable wave function (27), the particle dynamics for the subsystems depends on surfaces that are determined by the energy-momentum 4-vector either corresponding to ψ_s or ψ. But such a move does not seem amenable to the case where interactions are included and the separability of the universal wave function into subsystem and environment wave function is never exact.)

5 Lorentz invariant spontaneous collapse models

From the Lorentz invariant spontaneous collapse models [21, 22, 25], let us consider the one of Tumulka [21] here, since it is also formulated for non-interacting Dirac particles, like the Bohmian theories above. It will appear that both criteria for serious Lorentz invariance are satisfied.

In this model, the ontology in space-time is given by particular space-time points called *flashes*, which can be understood as the locations and times where collapses happen.[11] The flashes are denoted by X^i_k, where $k = 1, \dots, N$ is the particle label and $i = 1, \dots, n_k$ labels the flashes for the k-th particle. The probability distribution of these flashes is determined by a wave function ψ satisfying the multi-time Dirac equations (13). To each generation of flashes a new "collapsed" wave function can be associated, but these need not to be considered to specify the law for the flashes. For a wave function ψ and initial flashes $X^0_1, \dots, X^0_N$, the joint probability distribution for the flashes X^i_k to occur in the volume elements $d^4x^i_k$ is

$$P^\psi(X^i_k \in d^4x^i_k; k = 1, \dots, N; i = 1, \dots, n_k) = \langle \psi_0 | E^{(n_1)} \otimes \cdots \otimes E^{(n_N)} | \psi_0 \rangle, \tag{31}$$

where ψ_0 is the restriction of ψ to some surface $\Sigma_1 \times \cdots \times \Sigma_N$, with Σ_i, $i = 1, \dots, N$ arbitrary space-like Cauchy surfaces. The $E^{(n_k)}$ are operators acting only on the k-th particle subspace [21, p. 836]. Their explicit expression does not concern us here. For our purposes it suffices to note the following two properties.

First, for a Lorentz transformation $\psi' = U_\Lambda \psi$, $X'^i_k = \Lambda X^i_k$, which transforms $d^4x^i_k$ into $d^4x'^i_k$, the joint probability distribution satisfies

$$P^\psi(X^i_k \in d^4x^i_k; k = 1, \dots, N; i = 1, \dots, n_k) = \\ P^{\psi'}(X'^i_k \in d^4x'^i_k; k = 1, \dots, N; i = 1, \dots, n_k). \tag{32}$$

This implies that the theory is invariant under Lorentz transformations of the whole universe. Namely, given a dynamically possible history (ψ, X^i_k) and the probability

[11] Other choices are possible for the space-time ontology, like a matter density [24].

distribution P^ψ for the flashes given by (31), then a Lorentz transformation yields another dynamically possible history (ψ', $X_k'^i$), with probability distribution P' for the flashes given by

$$P'(X_k'^i \in d^4x_k'^i; k=1,\ldots,N; i=1,\ldots,n_k) = P^\psi(X_k^i \in d^4x_k^i; k=1,\ldots,N; i=1,\ldots,n_k). \quad (33)$$

From (32), it follows that $P' = P^{\psi'}$ and hence that Lorentz invariance holds.

Second, if the wave function is separable, i.e.,

$$\psi(x_1,\ldots,x_N) = \psi_s(x_1,\ldots,x_M)\psi_e(x_{M+1},\ldots,x_N), \quad (34)$$

then because of the form (31) of the probabilities,

$$\begin{aligned} P^\psi(X_k^i \in d^4x_k^i; k=1,\ldots,N; i=1,\ldots,n_k) = \\ P^{\psi_s}(X_k^i \in d^4x_k^i; k=1,\ldots,M; i=1,\ldots,n_k) \\ \cdot P^{\psi_e}(X_k^i \in d^4x_k^i; k=M+1,\ldots,N; i=1,\ldots,n_k). \end{aligned} \quad (35)$$

Together with the first property and applying a similar argument as for Lorentz transformations of the whole universe, it is clear that the relativity principle for subsystems is satisfied.

Furthermore, the theory has no absolute objects (beyond the Minkowski metric) and so the symmetry group is the Lorentz group according to Anderson's criterion.

6 Many worlds theory

Bohmian mechanics and spontaneous collapse theories concern an ontology in space-time,[12] like the particles or flashes or mass density, whose dynamics depends on the wave function. In the many worlds theory [52], the ontology is given completely by the wave function (or more abstractly by some state vector in a Hilbert space). There is no ontology in space-time on the fundamental level. There is not even space-time on the fundamental level. Rather, space-time and stuff in space-time are considered to emerge from the wave function. Hence, possible dynamical symmetries of the many worlds theory cannot have anything to do with space-time symmetries, at least not on the fundamental level, but only with how they emerge.

There are versions of the many worlds theory that *do* take space-time and stuff in space-time as fundamental. For example, one option is to postulate a tensor field $\mathcal{M}^{\mu\nu}(x) = \langle\psi|\widehat{T}^{\mu\nu}(x)|\psi\rangle$, with $\widehat{T}^{\mu\nu}(x)$ the energy-momentum tensor operator [50] (which equals $\sum_k T_k^{\mu\nu}(x)$ in the case of the non-interacting Dirac theory). Another option is the notion of state in algebraic quantum field theory that can be associated to each space-time region [53]. These ontologies do not introduce absolute objects and they satisfy a relativity principle for isolated subsystem. So, according to both Anderson's principle and the relativity principle, these versions of the many worlds theory qualify as seriously Lorentz invariant.

[12] This is sometimes referred to as the *primitive ontology* [50] or the *local beables* [51].

7 Conclusion

We have considered two possible ideas for characterizing what it could mean to have "serious Lorentz invariance" and found that there are both Bohmian and spontaneous collapse theories that qualify as seriously Lorentz invariant according to both characterizations. Nevertheless, some Bohmian models that do not seem seriously Lorentz invariant are not identified as such, suggesting the need for a more stringent criterion.

In any case, Lorentz invariance is just one property that one could demand from a fundamentally relativistic theory. Others have been considered, see e.g. [54] for a recent discussion. For example, one could demand that a relativistic theory only contains local beables. This would make any theory with a non-local beable like the wave function non-relativistic. Another possible demand is that the theory be local. But as such it would be in violation with the implications of Bell's theorem and hence a truly relativistic theory would seem impossible. Still others could be considered. For example, in the context of experimental tests, Bell's locality condition is sometimes viewed as the conjunction of parameter independence and outcome dependence, and it has often been argued that Bohmian theories are less relativistic than spontaneous collapse theories because they violate parameter dependence [26, 55–60]. But, on the other hand, as argued in [54], spontaneous collapse theories are less relativistic than Bohmian theories in that they are non-local even for a single particle, unlike Bohmian theories.

These criteria and the ones we have considered should perhaps not be seen as strict demands for a relativistic quantum theory, but could indicate what aspects — in addition to non-locality — may need to be given up in passing from classical to quantum physics [18]. Because of the different predictions of spontaneous collapse models and no-collapse theories, like Bohmian mechanics, this is not a purely metaphysical question. Perhaps it is simply the case that Lorentz invariance merely holds at the statistical level and not on the fundamental level. Nevertheless, given the success of relativistic wave equations in standard quantum mechanics (which figure unaltered in Bohmian theories), it is worth exploring to what extent Lorentz invariance is possible.

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