# A new theory of quasi-sets without atoms: a reply to Adonai Sant'Anna

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January 2024

**Abstract**

Adonai Sant'Anna made some criticisms to the theory of quasi-sets and in particular he asked why there is no a theory of quasi-sets that does not presuppose the existence of atoms. In this paper we present a sketch of such a theory. In between the text, we make some comments on Sant'Anna's arguments and try to answer at least part of them.

Keywords: quasi-set theory, indistinguishability, identity, permutation models, Adonai Sant'Anna.

MSC2020: 03A10, 03C99, 81P05, 81P10

> I would like to point out that (...) [set theory] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behaviour. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. (...) We should consider the possibilities of developing a totally new language to speak about infinity. [set theory is known as the theory of the infinite]
>
> Yuri I. Manin [Manin, 1976]

## 1 Introduction

Quasi-set theory was proposed to cope with collections of things that can be completely indiscernible without turning to be 'identical', the same thing. The clear motivation for such a theory is quantum physics, where in the standard interpretation, quantum entities

*Partially supported by CNPq.

are considered as completely indiscernible and in some situations (such as in bosonic condensates) where there is not even in principle something that can discern them. As the Nobel Prize winner Frank Wilczek said, "in the quantum world we need uniformity of the strongest kind: complete indistinguishability." [Wilczek and Devine, 1987, p.135]

The first theory of quasi-sets ('qsets' for short) was developed in 1990 (see [Krause, 1992]) and ever since them it has being improved; see [Krause et al., 2005, French and Krause, 2006] and [de Barros et al., 2024], [Wajch, 2023] for a more recent development.

The theory is not immune to criticisms. One of the most important is concerning the attribution of quasi-cardinals to a collection of indiscernible things; some people think that for attributing a cardinal to a collection of entities, the involved elements must present identity. Another criticism concerns the nature of the quantifiers; if some element of a collection of indiscernibles has a certain property, due to their indiscernibility, all of them would have the property as well. The references and the answers to these questions can be found in [Krause, 2024a, Krause, 2024b]. Here we address to other kind of criticisms.

Adonai Sant'Anna (I shall consider his more recent paper [Sant'Anna, 2023] where other references can be found)[1] put some remarks about the version of the Axiom of Choice used in the theory $\mathfrak{Q}$ (see [French and Krause, 2006, French and Krause, 2010]) and asked why there is no a theory of quasi-sets that does not presuppose the existence of atoms. His challenge has motivated us to develop such a theory, which is sketched here. In between the text, we make some comments on Sant'Anna's other arguments and try to answer at least part of them.

Some details about $\mathfrak{Q}$ seem to be in order since they will be considered below. The theory is compatible with the existence of two kinds of ur-elements, the m-atoms and the M-atoms. Notice that there are no postulates asserting that these entities do exist; as in may formulations of ZFA (Zermelo-Fraenkel with Atoms [Suppes, 1972]), atoms are simply admitted to exist but are not postulated to exist. The M-atoms play the role of the ur-elements of ZFA, and the m-atoms are conceived so that the defined notion of *extensional identity* does not hold to them. But, for the 'classical' entities, this identity coincides with the identity ascribed by the Standard Theory of Identity (STI) of classical logic ([Mendelson, 1997] for the 'first-order' theory of identity). The theory has a primitive unary functional symbol $qc$ so that if $x$ is a quasi-set (qset for short), then $qc(x)$ denoted the quasi-cardinality of $x$. Here we shall introduce a 'finitistic' version of this notion.

With these remarks in mind, we can start by presenting the theory $\mathcal{N}\mathfrak{Q}$, a version of $\mathfrak{Q}$ but any reference to atoms.

## 2 A theory of quasi-sets without atoms

In this section we sketch a theory of quasi-sets without assuming the possibility of existence of atoms; we call '$\mathcal{N}\mathfrak{Q}$' such a theory (for 'new' quasi-set theory). The underlying logic is classical first-order predicate logic *without* identity. The specific

[1] Some near point were also posed by Eliza Wajch [Wajch, 2023], but her arguments will not be considered here.

primitive symbols are standard symbols for Peano Arithmetics (PA), namely, 0, $S$, $+$, $\cdot$, $=$ and $<$ (with all the standard notions and definitions),[2] the binary predicates $\in$ (membership) and $\equiv$ (indiscernibility, or indistinguishability), and a binary functional symbol $qc$. Individual variables for quasi-sets ('qsets') are denoted by $x, y, z, \ldots$ and the natural numbers are denoted by $n, m, p, \ldots$; symbols for punctuation are used as usual. The axioms are those presented below plus some first-order formulation of PA [Shoenfield, 1967, p.22] (or [Franco de Oliveira, 2004, Chap.4] for a formulation involving all the chosen symbols) and those of the classical first-order calculus without identity ([Mendelson, 1997]). Notice from the start that *we do not want* to take arithmetics from inside the theory of qsets; PA will act as a 'step theory' for providing us natural numbers that are not ordinals in the same sense that tensor calculus is a step theory for General Relativity. In other words, our natural numbers are 0, $S0$, $SS0$ and so on, and not something like $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, etc. We use $n, m, s, p, \ldots$ for variables ranging over natural numbers.

If $x$ is a qset and $n$ is a natural number from PA, the term $qc(x, n)$ intuitively says that the qset $x$ has *quasi-cardinal* (q-cardinal) $n$, and this stands for the quantity of elements of $x$; we shall be restricted to qsets with a finite number of elements also (that is, their q-cardinals are natural numbers).[3] We present the specific postulates of the theory $\mathcal{N}\mathfrak{Q}$ with comments.

**1. Axioms for $\equiv$** This postulate says that the indiscernibility relation has the properties of an equivalence relation, that it, it is reflexive, symmetric and transitive.

So, if $x$ is a qset, we may form the *quotient qset* $x/_{\equiv}$ whose elements are equivalence classes of indiscernible elements of $x$. When we write $(\exists y \in x/_{\equiv})\alpha$ we mean $\exists y(y \in x/_{\equiv} \wedge \alpha$, being $\alpha$ a formula; the same holds for '$\forall$', that is, $(\forall y \in x/_{\equiv})\alpha$ means $\forall y(y \in x/_{\equiv} \to \alpha)$.

**2. Weak Extensionality Axiom** (WEA) This axiom states that qsets with the same quantities of indiscernible elements are indiscernible. This is expressed in terms of the quasi-cardinalities of the equivalence classes by the indiscernibility relation this way: for each equivalence class in $x/_{\equiv}$ with q-cardinal $n$ there exists an equivalence class in $y/_{\equiv}$ with the same q-cardinality and whose elements are indiscernible from those of $x/_{\equiv}$ and reciprocally:

$$\begin{aligned} \forall x \forall y((\forall z \in x/_{\equiv})(\exists w \in y/_{\equiv})(qc(z, n) \to qc(w, n)) \\ \wedge (\forall w \in y/_{\equiv})(\exists z \in x/_{\equiv})(qc(w, n) \to qc(z, n)) \to x \equiv y) \end{aligned} \tag{WEA}$$

The intuition is to capture the idea that we could write '$H_2SO_4 \equiv H_2SO_4$' or that '$H_2O \equiv H_2O$'. In what respects chemistry, the case of isomers deserve further clarifications, since involves the notion of *form*, which is difficult to treat set-theoretically.

[2] Note that identity (=) is introduced only for the natural numbers in PA.
[3] We have borrowed this notation from [Wajch, 2023].

**Remark** An important remark is in order. As we have said before, the natural numbers which are being used for expressing the q-cardinals of the qsets are taken from the Peano Arithmetics we have added to the theory. Of course the theory will enable us to prove that there exists a model of PA within $\mathcal{N}\mathfrak{Q}$, but this is another thing: *these* natural numbers (of the model) will be, as usual, ordinal numbers (the finite ordinal numbers). We do not wand that the q-cardinals are ordinals since this would imply that there is a bijection between the qset and the ordinal, something that implies that the elements of the qset can be discerned from one another. The attentive reader could argue against this claim by saying that if we have a qset with three indistinguishable things, in defining a bijection from this qset to $\{0, S0, SS0\}$ we are discerning them after all. But this is not what the attribution of a natural number to a qset means; notice that $qc(x, n)$ does not require that such a bijection exists. It is a primitive notion whose operational behaviour is given by the axioms.

So, I beg the reader to pay attention to this: the existence of q-cardinals *does not imply* identity or discernibility of the elements of the qset. The notation $qc(x, n)$ simply states that the qset $x$ has $n$ elements in the same sense that we can day that a neutral Helium atom has two electrons without any way to tell which is which (that is, defining a bijection from the 'set' of electrons to the von Neumann's ordinal $2 = \{0, 1\}$.

**3. Schema of separation** Let $\alpha$ be a formula and $x$ a qset. The schema says that we can 'separate' those elements of $x$ that satisfy the given formula to form a new qset $y$; in symbols,

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \alpha(z)). \qquad \text{(Sep Schema)}$$

For instance, given a neutral Sodium atom whose eleven electrons are thought of as forming a qset, we can 'separate' eight of them to form the subqset of those electrons in the second energy level.[4] Notice that despite we have the property they must obey, given by their quantum numbers, we do not have any means to specify *which* of the eleven electrons are these eight; electrons are not 'things' we can put our finger on. The most we can say is that there are eight electrons in the second energy level, but no particular identity is given to them.

The qset got from $x$ by the Separation Schema throughout the formula $\alpha$ is denoted

$$[z \in x : \alpha(z)].$$

**Theorem 1** *There exists a qset with no elements.*

*Proof — Let $x$ be a qset and $\alpha$ the formula defined by $\alpha(z) \leftrightarrow z \not\equiv z$. Using the Separation Schema and considering that the indiscernibility relation is reflexive, we get the required qset.* ∎

**4. The q-cardinal of the empty qset** Any qset with no elements has q-cardinal zero:

$$\forall x (qx(x, 0) \leftrightarrow \forall y (y \notin x)).$$

[4]We recall that the electronic configuration of a Sodium atom reads $1s^2\ 2s^2\ 2p^6\ 3s^1$.

We cannot prove that there is just one empty qset since the proof would require identity, which is not being considered here (except in the metalanguage). But this does not impede us of denoting *any* empty qset with the usual symbol '$\emptyset$'. The important thing is that they are indiscernible from one another by WEA and that their q-cardinal is zero. So, when we speak of the empty qset, we are speaking of any one of them.

**5. Axiom of union** Given qsets $x$ and $y$, there exists a qset $z$ denoted by $x \cup y$ such that for any $w$, $w$ belongs to $z$ iff it either belongs to $x$ or belongs to $y$. We can generalise this axiom by stating that for any qset $x$ there exists a qset $\bigcup x$ such that for any $z$, $z \in \bigcup x$ iff there exists $y \in x$ and $z \in y$.

Let $x$ and $y$ be qsets and $\alpha$ a formula defined as $\alpha(z)$ iff $z \in x \wedge z \in y$. Applying the Separation Schema in $x \cup y$, we get the qset of all elements that are in both $x$ and $y$; this qset is the *intersection* of $x$ and $y$. The definition can be generalised to $\bigcap x$ in an obvious way.

**6. Axiom for q-cardinals**

$$\forall x \forall y (qc(x, n) \wedge qc(y, 1) \wedge x \cap y \equiv \emptyset \rightarrow qc(x \cup y, n + 1)).$$

The meaning of this axiom is quite obvious: we can increase the quasi-cardinal of a qset by carefully chosen a new element.

**Definition 1 (Sub-qset)** $x \subseteq y := \forall z (z \in x \rightarrow z \in y)$.

This definition also deserves some care. How can we know if a qset $x$ is a sub-qset of the qset $y$? By the definition, when all elements of $x$ are also elements of $y$. But in order to know if some $z \in x$ also belongs to $y$, we need to find in $y$ some element that is *identical* to $z$, and this requires identity. So, the most the theory enables us to do is to *suppose* that $x$ is a sub-qset of $y$ and leave the proof to the particular theory to which $\mathcal{N}\mathfrak{Q}$ is being applied, for instance, chemistry. In this discipline, using the above example of the Sodium atom, we can say that there is a subqset of the qset of the electrons whose elements are those electrons of the second energy level.

The next two axioms will give us more ways for argumentation about sub-qsets.

**7. Axiom of the power qset** Let $x$ be a qset. Then there exists a qset termed $\mathcal{P}(x)$ such that for all $z$, $z \in \mathcal{P}(x)$ iff $z \subseteq x$ and further, being $qc(x, n)$, then the q-cardinal of $\mathcal{P}(x)$ is $2^n$.

**8. Axiom for q-cardinals** If the q-cardinal of a qset $x$ is the natural number $n$, then $x$ has a sub-qset with q-cardinality $p$ for any $p \leq n$. In symbols, we can write

$$\forall x \forall n (qc(x, n) \rightarrow (\forall p \leq n)(\exists y \subseteq x)(qc(y, p))).$$

**Theorem 2** *If $qc(x, n)$ and the elements of $x$ are indiscernible, then all sub-qsets of $x$ with the same q-cardinality are indiscernible.*

*Proof — Immediate consequence of WEA.* ▮

This result brings interesting consequences. For instance, let $qx(x, n)$, that is, the qset $x$ has $n$ elements and suppose they are indiscernible. How many possibilities we have of distributing the $n$ elements in $p \leq n$ sub-qsets ('states')? A simple reasoning will convince you that the number is done by the formula which expresses the Bose-Einstein statistics, namely,[5]

$$C_n^{n+p-1} = \frac{(n+p-1)!}{n!(p-1)!}.$$

We can fortify the theory with axioms corresponding to the axioms of infinite, regularity, replacement and choice.

**9. Axiom of regularity** We can formulate it as follows:

$$\forall x(x \not\equiv \emptyset \to \exists y(y \in x \land y \cap x \equiv \emptyset)) \tag{Regularity}$$

**10. Axiom of infinity** There exists a qset that have all the natural numbers as elements.

**11. Axiom of pairing** Given the qsets $x$ and $y$ as elements of a qset $z$, there exists a qset whose elements are those elements of $z$ that are indiscernible from either $x$ or $y$; we denote this qset by $[x, y]_z$. Notice that the q-cardinality of the 'pairing' qset may be greater than two.

One of the cute remarks given by Sant'Anna is that without the restriction to elements of $z$, the simple qset $[x, y]$ would be a proper class. He is right. This was corrected in [French and Krause, 2010] and used accordingly since them.

**Definition 2 (Singleton)** *If $x \in z$, the 'singleton' of $x$ (relative to $z$) is the qset $[x]_z$ defined this way:*

$$[x]_z := [x, x]_z \tag{Singleton}$$

The q-cardinality of $[x]_z$ may be greater than one. As just remarked, the reference to the qset $z$ is essential for on the contrary, something like $[x]$, the qset of *all* indiscernible from $x$, would be what we could call a *proper qclass*.

**Remark** This postulate has also interesting consequences. Let $x \subseteq y$ and take $z$ as the qset $y \setminus x$ (the difference of qsets, see the definition below) and suppose yet that the elements of $y$ are indistinguishable. Thus, how can we know which elements of $y$ belong to $z$? There is no 'objective' way to know that. But this is the same as questioning how can we know if some particular electron (if we could take one) belongs or not to a neutral Lithium atom or if some proton belongs to it; these questions have no sense. We simply know that there are three electrons and three protons there and this is enough; if our chose electron is in Mars and the Lithium atom is in the Moon,

[5] This result was previously discussed in [Sant'Anna et al., 1999]; see also [French and Krause, 2006, §7.6].

then *surely* our electron will be not in the atom, but we need care in dealing with a so quick conclusion.[6] What imports in quantum phyeics is that there are the kinds of things we are dealing with and their quantities, and the theory of quasi-sets enables us to consider this idea quite cutely. In the same vein, we can suppose that if there exists some element of $y$ that does not belong to $x$, then the qset $z$ is not empty and this is enough once the elements of $y$ are indiscernible.

**Definition 3 (Strong singleton)** *Let $x \in z$. A strong singleton of $x$ relative to $z$ is a subqset of $z$ whose element is indiscernible from $x$ and whose q-cardinal is one. We denote it by $[\![x]\!]_z$. If some qset $y$ is a strong singleton, we write $\mathcal{S}(y)$.*

In standard set theory, where STI holds, the strong singleton of $x$ would be its singleton set $\{x\}$. But this presupposes identity: $x$ is *the only* element of $\{x\}$. Since we are not considering the notion of identity, we cannot specify which is the only element of a strong singleton; what indicate that there is just one is its q-cardinal (but see below). But if all elements of $z$ are indiscernible and $[\![x]\!]_z$ and $[\![y]\!]_z$, then $[\![x]\!]_z \equiv [\![y]\!]_z$. The existence of strong singletons comes from the axiomatics; suppose $qc(z, n)$ with $n \geq 1$. Then, by Axiom 8, there exist (at least) one subqset of $z$ with q-cardinality one. Furthermore, it is consistent with the theory to suppose that the q-cardinal of the qset of all strong singletons of $z$ is $n$; we cannot differentiate them, but they count as more than one, so as bosons in a BEC or electrons in an atom.

An important remark concerns strong singletons. Eliza Wajch insist (private correspondence) that we cannot prove that a strong singleton has just one element. My answer is as follows. What we cannot do is to know which particular element belongs to a strong singleton $[\![x]\!]_z$, say to ascribe it a proper name which act as a rigid designator.[7] The only thing we can advance is that its element belongs to $z$ and is indiscernible from $x$. But since the q-cardinal of $[\![x]\!]_z$ is one, I suppose we can reason, at least in the metamathematics, that it has just one element.

### 12. Axiom for q-cardinals

$$\forall x \forall n (qc(x, n) \rightarrow (\forall y \subseteq x)(qc(y, 1) \rightarrow qc(x \setminus y, n - 1))).$$

This axiom says that we can 'eliminate' and element of the qset $x$ by admitting the existence of the qset $y$, despite we cannot identify the eliminated element. The situation is similar to what happens, say, when we ionise a neutral Helium atom. The He atom has two electrons and by given it some amount of energy, we can expunge one of the electrons getting a cation $\mathrm{He}^+$ and we could write $qc(\mathrm{He}^+, 1)$ while $qc(\mathrm{He}, 2)$. But nothing can tells us which electron was expunged.

[6] Really, in considering the joint system formed by the two electrons (one in Mars and another in the Moon), the wave-function is anti-symmetric and when we take its square to obtain some probability, an *interference term* does appear, which rigorously cannot be eliminated except for some practical purposes. To ignore the interference term, as physicists usually do in their 'practical physics' provide good results, but it is something similar to disregard infinitesimals in the earlier calculus; as George Berkely has shown, the results may be fine, but the logical foundations become weird; see [Krause, 2024a] for a discussion.

[7] This is an expression that came from Saul Kripke meaning a label that identifies an entity as *that* entity in every possible world where it exists. Proper names are supposed to play this role.

**Definition 4 (Difference of qsets)** *Let $x$ and $y$ qsets so that $x \subseteq y$. Then the difference $y \setminus x$ is defined as a qset whose elements are the elements of $y$ that do not belong to $x$.*[8]

**Theorem 3 ('Unobservability' of permutations)** *Let $x$ be a sub-qset of $y$ and $w \in x$. If $z \in y$, $z \equiv w$ but $z \notin x$, then*

$$(x \setminus [\![w]\!]_y) \cup [\![z]\!]_y \equiv x.$$

*Proof: If $qc(x, n)$, then $qc(x \setminus [\![w]\!]_y, n-1)$ by the Axiom 12. But $qc([\![z]\!]_y, 1)$ and so $qc(x \setminus [\![w]\!]_y \cup [\![z]\!]_y, n)$ by the Axiom 6. Since $w \equiv z$, then $[\![w]\!]_y \equiv [\![z]\!]_y$ and the theorem follows.* ▮

This is another polemic result of the theory of quasi-sets. The theorem is saying that if we 'exchange' an element of $x$ by an indiscernible one, the resulting qset is indiscernible from the original. This is analogous to the He atom of the above example. Suppose we ionise it by expunging one electron and later we make the cation to absorb an electron turning a neutral atom again. What is the difference between the new neutral atom and the original one? No one can tell us! They have exactly the same properties: they are indiscernible.

Sant'Anna criticises this result by saying that the theorem is not true in any permutation model of $\mathfrak{Q}$ minus the Axiom of Choice and minus the a-cardinals axioms. Well, as he emphasises, the proof depends on the notion of q-cardinal. Thus, if one drops the corresponding axioms as he did, the theorem cannot be proven and it would be a mistake to say that it is false. Notice that in no permutation model there are m-atoms as posed by the theory of quasi-sets $\mathfrak{Q}$; what one can achieve in such models is atoms that *pretend* to play the role of m-atoms of $\mathfrak{Q}$, but which behind the curtain are revealed to be individuals of ZFA. The wolf with a sheep face.

We think that we can reason as follows. For simplicity, take a reconstruction of Fraenkel's original second permutation model. In ZFA, let $x = \{a, \overline{a}\}$ be one of Fraenkel's *cells* [Fraenkel, 1967]. In the suitable permutation model, $\pi(a) = \overline{a}$, being $\pi$ an automorphism of the model, got from any permutation of the atoms. Hence the two elements are indiscernible *within the model*. Let $x' = x \setminus \{a\}$ and then $x'' = x' \cup \{a\}$. It follows that $x = x''$. These operations can be done because the elements are individuals endowed with identity, so we can *identify $a$* as being the same atom in both situations. This exactly what the above theorem is saying but using the indistinguishability relation instead of equality. I don't see why the theorem would be so problematic.

**Side remark** One question may intrigue the attentive reader. If $\alpha$ is a formula whatever and some $x$ satisfies $\alpha$, does any $y \equiv x$ satisfies it as well? In other words, does indiscernibility entails substitutivity? Notice that if this were true, since it is reflexive, it would be standard identity. So, we need to prove that this is not the case. An example is enough for that.

Let us consider a strong singleton $x$ whatever whose only element is denoted by $y$, so we can take $\alpha$ as $y \in x$. Of course we cannot know *which* entity is $y$, but since $qc(x, 1)$, we know (in principle) that *there is nothing else there*. Suppose that $z \equiv y$; can

[8] Usually the difference between sets $y$ and $y$ is denoted by $y - x$.

we conclude that $z \in x$ as well only because $z$ (whatever it is) is indiscernible from $y$? Of course not. This would entail that $x$ would contain as elements every indiscernible from $y$ and so its quasi-cardinality would not be one. In other words, from $y \in x$ and $z \equiv y$ we cannot conclude that $z \in x$. So, the indiscernibility relation is not a congruence, being distinct from the standard identity relation.

Since we are preparing the theory $\mathcal{N}\mathfrak{Q}$ for applications in the quantum domain, we think that we do not need, in principle, of a version of the Replacement Schema; anyway, we think that if necessary it can be assumed exactly as in $\mathfrak{Q}$ [French and Krause, 2006, §7.2.4], [de Barros et al., 2024]. More relevant seem to be a 'troublesome' version of the Axiom of Choice.

**13. The Axiom of Selection of Indiscernibles** This postulate was termed 'Axiom of Choice' in [French and Krause, 2006] and 'Axiom of Quasi-Choices' in [de Barros et al., 2024]. Due to the criticisms, which we find debatable, we prefer to change its name once again so that avoiding (so we hope) any association with the standard Axiom of Choice, despite the similarities. It simply states that some selections are possible in the theory.

The idea is quite simple and can be explained as follows. Suppose you have a qset whose elements are also qsets (since there are no atoms) but non-empty and without common elements (pairwise disjoint), say an imaginary collection of electrons, protons and neutrons. Then we wish to assume that there exists a qset whose elements are indiscernible from an element of each of the elements of the given qset, that is, a collection with one electron, one proton and one neutron.

In more specific talking, let $x$ be a qset so that if $y$ and $z$ are elements of $x$, then they are not empty (that is, not indiscernible from an empty qset) and $y \cap z \equiv \emptyset$. Then there exists a qset $u$ of the same q-cardinality than $x$ such that for any $w \in u$, there exists an element $s$ belonging to some element $y$ of $x$ that is indiscernible from $w$. The requirement that the q-cardinality of $u$ is that of $x$ grants that we are taking one element of each element of $x$.

We can formulate the axiom this way, where $|x|$ stands for the q-cardinality of $x$:

$$\begin{aligned}\forall x(\forall y \forall z(y \in x \wedge z \in x \to y \cap z \equiv \emptyset) \to \exists u(qc(u, |x|) \wedge \forall w(w \in u \to \\ \exists y(y \in x \wedge \exists s(s \in y \wedge s \equiv w))))\end{aligned} \tag{AQC}$$

The intuitive account of this axiom is so evident that it looks strange to question it. Anyway, formally of course things may be different.

Sant'Anna introduces a permutation model of ZFA which he says is a model for $\mathfrak{Q}$ minus choice and minus the axioms for q-cardinals (of the theory $\mathfrak{Q}$). Then he states that the Axiom of Choice of $\mathfrak{Q}$ is not true in his model; notice that he is not speaking of the above axiom. Well, if the Axiom of Choice of $\mathfrak{Q}$ is false in Sant'Anna's model, the conclusion if of course that the model does not model the Axiom of Choice of $\mathfrak{Q}$, endpoint. But he goes further. He defines another permutation model in which the axiom is true. Since it is supposed to be independent of the remaining axioms of $\mathfrak{Q}$, this would be not a surprise; all he has done, if his results are correct, is that the Axiom of Choice of $\mathfrak{Q}$ is independent of the remaining axioms (supposed consistent). Notwithstanding, Sant'Anna's conclusions says that "within the context of permutation models, quasi-set theory [that is, $\mathfrak{Q}$] is either inconsistent or equivalent to ZFU + Axiom of Choice [he means ZFA], provided that there are no micro-atoms [m-atoms]". This is grounded on the conclusion that in his new model, there are no m-atoms. The reason is not clear to me. In fact, the assertion is not clear firstly because ZFU (or ZFA) involves already an Axiom of Choice; so, which is the axioms being added to ZFU in his supposition? Secondly, the theory $\mathfrak{Q}$ is consistent with the existence of m-atoms, yet it does not postulate their existence. So, if there are no m-atoms, of course the theory collapses into ZFA, but this is a well-know result put already in [French and Krause, 2006]. The interesting fact is the supposition that they exist; this is what links the theory to quantum physics.

# 3 Models: just a sketch

The signature of $\mathcal{N}\mathfrak{Q}$ is the tuple

$$\langle 0, S, +, \cdot, =, <, \in, \equiv, qc \rangle.$$

So, in order to model the theory, we need to find interpretations to these symbols so that the axioms are satisfied. A question enters here, one which no philosopher I know has discussed it yet; it can be put simply this way: *where* these interpretations are constructed? It would be quite strange to construct a model for $\mathcal{N}\mathfrak{Q}$ in a set theory like ZFC or ZFA, since in thees theory all entities are endowed with identity and can always be discerned from one each other, even if at least in principle (atoms of ZFA inclusive). This move can be formally done, but would distort the ideas of $\mathcal{N}\mathfrak{Q}$.[9] So, there are two open possibilities: to construct an *informal* semantics, grounded on the natural language and to construct a formal semantics in the theory of quasi-sets $\mathfrak{Q}$.

The formal semantics can be elaborated on the following grounds. The notions corresponding to PA are interpreted in the standard model of PA that can be constructed in the 'classical' part of $\mathfrak{Q}$, that is, that part that does not involve m-atoms. The membership relation and the indiscernibility relation are interpreted in their corresponding relations restricted to qsets only. It is easy to see that such a structure models $\mathcal{N}\mathfrak{Q}$.

Notice that such a construction is not merely defining a model for $\mathfrak{Q}$ *within* $\mathfrak{Q}$, since there are differences between $\mathfrak{Q}$ and $\mathcal{N}\mathfrak{Q}$. But, of course, it seems that such a semantics is merely repeating things already said in $\mathfrak{Q}$. But, since no other theory can be found admitting things like m-atoms without attributing them identity, the only alternative would be to consider an informal semantics.

Of course the most interesting semantics for any theory of quasi-sets is the informal one. We have delineated it in the case of $\mathcal{N}\mathfrak{Q}$ in the previous discussion of the axioms, but turn to it again now. The motivation is, as said already, quantum physics, but we are free to imagine a situation involving absolutely indiscernible objects out of the quantum realm.

[9] For instance, the standard presentation of quantum mechanics presupposes Hilbert spaces which demand bases. But there are models of ZFA where there are vector spaces with no basis or vector spaces with basis of different cardinalities [Jech, 1977]. Can we chose a model like these ones to consider quantum mechanics? There are other situations where things show that we would pay attention to the metamathematics.

The natural numbers can be assumed as we do when reason about mathematics. The membership relation means 'to be an element of' as usual. The indiscernibility relation is the most controversial one. Can we assume that there are indiscernible entities? In our standard metaphysics, which has influenced classical logic and standard mathematics, so as classical physics, indiscernible things can be *relative* to one or to a bundle of properties, say when two objects share an equivalence relation, or *absolute*, when they share *all* possible properties and relations. In this last case, says our Leibnizian metaphysics, they are *identical*, that is, *the same object*. Indiscernibility entails identity, understood in the sense of sameness.

But, as Yuri Manin (and others) has suggested in the quotation put in the epigraph, quantum physics has presented us entities with a completely different behaviour. Bosons can share the same quantum state and in such a situation, there are no differences among them, *not even in principle*, for instance when we take a Bose-Einstein Condensate (BEC) [Ketterle, 2007]. This is reflected in their obedience to Bose-Einstein statistics. So, if we represent a BEC as a collection of entities in a framework encompassing STI, they would be *distinct* things and would present differences, contrary to what the quantum theory says. Thus, we have a strong motivation to introduce $\mathfrak{Q}$ and $\mathcal{N}\mathfrak{Q}$.

Concerning fermions, since they obey Pauli's Exclusion Principle, they cannot share the same state, hence one may say that there will be always a 'difference' between two fermions, so that they cannot be *completely indiscernible* even when sharing an entangled state. This brings a challenge. The challenge is that despite they present a difference, one cannot take, say, two electrons (which are fermions) and (using Hermann Weyl's example) state '*this* is Mike' and '*that* is Ike' in such a way that these names act as rigid designators [Weyl, 1950, p.241]. As Weyl said, "one cannot demand an alibi of an electron." That is, it is impossible that electrons retain their identities so that one electron could say 'I am Mike', while the other would say 'I am Ike' in different contexts. Any permutation between Mike and Ike will be not differentiated by the measurement of any observable; this is exposed in the Principle of Indistinguishability, which reads

$$\langle\psi|\hat{A}|\psi\rangle = \langle P\psi|\hat{A}|P\psi\rangle,$$

where $\psi$ is the state of the system, $\hat{A}$ is an Hermitian operator standing for some measurable quantity, and $P$ is a permutation operator. The expression $\langle\psi|\hat{A}|\psi\rangle$ stands for the *expectation value* of the measurement of the observable while the system in in state $\psi$; the rule says that this value does not change after a permutation of the entities involved in the system. So, even being fermions there are no detectable physical differences among them when they are of the same kind. But, if we are in a standard set theory such as ZFC or ZFA, even if there are no 'physical' differences, there are the logical ones, for instance to belong to their unitary sets. The discussion whether 'logical' properties count in discerning quantum entities is interesting and discussed in [Krause, 2024a]. Consequently, it would be a categorical mistake to assume that they can form *sets* in the sense of standard set theories, which are collections of *distinct* (hence *distinguishable*) things. In our opinion, the best way to represent collections of them is by using qsets.

## 4 More on Sant'Anna's conclusions

As seen before, Adonai Sant'Anna introduced permutation models for the theory $\mathfrak{Q}$ minus the Axiom of Choice and minus the axioms for q-cardinals of such a theory [Sant'Anna, 2023]. We agree that this can be done but such models are built in a theory where identity holds for every entity;[10] despite in the model some atoms *are made* invariant by automorphisms, standing for indiscernible things, when you leave the model you realise that even these atoms are individuals endowed with identity; recall that the whole universe of ZFA is a rigid structure since *every* entity in the universe has its singleton set which belongs to the universe.[11] So, it is not correct to say, as Sant'Anna did, that the axioms of ZFA do not distinguish among atoms. This sentence is surely reproduced from T. Jech's book [Jech, 2008, p.45], but should be read with some care. When Jech states that, he is referring to permutation models; thus, *inside* the models really the postulates do not distinguish among atoms (at least between 'conjugate' atoms) since they are constructed precisely for this aim; they are deformable structures, enabling the existence of nontrivial automorphisms. Consequently, these permutation models do not help us in representing entities that are indiscernible and cannot be identified by any means, as it seems to be the case of quantum systems in several situations, for the whole ZFA universe, being rigid, will enable the distinction when we are not confined to a permutation model.

Furthermore, concerning the standard permutation models of ZFA$^-$ (ZFA minus Regularity), one usually start with an infinite *set* $A$ of atoms and construct a hierarchy indexed by ordinals by positing $\mathcal{P}^0(A) := A$, $\mathcal{P}^1(A) := \mathcal{P}(A) \cup \mathcal{P}(\mathcal{P}(A))$, etc. (see [Jech, 2003, p.250]). But here we can return to an already posed question, namely, *where* these models are constructed? Even if you assume naïve set theory, the fact that $A$ is a set says already that its elements are distinct from one another, and you need to use the power set operation, which makes sense only for sets. That is, despite you make the trick saying that *inside* the model the atoms are indiscernible, really they are not! In 1922, Fraenkel, when constructed one of his models to prove the independence of the Axiom of Choice from the remaining axioms of ZFA (that time termed 'ZFU', the 'U' meaning *Urelemente*, atoms), assumed "a denumerable infinite number of *distinct* objects (...), none of which is considered as a set" (my emphasis) [Fraenkel, 1967], that is, atoms. Of course Fraenkel's paper can be said to have just an historical value, but the above argument prevails even today: $A$, the set of atoms, is a *set*. So, we conclude that permutation models can save the day FAPP (for all practical purposes, in John S. Bell's expression), but are not useful to cope with a metaphysics of non-individuals.

[10] Another approach admitting proper 'quasi' classes is presented in [Wajch, 2023] and developed 'from the scratch'.

[11] Of course, if the unitary sets did not belong to the universe, maybe we would have no other means to discern one object from another. This is what happens, for instance, with the additive group of the integers $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$, which is not rigid since $h(x) = -x$ is an automorphism of the structure. So, for instance, 3 and $-3$ are indiscernible *within* the structure, but not in the whole universe of sets. By the way, a set such as $\{3\}$ does not belong to the universe of the structure. But, in this particular case, the structure can be made rigid by adding the binary relation $<$. A strong result concerning standard set theories is that *every* structure can be extended to a rigid one by adding new predicates and relations so that if we report to the whole 'universe of sets', say the von Neumann's hierarchy of well-founded sets, we realise that *everything has identity*.

So, Sant'Anna's questionings, such as "the mere existence of permutation models [like those he has suggested] compromises the intended interpretation of quasi-set theory as [a possible formalisation of] a world [which] admits the existence of non-individuals" cannot be accepted at all! The permutation models, as argued before, one could say, at least can *save the phenomena*, or can be used FAPP but does not cope with the intended metaphysics, which is the main aim of the theory of quasi-sets. Furthermore, Sant'Anna claims things like this: "why can't we pursue another way [to cope with quantum entities]?" Of course one can do that, and this can be done, as is well known, for instance, taking Bohmian mechanics where the particles are individuals endowed with identity [Tumulka, 2022]. But no one of these attempts cope with *truly* indiscernible things, this being understood as entities that are taken as indiscernible *from the start*, as suggested by H. Post a long time ago [Post, 1973] and not made indiscernible *a posteriori*, by hand.

So, taking into account that the theory of quasi-sets was proposed to formally express a metaphysics of non-individuals (in the above sense), one can accept it or not, but it would be a mistake to criticise it for not doing what it was not proposed to do.


## Acknowledgement

I would like to thank Adonai S. Sant'Anna for the collaboration and discussions (and criticisms!) about quasi-set theory. It was Adonai who provoked me to develop a quasi-set theory without atoms. Perhaps the theory sketched in this paper fulfils his hopes. I also thank Eliza Wajch for several discussions and improvements with respect to the theory of quasi-sets.


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