The Continuum Hypothesis: schisms and other isms

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Overture. The present dialogue[1] is built around one of the most relevant debates in the contemporary philosophy of mathematics, namely, the analysis of the nature of the axioms. Indeed, the search for new axioms of set theory and the mere possibility of this process constitute quite deep concerns for the philosopher[2]. In the following, the reader will be surrounded by questions like *what constitutes an axiom?*, *can we come up with new axioms?*, *what is the nature of formal systems?*, which are equally important as fuzzy. We have tried to elude the main difficulty that arises when dealing with statements as these: being too wide in our scope.

What came to be the general plan of this article came to me during the conference *What Can Philosophy Do For Set Theory?*[3]. The sources of inspiration and, at the same time, the main goals intended to be achieved, are the following: (a) Present a refinement of an old interpretation of Wittgenstein's philosophy of mathematics through the fragment *Phenomenology* belonging to the *Big Typescript*, (b) Examine the role of phenomenology and historical analysis in the debate surrounding the nature of the axioms of ZFC, (c) Provide a brief interpretation of Kant's Architectonic as an exposition of central ideas of this debate. The success of the dialogue can be measured by how fruitfully this three tasks interact with each other.

Finally, some justification for the choice of form. The sudden idea of writing a dialogue was caused by my reading of the fictional conversation that appears in a well known paper of Feferman. Of course, the influence of *Proofs and refutations* by Lakatos is ubiquitous. My initial thought was that, in a dialogue, the presentation of the main theses would gain both clarity and appeal for the reader. I hope to have achieved such an effect[4].

[1] The first version of this article was finished on July 25, 2023. It is partially a continuation of *Wittgenstein on decisions and the mathematical practice*, from which it borrows some terminology (the draft of the latter was completed during January 2023).

[2] Maddy confesses that the program of finding new axioms for deciding undecidable statements in ZFC led her to study this kind of ideas more closely.

[3] Which took place in the University of Barcelona from the 29 to the 31 of March 2023. The redaction of this article began during April 2023.

[4] Keeping in mind the length of the present article, it is likely that further revisions and additions will be made in the future.

1. Presentation of the debate. We witness the debate between a group of mathematicians with a heavy philosophical background. The conversation is focused on the status of some phenomena that have arisen in set theory.

OMEGA: Let us start the debate by posing the question central to it, namely, *is the Continuum Hypothesis (CH) a proper mathematical problem and thus susceptible of solution?*

ALEPH: Of course it is!

BETH: Of course it is not!

DALET: Personally, I find this question uninteresting[5].

OMEGA: It is clear that, from our question, one should start by analyzing the use of the expressions *(proper) mathematical problem* and *solution*.

ALEPH: In my opinion, there is nothing special about CH beyond the (historically proven) fact that it is a *hard* problem[6]. But hardness does not undermine definiteness. Moreover, the hardness of mathematical problems is usually one of the reasons why they are appealing to us.

BETH: Indeed, we are used to deal with this open nature of conjectures when doing mathematics. Nevertheless, it is not so clear that CH *is* a definite mathematical problem, so we should not expect an absolute solution to it neither[7].

ALEPH: How so, Beth?

BETH: Well, CH certainly assumes some basic notions, such as *continuum*, *subsets of the continuum* and *mappings between such subsets*. But there may be some problem with how set theory usually describes such terms. For example, the definition of power set is *impredicative*, i.e. we cannot quantify over it without falling into some kind of circularity. This may prove that the power set is not a definite totality and, therefore, this kind of indefiniteness ends up affecting the meaning of CH[8].

ALEPH: So, in order to give a full answer to the question posed by Omega, what you are stating is a twofold thesis: '(A) CH is an inherently vague problem [and] (B) No new axiom will settle it in a convincingly definite way'[9].

BETH: Exactly, regarding CH, 'there is no way to sharpen it to a definite statement without essentially changing the meaning of the concepts involved in it'[10].

[5]This is the opinion expressed by some as Shelah in [1]. In fact, one can find in this kind of views a methodological critique of set theory and an explanation of the aggressiveness of the independence phenomenon in some of its branches.

[6]This position is held by Woodin. See [2].

[7]This is the opinion of Feferman in [7].

[8]See the previous footnote.

[9]See [4].

[10]Again, see [7].

ALEPH: I now see that you are a *predicativist*. Nonetheless, it is possible for me to reply you in a general manner: philosophical revisionism is far from reaching the usual concerns of the working mathematician and set theorists *are* mathematicians working in a certain field[11]. Moreover, if you want a metaphysical answer to your objection, let me say that the vicious circle principle is relevant 'only if one takes the constructivistic (or nominalistic) standpoint toward the objects of logic and mathematics, in particular toward propositions, classes and notions, e.g., if one understands by a notion a symbol together with a rule for translating sentences containing the symbol into such sentences as do not contain it, so that a separate object denoted by the symbol appears as a mere fiction'[12].

BETH: But Aleph, you are the one who is making exotic assumptions. I do not doubt that there are some well-known and natural standards of mathematical truth. The problem with your viewpoint is, in fact, that you have not given a full justification on why we should treat CH within those standards. In other words, 'the usual idea of mathematical truth in its ordinary sense is no longer operative in the research programs of Martin, Steel, Woodin, et al. which, rather, are proceeding on the basis of what seem to be highly unusual (one might even say, metaphysical) assumptions'[13].

ALEPH: So, in the end, what you propose is a revision of how set theory is developed on a metaphysical basis.

BETH: Or, perhaps, with other words, what I propose is a revision of the way you dangerously mix statements of different nature: what you are doing is *attaching metaphysical assumptions to mathematical practice and then claiming that we should leave the latter untouched*.

OMEGA: But, Beth, while I agree with you on the fact that Aleph *does make definite decisions and pretends to hide them under the label of mathematical activity*, don't you see that he can always find a flexible metaphysical justification of set theory (or, even better, a *wider* ontological domain)? Philosophical debate is not something that can be easily concluded. Moreover: you are also taking an explicit explanatory path, although different from the one taken by Aleph. Therefore, I suggest that we keep this kind of considerations aside for a while.

BETH: Alright. But then, allow me to ask Aleph for a better explanation of his solution. For example, how does the independence phenomenon fit with your conception of *solution*?

DALET: Exactly, why you don't accept the independence of CH as constituting *itself* the

[11] This is the classic argument used by naturalists. Of course, leaving mathematical activity as an untouchable totem has its consequences, as we will see.

[12] See [8].

[13] Again, see [7].

solution to the problem[14]?

ALEPH: For me, the independence phenomenon is not problematic, since it is clear 'not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set up so far'[15]. But, according to this, it seems an error to take the present status of CH as preventing any future elaboration towards its *real* nature. In fact, let me say that Dalet is being the true conservative here: he protects a petrifaction of ZFC, eliminating any further development...

OMEGA: ...while Beth is also a supporter of *progress*, although that progress differs from yours, Aleph.

ALEPH: For me, her conception of progress does not differ from what I am inclined to call *regression*.

GIMEL: I believe that Beth is in the right path, although I do not share her constructivist views.

ALEPH: Go on, Gimel.

GIMEL: Let me return to the beginning of our conversation. Aleph's *dream solution* to CH consists of two steps: 'first, one introduces the new set-theoretic principle, considered obviously true for sets in the same way that many mathematicians find the axiom of choice or the axiom of replacement to be true; and second, one proves the CH or its negation from this new axiom and the other axioms of set theory'[16].

ALEPH: Exactly.

GIMEL: But then, what I claim is that the second step undermines the possible achievement of the first one.

ALEPH: I don't see why.

GIMEL: Let me explain it to you. First, note that set theorists are used to work with some set theoretical universes in which CH holds and with others in which is false. As you defend the working mathematician, Aleph, I think you should agree on this[17].

[14]This is the opinion shared by some like Cohen.

[15]See [9]. In a footnote of this article one can find the following: 'the concept "property of set" (the second of the primitive terms of set theory) can constantly be enlarged, and furthermore concepts of "property of property of set" etc. be introduced whereby new axioms are obtained, which, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained in the axioms depending on the concept of set'.

[16]For this characterization of Gödel's program, see [5].

[17]Here is impossible not to read this ironic sentence of Gimel without the following connotation: the working mathematician is usually treated by philosophers of mathematics as the ideal worker that must be saved from rival methodologies or theories, she is the idealization of a sincere and silent laborer who does not have thoughts in her own. This kind of paternalism has obvious links with demagogy (let us not forget

ALEPH: And I do, proceed.

GIMEL: But now suppose that some set theorists propose a natural axiom that allows the decision between CH and its negation, say, that implies CH. On the one hand, this axiom is claimed to be natural in an absolute way and, on the other, the deep experience belonging to set theorists acquainted with the universes in which CH fails makes impossible for them to recognize it as such[18].

BETH: That is quite similar from what I was saying! The proponents of a solution of CH, based on exotic metaphysical assumptions, may elaborate plausibility arguments regarding their axioms which correspond to *their* experience, but which are not appealing to the mathematical community[19].

OMEGA: So, Gimel, what you are proposing is that the source of evidence is not unique or, in other words, that the dream solution to CH should unify a *natural evidence* and a *technical evidence* and that this is not so obviously feasible.

GIMEL: Exactly, the solution proposed by Aleph could only work if it incorporated a further step: *to show why the deep experience that we have of the negation of the settled statement is actually flawed*. That is, the followers of Aleph 'must not only argue for their preferred answer, mustering whatever philosophical or intuitive support for their answer as they can, but also they must explain away the illusion of our experience with the contrary hypothesis'[20].

BETH: This means that it is now Aleph who is against the commonplace activity of set theorists because they should give up their (already existent) work related with, say, the negation of CH. How ironic!

ALEPH: But your counterattack only *refines* what I claim to be possible, Gimel. I can now modify my original thesis and incorporate your third condition in my solution.

GIMEL: Don't you think that this third condition really goes against the *plausibility* of your desired solution?

ALEPH: Not if I *put more weight on the notion of naturalness*.

OMEGA: Please Aleph, explain yourself.

ALEPH: Of course. The third step is only a corollary from the naturalness of the axiom selected in the first step. But since the experiences with both CH and its negation are, as such, merely empirical, I can surpass their apparent appeal on the basis of the acceptance of

that a common feature of philosophers of mathematics is to vindicate themselves as the ultimate saviours of this -supposedly- neutral work carried out by the mathematical community).

[18] This argument can be found in [5].

[19] Again, [7].

[20] See [5].

a natural new axiom, not unlike the ones you mentioned as examples, which has more power of persuasion by its *a priori* nature. If we could find a set of conditions that new axioms had to meet (in order to constitute a non-arbitrary and necessary fulfilment of the conception of iterative set), then we could discard our intuitions on the negation of the decided statement by simply realizing that they are of the same kind as those one could have when working with all the usual axioms of ZF and interchanging the axiom of replacement with its negation, for example.

GIMEL: I'm not sure I agree with you on this.

ALEPH: Now, in your first step you talked about 'naturalness' in a somewhat imprecise way. I think that you were actually appealing to the idea of *intuition*. But the naturalness of an axiom is *not* the same as intuitiveness, since intuition can be defective and, at the same time, can allow different (and irreconcilable) ways of development of a formal system[21].

GIMEL: Where do you want to arrive at?

ALEPH: What I am pretending to make clear is that the term 'naturalness', which you are using quite freely, contains quite different and possible ramifications of meaning. For example, let me distinguish between two kind of criteria for 'naturalness'. I call a criterion *intrinsic* when it deals with the non-arbitrariness of a new axiom and *extrinsic* when it has to do with the pragmatic value of such axiom[22]. My proposal is, then, that we use the three intrinsic criteria of *consistency*, *maximality* and *fairness*[23], together with the extrinsic one of *success*, in order to decide the naturalness of a new axiom of set theory.

OMEGA: So, to bring our previous conclusions into this new terminology, what Gimel claims is that *intrinsic and extrinsic criteria are irreconcilable...*

GIMEL: ...or even that *only extrinsic criteria matter*[24].

OMEGA: Therefore, Gimel can conclude from his argument that the extrinsic criterion of *success* will never be fulfilled, which contradicts Aleph's template of a possible solution of CH.

ALEPH: But what Gimel says is not possible, Omega, since pragmatic considerations are idle when dealing with *a priori* principles that settle the status of the proposed axiom. In other words, 'the criterion of success can hardly be sufficient for accepting a new axiom. It should only be used to assess, *a posteriori*, the value of the axioms, which must be found according to other criteria'[25].

[21] Again, [11].
[22] This distinction first appeared in [9].
[23] The maximality principle is formulated in [11] as 'the more sets the axiom asserts to exist, the better'. The fairness one is stated there as 'one should not discriminate against sets of the same complexity'.
[24] In [13]§95: 'Every hypothesis is a heuristic method.'
[25] See [11].

OMEGA: Then I don't understand why you included it in the first place, Aleph, if you are willing to treat it as a subordinate principle.

DALET: He only established it in order to preserve his unconditional support for the working mathematician![26]

BETH: But Aleph, what you are proposing is that we can effectively find a set of 'meta-axioms of set theory'[27]. Doesn't this amount to shift the point where you place the problem while leaving it unchanged? Namely, your faith in finding non-arbitrary and natural axioms mutates into believing it possible to find meta-axioms with the same ideal properties.

GIMEL: That is true and, actually, I can reformulate my original argument for the decision of these meta-axioms. Let me tell you, Aleph, I still don't see how you could favor the adoption of a new principle over some well established and proven facts about sets (being those about universes in which CH holds or not) that contradict it.

ALEPH: I find confusing myself how both of you are willing to sacrifice our only source of evidence while favoring some *relative* facts.

OMEGA: It seems that we have reached an impasse. Let us all try to change the way we are focusing the problem.

[26] Again, one should read this sentence as an ironic depiction of Aleph's naturalist strategy.
[27] That is how they are named in [11].

2. Case study. Our group of mathematicians begins the analysis of two phenomena relevant for the present debate: the axiom of choice (AC) and the Church-Turing thesis (TCT).

(A) THE AXIOM OF CHOICE

ALEPH: Since last time Beth and Gimel seemed to have undermined my original claim, together with the epistemological and heuristic justifications I then provided, I think it appropriate to consider a *historical evaluation* of other problems which may be related to CH.

OMEGA: I agree. We should examine the problem from a different angle. Continue, Aleph.

ALEPH: Thank you, Omega. The first example I would like to examine is the following. As we all know, Zermelo first introduced AC as a natural axiom extending set theory in order to prove the well ordering principle (WOP). So, let me ask you: what makes this situation different from the one of deciding CH?

OMEGA: So, what you are implying is that WOP does not differ in nature with CH and, moreover, that if it was possible to extend ZF *successfully* with AC, one should expect that the same is possible with, for example, the proper forcing axiom (PFA)[28].

ALEPH: Exactly.

BETH: I disagree with Aleph's comparative. The situation has nothing to do with that of AC. This is because 'AC is both a simple intuitively true statement about the universe of arbitrary sets (granted the concept of such) and its use underlies many common informal mathematical arguments previously considered unobjectionable even by the critics of AC'[29]. My claim is that PFA does not share these features.

OMEGA: Beth, taking under consideration the first of your thesis would lead us to repeat the same conversation we had before since Aleph could reply again with his analysis of naturalness. So it seems that your true problem is that PFA, for example, is less likely to be found in informal proofs, for example, that it has a different status regarding its use by the working mathematician.

ALEPH: Well, that doesn't concern me at all: there have been found implications of higher set theory in other areas of mathematics[30] so there is no reason why this should not happen again here. Maybe a group theorist tries to make a working hypothesis which is actually equivalent to PFA. Why not? After all, this belongs to the extrinsic set of criteria for accepting PFA and, as I told you, provided the fact that PFA satisfies the intrinsic ones[31],

[28] This is, in a nutshell, what is argued in [11].
[29] This is what Feferman thinks the achievement of Zermelo's arguments in favor of AC were. See [7].
[30] See [4].
[31] Again, see [11].

I'm happy with accepting it as a new axiom of set theory.

GIMEL: Although not for the same reason, I agree with Beth: the two cases differ profoundly in one aspect.

ALEPH: Which one, Gimel?

GIMEL: The difference here is that the mathematical practice involving AC was widespread, as Beth said, *before* it was formalized as an axiom of set theory. That is, AC was implicitly used in the proof of some well-established mathematical facts, accepted by the mathematical community.

ALEPH: I don't see how this clashes with what I have already said.

GIMEL: Well, you have to note that PFA lacks the status of *fact* shared by AC, CH and its negation. The comparative fails because the epistemic force of the statements involved differs.

ALEPH: But Gimel, doesn't your classification of 'mathematical facts' and 'deep experiences' break the *natural order of evidence*?

OMEGA: What do you mean, Aleph?

ALEPH: Imagine that I agree with saying that, *seemingly*, CH and its negation are established facts at the same level of evidence. This would be a similar situation to the one of working with WOP and its negation (before Zermelo's work). The difference is that AC *emerged* as an axiom: not only it was a deep fact, as you say, but *more fundamental* than WOP. What you are claiming seems the same to me as merely adding CH and its negation to (some extension of) ZFC[32] and studying both systems as equally valid.

TAV: Do you really pretend to include yourself along *mathematical revisionists*, Beth? Aleph is extending his reactionary beliefs and you don't say anything! The problem is clearly that AC is *not* any kind of fundamental, natural or intuitive statement. *If you are willing to attack the set theoretic establishment, you should attack it all!*

ALEPH: Oh, so you are one of those that reject AC... In my opinion, 'the reluctance regarding its full acceptance by some mathematicians is due more to some of its counter-intuitive consequences, rather than to its otherwise very natural character'[33].

TAV: You are wrong, Aleph. My main problem with AC is that it is, as a principle, non-constructive. As I see it, the working mathematician only requires the infinite as a *tool*, not as an *object*. The problem of set theory is to pretend to establish deep facts concerning the infinite. What started as a *foundation* it is now a *branch* of mathematics. After all, it does

[32]This *overkill* method is inspired in Gödel's method of building increasingly big incomplete formal systems.

[33]See [11].

not seem so *wrong* to say that all there is nowadays in set theory is conceptual confusion and methods of proof[34]: one is certain about how to prove results but not about the nature of the constituents of these![35]

OMEGA: You have a point here, Tav. This *foundation/branch duality* of set theory links to the fact we have already discussed about how set theory should behave regarding its own results, that is, about the two types of evidence that set theory claims to deal with; here, the *natural/technical duality* arises from the previous one. In fact, let me add the following: when doing set theory, one usually confuses the *necessity* of formal derivations with the *artificially-imposed philosophical necessity* that their (supposed) content bears. Aleph: your idea of non-arbitrariness in axiom-adding is completely unfounded. It is closer to being a *dogma*.

TAV: And let me add more: constructive methods are the only way of eluding the *debasement of meaning* from which contemporary mathematics suffers[36]! In other words, 'the only reason mathematics is applicable is because of its inherent constructive content'[37]. The strange duality that Omega is describing is just a special case of the overall *schizophrenia*[38] that pervades the more abstract branches of mathematics.

ALEPH: How can you defend anything from what Tav is saying, Omega? He is attacking the principles of rational evidence themselves! And of course that I have dogmas, but at least mines are useful and do not conflict with mathematical practice. In fact, Tav's brand of intuitionism[39] can be refuted in the very same way as I already did with Beth's predicativism.

TAV: How can you refute me if I haven't yet specified the position I am willing to defend, Aleph?

ALEPH: Easy: you will give us the definition of an alternative formal system in which you will develop your *intuitionistic mathematics* and, later, you will try to vindicate this system as the only correct one. Of course, the resulting mathematics will be *weaker* than the ones pursued within the classical framework. Wasn't this strategy also Beth's?

TAV: You are wrong. Let me tell you: I do not favor the crystallization of intuitionism into any formal system[40], since *it is there where it loses its revolutionary power*. And you

[34] cf. [12]

[35] What Tav is saying is that set theory, by being both a foundation of mathematics and an active branch, seems to tend towards the impossible task of providing a foundation of itself. Moreover, if one adopts Aleph's naturalism, this foundation feature of set theory will then be undermined by how one decides to not interfere with the technical (that is, as opposed to foundational) goals of the theory.

[36] See [15].

[37] Again, [15].

[38] See [16].

[39] See [15]. There, Bishop confesses that he considers himself more brouwerian than others like Heyting.

[40] See [15].

should also keep in mind that, even when constructive mathematics can be seen from your perspective as a *reduct*, classical mathematics appear to me more as an *expansion*.

ALEPH: I don't follow, Tav. Exactly what are you proposing?

TAV: What I am saying is the following. Both Beth and you are trying to refute each other as if your views were incompatible, that is, as if your thoughts were essentially decisive for the working mathematician's *integrity*[41] Well, I do not identify myself with the *statu quo*[42] enforced by you, Aleph. But I don't certainly align with Beth's surreptitious dogmatism neither.

ALEPH: How can you seriously compare my views with those of Beth?

BETH: How can you say that I am being dogmatic when what I am trying to do is to debunk Aleph's reactionary thoughts?

DALET: How charming is to see the orthodox and the revisionist getting along so well!

TAV: In my opinion, Beth, once the views of Aleph are convincingly undermined, your revolutionary spirit will cease. Then, you will impose your own methodology and defend it in the same dogmatic way as Aleph is doing right now.

OMEGA: You are right about something, Tav: mathematical revisionism is interesting because it begins the *philosophical debate*. The *statu quo* is defied by the statement: *it is not necessarily so*. But, of course, the young spirit of a revolution only attacks the old regime; the task of building a new one differs essentially from it. This is why Beth and Tav ultimately do not defend the same set of beliefs: she claims that the constructive viewpoint is *the* only progress after the revolution has achieved its effect. On the other hand, Tav seems only interested -for the moment- in testing the limits and capability of the *statu quo*. In this philosophical debate we are not only interested in the *quantitative* content that supposedly each argument bears (for example, protecting the mathematical activity as it is), but also its *qualitative* value (what one uses to convince the other of the opinions expressed).

TAV: Exactly, Omega. Now, let me return to my previous idea of regarding classical mathematics as an expansion of constructive mathematics. The dispute between Aleph and Beth only interests the working mathematician at a *personal*, non-technical, level. My proposal, in the specific case of AC, is the following: 'Write the theorems that need [AC] in their proofs as implications, and be careful not to use [AC] for results that can be achieved without the use of it'[43]. Now, given this division of labour, each theorem 'A' that, say, Aleph can prove using AC, corresponds to my own theorem 'AC *implies* A'[44]. Thus, classical

[41] This term also appears in [16]. Compare with what we said above about philosopher's paternalism.
[42] This way of talking is inspired by [15].
[43] The original statement from [15] refers to LPO (limited principle of omniscience) but, in [16], the possibility of including (various forms of) AC is considered.
[44] Again, see [15].

theorems can be regarded as an expansion of constructivists ones.

BETH: But Tav, that doesn't *change* anything!

ALEPH: I agree, Beth, this so-called revolution claimed by Tav is a dialectical game at best.

OMEGA: That seems unfair to say, Aleph. From my point of view, Tav is aiming at the fact that we ultimately make hidden assumptions when dealing with axiom-adding, that each of these assumptions is itself a *decision which we have already accepted*. Every axiom or meta-axiom of set theory is itself a decision that one has to *execute*.

ALEPH: I don't really see how this puts my own view in trouble.

TAV: Well, Aleph, what I am saying tries to *relativize* in some sense the intended *meaning* you want to assign to (classically derived) theorems. What I am trying to argue is that we are acquainted with constructive meaning; classical theorems are results that may or may not be constructive and, for the shake of clarity, what my view defends is to keep a clear distinction between the domains of the constructive and the ideal[45]. You can *decide* to make classical mathematics in the manner expressed above. But, granted this, your theorems will be *less* meaningful than the constructive ones[46], 'because to trust the classical predictions one must believe in the theoretical validity of the concept of a God having the specified attributes [e.g. AC], whereas to trust the constructive predictions one must only believe in the theoretical validity of the concept of a being who is able to perform arbitrarily involved finite operations'[47].

ALEPH: So you need, after all, to develop a *constructivization* of mathematics.

TAV: Well, yes. But I don't think that this interferes with the working mathematician that has already committed to some set of decisions. This would be an enterprise different from Beth's and other formal (and thus reactionary) constructivists, who think that a monstrous *replacement* is needed in mathematics. If my project were completely fulfilled, the schizophrenia I mentioned before would dissipate. All that is needed is that mathematicians keep an *open mind* about the relationship between the classical and constructive[48].

BETH: Let me be sincere with you, Tav. I think that your brand of intuitionism allows Aleph to *continue developing a schism within his enemy*, that is, constructivism. In other words: the only thing you have achieved with your views is to actually validate what Aleph is trying to defend.

[45] This was Hilbert's conception of mathematics.

[46] See [16].

[47] cf. [16]. Bishop talks, in the context of arithmetic, about predictions of 'the results of certain finitely performable computations' (*ibid.*) in the context of arithmetic. One could mirror the same idea for the case of sets.

[48] On the *arrogance* of some mathematicians as a reason for the 'thoughtlessly' application of mathematics, see [15].

TAV: What are you proposing, Beth?

BETH: First, let me remind you how you said before that we should attack *all* the set-theoretic establishment.

OMEGA: And he did: he attacked the way in which set theorists usually regard their work and, in particular the axioms they work with. *The hardest pill to swallow is, perhaps, that the working mathematician carries with her determined philosophical views.* The goal of the philosopher is to distinguish these form mere *inclinations*[49]. We certainly have to come back to this topic later.

BETH: I agree, Omega. Still, this does not help. The separation made by Tav is insufficient and relegates the task of constructivism to a merely philosophical, even stylish, one.

ALEPH: Precisely, Beth. Although, in my case, the problem with Tav's explanation is that it relegates the meaning of classical theorems to an ethereal, even theological, one.

OMEGA: What each of you is saying amounts to a version of the same slogan: *a weak extreme view is preferable to a strong middle point one.* This view is common in the context of philosophical debate. Certainly, it makes things easier to reduce everything to the fight of *two* already specified sides, as it makes everything easier that the two armies wear different uniforms. In our case, Aleph and Beth -the *statu quo defender* and the *methodical revolutionary*- prefer to fight against each other to include Tav in their debate. This only proves that they are both, in some sense, defenders of the *same* establishment, namely: the *formalizative* one.

TAV: Exactly! They only represent the *a priori irreconcilable* intuitions about mathematical practice, namely, the duel between two apparently incompatible formal systems.

OMEGA: But don't you see that your *semi-anti-formalizative answer* gives Aleph and Beth the perfect excuse to keep fighting each other? Your *faith* is put in their open-mindedness: don't you see how *fragile* that is?

TAV: What are you thinking about, Omega?

OMEGA: My question, regarding your *relativization* strategy, is: *why stop there?* Why not consider each theorem as a consequence of a set of axioms, similarly as you described some classical theorems as consequences of, say, AC? Wouldn't this amount to merely relativizing its meaning to that of the axioms? This would extend the concept of *decision* sketched above, namely, that *we ultimately make actual, philosophically-charged, decisions within mathematical activity*[50].

[49] This terminology is inspired by [12].

[50] This is, essentially, the path that Omega has been building during this time. These are precisely the ideas I expressed time ago by other means.

TAV: But, Omega, I certainly *believe* that there is a well defined, absolute, evidence in constructive mathematical statements.

OMEGA: *There* is where you and I differ, Tav.

TAV: So what are you proposing? That axiom-adding is not guided by any principle?

ALEPH: Now, this is funny! No, Tav: Omega defends that the only (of course, non-)principle that regulates this process should be: *everything follows*[51].

OMEGA: Not so fast. I think that something certainly prevails: *philosophical debate*. One should embrace this debate without any aim of closing it *in an absolute way*. Debate is the only way we have in order to justify our decision-making. This is what I was trying to explain to Beth when she tried to argue for her own view as the only one, this is what Tav has helped me to visualize more clearly and this is what I am now arguing *contra* your idea of a methodologically-guided way of adding axioms.

TAV: I thought you and I were thinking similarly, Omega. Now I see that you are blind. Don't you think that your idea of 'decision-making' only underestimates mathematical activity? Do you think that Goldbach's conjecture is also relative to some 'philosophical background'?

OMEGA: It seems that you are identifying 'philosophical' with 'metaphysical': this is not the case. Also, I'm fine with accepting that there are more philosophically active branches of mathematics than others. *The problem of formalization is that it may lead us to think that, since the formal language is similar, so will be the described phenomena*[52]. Isn't this thesis similar to the one you were defending before, Tav? Also, don't you think that set theory, given the problems that we have discussed before, is quite near (and therefore quite susceptible to change due to) philosophical discussion? It seems unfair to compare the situation with CH, PFA and AC with the problem you mentioned.

ALEPH: My main problem now is that, after all this talking, I don't see how all these (misguided) things all of you are saying actually go against my thesis. 'Whether or not the above described metamathematical principles [such as PFA] are legitimate examples of [new natural axioms] can not be determined at this point. But at the very least they suggest that the search for new axioms is not futile and that there are rational methods for evaluating the evidence for and against particular candidates'[53].

GIMEL: This is where we ended last time!

[51] This idea is, after all, not that crazy: see [14]. Note that it is now Aleph who is being sarcastic.

[52] In [7], Feferman has written a dialogue in order to illustrate this situation, the difference between CH and the usual mathematical conjecture, although we have to remember that his conclusion differs deeply from ours.

[53] See [4].

OMEGA: But don't you see it, Aleph? Even your *rational methods for evaluating evidence* are themselves (although perhaps at a more profound level) decisions you have methodologically made, paths that you have already taken. In the end, all you have is *faith of progress*, derived from your *historical evaluation of a phenomenon, dependent of your philosophical beliefs*.

ALEPH: I can't conceive how you have fallen for Tav's side in the first place, Omega. You are now questioning the very same ideas we are supposing when trying to debate in a rational way about our original question. You are acting like the child who keeps asking 'why?', ignoring every possible answer that could be given to him.

BETH: Exactly, you are eluding the *true* nature of the problem. I don't see the point of this nonsense of relativizing mathematics. I mean, even Tav is no longer on your side, Omega!

OMEGA: Well, I'm sad to hear that from both of you. I hope that you understand how I developed my actual beliefs. I don't doubt the fact that we *make* definite decisions; what I am trying to make clear is the place of the corresponding hidden assumptions within mathematical activity. *The deeper a decision is executed, the more revolutionary power its evaluation will have*. But I don't claim that some of these deep facts *are* false: rather, that *it could be, from other perspective, otherwise*. The good thing in all of this is that, as I have already said, I firmly belief in philosophical debate as a source of shifting systems of ideas. So you may finally turn me to your side if you argue intelligently.

ALEPH: Okay then, since I see that my case study of AC has left you unmoved, let me present the second example confirming my position. Nevertheless, Omega, I would like to come back to this point and try to clarify some of the monstrous statements I have heard.

OMEGA: Of course Aleph. Now, please, continue.

(B) THE CHURCH-TURING THESIS

ALEPH: Again, let me present you a very famous case in support of my thesis. As we all know, the concept of *algorithm*[54] was central to Hilbert's program and, particularly, to his *Entscheidungsproblem*. It is also familiar to us how Turing's proposal for formalizing this notion, the *Turing machine*, became the (famous) representative of a very special equivalence class, namely, the one of all the models of computation that pretended to capture what 'algorithmic' meant. What I claim is that the way in which this formalization was achieved, that is, by some definite axioms, is also possible for sets, in order to settle CH[55].

[54] I will use the term 'algorithm' in order to refer to 'computable', 'effectively computable', etc. What is to be clear is that an algorithm is what a Turing's 'human computer' can do.

[55] Gödel saw this phenomenon as an example of 'an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen' (from [4]). Nevertheless, other remarks from Gödel seem to be opposed to this remark. We have restricted ourselves to this *attitude* of Gödel towards conceptual analysis. For more details, see [17]. In the case of sets, his *conceptual analysis* is less clear. It is always

OMEGA: So what you are proposing is to compare, in some way, the *Entscheidungsproblem* to CH and the definition of algorithm with some *ideal* (i.e. meeting your own requirements) extension of ZFC (say, +PFA) that ultimately allowed to decide CH.

ALEPH: Precisely.

OMEGA: And this definition is then what TCT ultimately expresses.

ALEPH: Exactly, here the meaning of 'algorithm' is completely *captured*, formalized. And more importantly, the equivalence of alternative models of computation to Turing machines only shows that it does not depend on any formalism[56].

OMEGA: Well, Aleph, I think that the first thing to do here is to acknowledge how this, let us say, *categoricity* is not mirrored by the case of ZFC.

ALEPH: Yes, that is completely true. In fact, as I said before, it is characteristic of the notion of set to be *open-ended*: I do not claim here to *end* the process of establishing its properties; rather, I only claim that this process is well justified and feasible *a priori*. The idea of the TCT is quite simple: (1) One specifies the requirements imposed by the notion of 'algorithm' by means of a conceptual, reflexive, analysis[57], (2) One shows that Turing machines satisfy those requirements and, thus, (3) One concludes that the notion of computability is captured by that of Turing machine[58].

DALET: Aleph presenting a threefold scheme... how Hegelian!

GIMEL: Note the similarity between this proposal and the characterization I made concerning Aleph's intended solution for CH.

OMEGA: So now Aleph is being more precise about his proposal: he claims that a further *analysis* of the notion of set is needed in order to arrive at, presumably, a formalization independent from any specific formal system or approach. That is, he is willing to keep investigating the concept of set in order to make it absolutely clear that it is an *objective invariant*, so to speak.

ALEPH: Exactly: the working mathematician activity is a constant tension between her freedom of thought, that is, her ability to develop new theories, new concepts, etc. and the objective constraints by which these are *necessarily* governed. To put it in a more pictorial way: the mathematician encounters herself with a *wall* against which she directs her work. This is another reason that supports speaking about *objective reality*.

OMEGA: If you don't mind, I think that we should let the assessment of *that* kind of expressions for other time. Now, let us examine your proposal again.

easier to make successful events coherent.

[56]See the previous footnote.

[57]Reflection was a very important notion in Gödel's philosophy. See [20].

[58]This is a way of condensing Turing's argument that can be found in [17].

GIMEL: I think that Aleph is once again committing an act of faith based on a history of success while, at the same time, disregards the potential flow of strange phenomena. TCT is not a definition *at all*, it is more close to be a working hypothesis[59].

ALEPH: Why? What makes TCT that special for you? I don't see any difference with other famous cases from *the history* of analysis. For example, the definition of the derivative was an immense achievement and we do not see it today as a working hypothesis in your sense.

GIMEL: Let me first ask you the following. Suppose that someone proposes a definition pretending to capture some notion. It may happen that, with further work, that definition is (not mathematically but pragmatically) proved to be *insufficient*. For example, imagine that the definition doesn't include some new cases that have become interesting. Would you still defend the definition as an achievement?

ALEPH: Of course not.

GIMEL: Then what you accept is that *bad definitions are working hypothesis*.

ALEPH: That are eventually discarded, yes.

GIMEL: So you do believe that there are good and bad definitions.

ALEPH: Of course! I find this quite fundamental, Gimel. TCT, on the other hand, presents a complete analysis of the notion of computability: it *exhausts* it.

GIMEL: But then, you are 'venturing [...] that that once in the future, somebody will define a function which is on the one hand, not effectively computable in the sense defined thus, on the other hand, its value obviously can be effectively calculated for any given arguments'[60]. You are again confronting the inherent dynamic nature of mathematical activity with the rigidness of formal axiomatizations!

ALEPH: But *precisely* what the TCT says is that no such function will ever be found.

GIMEL: Aren't you giving then TCT a *status* of empirical law?

ALEPH: ... explain yourself, please.

GIMEL: Of course. What is clear for both of us is that TCT 'is not a mathematical theorem that can be proved or disproved in the exact mathematical sense, for it states the identity of two notions only one of which is mathematically defined [say, 'Turing machine'] while the other is used by mathematicians without definition [i.e. 'algorithm']'[61].

ALEPH: That is completely true. In fact, as I said above, the independence phenomenon

[59] See [18].

[60] See [18]. The same applies, Kálmar argues, for the idea of 'solvable problems', but it is enough for us to study TCT.

[61] Again, from [18].

shows that we cannot expect to gain insight of any kind 'by trying to give explicit definitions for concepts and proofs for axioms, since for that one obviously needs other undefinable abstract concepts and axioms holding for them'[62].

GIMEL: Therefore, we both agree that the nature of TCT is *pre-mathematical*[63] or, say, *informal*[64].

ALEPH: Yes, although I still don't see what is your problem with my previous description of the situation.

GIMEL: My main concern with your idea is that you seem to overlook the potential behaviour of the informal notion of algorithm. You are convinced of its *petrifaction* within a formal definition, which does not distinguishes you from Dalet, who you accused of defending a bold conservativeness because of his apparent desire of keeping ZFC *as it is*.

ALEPH: Well, if for petrifaction of a concept you understand having a *clear* and complete analysis of it then of course I favor that view. As I have already said, the same situation does not apply for the concept of set that immediately.

GIMEL: My thought is that *here* we cannot aim for evidence, our only source of belief must be provided by *plausibility arguments*. In fact, one could even give arguments of this kind *against* TCT[65].

ALEPH: *Give me your degenerate cases, I will provide you a formal characterization that captures them*[66].

GIMEL: Again, that is an act of faith, faith of success... I think that it would be useful to introduce one distinction that we are unconsciously following.

OMEGA: Go on, Gimel. Let us see if this makes the debate to move forward.

GIMEL: Let me distinguish between what I call *euclidean* and *popperian* theories. During the debate we have been going back and forth between deciding whether giving the axioms or certain conjectures more *methodological* relevance. Let me call an statement *basic* when it is susceptible of receiving a truth value. 'Then a system is euclidean if it is the deductive closure of those of its basic statements which are assumed to be true. Otherwise it is [popperian]'[67].

ALEPH: I see. What else?

GIMEL: Euclidean and popperian theories differ profoundly in their historical development.

[62] See [10].
[63] This is the term used in [18].
[64] This term appears in [24] as a synonym for *inhaltlich*. We will adopt this usage.
[65] This is precisely the main idea that guides [18].
[66] Compare this with the famous quote of Leibniz.
[67] This distinction is due to [25]. I call 'popperian' to what Lakatos named 'quasi-empirical'.

An euclidean theory will follow a path similar to this one: (1) The naive prescientific stage, where there is no *homogeneity*; (2) The foundational period, which reorganises the discipline by providing a common framework for the multiplicity of facts from the previous stage; (3) The solution of problems within the formal framework[68]. On the other hand, the popperian theory 'starts with problems followed by daring solutions, then by severe tests, refutations. The vehicle of progress is bold speculations, criticism, controversy between rival theories, problem-shifts'[69].

ALEPH: So what do you pretend to argue by making this distinction?

GIMEL: I only want to compare more precisely your view with mine, Aleph. You accept the hard fact that we can only pretend to add new axioms to, say, ZFC, in order to settle questions like CH. This means that you believe in certain *quasi-empirical* (popperian) nature of set theory because methods like axiom-adding 'are fallible, not less fallible -and not less quasi-empirical- than the ordinary classical mathematics which was so much in want of foundations'[70].

ALEPH: Yes, this is the natural outcome of the independence phenomenon we have been discussing since we started the debate.

GIMEL: But, on the other hand, you believe in the possibility of finding *a priori* principles of set theory by means of reflection alone. Thus, you are *euclidean* in spirit. The problem with this approach is certainly that, as Omega has pointed out before, the set theory you want to pursue is, simultaneously, euclidean *and* popperian: its methodology and practice clearly reveal the second feature, while your faith in its role as *foundation* pretends to impose the first one.

OMEGA: I see, Gimel. As you say, we can then conclude that the previous foundation/branch duality mutes into the *spiritual* duality you are explaining.

TAV: Set theorists have become, in some sense, an *intellectual elite*, their own standards of justification lead to an absence of foundation for their own work for, if set theory is to be a foundation, what could legitimize its -at the same time- technical development apart from itself? This seems to me, more than a heroic and romantic enterprise, an *intellectual autocracy*.

OMEGA: Please, Tav, have patience.

GIMEL: In fact, what Aleph seems to postulate is a *convergence* of a contingent historical development, through some kind of *a priori* method, towards an immutable, eternal, abso-

[68] This idea is from [25]. Lakatos adds: 'The discovery of a decision method for theoremhood may abolish [...] [the third] stage altogether and put an end to the development'. One can compare this with Hamkins thoughts on 'finishing' one field of research (see [6]).

[69] See [25].

[70] Again, see [25].

lute truth. Dalet's joke could not be more precise: Aleph certainly believes in a rational *end* of history.

OMEGA: Let me return for a moment to the case we are studying now. We concluded that we can make a distinction between the *formal* notion of 'Turing machine' and the *informal* one of 'algorithm'. Gimel seems to favor that the inherently chaotic and potential development of the latter cannot be replaced properly and *a priori* by some form of the first. Aleph, on the other hand, while recognizes this distinction, is willing to declare Turing's formalization as *definitive* in some sense. I also want to point out something that we may be surreptitiously dealing with.

ALEPH: What is it, Omega? I hope that your analysis is not as *partial* as those of Gimel.

OMEGA: I want to generalize the template you followed when explaining TCT. Let me assume that we can characterize an informal notion with a (natural language) statement. In this statement, some *cases* will occur, that is, some *open-ended* features will appear in it. What I will call a *formal system* is nothing else that the formal specification of those cases[71]. This corresponds to the first two steps of your depiction, Aleph.

ALEPH: Of course, I agree with this template of formalization.

OMEGA: Now, let me satisfy Dalet once again by providing a *third step*[72]. Your conclusion is that 'Turing machine' captures, exhausts, formalizes 'algorithm'. But I may say, *contra* this, that -in general- two different specifications of the cases from the original statement are possible.

ALEPH: I don't see how this is may happen.

OMEGA: The historical development of mathematical theories can be seen, as Gimel does, as a chaotic process of refining systems in a *radical* way, that is, by proposing new ideas, going against pre-established notions, etc. Thus, there is a certain *historical enabling of mathematical pluralism*, given by the proliferation stages[73]. One may regard different alternatives as being, in the end, *subsumed*[74] under the unique, real one. And there is always a possibility for this. But, I claim, such a unity -in case one believes in it- cannot be *imposed*, it has to be *natural*. In other words, and as I have already said, one has to vindicate his own views by means of an open-minded, serious, respectful, philosophical debate. The problem with your view, Aleph, is that you impose something under the label of natural.

[71] Further stating that *there may be at least two different specifications of the cases* leads to what I call V-pluralism.

[72] There have always been third, middle terms in the history of philosophy. The infamous schematism from Kant qualifies as one of these, for example. Of course, third terms, the so-called synthesis, are more *obscure* than the others, since their positing adds something new to the previous distinction and therefore it always may seem arbitrary.

[73] This terminology is from Lakatos.

[74] Dalet murmurs: *Aufheben*.

ALEPH: Are you a defender of popperian theories? Doesn't this conception change only *where* the truth-value of our statements is injected, the form of the flow of truth inside the system[75]? That is, you were before arguing against how the axioms do not express absolute facts. How does this fit with your popperian derive?

OMEGA: That doesn't concern me at all: the distinction between euclidean and popperian is only *methodological*. What I wanted to point out before was the way in which the statements of formal systems (axioms or theorems) are interpreted. So I don't see the tension here: I simply don't believe in the absolute truth of basic statements in general. My line of argument would be similar here, since the relationship of basic statements with the informal is analogous.

ALEPH: I see. But, Omega, returning to your pluralism, let me say that you are the one *imposing* your own views, namely, you are making me to accept pluralism in some way. *The anarchist has become the enforcer of a new establishment*[76].

OMEGA: The way I see it, Aleph, pluralism is the only way of keeping debate alive and, as I told you, debate is the only way of staying in constant revolution. Or, equivalently, pluralism sacrifices itself (as you are implying) for the shake of debate, in the same way as a truly democratic system must position itself against ideas which endanger some members of the community and go against *integrity*. Would you say then that this system is *less* democratic for having these kind of means of guiding discourse? If you mean that I am making you to respect other views and thus avoid that you preserve *statu quo* at any cost then yes, I am willing to fight for pluralism, not as a new, untouchable idea, but as a *tool of criticism of any proclaimed legitimate and unique approach to truth*[77].

ALEPH: But, as I see it, pluralism does not go beyond any possible debate itself, it is not something *above* the other views: it is a definite position itself.

OMEGA: No, Aleph. The difference between you (together with, say, Beth) and me is that I do *not* subscribe pluralism, I don't pretend to give any arguments in favor of it. In fact, it seems to me that it can be badly developed by overlooking essential philosophical questions[78]

BETH: What are you claiming, then?

OMEGA: What I am willing to do, in order to keep the debate alive, is to adopt a *tactical pluralism*. Instead of using it as an ideal that should be preserved in order to, for example,

[75]These expressions are used by Lakatos.

[76]This is a perfect example of how plain reactionary ideals can be masked as *revolutionary* ones. Undermining the revolution is a demonstration of the force of the state (the *statu quo*); going against the state is channeling the force of history.

[77]The terminology is due to Lakatos, although is used in a slightly different sense. See again [25].

[78]Namely, when we work with different formal systems without acknowledging how we make some well-defined decisions in the process. This is something I have argued somewhere else.

protect the mathematical activity or whatever other *highly regarded ideal*, I will use it to *attack* any new position that the state may have conquered and to *defend* every idea that the revolutionaries may want to weaken after having *their* revolution concluded. The first will attack the imposition of the *statu quo*, the latter will do the same with the imposition of a new one. What both movements have in common is the deep desire of *ceasing* the debate, the movement of ideas.

ALEPH: But then, you will have become no less than a *sophist* while I will stay as a *platonist*[79].

OMEGA: Well, I *do* arrive at conclusions that may be as wrong or correct as yours, Aleph. But our methods of work radically differ. I *do* have ideals, but these are not achieved fighting *inside* a determined position.

ALEPH: Yes, that is for sure.

OMEGA: Let us continue. Has anyone anything new for approaching the general question again?

[79]Of course, Aleph's intelligent use of words doesn't necessarily imply that he is a platonist in the usual term of the ontology of mathematics.

3. Crisis and phenomenology. Some of our mathematicians are uneasy with the present stage of the debate. It seems that the conversation is gradually becoming more difficult but no clear answer or partial conclusion is provided.

OMEGA: Well, since no one wants to start, let me make a brief recapitulation of the debate. So far, Aleph has presented two historical examples in support of his general view regarding mathematics, which we have seen particularized for the case of CH. A question arises naturally: *can we arrive somewhere following this path?*

ALEPH: Of course not if you are not willing to listen carefully to what I want to extract from the examples.

GIMEL: So you are supposing that something neutral resides within them. Then (how surprising) your faith springs from this apparent neutrality and motivates your view as an absolute truth.

ALEPH: Exactly. Now let me show you how I *interpret* these kind of examples in general. 'I believe that the most fruitful principle for gaining an overall view of the possible world-views [*Weltanshauungen*] will be to divide them up according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics (or religion)'[80].

BETH: How does this apply here, Aleph?

ALEPH: Let me give you a few concrete examples. Although there may be mixed cases, I distinguish *left* and *right* characters in the following sense. I claim that 'we immediately obtain a division into two groups: scepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the other [...] Furthermore one sees also that optimism belongs in principle toward the right and pessimism toward the left'[81].

GIMEL: So you are certainly identifying yourself with the right side of your duality.

OMEGA: Now, we have a new duality which may be connected with the ones above. But Aleph has only proposed a general, philosophical, distinction; he needs to put it in practice by interpreting some historical content, for that's how his notion of history will start *walking by itself*[82]. Don't you, Aleph?

ALEPH: Precisely, my main thesis is that 'the development of philosophy since the Renais-

[80]See [10]. Gödel *does* have a philosophy of history. Note the relevance that theology gains. Hegel also placed *religiosity, ethics*, in a special place, as something itself eternal, not subject to the chaotic flow of history, see [28].

[81]See [10]. Gödel continues: 'For scepticism is certainly a pessimism with regard to knowledge. Moreover, materialism is inclined to regard the world as an unordered and therefore meaningless heap of atoms. In addition, death appears to it to be final and complete annihilation, while, on the other hand, theology and idealism see sense, purpose and reason in everything'. This division reminds us of the one that William James between *rude* and *selective* spirits ([29]). Once again, James' pragmatism comes as the *third term* between the two, apparently confronting, characters.

[82]The march of universal history (*Gang der Weltgesichte*).

sance has by and large gone from right to left - not in a straight line, but with reverses, yet still, on the whole. [...] [M]athematics, by its nature as an a priori science, always has, in and of itself, an inclination toward the right, and, for this reason, has long withstood the spirit of the time [*Zeitgeist*] that has ruled since the Renaissance'[83].

GIMEL: So you now admit that you are euclidean in *spirit*!

OMEGA: Not so fast, Gimel. Aleph is depicting empiricism as a *pernicious tendency* in the history philosophy; he claims that it has become the dominant form in which the spirit reveals in this portion of universal history[84]. Therefore, under this view, he claims that mathematics are the ultimate resistance against the inexorable *Zeitgeist*. Undoubtedly, this pumps some *romantically defiant* character into mathematics; what is less clear is the degree of *anti-revolutionary* spirit that accompanies this process.

GIMEL: Even if I agreed with your interpretation of history, Aleph, I could still ask: why can't mathematics resist anymore? Why does your optimism end here?

ALEPH: Well, the main reason is that, not so long ago, mathematics reached a *crisis*.

OMEGA: Oh, I see. Aleph's reading of history consists, for the moment, of an abstract conception of different stages -which is more or less *stable*- together with a conflict that stands in need of resolution.

ALEPH: Exactly. We all know something about the so-called 'crisis of the foundations of mathematics'. What is *clear* is that 'the result was that many or most mathematicians denied that mathematics, as it had developed previously, represents a system of truths; rather, they acknowledged this only for a part of mathematics (larger or smaller, according to their temperament) and retained the rest at best in a hypothetical sense namely, one in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions'[85]. This approach fits quite naturally with your views, Omega.

BETH: You are right, it fits like a glove. Although it could be argued that my constructive temperament makes me recognize as *truths* only a smaller portion of what you would label as such, Aleph, there is a huge difference between *our* positions and the complete nihilism that Omega wants to explore.

OMEGA: Once again, Beth, you have misinterpreted my claims. What Aleph is describing is just a form of *ifthenism*[86]. I stand against it because it overlooks the philosophical background that ultimately guides our execution of decisions. To put it more briefly, your

[83] Again, [10].
[84] Of course, this terminology is Hegelian.
[85] Again, from [10].
[86] That is how I like to call it.

accusation is scarcely grounded. But please, Aleph, continue.

ALEPH: The natural consequence of ifthenism is that mathematics becomes an empirical science[87]. Moreover, this conception is the result of a *wrong* interpretation of well-known historical facts: (a) The impossibility of proving consistency within the formal system, from which is concluded that the fact that we haven't arrived to inconsistency is contingent; and (b) The independence phenomenon in general, from which 'many questions lose the form "Does the proposition A hold or not?" For, from assumptions construed as completely arbitrary, I can of course not expect that they have the peculiar property of implying, in every case, exactly either A or not A'[88]. But this view falls by itself, since 'one thus aims to prove, for inherently unfounded rules of the game with symbols, as a property that attaches to them so to speak by accident, that of two sentences A and not A, exactly one can always be derived'[89].

OMEGA: And this is what, for you, the ultimate impossibility of this view consists in. Note that, for Aleph, any view that pretends to refine how the *Zeitgeist* interferes with the character of mathematics is *inherently impure*[90].

ALEPH: Exactly: 'it is impossible to rescue the old rightward aspects of mathematics in such a manner as to be more or less in accord with the spirit of the time. [...] One must either give up the old rightward aspects of mathematics or attempt to uphold them in contradiction to the spirit of the time'[91].

GIMEL: But *then*, you have already said it: we could give up the rightward nature that you pretend to impose on mathematics. *Why not?*

ALEPH: Let me be clear: if I was able to actually preserve that nature, would you oppose? This would amount to denying true progress, to look in the opposite direction of evidence and truth: it would be a purely *negative* view only held in virtue of philosophical considerations, not as a *result* of mathematics itself[92]. Omega's claims about a serious and impartial philosophical debate are as ideal for me as my positing of evidence is for him: the *Zeitgeist* shows us that the situation is quite other.

GIMEL: Well, I don't know how to imagine what you are proposing. You want to *surpass* every possible duality by means of an evident, clear, mediate yet *a priori* method.

[87] See [10].
[88] See the previous footnote.
[89] See the previous footnotes.
[90] Gödel puts the example of Hilbert's program: 'And thus came into being that curious hermaphroditic thing that Hilbert's formalism represents, which sought to do justice both to the spirit of the time and to the nature of mathematics'. See the cites above.
[91] Again, see [10].
[92] This is what Gödel says, in virtue of his (optimistic) principle: 'for clear questions posed by reason, reason can also find clear answers'. Compare this with the Leibnizian sentence above. The spirit of Illustration is still visible nowadays.

ALEPH: Precisely, Gimel: 'the correct attitude appears to me to be that the truth lies in the middle or consists of a combination of the two conceptions'[93], the right and the left. We do not only need a systematic knowledge of syntax, which formalization achieves, but a deep knowledge about the abstract concepts the mathematician -and, more prominently, the set theorist- is dealing with. We need an *informal* method with the properties I have been anticipating during the debate, 'a clarification of meaning that does not consist in giving definitions'[94].

BETH: Can you be more precise, Aleph?

ALEPH: My main thesis is that *phenomenology* can provide such a method and, therefore, a solution to the crisis presented above[95]. In fact, my proposal of solution for CH must necessarily make use of the phenomenological method: it is through this method how one can make non-arbitrary extensions of ZFC. Moreover, '[i]t is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate'[96]

OMEGA: What do you understand by 'phenomenology', Aleph?

ALEPH: I want to identify phenomenology with some 'procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us'[97]. Now, this is actually done by directing our attention to our own acts when manipulating abstract concepts. We may carry out some (mental) *act* with a determined (possibly psychological[98]) *content* in order to, for example, work properly with an abstract *object*[99]. This is what we may study in order to get a deeper knowledge of how concepts are *forced upon us*.

OMEGA: So there is something like *informal rigour* with your proposal. What is clear from what you say is that you want to make the original, already lost, *sense* of mathematics explicit or, in other words, you want to distinguish between the *theoretical art* that ifthenism makes unavoidable from the *true, philosophically-rooted, informally motivated, science*[100].

[93] See above. Hilbert's program was 'obviously too primitive and tending too strongly in one direction'.

[94] See the previous footnotes. Also, note how Aleph tries to make his view more appealing by adopting the usage of expressions from the opposition.

[95] This is the main thought presented by Gödel in [10]. For a study of this link, see [3], for the development of Gödel's philosophy and its relation to phenomenology, see [20].

[96] From [10]. This was a recurring thought for Gödel from his Gibbs Lecture. Again, see [20].

[97] See the previous footnote. This identification is not exempt of controversy.

[98] Gödel liked to talk about 'psychological facts', see [3]. Husserl also treats the connection between phenomenology and psychology extensively in [31].

[99] In [30], it is suggested that the general picture of intentionality is of the form 'Act(Content)→[object]'. The bracketing means, of course, the possibility of *bracketing* the object during some phenomenological reduction.

[100] This idea, applied to all science in general, is one of the main theses defended by Husserl in [31]. It is

ALEPH: One could put it that way, yes.

OMEGA: Then, Aleph, you must see that we are not that far way from each other: note how this idea of informal motivation is a natural consequence of the decision-making I presented above. One could even compare your 'acts' with my 'executions'. Nevertheless, there is one key difference: you are willing to concede that phenomenology is a science, with a definite outcome, that will show how decision-making is ultimately guided by some *a priori force*.

ALEPH: Well, duh: that's why I wanted to make use of it in the first place! Now, the scheme above can be particularized for the case of *mathematical objects*, in the sense that we can direct our acts to contents themselves. The need for doing so is precisely the need of analyzing phenomena within their *eidetic domain*[101]. Denying this would amount to trying to preserve the widespread prejudice that compels us to assert that we cannot have something like an intuition of an abstract concept[102]. Now, one of my further claims is that we do have something that we could call *mathematical intuition*.

GIMEL: I still don't see how this method would be plausible. Do you have any examples?

ALEPH: Of course! Consider the TCT once again. Remember how this statement linked some *informal* notion, 'algorithm', with a *formal* one, 'Turing machine'. In fact, I do not feel comfortable with this terminology since, in my view, the first one becomes, through *clarification*, the latter. As I put it before, this process consisted in establishing the essential properties of 'algorithm'. Let me be more precise: what we actually do is to make an *eidetic variation* of the properties we ascribe to the notion of 'algorithm'. 'The concept will not survive the addition, subtraction or substitution of some properties but it will survive in the case of others. The properties that cannot be subtracted, or for which substitutions cannot be made, are essential to the concept, and otherwise they are accidental'[103]. This is how we arrive at a *rule* governing the application of the original notion, and this rule is what is ultimately formalized[104]. In this example, it corresponds to the concept of 'Turing machine'.

OMEGA: But you are now interpreting an *achievement*. It is always easy to make an

funny that Gödel disliked Husserl's approach in this work, since here the project of a 'scientific philosophy' was put aside (see [30] again).

[101]For Husserl, the 'given' has multiple dimensions, through different aspects of 'giveness'. A theory can be thought as studying one of these aspects. Notice how near this idea is to the Quine-Putnam notion of 'ontological commitment'.

[102]This is Husserl's *categorial intuition*. For how Gödel's conception of this notion differs from that of Husserl, see [33]. I know that there is a deep bibliography on Gödel's idea of mathematical intuition (and on Gödel's philosophy of mathematics in general). I do not want the debate to be directed towards that, specialized, topic (only when it is necessary for the shake of clarity). For more details, see any of [3], [30] or even [20]. Moreover, Aleph is not necessarily a Gödelian, he only uses the phenomenological approach for the moment since this is one of the philosophical basis of Gödel's program, that is, of the conception that defends a dream solution for CH.

[103]This is the description of the method of free variation given in [30]. Husserl makes use of it in [32].

[104]See the previous footnote.

argument appealing by doing so, since *the historical force of the success is conveyed to the robustness of the argument*. How would you apply this situation to the CH one, in which the future development is not that clear?

ALEPH: For the CH case I claim that is possible to follow an eidetic variation of the iterative conception of set: those properties that become essential to us will actually be condensed in future axioms which, in order, will express some non-contingent, non-arbitrary features of that notion.

GIMEL: But how does phenomenology ultimately help you to justify your optimism? That is, what fundamental reason does it provide apart from another philosophical, historically motivated, ground for it? Without any clear criterion, I can make my previous arguments work again.

OMEGA: Wait, Gimel: phenomenology is not something *positive* or *negative*, it only should make us see a clear picture of what we already know, it should provide us a new *attitude* towards well known knowledge.

ALEPH: I find it surprising that Gimel, who has been suggesting the connection of empirical science and mathematics, does not follow my train of thought. Such a connection reveals itself when we realize how our structures of consciousness directed when perceiving and when dealing with mathematical concepts are quite similar: 'I cannot see anything irrational in the proposal to investigate this 'givenness as real' that we encounter in perception as well as in mathematical intuition'[105]: *evidence stems from the fulfillment of the contents of our acts*[106].

OMEGA: Namely, this connection tries to make justice to your -immediately recognizable for the working mathematician as well as for the, say, physicist- picture of mathematical (in general, scientific) cognition as an objective invariant, as the wall against which we should make use of every battering ram at our disposal in order to break through it and achieve an eternal truth.

ALEPH: Precisely. In fact, let me return to my previous discourse on mathematical objects.

TAV: Careful, Omega! Now Aleph will finally reveal the *true face* of his optimism... he will make it untouchable by means of *metaphysics*! What he won't explain is how we *access* these supposedly abstract objects that so certainly *exist*[107]...

GIMEL: I also want to know what you have to say regarding these ontological issues, Aleph.

ALEPH: Let me show how phenomenology is useful to elucidate the nature of abstract

[105] See [4].
[106] See [30].
[107] Here Tav has in mind all the challenges that Benacerraf poses for the realism of abstract entities: *identification* and *causal interaction*.

objects. Let us, as I said before, turn our attention to the way in which we deal, we *act*, for example, with mathematical statements. 'In a living, many-membered thinking action I produce a structure: a theorem [...]. Subsequently I repeat the producing, while recollecting my earlier producing. At once, and by essential necessity, an identifying synthesis takes place; furthermore a new identifying synthesis occurs with each additional repetition [...]: It is identically the same proposition, identically the same numerical structure, *but repeatedly produced* or, this being equivalent, repeatedly made evident. Therefore in this case, through the medium of recollective presentiations, the synthesis [...] makes a *connexion* between my present and these pasts'[108].

GIMEL: I don't see where you want to arrive with this monologue, Aleph.

ALEPH: The intentional analysis I have provided really shows that it is wrong to talk about the *supertemporality* of ideal entities that Tav possibly has in mind. He is trying to identify my view with this form of *naive* platonism which he thus so easily attacks. In fact, '[t]heir supertemporality turns out to be *omnitemporality*, as a correlate of free produceability and reproduceability at all times. After constitution of the Objective world with its Objective time and its Objective men as possible thinking subjects, that obviously carries over to ideal structures, as themselves Objectivated, and to their Objective omnitemporality'[109]. And, again, this is quite a natural *feeling* for the working mathematician.

TAV: This certainly presupposes the *existence* of the objects we want to ontologically examine in the first place, Aleph...

ALEPH: Well, Tav, if you want me to fight against your nominalistic ontology, I can simply make use of my account of intentionality, that is, I can simply point out that the object at which we direct our acts is not to be thought in terms of causality, like in the *natural attitude*[110]...

DALET: An act *without* object, how Rilkean!

ALEPH: ...but even if I achieved arguing against you in that way, you should first be aware of the following: '[...] the question of the objective existence of objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere *psychological fact* of the existence of an intuition sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis'[111]. This intuition is none other

[108] See [32]. Gödel seemed to be very satisfied with Husserl's analysis of reflection in [32] (see [20]). It is thus very likely that he found passages as this one appealing.
[109] See above.
[110] For this idea, see [30].
[111] From [3]. See [9]. Italics are mine. Husserl admits the possibility of fruitful connections between psychology and phenomenology in [31].

than the mathematical one.

OMEGA: I see, Aleph, so you deny the ontological form that naive platonism supports and you defend some kind of *epistemological platonism*[112]. You are not that interested in possibly existent entities but in our mental, transcendental, faculties.

ALEPH: Yes, one could say that. And note that this is a consequence of my previous slogan: *make the intentional acts explicit*.

TAV: Now I see it very clear! Aleph presented phenomenological analysis as the *natural* consequence of a *widely accepted* interpretation of history, as a solution of some *foundational crisis*[113]. Let me put an end to your previous doubts regarding the phenomenological method, Gimel: not only it does not provide a truly neutral, philosophical justification for Aleph's optimism but it also strengthens the *myth* derived from the lecture of history made by the *statu quo*[114].

OMEGA: You are right about something, Tav: the working mathematician has learnt that the so-called foundational crisis happened time ago, and she also is aware of the main (mathematical) consequences that derived from it. But providing the sense in which the crisis was truly a turning point in the history of mathematics, that is a philosophical task. Aleph has tried so far to give an interpretation of that event that *enables* his views. Therefore, a natural condition one should impose him is captured in the following slogan: *make the philosophical assumptions explicit*. Otherwise, one should not be afraid of labeling his methods of argumentation as *demagogical*.

TAV: But now, Omega, we have fallen into Aleph's *web of beliefs*[115]: the whole debate is now contaminated by phenomenological considerations and these are completely idle in relation to our purposes, for they only seem to satisfy the standards acquired by success, that is, they only seem to fit a *blind optimism*, the one exhibited by Aleph's discourse.

OMEGA: No, Tav. For me, phenomenology is only a name we give to a way of *making* philosophy, of grounding philosophical arguments. One could disagree with how Aleph has made use of it but I certainly believe that this approach is interesting and can clarify the nature of our main concerns. Since phenomenology has emerged in our debate as a method of elucidating some philosophically problematic features of our usual (mathematical) knowledge, the way we characterize its use ultimately depends on the account we have given of the initial problem. Therefore, from a different conception of history one should expect

[112]This way of acknowledging this change in Gödel's strategy is due to [3].

[113]Compare this with the historical analysis that one finds in [31]. There, Husserl provides a justification of phenomenology on a historical basis and, moreover, he argues that it is the natural step that philosophy is to make. Moreover, the comparative that Tav made between psychology and set theory can be connected with the fact that one of Husserl's main concern here is the crisis in which the psychology of his day was. Of course, Aleph would dismiss such a link.

[114]This terminology is due to [14].

[115]Of course, Beth is here making a word game by mentioning Quine's web of belief as a trap.

a different explanation of the 'foundational crisis' and thus a different solution to it.

BETH: Exactly! All of you are acting as if Aleph's interpretation of history were the only one possible. Were you not an *instrumental pluralist*, Omega? Why don't you say something *else*? Let me give you a different interpretation of the crisis that Aleph so readily accepts as a neutral fact.

OMEGA: Of course, I was expecting you to eventually say something, Beth.

BETH: Well, first of all, I want to ask Aleph for the nucleus of the foundational crisis, namely, the fact that many paradoxes were discovered within the domain of classical mathematics, in particular, of set theory as developed by Cantor. Why couldn't it be the case that, in fact, some dangerous ideas were introduced in mathematics without an exhaustive examination?

ALEPH: In my view, the apparently destructive consequences of the paradoxes were completely *exaggerated*[116] by those mathematicians and philosophers with a *leftist* inclination: 'in the first place, these contradictions did not appear within mathematics but near its outermost boundary toward philosophy, and secondly, they have been resolved in a manner that is completely satisfactory and, for everyone who understands the theory, nearly obvious'[117] But, as I said before, the *Zeitgeist* clearly surpasses well founded[118] arguments as these.

BETH: How can you say that these issues were completely resolved when you advocate for a notion of set that possesses inherently dangerous features such as that one of impredicativity? Let me return to the beginning of the twentieth century. From the Cantorian viewpoint, together with the aid of Hilbertian freedom, mathematics acquired a strong tendency towards abstraction. One could even say that the modern mathematicians of the day extended the methods of proof beyond the usual limits of human cognition: this is the only sense in which there was a Cantorian *paradise*[119]. Couldn't this fast development have been not careful enough and thus dangerous for mathematics as a whole?

ALEPH: Beth, I think that you don't understand the *true progress* of mathematics. Abstraction is something *desirable*!

BETH: Maybe, Aleph. It is clear is that this *far-right, abstractionist* spirit introduced new concepts and techniques, but the potential dangers carried by these remained hidden. Nevertheless, the foundational crisis, incarnated in the paradoxes, truly showed that this danger was *real*. Therefore, an exhaustive revision of the problematic ideas was the only natural outcome of the crisis. In other words, a re-examination of the foundations of mathematics

[116] sic. See [10].
[117] See the previous footnote.
[118] Dalet smiles.
[119] Beth bears in mind the infamous quote: *This is not mathematics, this is theology.* Bishop talks about *God* in [16].

was made *urgent*[120]. You have admitted this, but I profoundly differ with your opinion that this process was completely achieved; without a constructive re-construction of some notions, many of the issues remain and interfere with actual mathematical practice... but you prefer to just look the other way. Thus, you seem to favor not a middle position between left and right but the far-right ideals of a *careless abstractionism*.

OMEGA: So you differ with Aleph's historical analysis, Beth.

BETH: Precisely. I find Aleph's analysis on the solution to the paradoxes rather unsatisfactory. Not only that, this implies that his whole historical interpretation is *incomplete*.

TAV: I think that the main conclusion we should extract from our debate is the following: the history of mathematics or, better, *the history of philosophy of mathematics is very dangerous*. Here, 'the superficial danger is that it will be and in fact has been systematically distorted in order to support the *statu quo*'[121]. Once again, I think that Aleph's interpretation of the foundational crisis is a mere tool of preserving intellectual authority and, at the same time, Beth's response is a watered-down and conservative caricature of the vivid constructivism I wish to represent.

ALEPH: A *tool*? I was just making a rational argument!

BETH: Please, Tav, go ahead and share your *brilliant* ideas with us.

TAV: Well, let me start by saying that the crisis is not an isolated historical event; the schizophrenia of mathematics is an actual fact, it is *alive and well*. In my view, Aleph and Beth only represent the two sides of a misguided controversy. 'The point is not whether a particular statement is true, but what do we mean by the statement'[122]. Turning our attention to this is the only way to actually treat the contemporary debasement of meaning: 'There is only one basic criterion to justify the philosophy of mathematics, and that is, does it contribute to making mathematics more meaningful'[123].

ALEPH: But a clarification of meaning *is* what I ultimately proposed with the phenomenological method. What is new in your approach?

BETH: Exactly what do you mean, Tav?

TAV: Let me explain you. It is true, Omega, that phenomenology may be, after all, useful to the intuitionist, useful to clarify the constitution of a radical form of constructivism[124]. But my employment of the phenomenological method would differ from yours, Aleph, since I would be willing to accept it only as an exploration of the constructive content of our

[120] See [15] for this terminology.
[121] See [15].
[122] Again, see [15].
[123] See [15].
[124] There have been a number of commentators of Husserl that have established some connections between Husserl and Brouwer. See [36].

expressions. In my case, I could arrive at *alternative* interpretations of the logical connectives[125]. Thus, a constructive statement would then receive *full*, clear, meaning. On the other hand, remember that your (strictly) classical theorems can be expressed as, for example, AC *implies* A, so it is not surprising that the meaning of your statements could only aspire to be *partial* under these semantic considerations. This is the only way to eliminate any controversy as the one held by you and Beth: this should be the outcome of the so-called foundational crisis.

ALEPH: No, Tav, I will *never* accept that you declare my theorems as having no clear meaning, or as being relative to your semantics. Moreover, my phenomenological analysis proves that this is not the case, that there is a definite way of deepening our grasp of abstract concepts, and these are not restricted to purely constructive ones.

BETH: No, Tav, I will *never* accept that you admit classical theorems as having a definite meaning, even when they involve problematic and confusing concepts within them. Moreover, your phenomenological analysis, carried with absolute seriousness, should end up providing the constructive framework with a formal system, in a similar manner as classical mathematics are pursued. After all, this is what makes Aleph's phenomenological account so attractive.

TAV: Once again Beth's *true opposition* to Aleph tacitly assumes that formalization is indispensable!

OMEGA: Well, it certainly seems that the original question I posed earlier, namely, *can we reach an agreement by means of the historical analysis provided so far?* must receive a negative answer.

ALEPH: I have tried my best, Omega! But Beth pretends to impose her views disregarding historical facts and Tav dismisses history as a source of learning form the past while, on the other hand, Gimel is completely sceptic about the phenomenological method. *There is no possible progress with conservativeness, nihilism and scepticism*, not even for this debate.

OMEGA: In my opinion Tav does have a point, Aleph. You are talking as if your historical interpretation made, for example, epistemological platonism as the *inevitable conclusion*, you depict the chaotic flow of history as guided in some way by a definite path that naturally leads to the establishment of the *statu quo*. It is then trivial to accept your optimism, since *there is no alternative or possibility of improvement when a unique, self-founded idea is the only course of action available*. In that case, Beth and Tav suspicions of the authoritarian character that your means of justification bear would be grounded.

ALEPH: For me it just seems as if you were trying to *alienate* the working mathematician with your destructive ideologies. My historical interpretation is widely accepted: it is a *fact*

[125]It can be argued that this is what Brouwer ultimately did, see the footnote above.

that the foundational crisis happened and that the paradoxes in set theory have been solved, as it is also *true* that the content of mathematics is no longer limited or endangered in any sense, etc. In the end, there are better interpretations of history than others, and that's what is to be decided. Not wanting to fully accept this only weakens even more the spirit of contemporary mathematics and limits our capability of reaction against some new crisis in the future. The problem is that it is *easy* to surrender to the spirit of the time, as Beth and Tav show. I think you should agree with me on this, Omega, or does your 'decision-making' also apply here?

OMEGA: Well, you now pretend to acquire some *objective* knowledge from your own historical interpretation and, simultaneously, to take it as a prescription for the future. 'But what experience and history teach us is this, that nations and governments have never learned anything from history, or acted according to rules that might have derived from it. Every period has such peculiar circumstances, is in such an individual state, that *decisions* will have to be made, and *decisions* can only be made, in it and out of it'[126]. Moreover, your phenomenological method may be regarded, after all, as the desperate search for an underlying justification of the historical achievements of the past, like the one concerning the formalization of 'algorithm'.

ALEPH: But I still don't see anything *methodological* in your proposal. At least my phenomenological method (which you all so heavily despise) is a serious attempt of providing a philosophical account of the development of our mathematical understanding. In my opinion, it makes my optimism *grounded*. But your bland defense of pluralism and decision-making seems to enable every -possibly irrational- position in a irresponsible way.

OMEGA: Let me summarize briefly our conclusions up to this point, since this will make clear our differences. In general, *a depiction of a crisis is only possible by means of a philosophical position*. For you, the crisis was caused when the leftist mathematicians felt legitimated to claim ifthenism as the natural spirit in mathematics while, for Beth, the crisis consisted in the careless methods of the mathematicians more close to the right. Tav, on the other hand, believes that the crisis is a process that happens nowadays, in a obscure manner, and that it has to do with a misunderstanding of previous controversies. *Once a crisis is restated within a philosophical position, it feels natural to provide a solution to it, as well as to give an account of what we should learn from it.* Normally, the problem is identified with a semantic one. For Aleph, the crisis is only solved properly if we attach content to classical mathematics in a non-arbitrary, absolute way. For Beth, the crisis is solved if we restrict ourselves to more secure methods. For Tav, the solution consists in a relative clarification of meaning. *Reflecting on how we have acted before the crisis and how we ought to act now is identified with phenomenology as the method of elucidation of the*

[126]Italics are mine. I read this quote (from Hegel) for the first time in [14]. See there the footnote 4 for details.

essential issues of the crisis. One could say that Aleph's phenomenology is absolute, while that of Tav is relative.

BETH: Okay, Omega, we already know *that*. My question is: where do you place yourself in this picture? Do you believe that there is a *crisis* in mathematics or not?

4. A Wittgensteinian manifesto[127]**.** Omega now begins the exposition of his answer to Beth's question[128]. Our mathematicians listen carefully but with secret suspicions that Omega has finally fallen to skepticism.

OMEGA: Let me begin a systematic presentation of some of my views. Of course, no finite fragment of ideas is complete, since these only gain their force and vividness through debate and confrontation. Of course, my own point of view is susceptible of further improvement and elaboration, so I encourage you to please point out what you find the most controversial.

1. Mathematics are at crisis only when we pretend to consider its development and results independently from the decision-making that necessarily accompanies its practice. Therefore, my conception of crisis is quite near that of Tav: we are not talking about discrete historical events but about a slow process of propagation of an illness or even a reactionary attitude shared by a determined group of professionals.

> *1.1. Mathematicians have inclinations that the philosopher should treat*[129]*.* It is remarkable how mathematicians usually make a use of language that is quite interesting for philosophers[130]. Of course, both mathematicians and philosophers are working specialists in their respective areas. Nevertheless, a problem of *professional competence* may arise in areas of mathematics which are more philosophically-charged[131], like set theory or category theory. I will be talking about *mathematicians* in order to refer to the professionals of these 'foundational branches'[132].
>
> *1.2. It may be the case that mathematicians identify philosophical notions with the formal concepts they are used to deal with.* In other words, it may happen that mathematicians feel like their definitions have gained precision with respect to the *traditional* management and treatment of informal notions. This may apply easily for the case of 'set' in contraposition with 'collection' or 'extension of a concept' or for the identification of metamathematics with mathematical activity as a whole. Moreover, as we have already seen, the work of these mathematicians is ambiguous, something that lies in the middle point between a *branch* and a *foundation* of mathematics, in some sense. It is in this sense in which we argue that a *schizophrenia* is taking place, namely, in

[127] What follows is an exposition of what I have already defended somewhere else. The adjective 'wittgensteinian' has nothing to do with the fact that Omega's presentation is ordered by enumeration of sentences but with the one that we will cite extensively the section 'Phenomenology' from the 'Big Typescript', [13].

[128] Of course, still more ideas are to be presented later, as Omega may recognize in a moment.

[129] This is close in spirit to some paragraphs from [12].

[130] One could even go further and assert, like some have done before, that the history of philosophy is precisely the history of philosophy of mathematics.

[131] Compare with Aleph's remarks above about how the paradoxes only had an effect on these branches of mathematics.

[132] The axiomatic presentation of formal systems makes easier to forget how the corresponding original intuitions that are being formalized, and thus the corresponding motivations of work, may profoundly differ.

the form of a 'surreptitious shift in meaning'[133].

1.3. It may be the case that mathematicians employ expressions from other sciences in their discourse. These sciences include, more prominently, physics and psychology. As an example, we may consider how Aleph used to talk about mathematical *objects* and transcendental *faculties*. This employment, at best, contributes to the proliferation of misleading *similes*[134]. The philosopher, then, feels legitimized to question the use of language made by the mathematician and, thus, the similes become *explanations* in response to such interrogations. But here the dissatisfaction cannot be removed by explanation, as metaphysics pretends to do[135]; rather, what we find disturbing is the lack of clarity of the grammar of the expressions used by the mathematician in the first place[136]. We may here include the infamous the construction 'exists x' as the pointing example of a misguided use of language motivated by mathematical practice[137]. We cannot fall into the trap of metaphysics pretending to follow the bright light of explanation.

1.4. As a conclusion, there is a formalizative tendency among mathematicians: an inclination towards the identification of the formal systems and the informal content they formalize. In this sense, mathematicians take their results as something *absolute*, and philosophers try their best to provide a *clear* interpretation of those or even accept the new, philosophically privileged position of mathematicians. Mathematics are thought as something *hard*, as a disk made of steel[138] while our informal notions remain as *fuzzy*. Of course, this may correspond to a useful insight for the mathematician, but the following confusion seems to have pervaded her philosophical imaginary: the supposedly universal necessity of the formal derivations within a calculus and the necessity that the calculus should bear by itself, as a model of intuitive notions, are being established as equivalent. This inclination, when philosophically refined, mutes into what we may call the *dogma of hardness*. On the other hand, and as a natural consequence of this, philosophy becomes into *mathematical philosophy*[139], which undermines the need for philosophical debate. Therefore, the crisis extends also to the integrity of the working philosopher.

[133] In the sense of [24].
[134] See [13]§100.
[135] See [13]§94.
[136] See [13]§94: 'The confusion arises from our believing that we have to decide about the presence or absence of an object (a thing) – of the spot; as when I decide whether what I'm seeing (in a physical sense) is a red coat of paint or a reflection.'
[137] In [13] can be read: 'The linguistic form "I perceive x" originally refers to a phenomenon (as an argument) in physical space (here I mean "in space" in the ordinary way of speaking). Therefore I can't automatically apply this form to what is called "sense data", say to an optical after-image'.
[138] See [13]§96.
[139] See [27]. Bishop, in [16], says the following: 'Mathematical philosophy consists of the creation, comparison, and investigation of formal systems. Consistency is the goal. In consequence meaning is debased, and even ceases to exist at a primary level'.

2. Against the statu quo, we should acknowledge different ways of doing mathematics. Before, I called *V-pluralism* to the view that defends that different formal specifications may be provided for the same intuitive or informal content. The dogma of hardness then goes against this, because it establishes formal systems as *essentially identical* to the notion they formalize. From here, then, it follows that only a true, real characterization is possible for each informal content and that, moreover, the first is an ultimate improvement, with respect to its rigour, of the latter. Quite on the contrary, this 'is completely analogous to the relationship between ordinary language and a mode of expression that is "logically clarified". Both are completely equivalent to each other; it's just that in the one the rules of grammar are expressed simply by its outward appearance'[140].

> *2.1. Since we may encounter ourselves with different proposals, philosophical debate is needed.* Formalization is not a formal process. First, some informal statements must be taken as the *natural* analysis of the informal notion we are studying in the first place and, later, a formal system is provided as a *reliable encoding* of the properties made explicit by the analysis. Therefore, arguments from all kinds (epistemological, metaphysical, methodological, aesthetical, historical) may be used to make the desired proposal appealing. It is only in this sense in which one could say that *everything follows*. The election of one of the proposals is the first decision that one executes. The set of beliefs that one uses as favoring their selection is what we have called a *philosophical background*.
>
> *2.2. Even when we have decided to study some formal system, decisions are needed at each step of development.* Aleph's phenomenological method was a tool for achieving non-arbitrary expansions of formal systems. For example, his dream solution for CH relied on the fact that we can expand ZFC with necessary properties for the iterative conception of 'set'. The idea that, in general, each formal system only has a canonical expansion is the *dogma of original intuition*. Why this dogma? It clearly allows to defend a position between V-pluralism and ifthenism, in the sense that one could simply commit to a determined philosophical background incarnated in some formal system and then simply study that system. In other words: the dogma of original intuition is a more concrete form of the dogma of hardness, which tries to legitimize mathematical philosophy. This of course would degrade the working philosopher even more. But Gimel has shown, through the examples on axiom-adding, that this idea is no longer possible: one should either admit that we have to make a *decision*[141] or that there are no decisions to be executed at all that and hence we can only talk

[140] See [13]§100.

[141] Compare with [13]§96: 'But then I can decide: either to say that there is no further division, i.e. that to speak about one is senseless – and in this case I have obliged myself to give a different description of any phenomenon that might occur and that I might be tempted to call a further division – or: to allow the divisibility to continue within my symbolism'.

about *imposition*. This imposition is precisely what the previous dogma achieves. This imposition is the true face of the *statu quo*[142].

2.3. Radical pluralism: pluralism is a tool of criticism. We could simply use our form of pluralism in order to attack the *statu quo* as, for example, represented by Aleph. But we have seen that the revolutionary opposition to it, namely, Beth's thought, can be reactionary on its own. What distinguishes the *instrumental pluralist* from the enforcers of views as those is precisely the fact that she does not pretend to begin *a* revolution against some concrete, technical, features defended by the *statu quo* which could therefore simply rule out valuable others; rather, the instrumental pluralist must *be*, with her critical attitude, the revolution. The instrumental pluralist must position herself against any limitation of the philosophical debate, since it is the only way of progress. The instrumental pluralist must be of course against any clean-cut division and simplification of this debate. The instrumental pluralist must, thus, attack any petrifaction of a philosophical background.

3. Phenomenology must be identified with the 'investigation of the rules of the use of our language, the recognition of these rules, and their clearly surveyable representation amounts to'[143]. This is where Aleph's conception of the phenomenological method differs from mine[144]. Philosophical debate, by itself, by no means can clarify the problematic attitude we are pointing at and which is carried by some dangerous inclinations of mathematicians. It is here where phenomenology must fulfil its role. It should help us to clarify, to analyze our use of language. But note, then, that there is no *new* phenomenon[145] involved in its processes at all. Phenomenology provides modes of representation of already known phenomena, 'each time we recognize that such and such a mode of representation can be replaced by another one, we take a step toward that goal'[146], that is, the goal of the phenomenological method. This use of philosophy is not speculative: it leaves everything as it is, but makes explicit what the possible decisions amount to. It is then the task of the professional to actually *decide* and take a definite path.

3.1. Aleph's use of phenomenology is just a way of extending the range of imposition. Remember that his phenomenological method was firstly designed in order to

[142] In fact, I have argued in other place that Krikpe's rule-following paradox should be understood as defying the dogma of original intuition, not the basic aspects of rule-following that are more or less natural.
[143] See [13]§94. Now Omega has adopted the role of *The Mouth of Wittgenstein*.
[144] Can both the phenomenology of Husserl and Wittgenstein be compared with each other? This question clearly exceeds the scope of the present dialogue. See [34] for some answers. The question here is even more difficult, since we pretend to contrast Gödel's interpretation of Husserl's phenomenological method (applied to mathematics) with what Wittgenstein calls 'phenomenology' in [13].
[145] See [13]§96: 'What we're missing isn't a more precise scrutiny (say of A, B and C), nor the discovery of a process *behind* the one that is observed superficially (that would be the investigation of a physical or psychological phenomenon), but clarity in the grammar of the description of the old phenomenon. Because if we looked more closely we would simply see something *else*, and would have made no advance on our problem. *This* experience, and not another, is what needs to be described'.
[146] See [13]§94.

justify the dogma of original intuition. The problem with this approach is the following: phenomenology is not a positive (nor negative) task[147]. Pretending otherwise would amount to the problematic attitude explained above. Aleph's phenomenological method, indeed, made reference to a *psychological fact*, to the existence of a mental faculty. But phenomenology does not consist in the positing *philosophical images or theses*; phenomenology is a an analytical *activity*, a process of recognizing the rules of the *grammar* in our language. When we commit to some decision, what we leave out is simply *senseless*[148] in the new, accepted grammar[149]. In other words, with each execution of a decision, we accept some grammatical elements as *simple*[150]. We cannot pretend to uncover further properties of these elements. Gimel's overall suspicions regarding Aleph's phenomenological method may be rephrased as follows: 'One might want to describe what is obvious, but one can't »want to examine what is obvious«'[151]. The phenomenological method presented by Aleph is very close to be something like an *absolute experiment*, that is, a process that cannot be mediated (with formal rules) but that is capable of reaching *a priori* results or, as he puts it, a special state of consciousness[152]. One may then argue that the notion of 'set' is a grammatically simple one, in the sense above; we are only expected to execute definite decisions regarding its potential, historical, development. There is no imposition but an *active use* of it. Thus, the non-arbitrariness of which Aleph used to speak can be only *intrinsically historical*. There is no help from the past or the future. Even if we accepted Aleph's phenomenological method, it is likely that, in order to achieve its ideal goals, we would have to accept a new set of dogmas designed as a non-arbitrary guide for its application. Therefore, Aleph has converted phenomenology into the epitome of a schizophrenic duality: a founding method that has no further foundation itself, an a-historical method that must take place in history[153].

3.2. The outcome of the phenomenological method is a gradual change in the attitude towards formal systems. That is, in a similar spirit as Tav's brand of intuitionism, the grammatical conception of phenomenology that we defend should help us clarifying mathematical discourse without falling into misguided uses of linguistic expressions by

[147]More explicitly, compare this with [13]§96: '[...] it actually isn't a question of the difficulty of calling up a particular image before my mind's eye, nor is it a question of something that I can try but fail at; rather, it's a question of acknowledging a rule for a mode of expression.'

[148]Again, [13]§96.

[149]Compare with [13]§96: 'I can set up countless different stipulations for what is to be the criterion for sameness of length in a visual image, and accordingly different meanings of the measurements will result'.

[150]See [13]§100.

[151]See, also for the non-penetrability of visual space, [13]§99.

[152]Compare this with [13]§94: 'What Mach calls a thought-experiment is of course not an experiment at all. At bottom it is a grammatical examination'. See [13]§94, 100 for further discussions on the difference between proof and experiment.

[153]This is related with the basic scheme of meaning platonism: the chain of meaning dependence should end somewhere, but then it is arbitrary to choose where it does end.

making each execution of decisions *explicit*. The outcome of this method is, therefore, a change of attitude[154]. If we have succeed in communicating our beliefs, the working mathematician should feel, when asserting the sentence 'sets (do not) exist as mathematical objects', something similar to what she feels when stating that 'there is never a high C within a visual field'[155].

[154]This attitude can be condensed in the slogan: *it depends*, in the sense of [13]§96: »Does something *force* me into the interpretation that the tree that I see through my window is bigger than the window? That depends on how I use the words "bigger" and "smaller" [italics are mine]«. Here the difference with Aleph's impositions is even clearer.

[155]Compare with [13]§94: »If a true circle within one's visual field is in some sense inconceivable, then the sentence "There never is a true circle within a visual field" must be the same kind of sentence as "There is never a high C within a visual field"«.

5. Schisms and other isms. There is a generalized disquiet along our group of mathematicians. Some of them differ deeply with Omega's views, others simply want to return to the main points of the debate.

SHIN: Omega's anarchism is truly a disgrace... don't you all agree? What are we supposed to learn with all of this? Where is the philosophical debate that Omega so firmly defends? I don't see nothing but a theatrical representation of what should be a true rational debate; no: what we are doing looks more like a *caricature* to me. Omega has finally succeed in turning a beautiful debate into a philosophical quagmire. Let me remember the original question, since many of us have been completely lost for a while: *is CH a proper mathematical problem and thus susceptible of solution?*[156]

ALEPH: Exactly, Shin! It is always easy to point at some insufficiency in the best approach available towards the solution of a problem like ours. I propose that we restart now the debate.

SHIN: All of this is also your fault, Aleph. You have been trying to fight fire with fire from the beginning. And you have got burnt. We have all forgotten a basic aspect about the debate: should it be philosophical or mathematical?

BETH: How is *this* more interesting than what we were debating before, Shin?

SHIN: Ah, my dear Beth... of course that you tacitly assume that philosophy may have relevant consequences in mathematical practice... you were, after all, a mathematical revisionist, huh?

BETH: I don't get why you talk to me like that. What is incorrect in my views that is so *obvious* for you?

SHIN: Well, for me mathematics and philosophy are to be separated completely: philosophical *debate* has no relation whatsoever with mathematical *practice*. Mathematics are *autonomous*: they do not respond to extra-mathematical agents, such as philosophical standards.

GIMEL: So what do you propose?

SHIN: What I claim is that mathematics has its own methods of methodological justification – we may call these *internal*. Of course, there may be philosophers willing to attack mathematics by means of metaphysics... this is a foolish strategy, in my opinion.

GIMEL: Could you please give some examples of these internal criteria?

SHIN: Well, I may as well name the same criteria that Aleph defined at the beginning of

[156]Shin is of course *Naturalism's mouth*. I have decided to put in Shin's mouth much of Maddy's thoughts. I do this only to keep the debate readable and to make the positions more flexible. I will also make reference to some comments I wrote down during the conference *What can philosophy do for set theory?*

the debate, but without any metaphysical pretension at all. These are to be regarded as what mathematicians find *natural* only in virtue of their profession itself.

BETH: But Shin: is not your view also a philosophical standpoint? Why we should take your remarks as something absolute?

SHIN: You can do whatever you want, Beth. I am only stating that your philosophically-charged strategy is superfluous. Some people would surely take this as a useful advice... My philosophy of mathematics is completely *naturalized*. There has always been a true difference between mathematics and philosophy: in the first, the main protagonists are (like in the natural sciences) general agreement along its practitioners; in the latter, however, there is no consensus, even a small one, on what its general aims and tasks should be. This is truly the sense in which the first is *hard* while the other is *fuzzy*. And this can be learned by anyone who understands history properly.

ALEPH: Exactly! This is similar to what I have been arguing all the time. We cannot let philosophical debate enter in the legitimate domain of mathematical practice.

SHIN: If I were you, Aleph, I would not be so excited. What I am defending is not that philosophical debate is pointless because some philosophical position is the only correct one, but that this kind of debate is pointless *per se*, at least for what mathematicians are interested in. What we should be doing is to have a *mathematical debate* instead, if you actually feel encouraged to call it that way. Before, you have been fighting each other with the imprecise tools of *first* philosophy, now we need to evolve towards *second* philosophy and more refined means of argumentation[157].

TAV: Oh, so we have here a philosophy of mathematics that is like a transparent covering around mathematical practice: it wants to vindicate its philosophical interest while claiming at the same time that philosophy is pointless in some sense!

SHIN: Your infamous dualities are of no use against my naturalistic arguments, Tav. I find it really funny that you were once considered to be the anarcho-constructivist revolutionary. You seem to me now more like a mathematical complainer with no aim nor justification and, at the same time, a philosopher who belongs more to the *Kantian old guard*[158] than to the intellectual vanguard...

TAV: So not only you are a rigid defender of the *statu quo* but you are also dogmatic in your beliefs. Why should we all be listening to your rather regressive views, Shin? Do you really think that you are making a valuable contribution to the present debate?

SHIN: Quite on the contrary, I am being truly revolutionary regarding philosophy of mathematics. Omega has been talking about how mathematicians have inclinations that should

[157]This way of expression is custom of Maddy.
[158]For a classic comparative of Brouwer's and Kant's views, see [19].

be treated by philosophers... Well, this is *wrong*: these inclinations are not the beginning or the subject-matter of the philosophy of mathematics but its *conclusion*.

BETH: Well, of course you are being revolutionary regarding philosophy: your ingenious idea is to destroy its links with mathematics. This is not a revolution but an act of *state violence*!

SHIN: Look, we are all mathematicians here. It seems ridiculous to me that we pretend to be talking about philosophy. And I am pretty sure that there are philosophers that agree with me. We must accept the kind of professionals we are, just as philosophers should accept their own status. It is time to start talking seriously about mathematics.

ALEPH: So are you implying that, maybe, mathematics *are they way that they are* and that any revisionism based on philosophical consideration must be given up?

SHIN: Yes: if philosophy conflicts with mathematical practice, it is the former what we should give up. Moreover, what I propose as an alternative may be described as a *working philosophy*, that is, a purely methodological, even pragmatical, counterpart of the metaphysical considerations that you have tried to clarify before. This is, in fact, a new form of discourse that has to be explored. Philosophy cannot interfere with mathematics and, more precisely, set theory. Internal philosophy is merely methodological, everything else can be considered metaphysics[159].

ALEPH: Shin, I really think that you should distinguish between philosophical *justifications* and *inspirations*. I would say that the realist viewpoint is 'the most common by far among mathematicians'[160]...

DALET: Oh, so we are also touching *sociology* in the debate! How nice!

ALEPH: ...so, if you wanted to fully commit to your ideas, you would have to at least accept some kind of realist *motivation* behind, say, in order to pursue of a canonical extension of ZFC that would allow the decision of CH.

SHIN: Well, I could do so, if you are happy with that. I don't see that this makes any effective difference at all.

TAV: But then your views are just as partial as those of Aleph! In fact, I could develop my own brand of *intuitionistic naturalism* by stating that, for example, in computer science and the study of numerical methods the working philosophy is heavily inspired by constructivist philosophy. If you wanted to attack this view then you should explain why you prefer some form of naturalism over the other.

SHIN: No, Tav: you are misinterpreting my claims. What I was investigating was whether

[159] An example of working philosophy? Some say that Woodin's work.
[160] See [11].

mathematical activity, by itself, had its own standards of justification.

TAV: Precisely, so I was. The problem is that you are only taking under consideration a *branch* of mathematics as the preferred representative of mathematics as a *whole*, whatever this should mean. That is, I am pretty sure that you have set theory in mind when talking about mathematics in general.

SHIN: Of course, set theory surely has a special place within mathematics as its *foundation*.

TAV: Ah, 'foundation'... in which sense?

BETH: Exactly, don't you need a set of criteria? And what reason could you provide in favor of its absoluteness?

SHIN: Well, I think that it is quite clear in the mathematical imaginary what a 'foundation' should do. For example, it should enable metamathematics, it should provide us a way of elucidating of *old* notions, it should establish some standards of proof and it should allow us to unify different branches of mathematics, in the sense of a shared language. Note that I have not included any metaphysical or epistemological conditions here: this was Aleph's main error[161].

BETH: At least Aleph's observations, while essentially authoritarian, were deep enough to keep the debate interesting!

ALEPH: But then your persistent inclination towards set theory is not clear . Why could not category theory serve as well? Or even a liberal constructivist system as that of Tav? My aim before was to actually make my choice *founded* in some sense.

SHIN: What you found deep in the debate, Beth, was Aleph's misunderstanding of your own views and, dear Aleph, I will not *succumb* and provide any philosophical argument for my choice of set theory. Even more, I am willing to accept any formal system as foundational provided that it fulfils the set of criteria presented above. My claim is that, for the moment, no other has been presented.

TAV: But Shin, how can you be certain that the foundational properties of set theory will not ever be in tension with its features as a branch of mathematics, active as any other? You can not assume that the working set theorist will bear these criteria in mind while trying to prove a theorem in her field of research.

SHIN: As I see it, the way in which set theory is usually regarded as a branch of mathematics can be explained as follows: we study the consequences of the axioms (potential and actual) that actually make set theory a serious foundation. So, yes, there is some mathematical work developing independently from the general picture, but at the same time it is the own

[161] This set of criteria are presented in [21].

framework of research that keeps this work within the foundational region of mathematics.

BETH: So we return here to the problem of axiom-adding and to the criteria provided by Aleph in the first place...

GIMEL: Yes, Shin: you have not explained clearly why my *multiverse* conception of set theory is flawed. You are basically defending the *statu quo* but eluding at the same time any direct confrontation. You are talking about how the mathematical practice should guide the working philosophy but you are not explaining the source of your ideals regarding such practice. Your view lacks any motivation[162].

SHIN: Of course, Gimel. Your multiverse proposal attacked the possibility of arriving at a dream solution of CH. You talked about how mathematical practice has developed in universes in which CH holds as well as in other in which its negation does. I believe that, therefore, there is a need of further determination for the meaning of 'set', since different theories have been motivated by it. But, 'even if it's true that our current meaning or concept is indeterminate in this way, there remains the possibility that it might be more *mathematically productive*, not to give up the quest for an answer to CH, but to seek out a fruitful successor to our current meaning or concept'[163].

TAV: So let me put it straight: you don't accept any kind of philosophical discourse in mathematics except the methodological, internal one. That is, you pretend to take the usual discourse of mathematicians at face value, philosophically speaking. From here, you deny my intuitionism on the basis that it involves a philosophically-motivated reform of mathematics.

SHIN: Exactly, Tav.

TAV: But then, I don't really see how you have arrived at your views opposed to Gimel's multiverse from your vehement defense of the mathematical practice *by itself*, for are not those views different, in essence, from the merely methodological ones?

SHIN: In which sense?

TAV: Maybe one could not understand mathematical practice without the working mathematician asking the latter kind of questions, but we can perfectly imagine him -as Gimel has argued- without the need of making reference to the first. I don't know, Shin, your views seem very close to Aleph's phenomenological method.

SHIN: Well, this only happens because, indeed, I am willing to accept the universist conception of set theory as the default one. For me, what Gimel is proposing is to *replace* the standard way in which we reason about sets by the multiverse. I only claim that the grounds

[162] A usual argument against naturalism.
[163] See [21]. Italics are mine.

for doing so are inconclusive[164].

ALEPH: I am in shock but I have to agree with Tav in the following thing. You, Shin, accept some kind of *ideal mathematical practice* in which the working mathematician is silently immersed. I know that Dalet made fun of me because I was trying to argue in sociological terms... and he was certainly right then. Now, I have seen that Shin pretends to embrace the mathematics of the *majority*, against what is practiced by some *lobbyists*[165]. This is the main difference between her view and mine: she pretends to advocate for a set of ideals with the contingent, *a posteriori*, historical development of mathematics as the only source while my aim was to pursue some kind of a-historical and rational properties that mathematical activity essentially bears.

BETH: In other words, you are willing to acknowledge that there may be other ways of conceiving mathematics, but that these arise from the *wrong* philosophical background. Shin, on the other hand, pretends to deal with the historical process of mathematical activity and must therefore choose a determined stage in its development as the one to defend. This is the main point of analyzing a historical *tendency* instead of a historical *phenomenon*.

ALEPH: Exactly![166] *That something is not historically sound does not mean that it is inconceivable*. This is what philosophical discourse allows us to examine more closely. This is the path for truth.

SHIN: No, Aleph: your realism only 'plays an inspirational role, like Einstein's verificationism'[167]. The truly relevant standards that will actually have fruitful consequences are the ones you presented before but, as I said, with no mystical nuances at all. Also, we should call to mind that another reason for the separation between mathematical and philosophical discourse (in the sense above) is that mathematicians deal with *highly hard and sophisticated* problems. That means that there is no question that one could ask, say, regarding CH which could be *merely philosophical*: there is a great deal of *technical knowledge* behind a true, methodological question.

ALEPH: So you are meaning that the working set theorist is an *opportunist*: he makes use of the philosophical, idealistic defense of realism and then he simply decides to not recognize its proper weight![168]

[164] See [21].

[165] This terminology is inspired in [22].

[166] Here we see how Aleph and Beth form a lobby by themselves, now excluding Shin from their own revolution-reaction dialectic.

[167] See [22].

[168] Woodin has said that we know that there is a physical world but we seek its laws while we know the set-theoretical laws but we seek for a set-theoretical world. This kind of remarks show that the working mathematicians that naturalism wishes to defend do not necessarily share its own opinions regarding the separation between mathematics and philosophy. This evidences the fact that naturalism is a reaction within philosophy.

TAV: I don't really understand what you are implying with your last remarks, Shin. I can only interpret them as making the schizophrenia of set theory even more *severe*: on the one hand, set theorists are the guardians of the common language of mathematicians and the shared standards of proof while, on the other, they deal with problems that cannot be understood, even superficially, by the working mathematician.

BETH: That is true, Tav! What you have been saying from the beginning is now being confirmed by Shin: she is describing set theorists as a group of *privileged* intellectuals.

SHIN: No, I was just saying that if we wanted to debate properly about the status of CH, we should at least have the required knowledge to do so, just as we would not be debating on the existence of a determined class of groups if we did not know some technical details.

TAV: But *that* is precisely the point of the so-called foundational frameworks... the corresponding internal questions soon start getting mixed with what you consider to be metaphysical ones...

GIMEL: As your attacks against my multiverse views have already shown!

DALET: The problem with your Naturalism, Shin, is that it is a philosophical view like any other but, at the same time, you try to make it appealing for mathematicians. *This* is not one of Tav's dialectical pairs but the tension that lives within your naturalist.

TAV: Y'all, look who has stopped joking around!

SHIN: As I said before, my intention was to argue against those who place philosophy *above* mathematics. Philosophy cannot decide anything of relevance for the mathematical practice.

DALET: I mean, you are right about something: it is true that mathematics had acquired a greater degree of rigour during the last century, and that this certainly has to do with the refinement of our study of formal systems. But you cannot say that philosophers should learn technical results in order to philosophize about mathematics. That would be an arbitrary imposition when compared with your adoption of naturalism.

SHIN: Why not? In my opinion, *there are no inherently philosophical questions* except –one could say– those which are uninteresting for the mathematician, namely, the metaphysical ones.

DALET: But the mathematical community is, in fact, more diverse than you assume. For example, there may be philosophers who, at some point of their work, start doing mathematics... and I don't see how the converse situation is so difficult to imagine. These cases are of course problematic for your separation between mathematics and philosophy.

SHIN: I don't see why, since in those cases the discourse of the philosopher would then

become methodological and the one of the mathematician would degenerate into the philosophical one.

DALET: Okay, now let me analyze your main claim. For me, there is only one way in which one could understand your separation between mathematics and philosophy. You assume that, in fact, philosophy can collide with mathematics, but it is not clear the nature of this potential collision.

SHIN: The example of intuitionism works well for me.

DALET: So you claim that, in that case, mathematical practice was interrupted by non-methodological philosophical considerations. But *the conflict can only arise between two entities with the same status*, that is, in case intuitionism could have changed radically mathematical practice, the latter would have some kind of link with a determined philosophical background.

SHIN: Not necessarily, it could be that intuitionists secretly placed some metaphysical charge in classical mathematics so that their philosophical hostilities could be justified.

TAV: Well, it also could be the case that the defenders of classical mathematics placed that kind of content within intuitionistic mathematics in order to weaken any criticism.

DALET: Then the question remains: why do you, Shin, take as a relevant sociological fact that mathematics are pursued *in some way* but not that a great number of mathematicians have also *metaphysical inclinations*?

SHIN: Maybe the problem is that philosophers try to take what mathematicians say (usually without formal philosophical knowledge) at face value. Thus, what you call metaphysical inclinations are no other thing than methodological ones (or unfortunate reformulations of such).

TAV: The only fact is that there are mathematicians who not only are inspired but *moved* by philosophical aims. The naturalist should then be able to distinguish between both kind of mathematicians... that's quite a task, if you ask me.

DALET: The problem that I see with your account, Shin, is that accepting naturalism would be a *huge* turning point for the philosophy of mathematics. But, often, huge change makes impossible to evaluate any potential consequence with detail. For example, the professional situation of the working philosopher could get compromised: what if her work lacked any purpose after all?

SHIN: Well, I think that it is better for both philosophers and mathematicians to diverge in their paths. Of course, I do not exclude the help that they may bring regarding fruitful distinctions, as that of methodological and metaphysical questions, but I am willing to label philosophical work as mathematically impotent when it has got drowned in metaphysics.

DALET: But we have seen that your foundational working mathematician has a mixed, problematic status. One could even talk about your proposal as the gradual replacement by a *New Class* (foundational mathematicians) of the *Old Class* (philosophers)[169]. That is, naturalism could not prevent a situation like this because it would side mathematical activity blindly.

SHIN: What do you mean?

DALET: Well, take the notion of infinity as an example. Traditionally, mathematicians dealt with some kind of vague notion of infinity (of course, if you want, a *naturalized* version of it). Philosophers felt naturally legitimized to analyze this notion, debate about its properties, etc. During those times, naturalism would have condemned these kind of discourse on the infinite while supporting its (imprecise) use made by mathematicians, even when there was no clear distinction between the two.

SHIN: Exactly. In those times, the infinite was usually mixed, by philosophers as well as by mathematicians, with metaphysical questions, or even theological ones. The only thing that mattered, nevertheless, was the use of it in mathematics, which usually translated into some properties of functions, collections of points, etc.

DALET: Then, Cantor provided a mathematical theory in which the previous use of the infinite by mathematicians was made formally clear. Not only that, he even developed a formal theory of the infinite that allowed to finally put end to some philosophical questions that had the lack of conceptual clarity as their main source for their proliferation.

SHIN: Precisely. Here the distinction between methodological and metaphysical questions was made even more evident!

DALET: In other words, a greater portion of the *philosophical capital*[170] generated by philosophical discourse was now in hands of mathematicians. Concretely, of the New Class of foundational mathematicians.

SHIN: Yes, I agree with your description.

DALET: The replacement I was talking about took place, then, through a process of formalization or *mathematization*. The problem here is that, for example, an intuitionistic naturalist such as Tav's could consider this process as inherently philosophical, in the sense

[169]This terminology is due to [23]. Gouldner talks about the class of intellectuals as increasingly powerful and morally ambiguous. Some mathematicians and philosophers of mathematics share this features, for they describe their own work as substantially relevant and helpful for the working mathematician and, standing on this premise, they pretend to argue in an *absolute* way. One could naturally think of naturalism as such a tendency: philosophy is confusing, mathematics is inexorable. *This* is what is dangerous for Omega. The inexorability of mathematics is closer to that of a mechanical procedure. The mysterious features and miracles that others want to inject into them is precisely what motivates philosophical debate. Surprise is only possible when we assume a determined philosophical background.

[170]See [23]. In this sense we can also quote [37]: every document of civilization is also a document of barbarism.

that methodological questions would just consist in the traditional, philosophical ones, but restricted to a fixed formal system arbitrarily chosen. From here, it would seem natural to say that the replacement of the flexible and critical Old Class of philosophers by the New Class of, say, set theorists could take place.

TAV: And arbitrarily means chosen by historical, yet no rational, force. *Naturalism is just a sublimation of an interpretation of history.*

SHIN: No, Dalet: the process of formalization is not at all philosophical. Philosophers can be, nowadays, speaking about different conceptions of the infinite. This has nothing to do with the naturalized infinite with which set theorists work.

TAV: Then it is not clear *why* mathematicians usually motivate set theory as treating the traditional concept of infinite, *why* they use the same term for two, so deeply different, notions and *why* there is in fact a collision of mathematical and philosophical practice. Here we return to the tensions Dalet was mentioning before.

DALET: In fact, let me add the following. The result of your separation between mathematics and philosophy is likely to consist in a further disagreement between philosophers and mathematicians. You already verbalized it: mathematicians will continue thinking that their notion of the infinite is the rigorous version of the traditional one and, respectively, philosophers will still believe that mathematicians have already made a decision and thus stayed in a fixed stage from the development of that notion.

SHIN: I still don't see anywhere where my ideas are *mathematically dangerous*. Even when that separation sounds sad as you have put it, we will have at least have eliminated all sign of revisionism founded on some spurious cause.

DALET: Well, you already admitted before how the Old Class is gradually more in need of the technical knowledge of the New Class to fulfil its work. This will obviously lead to a greater accumulation of philosophical power and capital by the latter[171], even if your distinction between philosophical and methodological questions were sound. Again, naturalism cannot *prevent* this (possible) phenomenon.

TAV: Not bad for an old jokester, Dalet! What you are meaning is that one can always pretend the others to take his own claims as something philosophical, as something *deep*, while asking methodological questions is only possible for a specific group of professionals. The problem, thus, is that in foundational branches is too easy to pretend that one is *making philosophy*. This is how philosophical capital is at stake.

SHIN: But then I can simply argue that I do not agree with this picture: I still cannot see anything pernicious inherent to the *common*, methodologically induced, inclinations of

[171]This is a corollary of the main point developed in [23].

mathematicians.

DALET: And, once again, someone like Tav could reply to you by saying that *a historical imposition is always political*, and that *a political imposition may always find resistance by means of political opposition.*

TAV: Exactly. Moreover, I don't understand the fear for revisionism that naturalism takes as grounded. I have already depicted a way in which one could reconcile intuitionistic and classical reasoning. Of course, this idea was attacked by Aleph's pride and Beth's blind radicalism.

SHIN: Oh, right, your proposal... I don't find it mathematically interesting nor fruitful at all. Although your revisionism is quite moderate, I won't tolerate any of it (on metaphysical grounds).

TAV: But don't you see that your *attitude* towards my idea is irremediably related with how you have characterized the problem that naturalism pretends to solve? It seems that you want to keep playing the game with your own set of rules. You have converted mathematical revisionism into a monster that one should avoid under any circumstance, even when there is no monster at all or when the means to do so are highly suspicious and dangerous for other professional work.

OMEGA: It seems that, once again, we have reached a deadlock.

BETH: Omega, where were you all this time? Do you have something to add to the conversation?

OMEGA: Only that Shin's naturalism is a reasonable point of view, at least in spirit. As I have argued before, we should distinguish between phenomenology and metaphysics, and this is what Shin achieves by separating the methodological from the metaphysical. My problem with her approach is that she takes this *open task* as an essential *distinction.*

SHIN: What do you mean?

OMEGA: I think that the working mathematician should, of course, be able to reflect on the grammatical rules of which his formal system consists. You have labeled this as an *internal* goal.

SHIN: Exactly.

OMEGA: But, for me, this belongs to *philosophical activity.* Your picture regarding a separation of mathematical and philosophical spheres really presupposes some kind of characterization of these as a fixed set of sentences or as a determined property that some statements should bear.

SHIN: Well, more or less. This picture is only a mere recourse of argumentation.

OMEGA: Let me return to my manifesto. There, I tried to argue that, through phenomenology, we can achieve a *surveyable* representation of our grammatical rules, that is, of all the (methodological) decisions we have executed through the development of our formal system. But this analysis is not restricted to the adoption of this or that formalization: the ultimate tool of criticism, the *why not?* slogan helps us to make explicit deeper, philosophical backgrounds.

SHIN: But which is the nature of these backgrounds? Isn't it metaphysical?

OMEGA: No, you have not understood me fully. Through our criticism we arrive at a point in which we have made clear each execution of decisions. Now, axiom-adding can be regarded as an *internal* decision (for example, as an answer to the question 'which axiom extends naturally such and such property in such and such context?') but also as an *external* one, in the sense that, with the acceptance of one axiom or another, the overall formalization provided by the system differs. This is why I said above that *the depth at which a decision is executed is a measure of its revisionist character*: multiversism is an *internal* revisionism while, say, a structuralist mereology (as a possible formalization of 'collection') would be an *external* one. In other words, we can measure revisionism with respect to a formal system by looking at the set of decisions that are preserved and comparing it with the set of divergent ones.

SHIN: So you are implying that the true heart of mathematics consists of metaphysical considerations, and not methodological ones. Now I don't even need sociological induction in order to believe that your ideas are incorrect!

OMEGA: No, Shin: by means of phenomenology I want to clarify all the methodological decisions that have been taken and separate them from metaphysics or philosophical debate in general. I want to be able to say: *these are the decisions we have executed, this is all what may inspected from here, this is all that can be said.*

SHIN: So you are also against the imposition of philosophical standards, huh?

OMEGA: I wish to present the configuration of formal systems as what is showed by the repeated execution of a set of decisions. Now, it is a fact that we *act*, but it is also a fact that we argue for *how we decide to act.* It is in the philosophical debate where some will try to make their decisions look as grounded and, therefore, to turn them into impositions and, at the same time, where others will try to remove any kind of metaphysical speculation from the executions made by the first ones. This last task is what I claim the one of the working philosopher to be.

SHIN: But *this* kind of vision is what I pretend to give up from the beginning: there can be no philosophical debate interrupting mathematical activity.

OMEGA: And *this* is where you and I differ, Shin. It seems to me that the use of slogans such as *mathematics for the mathematicians* are only a form of demagogical paternalism. After all, naturalism *is* a philosophy of mathematics. Of course, you can claim that it is a form of *anti-philosophy* of mathematics[172] –as I also claim about some of my views– but this is not far away from Tav's transparent covering metaphor. You cannot pretend to embrace the deep problems that your ideas face, as the ones presented by Dalet, just because your own view collapses philosophical debate by itself. *There is no going up the ladder and then throwing it away*, at least not in this sense[173].

SHIN: But then, what are you proposing? I still don't see the problem in simply letting the mathematician do his work. It is unreasonable to believe that, at each step, a philosopher should guide the growth of a technical idea with which he is not acquainted: it is the mathematician who does so.

OMEGA: Well, yes! Then you are with me against ifthenism. In fact, we think similarly about the relationship of mathematics and philosophy: the mathematician can only make useful philosophy within a formal system, that is, regarding the rules of that system and trying to get new perspectives from what has already been decided. The philosopher –or the working mathematician in her philosophical moments[174]– deals with general properties of concepts. This last task for you is useless, since there is *one* approach to mathematics. But my instrumental pluralism forces me to accept at least some utility of philosophical debate: it enables rational argumentation regarding the decisions executed, once these are made explicit. Now, philosophical debate cannot be idealized: it is *free* in its forms, so we cannot expect that it is carried on in terms of purely methodological arguments. Actually, Shin, you pretend to forget about this aspect of mathematical activity, even when philosophical debate may remain independently as an ongoing battle of ideas. The fact that you *can* elude it from your naturalist viewpoint is a *proof* that the *statu quo* is being effective in its defence. But we do not deal with possibilities here. Classical mathematics are, at the same time, the canon that allows the establishment of the *statu quo* and the main source for revisionism: *it is a fact that we have acted in such and such way, but why not in this other?* It is the huge landscape of possibilities what we wish to make explicit by bringing the executed decisions to light. The *profoundness* of classical mathematics hides the *richness* of its alternative possibilities.

[172] As Maddy labels Wittgenstein's philosophy of mathematics. We understand this adjective as depicting some methodological attitude: we do not wish to develop a system for philosophical cognition. Rather, philosophy is a continuous task, and each problem requires different viewpoints.

[173] Of course, this expression is not arbitrary: remember that Omega is quite fond on being a Wittgensteinian.

[174] As Quine puts it.

6. A hermeneutic digression. Dalet wishes to use a philosophical example as a way of condensing the main points of the debate. He politely asks the others for permission.

DALET: Following Shin's suggestions, I wish to make a brief comeback to the Kantian old school. More specifically, I would like to analyze a very important section of the *Critique of Pure Reason*[175]: the Architectonic of pure reason[176].

SHIN: So I see that you still don't want to listen to what I said... it is easier to keep debating about philosophy, isn't it?

DALET: Well, in fact, I think that this will shed light on the positions that have been presented during our debate.

OMEGA: Please Dalet, go on.

DALET: When reading the KrV for the first time, I was puzzled by the use of the technical term *schema*, one of Kant's most relevant and, at the same time, obscure notions[177].

SHIN: Oh, how surprising! This lack of clarity of technical philosophical terms is very common among first philosophy... this is precisely why I argue that this debate is a loss of time.

DALET: Patience, Shin: the reward awaits us ahead. Let me first provide you some context. Kant defines the architectonic as 'the art of systems'[178], where a *system* is 'the unity of the manifold cognitions under one idea. This is the rational concept of the form of a whole, insofar as through this the domain of the manifold as well as the position of the parts with respect to each other is determined *a priori*. The scientific rational concept thus contains the end and the form of the whole that is congruent with it'[179].

ALEPH: I could really see the analogy between this notion of system and the concept of *formal system* that we have been using all this time. For example, one could say that set theory is the systematic, *a priori* presentation of a manifold of cognitions under the idea (of the iterative conception) of 'set'.

DALET: Precisely. Moreover, Kant seems to be talking about your non-arbitrary development of systems: 'The whole is therefore articulated (*articulatio*) and not heaped together (*coacervatio*); it can, to be sure, grow internally (*per intus susceptionem*) but not externally (*per appositionem*) [i.e. from an internal cause, not by juxtaposition]'[180].

[175] In what follows, KrV (see [26]). We will follow the usual convention for quoting the book.
[176] A832/B860-A851/B879.
[177] This is not some personal opinion held by Dalet: it is a recognized historical fact.
[178] A832/B860. And 'systematic unity is that which first makes ordinary cognition into science'.
[179] *ibid.*
[180] A833/B861. Kant presents the first of the biological metaphors that appear during the Architectonic: 'like an animal body, whose growth does not add a limb but rather makes each limb stronger and fitter for its end without any alteration of proportion'. It is interesting how, in [31], Husserl also makes use of this kind of comparatives in a paragraph that deals with similar questions as these.

TAV: All of this is very interesting, Dalet. But this idealization of the supposedly *a priori* harmony of reason (as related with, say, the development of mathematics) only reflects the fact that Kant philosophy is certainly limited in some way by the spirit of his time.

DALET: I thought that you were an unconditional acolyte of Kantism, Tav!

ALEPH: My opinion on the Kantian conception of mathematics is that '[t]he relevant utterances by Kant are, it is true, incorrect if taken literally, since Kant asserts that in the derivation of geometrical theorems we always need new geometrical intuitions, and that therefore a purely logical derivation from a finite number of axioms is impossible. That is demonstrably false. However, if in this proposition we replace the term "geometrical" - by "mathematical" or "set-theoretical", then it becomes a demonstrably true proposition'[181]. It is *there* where his ideas have not stood the test of time.

OMEGA: Dalet, could you elaborate further?

DALET: For sure! Our main concept makes now its enigmatic entrance: 'For its execution the idea needs a *schema*, i.e., an essential manifoldness and order of the parts determined *a priori* from the principle of the end. A schema that is not outlined in accordance with an idea, i.e., from the chief end of reason, but empirically, in accordance with aims occurring contingently (whose number one cannot know in advance), yields *technical* unity, but that which arises only in consequence of an idea (where reason provides the ends a priori and does not await them empirically) grounds *architectonic* unity'[182].

BETH: So one could then make the distinction between technical and architectonic schemata.

DALET: Of course. Kant's conclusion is that '[w]hat we call science [...] cannot arise technically [...] but arises architectonically'[183].

OMEGA: So what exactly do you find strange about this use of the term 'schema'?

DALET: Well, earlier, during the Trascendental Doctrine of Method[184] Kant makes use of it in order to characterize mathematical cognition as radically different from philosophical one. Roughly, he argues that '[p]hilosophical cognition is rational cognition from concepts, mathematical cognition that from the construction of concepts'[185]. Now, 'to construct a concept means to exhibit a priori the intuition corresponding to it'[186].

OMEGA: Does this imply that Kant is arguing for a constructivist (almost finitistic) conception of mathematics?

[181] See [10].
[182] *ibid.*
[183] *ibid.*
[184] A708/B736.
[185] A7I3/B741.
[186] *ibid.*

DALET: Not necessarily, for intuition does not mean the same as (sensible) perception[187]. What is important from all of this is that '[p]hilosophical cognition thus considers the particular only in the universal, but mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined'[188].

BETH: But what could be this 'particular' in the case of sets? One could argue that we access the concept of set through the set of formal axioms that condense its essential properties... But this does not seem enough to relate this with the previous conception of 'schema'.

DALET: Let me get back to the Architectonic. Kant admits that 'in its elaboration the schema, indeed even the definition of the science which is given right at the outset, seldom corresponds to the idea [...] For this reason sciences, since they have all been thought out from viewpoint of a certain general interest, must not be explained and determined in accordance with the description given by their founder, but rather in accordance with the idea, grounded in reason itself, of the natural unity of the parts that have been brought together. For the founder and even his most recent successors often fumble around with an idea that they have not even made distinct to themselves and that therefore cannot determine the special content, the articulation (systematic unity) and boundaries of the science'[189].

OMEGA: Well, this is interesting! It seems that both technical and architectonic schemata are related in some sense... that the first kind is indistinguishable from the second at some stage of the development of science...

TAV: Exactly, Omega! Moreover, it seems to make a separation between the *direct history* as presented by the working mathematicians and the *analyzed history* that one has to elucidate through philosophical debate[190].

BETH: Precisely! Kant's advice is to double check the philosophical reading that set theorists give from their own work (i.e. how do they place it architectonically).

GIMEL: Not only that: the technical/architectonic division within schemata can be easily translated into our previous distinction between technical and natural evidence!

[187]This is what is argued in [35], that an intuition is merely a singular representation while a concept is a general one.

[188]A714/ B742. In [35] Kant's conception are linked with the euclidean tradition of geometrical proof. This comparison is at least plausible when reading the explanation from A713/B741. I will not make references to the Schematism chapter, since it may be assumed that the occurrence of 'schema' in the quoted paragraph follows the meaning given there.

[189]A834/B 862.

[190]This separation is inspired by a similar one from [28].

DALET: In fact, Kant adds: 'it is first possible for us to glimpse the idea in a clearer light and to outline a whole architectonically, in accordance with the ends of reason, only after we have long collected relevant cognitions haphazardly like building materials and worked through them technically with only a hint from an idea lying hidden within us'[191].

GIMEL: So, actually, Kant presents architectonic unity as arising from the technical one! The problem here is that he does not explain *how* this can be achieved... Once again, there is an inevitable tension between an *ideal* and its *fulfilment*.

TAV: The spirit of Kant's time crystallizes as a blind faith in the ideal distinction between the technical and the architectonic, and this resembles Aleph's mythical idea of providing a definitive set of criteria for detecting the non-arbitrary (architectonic) expansions of formal systems as opposed to contingent (technical) ones.

DALET: But let me continue. I think that the following will be interesting for Omega. Let us focus on the case of philosophy. In general, Kant claims that 'all cognition, considered subjectively, is either historical or rational. Historical cognition is *cognitio ex datis* [cognition from what is given], rational cognition, however, *cognitio ex principiis* [cognition from principles]. However a cognition may have been given originally, it is still historical for him who possesses it if he cognizes it only to the degree and extent that it has been given to him from elsewhere, whether it has been given to him through immediate experience or told to him or even given to him through instruction (general cognitions)'[192].

OMEGA: Could you provide some example?

DALET: As Kant does, consider the case of a philosopher who has learnt a system of philosophy. This philosopher only has a historical cognition, regardless of the rational aspirations that the system of philosophy is intended to achieve: 'the faculty of imitation is not that of generation, i.e., the cognition did not arise *from* reason for him, and although objectively it was certainly a rational cognition, subjectively it is still merely historical'[193]. Therefore, philosophical cognition may be regarded as objectively rational and, simultaneously, subjectively historical[194].

ALEPH: Where do you want to arrive at, Dalet?

DALET: As a conclusion, philosophy can only be *learnt* in a historical manner: 'we can at best only learn *to philosophize*'[195].

OMEGA: This is the same as saying: *philosophy is an activity*.

[191] A834/B862.
[192] A836/B864.
[193] *ibid.*
[194] A837/B865.
[195] *ibid.*

DALET: Or, alternatively, that it 'is a mere idea of a possible science, which is nowhere given *in concreto*, but which one seeks to approach in various ways until the only footpath, much overgrown by sensibility, is discovered, and the hitherto unsuccessful ectype, so far as it has been granted to humans, is made equal to the archetype'[196].

OMEGA: I see, Dalet. So now you are expecting me to explain my previous thesis *mathematics is an activity* in the light of these observations.

DALET: Precisely.

OMEGA: Well, Kant keeps arguing that one can rationally (and not only historically) learn mathematics: 'it is strange that mathematical cognition, however one has learned it, can still count subjectively as rational cognition, and that the difference present in the case of philosophical cognition is not present in this case. The cause of this is that the sources of cognition on which alone the teacher can draw lie nowhere other than in the essential and genuine principles a of reason, and consequently cannot be derived from anywhere else by the student, nor disputed in any way, precisely because reason is here used *in concreto* though nevertheless *a priori* founded, that is, in pure and therefore error-free intuition, and excludes all deception and error'[197].

ALEPH: I personally don't see anything strange with this: my phenomenological method could be seen as a way of sharpening these deep principles of reason. This is the source of the objectivity and correctness of mathematics, what Omega has come to denote by *hardness*.

OMEGA: In my opinion, this is the only point -from the ones that Dalet has mentioned before- in which Kant closely follows the spirit of his time. My suggestion is to question this difference that Kant draws between mathematics and philosophy: *we can at best only learn to mathematize*.

BETH: That does not make any sense, Omega!

OMEGA: Why not, Beth? I am not claiming that the essential features that allow Kant to distinguish the philosophical from the mathematical are idle. It seems correct to say that mathematicians *schematize* (in the sense of the Doctrine), that they reason *a priori* and *in concreto*. From here, it is reasonable to think that the ethereal conception of a general system of mathematics can only be a *regulative* ideal, as we have already seen for the case of philosophy above.

TAV: And, even more, you are admitting that mathematical cognition may be subjectively historical!

SHIN: I see... now you are once again arguing for a *philosophization* of mathematics, don't

[196] *ibid.*
[197] A837/B865.

you, Omega?

OMEGA: How is that?

SHIN: Well, all what Kant says is that, *because* there is an essential difference in the source of philosophical and mathematical cognition, one should expect both activities to develop differently. What you are arguing is that mathematics and philosophy, while having essentially different features, share some similarities in their fulfilment. In other words, you are imposing certain properties -which one may reasonably call philosophical- to mathematical cognition. Hence, you are still fully embracing some exotic form of anti-naturalism.

OMEGA: Let me answer your accusations by quoting Kant. Your naturalism arises from a misconception of the diverse relations that may occur between different *cultural activities*: you keep saying that mathematics should not be subordinated to philosophy, that philosophers do not have any right at all for deciding relevant and deep questions of mathematics, etc. In other words, you are giving primacy to the relation of *subordination*.

SHIN: That's right! Is there anything fishy here?

OMEGA: Let me quote two different paragraphs: 'a science must be distinguished from others with certainty and in accordance with principles'[198], 'the mere degree of subordination (the particular under the universal) cannot determine any boundaries for a science, but rather, in our case, only the complete heterogeneity and difference of origin can'[199]. And this heterogeneity between philosophy and mathematics is what I wish to concede in the first place.

SHIN: But then you agree with my distinction between the methodological and the metaphysical; you agree that there is some kind of *a priori* distinction that one should make.

OMEGA: Yes, the point that Kant makes here is that 'what obscured the fundamental idea of metaphysics from yet another side was that, as *a priori cognition*, it shows a certain homogeneity with mathematics, to which, as far as *a priori* origin is concerned, it is no doubt related; but the comparison between the kind of cognition from concepts in the former with the manner of judging *a priori* through the mere construction of concepts in the latter requires a difference between philosophical and mathematical cognition - thus a decided heterogeneity is revealed, which was always felt as it were, but was never able to be brought to distinct criteria'[200]. This homogeneity is what I claim that we should keep examining and analyzing; the distinction that Kant makes regarding the essential difference between mathematics and philosophy should only be considered -at best- as a provisional criterion, not as evidence of absolute victory in our continuous activity.

[198] A834/B862.
[199] A844/B872.
[200] *ibid.*

Coda. Perhaps by now the reader is already tired of terms like *revolution*, *anarchism*, *manifesto* or *conservatism*, in the sense that they have received during the dialogue. But the most interesting catchphrase used by our mathematicians is, surely, *such and such statement is a dogma*. Labeling some proposition as a dogma is equivalent, in the context of rational debate, as showing that one's position is ultimately untenable and therefore susceptible of further improvement. We have seen how Aleph, Tav, Dalet and Omega have all tried to show the dogmatic character presented by some of their opponents beliefs. We have also seen how they have attacked *faith* as an enigmatic black box that allows the others to derive some unacceptable conclusions. From this perspective, one could say that the dialogue has a subtle theological component (even more intense than the political one). But let us be more precise.

Philosophers have taken some discourse of mathematicians at face value. As happened with Aleph's platonism, when they deny the existence of things (ontology) they turn to the existence of faculties (epistemology). This hard fact then constitutes the axis of a belief, and a belief without rational justification stands in need of some degree of faith. One could argue for such set of beliefs (positions) with the aid of pragmatical or historical reasons but, as we have seen, the revisionism to which these are subject is indefatigable. Some positions have been historically enabled by their precedents; when questioned, the faith of their followers is translated into an imposition disguised as a rationally inevitable conclusion.

Omega is heavily inspired by two Benjaminian theses against the faith in the progress: *an institution is corrupted when it loses conscience of the latent violence within it* ('make decisions explicit'), *against a solution of compromise one should always say that 'it would have been better otherwise'* ('why not otherwise?')[201]. These claims try to provide a desmythologizing picture of mathematical activity. There is a point at which the philosopher (or the mathematician that philosophizes) is faced with her own background, as the heart of all her decision-making. It is here where faith arises naturally. But even this notion of faith can be understood in several ways.

The constant appearance of the word *activity* has deep links with this religious flavour of our debate. Against the monolithic and mythological existence of a realm to which we shall (somehow) access, one may say that we defend the *liturgical* character of activities: it is not that abstract objects or ideal faculties exist or not; the relevant factor here is rather the *context* in which we develop our activities. One may say that it is praying *itself* what represents the act of faith, independently from the God that is to hear us[202]. Here, as

[201] See [37].

[202] Does the link of accessibility between a temporal, spatial being and an ethereal one take place in the world? It could be said, additionally, that we should not be so *proud*, that if there was a sublime being then there would be no reason for it to interact with earthy beings like us. We act, on the other hand, *as if* this was the case and *as if* the liturgical objects were themselves this being.

Omega would like to claim, the point is that we *show* something by how we have acted, by the act itself (compare this with, say, the schematic representation of Husserl's acts), while the philosopher tends to *say* something. The same happens with the common discourse on mathematical objects and whatnot. We see, then, that the questions that Omega initially condemned in the field of rational debate have a precise formulation within the sea of faith and that, here, they are not to be eluded.

Now, the title for the dialogue looks even more appropriate, for it can be seen as representing a passionate debate surrounding dogmas, messianic revelations and mathematical miracles rather than revolutions, oppositions and manifestos.

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