Why the wavefunction already is an object on space

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Abstract

Since the discovery of quantum mechanics, the fact that the wavefunction is defined on the $\mathbf{3n}$-dimensional configuration space rather than on the 3-dimensional space seemed uncanny to many, including Schrödinger, Lorentz, and Einstein. Even today, this continues to be seen as an important problem in the foundations of quantum mechanics.

In this article it will be shown that the wavefunction already is a genuine object on space. While this may seem surprising, the wavefunction has no qualitatively new features that were not previously encountered in the objects known from Euclidean geometry and classical physics. This will be shown to be true also in Felix Klein's *Erlangen Program*. This fits naturally in the classification of quantum particles by the representations of the spacetime isometries realized by Wigner and Bargmann, adding another layer of confirmation. Once we realize that all quantum experiments take place in space, they can be interpreted naturally and consistently with the wavefunction being an object on space.

1 Introduction

In *nonrelativistic quantum mechanics* (NRQM), the wavefunction of $\mathbf{n}$ particles is defined on the $\mathbf{3n}$-dimensional configuration space $\mathbb{R}^{\mathbf{3n}}$,

$$\begin{aligned} &\psi : \mathbb{R}^{\mathbf{3n}} := \underbrace{\mathbb{R}^3 \times \ldots \times \mathbb{R}^n}_{\mathbf{n}\text{ times}} \to \mathbb{C} \\ &\psi(\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}) \in \mathbb{C}. \end{aligned} \tag{1}$$

While $\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n} \in \mathbb{R}^3$ are points in space, at least in this representation, the wavefunction cannot be understood as defined at each point of space, but as a complex function defined on the configuration space $\mathbb{R}^{\mathbf{3n}}$.

Since the dawn of quantum mechanics, this was seen as problematic by many, including Schrödinger (Bacciagaluppi and Valentini, 2009), Lorentz (Przibram, K. (ed) and Klein, M.J. (trans), 1967), p. 44, Einstein (Howard, 1990; Fine and Brown, 1988), Heisenberg, Bohm (Bohm, 2004) *etc.* For example, in a letter to Schrödinger, Lorentz wrote how satisfied he is with Schrödinger's *wave mechanics* over Heisenberg's *matrix mechanics*, but complained about the apparent impossibility of interpreting the wavefunction as a physical wave (Przibram, K. (ed) and Klein, M.J. (trans), 1967), p. 43–44. Schrödinger himself was deeply dissatisfied, considering the representation of the wavefunction on the configuration space "only as a mathematical tool" (Bacciagaluppi and Valentini, 2009), p. 477.

Many modern physicists do not seem to worry about this, because the theory can be used successfully to explain the results of experiments, and maybe this gives us the feeling that there is no problem and no need for deeper understanding. Others are using this problem to make the case against interpreting the wavefunction as providing the complete ontology, in particular against Everett's interpretation of QM, while arguing that the point-particles in the pilot-wave theory or the Ghirardi-Rimini-Weber (GRW) interpretation of QM with the *flash ontology* achieve the desired space ontology (Maudlin, 2010, 2019; Norsen, 2017).

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Arguments that we should simply embrace the fact that the wavefunction is defined on the configuration space were brought in (Albert, 1996; Loewer, 1996; Albert and Loewer, 1996; Lewis, 2004; Ney, 2012, 2013; North, 2013; Albert, 2019), in particular in the context of Everett's interpretation (Barrett, 1999; Vaidman, 2018; Wallace, 2002, 2003; Brown and Wallace, 2005; Barrett, 2017). Objections were raised in (Monton, 2002, 2006; Maudlin, 2007; Allori et al., 2008; Maudlin, 2010, 2013; Monton, 2013; Chen, 2017; Emery, 2017; Maudlin, 2019). More discussions of the problem can be found in (Ney and Albert, 2013).

A simple way to understand the wavefunction as an object in space is to interpret $\psi(\mathbf{x_1}, \ldots, \mathbf{x_n})$ as a *multifield*, which is similar to a field, but it depends on multiple positions in space (Forrest, 1988; Belot, 2012).

In this article I will show that the wavefunction *already is* such an object, in precisely the same way as objects in Euclidean geometry or in classical physics.

The article is organized as follows. Section §2.1 makes an inventory of the types of objects on space in geometry and in classical physics. Section §2.2 contains a proof that the wavefunction is such an object, and it does not have qualitatively new properties compared to other objects on space. In Section §3.1 it is shown how this is true also in the precise sense of Klein's *Erlangen Program*. Section §3.2 discusses this result in relation to the classification of quantum particles by the representations of the spacetime isometries realized by Wigner and Bargmann. Section §4 explains that the already existing empirical data is consistent with these conclusions and they are naturally interpreted as about objects on space. The final Section §5 discusses these results in the context raised by critics of the idea that the wavefunction is sufficient in quantum mechanics.

2 Euclidean geometry explanation

This section contains an intuitive explanation of the fact that the wavefunction already is an object on space, based on Euclidean geometry. Symmetry groups, discussed in Sec. §3, will deepen this insight.

2.1 Types of classical objects on space

Based on Euclidean geometry and classical physics, let us see what kinds of objects count as objects on space.

Type 1 (Subsets of space)**.** The most commonly used objects in space are subsets of space.

Example 1 (of Type 1)*.* Lines, segments, and circles, are subsets of the Euclidean plane $\mathbb{R}^2$. Technically, points are elements of the plane, not subsets. But since statements of the form "let the point A be the intersection of the lines b and c", meaning in fact that $\{A\} = a \cap b$, are usual, we will include both these usages in this same type.

In classical mechanics, particles are represented as points, and solid objects as subsets of space. □

Type 2 (Composite objects)**.** A geometric object may be *composite*, *i.e.* it may consist of a collection of subsets of the geometric space, without being identified with the union of those subsets.

Example 2 (of Type 2)*.* Consider a triangle ABC in the Euclidean plane. Let $M \in [BC]$ be the middle of the segment $[BC]$, and let $N \in [BC]$ so that the half-line $[AN$ is the bisector of the angle $\angle CAB$. Then the half-lines $[AM$ and $[AN$ are distinct, even though in the case when $|AB| = |AC|$ they coincide as subsets of the plane. Therefore, they should be considered as distinct objects composing the figure, and considering instead their union would miss this fact.

Even though the edges $[AB]$ and $[AC]$ of the triangle ABC are subsets of the plane that share the point A, they are distinct objects. If the triangle is degenerate so that $B \in [AC]$, $[AB] \subset [AC]$, and it is even possible that $B = C$, in which case $[AB] = [AC]$ as sets, but theorems that apply to general triangles remain valid only if we consider the edges $[AB]$ and $[AC]$ as distinct objects.

Such extreme and degenerate situations occur for instance in problems of geometric locus, where it is essential to consider that distinct points remain distinct even when they happen to coincide. For example, given a segment $[BC]$ in plane, the geometric locus of the points A for which the angle

$\angle CAB$ is right is the circle having $[BC]$ as diameter. But if we exclude the cases when $A = B$ and $A = C$, the locus is no longer a full circle.

Classical mechanisms can be composed of rigid bodies connected by gears, belts, cams, linkages *etc.* From a geometric point of view they are Type 2 objects too. Classical waves and fields can occupy common regions in space and interfere, yet they remain distinct and continue to propagate unaffected. In particular, gravity is not affected by the presence of electromagnetic waves. □

Composite geometric objects from Example 2 are very common in geometry and classical physics. Identifying composite objects with the union of the subsets would collapse Type 2 into Type 1. This would reduce the generality of the theorems in geometry and classical physics, so they would be broken into many very particular theorems with reduced range of applicability, undermining universality. Often geometric reasoning starts with extreme or degenerate cases, the general solution resulting from continuity. But Type 2 objects are not merely useful, they capture general geometric facts.

Type 3 (Objects with additional properties)**.** If we associate to a geometric object an element s from a set S, the result is still a geometric object. This also applies to the components of an object of Type 2.

Example 3 (of Type 3). Two lines are congruent, but they are distinct objects, so *labeling* them helps us track their identity. But it is not merely a matter of utility, labeling them reflects their distinct identities and meanings, like the half-line and the bisector from Example 2.

In classical mechanics, an object with a mass and an electric charge is still an object on space. A composite object can have a total mass and a total charge, but at the same time its constituent parts have their own masses and charges. Therefore, labeling geometric objects is not just a convention, it reflects real properties. In physics it is arguable whether these properties are geometric in nature, but they are nevertheless "on space". □

Remark 1. One may argue that, except for tracking the identity of geometric objects, additional properties are not genuine properties of Euclidean geometry, because they are neither Type 1 nor Type 2, which is true. Type 3 includes additional elements that cannot be reduced to points in space. But this was never a reason to consider that properties like mass and electric charge in classical physics are not on space.

Consider the charge density. It depends on position, so it is a scalar field $\rho : \mathbb{R}^3 \to \mathbb{R}$. In general, classical physics contains among its objects various scalar, vector, and tensor fields. This justifies the geometric interpretation of these fields as *sections* of various *fiber bundles* over the space $\mathbb{R}^3$. For example, scalar fields are sections of the trivial bundle $\mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$, vector fields are sections of $\mathbb{R}^3 \times \mathbb{R}^d \mapsto \mathbb{R}^3$ *etc.* Here, "$\mapsto$" represents the projection on the first component of the Cartesian product, and "sections" are continuous subsets of $\mathbb{R}^3 \times \mathbb{R}$ or $\mathbb{R}^3 \times \mathbb{R}^d$ whose points are put into a one-to-one correspondence with the points of the space $\mathbb{R}^3$ by the projection. Hence, the notion of section captures the idea of scalar or vector field in a geometric way. Complex fields are also possible in classical physics, and they can be trivially interpreted as real fields, since any d-dimensional complex vector space $\mathbb{C}^d$ has an underlying real $2d$-dimensional vector space $\mathbb{R}^{2d}$. Moreover, it turns out that some of these fields are more appropriately interpreted as special geometric objects defined on these bundles. Notably, the electromagnetic potential is understood as the components of a *connection* of the bundle $\mathbb{R}^{3+1} \times \mathrm{U}(1) \mapsto \mathbb{R}^{3+1}$ where $\mathbb{R}^{3+1}$ is the Minkowski spacetime, while the electromagnetic field is its *curvature*. This is the geometric formulation of *gauge theory*, and it generalizes to non-Abelian groups corresponding to the electroweak and strong interactions, resulting in the (classical) *Yang-Mills theory* (Yang and Mills, 1954).

We see that the additional properties appearing in classical physics, while not being geometric objects inherent of the 3-dimensional space, can be interpreted as objects of Types 1 and 2 of extended spaces – the fiber bundles. While fiber bundles extend the geometry of space or spacetime, the projection "$\mapsto$" makes their sections, connections, and curvatures genuine geometric objects *on* the base manifold, that is on the Euclidean space $\mathbb{R}^3$, on the Minkowski spacetime $\mathbb{R}^{3+1}$, or on the four-dimensional curved spacetime of general relativity,. □

The discussion from Remark 1 justifies the following geometric version (and generalization) of Type 3 objects:

Type 3′ (Geometrization of additional properties)**.** Geometric objects of Types 1 and 2 of a fiber bundle over a manifold M (which can be the space or spacetime) are objects on the base manifold M.

Example 4 (of Type 3′)*.* All examples from Remark 1. Note that objects of Type 3′ don't have to be sections of the bundle. For example, for a charged classical point-particle, the charge is localized at a point, so if we interpret it as a section of the bundle $\mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$, it is a discontinuous section. In general, geometric objects of Types 1 and 2 of a fiber bundle over space are not sections, although can be interpreted as discontinuous sections or collections of such sections.

Remark 2*.* Any Type 1 object can be understood as a Type 2 object, where the collection of Type 1 objects has only one element. Any Type 2 object can be understood as a Type 3 object, where the additional property is the same for all elements of the collection defining the Type 2 object. Any Type 2 object can also be understood as a Type 3′ object, where the fiber bundle is $\mathbb{R}^3 \times L \mapsto \mathbb{R}^3$, where L has one element per set of points composing the Type 2 object. □

Throughout this article, I refer to Type 1 and 2 objects as objects *in space*, and to Type 3 and 3′ objects as objects *on space*, to distinguish that the former consist of subsets of space, and the latter consist of Type 1 and 2 objects with additional nongeometric properties, or as geometric objects in a fiber bundle over space.

2.2 Wavefunction on space

We are ready to prove that the wavefunction already is an object on space.

Theorem 1. *The wavefunction in NRQM is an object on space of the classical Types described in Sec. §2.1.*

Proof. Any configuration of $\mathbf{n}$ points in space $(\mathbf{x}_1, \ldots, \mathbf{x_n})$ is an object of Type 1. The collection of all such $\mathbf{n}$-tuples is an object of Type 2. But to specify the wavefunction ψ from eq. (1), to each of these $\mathbf{n}$-tuples has to be associated a complex number $\psi(\mathbf{x}_1, \ldots, \mathbf{x_n}) \in \mathbb{C}$. Therefore, the wavefunction ψ is an object of Type 3. More precisely, the wavefunction is an object of Type 2 in the fiber bundle $\mathbb{R}^3 \times \mathbb{C} \mapsto \mathbb{R}^3$. To see this, we can interpret geometrically the pair $((\mathbf{x}_1, \ldots, \mathbf{x_n}), c)$, where $c = \psi(\mathbf{x}_1, \ldots, \mathbf{x_n}) \in \mathbb{C}$, as a configuration of $\mathbf{n}$ points in $\mathbb{R}^3 \times \mathbb{C}$, $((\mathbf{x}_1, c), \ldots, (\mathbf{x_n}, c)) \subset \mathbb{R}^3 \times \mathbb{C}$. This is a Type 1 object in $\mathbb{R}^3 \times \mathbb{C}$. The collection of such objects corresponding to all possible choices of $\mathbf{n}$ points in $\mathbb{R}^3$ is therefore an object of Type 2 in $\mathbb{R}^3 \times \mathbb{C}$. This makes it an object of Type 3′ on the 3-dimensional space $\mathbb{R}^3$.

Another way to see this is the following. We can interpret a pair $((\mathbf{x}_1, \ldots, \mathbf{x_n}), c)$ as a function $f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}} : \mathbb{R}^3 \to \{0, 1\} \times \mathbb{C}$,

$$f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}}(\mathbf{x}) := \left(\chi_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}}, c\right), \tag{2}$$

where χ_M is the characteristic function of the set M, that is, $\chi(\mathbf{x}) = 1$ if $\mathbf{x} \in M$ and $\chi(\mathbf{x}) = 0$ otherwise. So $f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}}$ is a section of a bundle $\mathbb{R}^3 \times (\{0,1\} \times \mathbb{C}) \mapsto \mathbb{R}^3$. The collection of such sections, corresponding to all possible choices of points in $\mathbb{R}^3$ is therefore a section $\mathcal{F}$ in a bundle with a very large fiber over $\mathbb{R}^3$, obtained as the bundle direct sum $\bigoplus_{\mathbf{n}\in\mathbb{N}} \bigoplus_{(\mathbf{x}_1,\ldots,\mathbf{x_n})\in(\mathbb{R}^3)^{\mathbf{n}}} \mathbb{R}^3 \times (\{0,1\} \times \mathbb{C}) \mapsto \mathbb{R}^3$, namely $\mathcal{F}$ has as components all functions $f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}}$, for all $(\mathbf{x}_1, \ldots, \mathbf{x_n}) \in \left(\mathbb{R}^3\right)^k$, for all $\mathbf{n} \in \mathbb{N}$. So the wavefunction can be interpreted as well as a section with infinitely many components over the 3-dimensional space $\mathbb{R}^3$. □

Remark 3*.* Note that the restriction of the function $f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}(\mathbf{x})}$ from equation (2) to a subset of points $\{\mathbf{x}'_1, \ldots, \mathbf{x}'_k\} \subsetneq \{\mathbf{x}_1, \ldots, \mathbf{x_n}\}$ is not the function $f_{\{\mathbf{x}'_1,\ldots,\mathbf{x}'_k\}}$ corresponding to the subset $\{\mathbf{x}'_1, \ldots, \mathbf{x}'_k\}$. These functions are rigid objects, just like rigid bodies in classical physics, or like Type 2 objects in Euclidean geometry. One may object that the component c of $f_{\{\mathbf{x}_1,\ldots,\mathbf{x_n}\}}$, which is $\psi(\mathbf{x}_1, \ldots, \mathbf{x_n})$, is not localized anywhere in space, since it depends on the configuration of points $\{\mathbf{x}_1, \ldots, \mathbf{x_n}\}$. But consider for example a classical plane wave: can we say that its wavelength is localized in some point in space? Its wavelength depends on the values of the plane wave at all points in space, or at least an extended region of space, and yet, nobody would doubt the fact that the plane wave is a classical object.

However, it is possible to have a representation of the wavefunction on space as a section in a very large fiber bundle, so that the section restricts in this way, to obtain local separability. This can be achieved by using a very large local gauge group on that very large bundle, as shown in (Stoica, 2019). ☐

Remark 4. Theorem 1 shows that the relation between the wavefunction and the 3-dimensional space is not qualitatively different compared to the relation between objects in Euclidean geometry or classical physics and space. Quantitatively speaking, the wavefunction is in some sense an "extreme" case of Type 3 because, as a collection of configurations of points in space, it includes all possible tuples of points. ☐

Remark 5. Theorem 1 can be interpreted as providing additional grounding on space for the multifield interpretation of the wavefunction (Forrest, 1988; Belot, 2012). However, there is no need to adopt multifields, once we realize that the wavefunction-as-it-is already is an object on space. ☐

2.3 Quantum fields on space

What about quantum fields? After all, the correct quantum theory is not NRQM, but quantum field theory. A simple way to show that quantum fields can as well be understood as objects on space follows from Theorem 1, as we will see now. But it is also possible to show it directly for fields in the *wavefunctional representation*, as we will see in Theorem 2.

Corollary 1. *The result of Theorem 1 extends to quantum field theory as well.*

Proof. We will use the Fock representation of quantum states. Quantum fields are usually represented as operator-valued distributions, expressed in terms of creation and annihilation operators $\hat{a}_D^\dagger$ and $\hat{a}_D$. These in turn are subject to commutation (in the case of bosons) or anticommutation relations (in the case of fermions). The subscript index D represents the particle type, and it may include spin degrees of freedom and internal degrees of freedom (like color and hypercharge). Let $\mathcal{D}$ be the set of all types of particles. We do not need to detail all possible types of Standard Model particles, as they all can be expressed by creation and annihilation operators for various types of particles from $\mathcal{D}$.

By repeatedly applying various operators $\hat{a}_D^\dagger$ and $\hat{a}_D$ and linear combinations to the vacuum state $|0\rangle$, any Fock state can be constructed. Since we are interested in the position basis, we apply creation operators of the position eigenstates,

$$|\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}\rangle_D := \hat{a}_{D_1}^\dagger(\mathbf{x}_1) \ldots \hat{a}_{D_\mathbf{n}}^\dagger(\mathbf{x}_\mathbf{n})|0\rangle, \tag{3}$$

where $D = (D_1, \ldots, D_\mathbf{n})$, obtaining the Fock representation in the position basis. Then, any quantum state $|\psi\rangle$ can be expressed as a superposition of $\mathbf{n}$-particle wavefunctions for all possible positive integer values of $\mathbf{n}$,

$$|\psi\rangle = \sum_{\mathbf{n}\in\mathbb{N}} \sum_{D\in\mathcal{D}^\mathbf{n}} \int_{(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in\mathbb{R}^{3\mathbf{n}}} \psi(\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}, D)|\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}\rangle_D \, \mathrm{d}\,\mathbf{x}_1 \ldots \mathrm{d}\,\mathbf{x}_\mathbf{n}. \tag{4}$$

Hence, quantum states from quantum field theory are just superpositions of $\mathbf{n}$-particle states, already known from Theorem 1 to be objects on space. In the general case from eq. (4), they are collections of configurations of $\mathbf{n}$ points in the fiber bundle $\mathbb{R}^3 \times \mathbb{C} \times \mathcal{D}$. So the infinite collection of wavefunctions from eq. (4) form an object of Type 2 in $\mathbb{R}^3 \times \mathbb{C} \times \mathcal{D}$, and therefore an object of Type $3'$ on the 3-dimensional space $\mathbb{R}^3$. ☐

In the standard quantization scheme used in NRQM, one starts from classical point-particle configurations, and obtains wavefunctions defined on that configuration space. The basis of the resulting Hilbert space corresponds to points in this configuration space, so they can be labeled by the positions, $|\mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}\rangle$. To obtain field quantization, the resulting wavefunctions are then again quantized (the *second quantization*).

But we can apply the same quantization idea directly to classical fields (Hatfield, 2018), leading to another way to understand quantum fields as objects on space, at least for bosonic fields.

Theorem 2. *Quantum field configurations in the wavefunctional representation are, in the case of bosonic fields, objects on space.*

Proof. A classical field φ configuration is defined on the three-dimensional space, and valued either in $\mathbb{C}$, or in some vector space $\mathbb{V}$, whose dimension depends on the spin and internal properties of the field. The values of the field at different points are required to satisfy canonical commutation or anticommutation relations, according to the spin, but this does not impact this proof. Let $\mathcal{C}(\mathbb{R}^3, \mathbb{V})$ be the classical field configuration space of classical bosonic fields defined on space and valued in $\mathbb{V}$. In the wavefunctional quantization (Hatfield, 2018), the Schrödinger wavefunctional, or its relativistic versions, is defined on $\mathcal{C}(\mathbb{R}^3, \mathbb{V})$. The Schrödinger wavefunctional $\Psi : \mathcal{C}(\mathbb{R}^3, \mathbb{V}) \to \mathbb{C}$ depends on the classical field configuration φ, and associates a complex number $\Psi(\varphi)$ to each φ. The basis of the Hilbert space of wavefunctionals is in one-to-one correspondence with the classical field configurations, so we will label them by $|\varphi\rangle$. The state vector has the form

$$|\Psi\rangle = \int_{\mathcal{C}(\mathbb{R}^3,\mathbb{V})} \Psi(\varphi)|\varphi\rangle \mathcal{D}\varphi, \tag{5}$$

where $\mathcal{D}\varphi$ is the measure on $\mathcal{C}(\mathbb{R}^3, \mathbb{V})$.

The classical field φ is a section in a bundle $\mathbb{R}^3 \times \mathbb{V} \mapsto \mathbb{R}^3$. The wavefunctional (5) associates to each section of the bundle a complex number. The sections of $\mathbb{R}^3 \times \mathbb{V} \mapsto \mathbb{R}^3$ are Type 1 objects in the bundle's total space $\mathbb{R}^3 \times \mathbb{V}$. A section with a label from $\mathbb{C}$ is a Type 1 object in $\mathbb{R}^3 \times \mathbb{V} \times \mathbb{C}$, which is the total space of a bundle $\mathbb{R}^3 \times \mathbb{V} \times \mathbb{C} \mapsto \mathbb{R}^3$. The wavefunctional (5) is thus an object of Type 2 in $\mathbb{R}^3 \times \mathbb{V} \times \mathbb{C} \mapsto \mathbb{R}^3$, and, by Remark 1, an object of Type 3′ on the 3-dimensional space $\mathbb{R}^3$. □

For fermionic fields there are important differences, that prevent a straightforward interpretation like that from Theorem 2 for the case of the bosons (Jackiw, 1988). But the representation from (Stoica, 2019), while not in terms of wavefunctionals, works for both bosons and fermions.

3 Symmetry group explanation

In the previous section, we have seen that the wavefunction does not have qualitatively new features that were not already encountered in Euclidean geometry or in classical physics. The difference is at best quantitative, by including all configurations of points (see Remark 4).

By appealing to symmetry groups, this section provides a deeper understanding of what was presented in Section §2. The geometric nature of the wavefunction is better understood in terms of spacetime symmetries, and, as shown by Wigner and Bargmann, this approach leads straightforwardly to the classification of particles based on spin and mass. Additional properties of particles, like electric charge and color, come from the local gauge symmetries, which also account for interactions. I will explain how Wigner's approach is a natural application of Klein's *Erlangen program* for geometry. As such, spacetime and gauge symmetries are not simply properties of quantum theory, but they determine its very structure, including the properties of the wavefunction.

3.1 In the light of Klein Geometry

In his 1872 *Erlangen Program* paper, Klein explained what various *homogeneous geometries* have in common (Klein, 1893). Homogeneous geometries include Euclidean, non-Euclidean, affine, projective *etc*, but not Riemannian geometry, which in general is not homogeneous (and for which we can use a generalization of Klein's idea named *Cartan geometry* (Sharpe, 2000)). Klein's major insight is that at the core of each of the homogeneous geometries is a group G (in general a *Lie group*) acting transitively, and in general effectively, on a space S. An action of G on S able to transform any point of space into any other point is called *transitive*. This makes S into a *homogeneous space* for the group G. An action is called *effective* if the only identity transformation of S by elements of the group G is due to the identity of G. The group G is a *transformation group* for the space S. Its action on the space S is a *representation* of G on S.

Geometric objects consist of subsets of the space S, so in general they are objects of Type 1 or 2. In Klein's view, geometry studies the *invariant* properties of geometric objects, *i.e.* those properties that remain unchanged under the action of the group.

Two objects related by a symmetry transformation are *congruent*, or in general *isomorphic*. This generalizes the notions of *congruence* and *isometry*, which in Euclidean geometry are established by translations and rotation, to other geometries, including those that do not have notions like distance or angle, *i.e.* without a *scalar product*, or which have different kinds of scalar products. For example, affine geometry does not have a scalar product. Minkowski geometry has a scalar product, but this has a different signature from that in Euclidean geometry.

In Euclidean geometry for example, a triangle can be transformed into another one by translations, rotations, and reflections. Such a transformation exists only if the two triangles are congruent, so this time the group action is not transitive, but can be decomposed into orbits, and the group action is transitive on each orbit. The set of all geometric objects of a given kind forms itself a space on which the group acts. Each element of the group transforms a point of that space into another one, corresponding to a transformation of an object in that space into another one.

Let us review the mathematical grounds of Klein geometry in a more precise way.

Definition 1 (Orbits and stabilizers)**.** *Let (G, S) be a left action of a group G on a set S. The orbit of a point $s \in S$ is the subset* $\operatorname{Orb}(s) \subseteq S$ *of S defined as*

$$\operatorname{Orb}(s) := \{g \cdot s \in S | g \in G\}. \tag{6}$$

The stabilizer subgroup $\operatorname{Stab}(s)$ *(also* isotropy group*) of G with respect to s is defined as*

$$\operatorname{Stab}(s) := \{g \in G | g \cdot s = s\}. \tag{7}$$

Theorem 3 (See *e.g.* (Helgason, 1979))**.** *Let $s \in S$. Then, there is a one-to-one map between the* left cosets $G/\operatorname{Stab}(s)$ *and the elements of the orbit of s, defined by*

$$\begin{aligned} \varphi &: G/\operatorname{Stab}(s) \to \operatorname{Orb}(s) \\ \varphi &(g \operatorname{Stab}(s)) = g \cdot s. \end{aligned} \tag{8}$$

The map φ defines an action isomorphism between the left action of G on its left coset space, and the action of G on S. If G is a Lie group, φ is a diffeomorphism*.*

Proof. See for example Theorem 3.2 (p. 121), Theorem 4.2 (p. 123), and Proposition 4.3 (p. 124) in (Helgason, 1979). □

Definition 2 (Klein geometries)**.** *A Klein geometry (G, H) consists of a* Lie group G *and a* topologically closed *subgroup H of G (Sharpe, 2000). The left* coset space $G/H = \{gH | g \in G\}$ *is identified as the space on which G acts. This results in a principal fiber bundle $G \mapsto G/H$ with typical fiber and structure group H (Cohen, 1998), Cor. 1.4.*

Proposition 1. *Any* homogeneous space *S of a group G can be obtained as the coset space G/H, where the subgroup H of G is the stabilizer of a point $s \in S$.*

Proof. Follows from Theorem 3. □

Example 5. For the Minkowski spacetime, G is the Poincaré group $\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3)$, where $\mathbb{R}^{1,3}$ is in fact the vector space $\mathbb{R}^4$, and H is the Lorentz group $\mathrm{O}(1,3)$, the group of transformations of the vector space $\mathbb{R}^{1,3}$ which preserves the symmetric bilinear form

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

An element of the Poincaré group has the form $(\mathbf{v}, L)$, where $\mathbf{v} \in \mathbb{R}^{1,3}$ corresponds to a translation by a vector $\mathbf{v}$, and $L \in \mathrm{O}(1,3)$. The Poincaré group admits the matrix representation $\lambda : \mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3) \to \mathcal{M}(5,5)$,

$$\lambda(\mathbf{v}, L) = \begin{pmatrix} 1 & 0 \\ \mathbf{v} & L \end{pmatrix}. \tag{10}$$

The element corresponding to the translation with a vector $\mathbf{v}$ has the form $\lambda(\mathbf{v}, I_4) = \begin{pmatrix} 1 & 0 \\ \mathbf{v} & I_4 \end{pmatrix}$. The element corresponding to the Lorentz transformation L has the form $\lambda(\mathbf{0}, L) = \begin{pmatrix} 1 & 0 \\ \mathbf{0} & L \end{pmatrix}$.

Let us denote the subgroup of translations and the subgroup of Lorentz transformations by $\lambda(\mathbb{R}^{1,3}, I_4) := \begin{pmatrix} 1 & 0 \\ \mathbb{R}^{1,3} & I_4 \end{pmatrix}$ and $\lambda(\mathbf{0}, \mathrm{O}(1,3)) := \begin{pmatrix} 1 & 0 \\ \mathbf{0} & \mathrm{O}(1,3) \end{pmatrix}$. Then, the left cosets from G/H in Definition 2 have the form $\lambda(\mathbf{v}, L)\,(\mathbf{0}, \mathrm{O}(1,3))$, hence they have the matrix form

$$\begin{pmatrix} 1 & 0 \\ \mathbf{v} & L \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbf{0} & \mathrm{O}(1,3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{v} & \mathrm{O}(1,3) \end{pmatrix}. \tag{11}$$

Since in the RHS of eq. (11) the only free parameter is $\mathbf{v}$, the cosets $\begin{pmatrix} 1 & 0 \\ \mathbf{v} & \mathrm{O}(1,3) \end{pmatrix}$ form a four-dimensional space identifiable with the Minkowski spacetime $\mathbb{R}^{1,3}$.

Similar interpretations as left coset spaces can be given to the Euclidean geometry on $\mathbb{R}^n$, and to the hyperbolic, elliptic, projective, affine, and conformal geometries. □

Let us analyze Type 1–3 objects from §2.1 from the point of view of Klein's approach to geometry.

Remark 6. According to Klein, the transformation group also acts on manifolds whose points are configurations of points or geometric objects from the original space (Klein, 1893):

> we may use instead of the point any configuration contained in the manifoldness, – a group of points, a curve or surface, *etc.* As there is nothing at all determined at the outset about the number of arbitrary parameters upon which these configurations shall depend, the number of dimensions of our line, plane, space, *etc.*, may be anything we like, according to our choice of the element.

These geometric objects are Type 1 objects. For example, the set $\mathcal{T}$ of all triangles congruent to a general fixed triangle $\triangle ABC$ in plane form a 3-dimensional manifold, even though the plane itself is 2-dimensional. The reason is that we can parametrize these triangles by their center of mass and the rotation angle. Since these triangles are all congruent, the group of isometries of the Euclidean plane acts transitively on $\mathcal{T}$. □

Remark 7. Klein points out that the essential structure defining a geometry is the group structure (emphasis in the original text) (Klein, 1893):

> *But as long as we base our geometrical investigation upon the same group of transformations, the substance of the geometry remains unchanged.* [...] The essential thing is, then, the group of transformations; the number of dimensions to be assigned to a manifoldness appears of secondary importance.

But the action of the Euclidean group on the set $\mathcal{T}$ of triangles in plane is determined by its action on the plane $\mathbb{R}^2$, which provides the *standard* or *defining representation* of the Euclidean group. So $\mathbb{R}^2$ retains its fundamental character, despite the existence of other associated representations of higher dimension. Klein gives as examples more types of manifolds associated to space, in particular consisting of lines, planes, curves or surfaces, following ideas introduced by Plücker and Grassmann. □

Remark 8. Klein also considers composite objects, for example he wrote (emphasis in the original text) (Klein, 1893) *we must evidently choose our space-elements in such a way that their manifoldness either is itself a body or can be decomposed into bodies.*

Therefore, Type 2 objects are geometric objects in space in Klein's geometries too. □

Remark 9. Other types of manifolds on which the principal group acts can be obtained as the product between the original space and another space. This is the case of fiber bundles. But are fiber bundles over space or spacetime present in Klein geometry? By applying Definition 2 to the Poincaré group $\mathbb{R}^{1,3} \rtimes O(1,3)$, we get the fiber bundle $\mathbb{R}^{1,3} \rtimes O(1,3) \mapsto \mathbb{R}^{1,3}$, whose base manifold is the Minkowski spacetime, and whose typical fiber is diffeomorphic with the Lorentz group. Since the Poincaré group acts on this bundle, it also transforms its sections and in fact all of its subsets, so they form representations. In general, since fiber bundles are themselves manifolds on which the isometry group of the base space acts, Type $3'$ objects qualify as objects on space as well, according to Klein's description of "manifoldness". □

Let us summarize these remarks:

Observation 1. *Types 1, 2, and $3'$ are geometric objects in Klein geometry, as described in (Klein, 1893). In particular, by Theorem 1, the wavefunction is an object of Klein geometry.*

Proof. Follows from Remarks 6, 8, 7, and 9. □

3.2 Wigner's theorem

In the following, we will see more precisely how the wavefunction is understood based on Wigner's classification, and how this fits in Klein's Erlangen program.

Physical systems can be subject to symmetry transformations of space or spacetime, of the internal spaces of the gauge groups, and of permutations of identical particles. Since the quantum states of physical systems are represented by vectors in a Hilbert space $\mathcal{H}$, these symmetries induce transformations of the Hilbert space $\mathcal{H}$.

A quantum system is not uniquely represented by a state vector $|\psi\rangle \in \mathcal{H}$, but by a ray in $\mathcal{H}$. Therefore, the actions of the symmetry groups are in fact representations on the projective Hilbert space defined by the rays of $\mathcal{H}$, $P(\mathcal{H}) := (\mathcal{H} \setminus \{0\}) / \sim$, where $|\psi\rangle \sim |\psi'\rangle$ iff there is some $\lambda \in \mathbb{C} \setminus \{0\}$ so that $|\psi\rangle = \lambda|\psi'\rangle$. Let us denote by $|\psi) := |\psi\rangle / \sim \in P(\mathcal{H})$ the ray corresponding to a state vector $|\psi\rangle \in \mathcal{H}$, and by $(\psi| = |\psi)^\dagger \in P(\mathcal{H}^*)$ its dual, which is the ray of $\langle\psi| \in \mathcal{H}^*$.

Projective transformations preserve the *ray scalar product*, defined as

$$(\psi_1|\psi_2) := |\langle\psi_1|\psi_2\rangle| \,. \tag{12}$$

We are interested in the representations of a symmetry group G on the projective Hilbert space $P(\mathcal{H})$. Let T_g be the symmetry transformation of $P(\mathcal{H})$ corresponding to the element $g \in G$. Since the action of each transformation T_g has to preserve the transition probabilities, it follows that for any $|\psi_1), |\psi_2) \in P(\mathcal{H})$,

$$(T_g\psi_1|T_g\psi_2) = (\psi_1|\psi_2) \,, \tag{13}$$

where $(T_g\psi| := (T_g\,|\psi))^\dagger$.

Wigner's theorem shows that, for $\dim \mathcal{H} \geq 2$, any transformation T_g of $P(\mathcal{H})$ satisfying (13) is induced by a transformation $\widehat{T}_g$ of $\mathcal{H}$ which is either unitary or *antiunitary* (Wigner, 1931, 1959). If $\dim \mathcal{H} = 1$ there is a unitary transformation and an antiunitary transformation. A transformation $\widehat{T}$ of a Hilbert space $\mathcal{H}$ is *antiunitary* iff $\langle\widehat{T}\psi_1|\widehat{T}\psi_2\rangle = \langle\psi_1|\psi_2\rangle^*$. An antiunitary transformation T is *additive*, *i.e.* $\widehat{T}(|\psi_1\rangle + |\psi_2\rangle) = \widehat{T}|\psi_1\rangle + \widehat{T}|\psi_2\rangle$, and *antilinear*, *i.e.* $\widehat{T}(\lambda|\psi_1\rangle) = \lambda^*|\psi_1\rangle$, for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$.

Given the projective transformation T_g, the corresponding unitary or antiunitary transformation $\widehat{T}_g$ is unique up to a phase factor $e^{i\vartheta}$ for each $g \in G$. Since the product of two transformations $\widehat{T}_g$ and $\widehat{T}_{g'}$ satisfies $\widehat{T}_{gg'} = e^{i\vartheta(g,g')}\widehat{T}_g\widehat{T}_{g'}$ for some $\vartheta(g,g') \in \mathbb{R}$, the freedom given by ϑ is useful to ensure that the projective representation corresponds to a representation of G on $\mathcal{H}$. This is not always possible, but Wigner showed that it is possible locally around the identity (Wigner, 1939).

Bargmann showed that representations that induce the projective representations do not exist for the rotation group SO(3), the Lorentz group and the Poincaré group, but that they do exist for the *universal covers* of these groups, and they induce the projective representations of these groups (Bargmann, 1964). The universal cover group of the *proper orthochronous* Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$ – which is the connected component of the Lorentz group containing the identity – is its double cover $\mathrm{Spin}(1,3) \equiv \mathrm{SL}(2\mathbb{C})$ (which is the *spin group*). The full Lorentz group is recovered by combining it with the time reversal operator T and the parity operator P, which generate a group $\{1, P, T, PT\}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (incidentally named the *Klein four-group*). If we denote by $\widetilde{\mathcal{P}}_0 := \mathbb{R}^{1,3} \rtimes \mathrm{Spin}(1,3)$ the universal cover of the connected component of the Poincaré group containing the identity, the universal cover of the full Poincaré group is $\widetilde{\mathcal{P}} := \widetilde{\mathcal{P}}_0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The fact that in this case the universal cover is the double cover explains why the wavefunction of a spin $\frac{1}{2}$ particle changes its sign under a 2π rotation.

Thus, Wigner and Bargmann showed that the projective representations of the Poincaré group on a Hilbert space $\mathcal{H}$ correspond to representations of its universal cover by unitary and antiunitary transformations on $\mathcal{H}$, and that particles, both fermions and bosons, are classified by spin and mass. Moreover, the evolution equation for free quantum states follows automatically from the symmetry, according to the spin and mass. In particular, for massive free spin $\frac{1}{2}$ particles they recovered the Dirac equation.

Other properties of quantum particles, like electric charge, color, and weak isospin, require the existence of fiber bundles, whose fibers are representations of other groups like $\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{SU}(3)$. The idea of symmetry is related in their case to the Maxwell and Yang-Mills equations, and the charges result as invariants of the representations. All these groups and the group $\widetilde{\mathcal{P}}_0$ act on the full Hilbert space, and their actions commute with each other. This means that the Wigner classification and the classification due to gauge theory complement each other in giving a full classifications of the particles of the Standard Model based on representations.

It is not evident that merely making use of symmetry groups and their representations in the way Wigner and Bargmann did actually is Klein geometry. The following Theorem clarifies the relation between the two.

Theorem 4. *The particle wavefunctions in the Wigner classification are geometric objects on spacetime, as prescribed in Klein geometry.*

Proof. We apply Theorem 3 for the Wigner representation of the universal cover of the Poincaré group $\widetilde{\mathcal{P}}$ on the Hilbert space $\mathcal{H}$. This action is not transitive, but from Theorem 3, all of its orbits are classified by the cosets in $\widetilde{\mathcal{P}}$ of the form $\mathrm{Stab}(|\psi\rangle)$, where $|\psi\rangle \in \mathcal{H}$.

The representations of $\widetilde{\mathcal{P}}$ on $\mathcal{H}$ are ultimately "made of" homogeneous spaces, which are identifiable with the orbits of the action of $\widetilde{\mathcal{P}}$, and, by Theorem 3, with cosets of $\widetilde{\mathcal{P}}$. The Wigner representation is decomposable in an invariant way into orbits, which are homogeneous spaces for $\widetilde{\mathcal{P}}$, and therefore are isomorphic to coset spaces of $\widetilde{\mathcal{P}}$. By Remark 8, they are objects of Klein geometry, so the particle wavefunctions in the Wigner classification are geometric objects on spacetime. □

From Theorem 4, the Hilbert space $\mathcal{H}$ is decomposable into infinitely many orbits, all of which have dimension of maximum the dimension of the group $\widetilde{\mathcal{P}}$, which is 10. Since $\mathcal{H}$ is infinite-dimensional, and the coset spaces of $\widetilde{\mathcal{P}}$ are finite-dimensional, it follows that $\mathcal{H}$ is decomposable into infinitely many orbits. Each part of the wavefunction from an orbit is transformed "rigidly" (isometrically) by $\widetilde{\mathcal{P}}$ within its own orbit, just like rigid geometric objects transform in spacetime. At the same time, even if they transform in the same way, the structure of each such part of the wavefunction is more complicated than that of subsets in the Minkowski spacetime. They cannot be interpreted as such subsets, but they can be understood as geometric objects of Type 3′, as in Theorem 1, Corollary 1, and Theorem 2.

Theorem 4 applies even when gauge fields and the corresponding charges are present, because the universal cover $\widetilde{\mathcal{P}}_0$, by commuting with the gauge groups, acts on the Hilbert space in the same way as $\widetilde{\mathcal{P}}$ in Theorem 4. Here we should take $\widetilde{\mathcal{P}}_0$ rather than the full group $\widetilde{\mathcal{P}}$ because the Standard Model is not invariant under the P, T, or C (charge conjugation) transformations individually, but only under the combined CPT transformation.

4 Consistency with experiments

The consistency of the wavefunction being an object on space with experiments was empirically verified countless times, implicitly. The following examples are intended to explain how experiments can be understood as being about wavefunctions as objects on space.

Experiment 1. Consider any macroscopic rigid object, for example a rock. The only transformations it can undergo without breaking it or changing its structure are spacetime isometries, *i.e.* Galilei transformations in the nonrelativistic case, and Poincaré transformations in the relativistic case. The rock being a rigid object, its symmetries are reduced to isometries of spacetime. We perform this experiment implicitly numerous times every day, whenever we manipulate rigid objects.

But the rock is made of atoms, and in each atom, electrons and nucleons are entangled. Even in the Helium atom, in the ground state energy level, the two electrons are entangled. In fact, already in Hydrogen there is entanglement between the electron and the proton, which can be seen in a more refined model than the standard Schrödinger solution in which the proton has infinite inertia and no size (Tommasini et al., 1998). This is even more the case for heavier atoms. Moreover, molecular and crystalline structures involve entanglement between the atoms. This is not macroscopic superposition as in the Schrödinger cat experiment, and it is not likely that it could be harnessed for quantum computing, but it clearly shows that even a rock is a quantum system whose wavefunction requires a high-dimensional configuration space.

However, the possible transformations we can apply to the rock without changing its structure are very limited. While of course we can imagine all sorts of unitary transformations of the Hilbert space of the rock, the ones leaving unchanged its spatial structure are only the phase changes, permutations of identical particles, gauge transformations, and the spacetime isometries. From these, the only ones we can apply to the system without changing its structure, and whose effects are observable, are the spacetime isometries, under which the rock transforms like a classical rigid object, as in Theorem 4.

One may object that with all the entanglement already present in the atoms and molecules of a rock, this is still limited to a very small subset of the Hilbert space necessary to describe it. And indeed, the total Hilbert space contains infinitely many more Schrödinger's-cat-like states that we do not observe in reality. But even quantum superpositions of different "rock states" transform like rigid objects, because spacetime isometries apply simultaneously to each term in the superposition. □

Experiment 2. Consider a quantum computer, or a laboratory in which a quantum experiment is performed. Let us assume it to be mobile, for example that it is placed on a ship moving with constant velocity. Due to the principle of relativity, the experiment is not affected by this motion. In general, it is not affected by an isometry of spacetime, and this holds for both a quantum laboratory and a quantum computer. But apart from transformations representing spacetime isometries, all other unitary or anti-unitary transformations are either not observable, or they affect the structure of the experiment or the computation. Now, it is clearly true that interactions are not captured in the Wigner-Bargmann approach, which is based solely on the Poincaré group. But they are accounted for by gauge theory, and local gauge transformations commute with the spacetime isometries, which is why even when interactions are present the wavefunction is still an object on space. □

Experiment 3. When we perform experiments, we manipulate the measuring apparatus and the system on which we experiment. This manipulation happens in space. The human experimentalists, the measuring devices, as well as the apparatuses utilized to prepare the observed system, are manipulated in space. Even when we perform quantum mechanics experiments, the preparation and the measurement take place in space. The observed system may consist of more subsystems in entanglement, and the experiment may involve spacelike separated measurements, like in the EPR experiment (Einstein et al., 1935; Bohm, 1951). But even in this case, we measure the state of the observed system in space. Experimental protocols include avoiding the contamination of the experiment with other systems, for example when we detect interference or we measure the spin of an atom we have to make sure that the particle or atom is the intended one, and not an intruder. The reason is that if more particles arrive at the same region of space, the detection can be compromised. The detector is unable to distinguish which copy of the three-dimensional space from the configuration space is the one corresponding to the observed particle, even though the wavefunction of all particles is interpreted as propagating on the configuration space. This means that ultimately the detection confirms that the particle occupies the same space as the detector, and not a different copy of space in the configuration space. This is true in general, for all interactions between particles. □

5 Discussion

Some authors claim that the fact that the wavefunction propagates on the configuration space means that quantum systems cannot be on the three-dimensional space, and in particular objects like chairs and tables cannot make sense in a theory in which only the wavefunction exists and is governed by the Schrödinger dynamics (Bell, 2004b; Maudlin, 2010, 2019; Norsen, 2017). This argument is often used to claim that the wavefunction either needs to be supplemented by point-particles, as in the pilot-wave theory (Bohm, 1952), or the positions of the spontaneous localization as in the GRW proposal (Ghirardi et al., 1986) with the flash ontology (GRWf) (Bell, 2004a). And that, by failing to provide such spatial features, wavefunction-only interpretations like Everett's interpretation (Everett, 1957, 1973; Wallace, 2012) and even the GRW interpretation with the mass-density ontology (GRWm) are insufficient.

Let us consider first how the pilot-wave theory is supposed to address the problem of how objects like chairs and tables are localized in space. The claim is that the point-particles do this job naturally, since they are located in space. But it is not sufficient for the point-particles to be located in a configuration of positions in space that look like a chair – the chair-configuration should remain a chair-configuration after a reasonable amount of time has passed, assuming nothing violent happens in the meantime. And since the point-particles are guided by the wavefunction, and since the probability of the point-particle configuration to be located at a certain set of positions is determined by the wavefunction's amplitude at the point in the configuration space corresponding to that set of positions, the wavefunction itself should retain a stable "chair-like" configuration. So, to solve the problem for point-particles, we should solve it first for the wavefunction itself, and the pilot-wave theory turns out to have the same problem attributed by these authors for instance to Everett's interpretation, which has to appeal to decoherence. In fact, Bohm realized this, and anticipated the necessity of decoherence for the pilot-wave theory (Bohm, 1952).

I the case of the GRWf interpretation, decoherence seems to be unnecessary. But the probability that the collapse localizes the wavefunction at a certain point in the configuration space, corresponding closely to a chair-like configuration of points in space, also depends on the wavefunction's amplitude at that point in the configuration space. To ensure that successive spontaneous localizations, and therefore "flashes", happen in such a way that the chair is stable, the wavefunction itself must be "chair-like", and the spontaneous collapse should be sharp, but not too sharp to sabotage this. Banning the wavefunction from the ontology of the GRW interpretation, and retaining only the flash ontology, does not change the need for the wavefunction to be "chair-like".

So if this problem exists for standard quantum mechanics or for Everett's interpretation, it exists as well for the pilot-wave theory and GRW, regardless of the additional elements like point-particles or positions of the flashes.

The pilot-wave theory's point-particles and the GRWf flashes are called by Bell *local beables* (Bell, 2004c). These are elements of the theory that are localized in space. Can standard quantum mechanics have local beables? Bell himself considers that it has – and he gives as example "the settings of switches and knobs and currents needed to prepare the initial unstable nucleus" (Bell, 2004d). But these are not the only local beables. Suppose we measure a complete set of commuting observables of an elementary particle, and find definite values. The wavefunction of the particle is then separable for the rest of the universe, so it can be understood straightforwardly as an object on space. But if we measure an incomplete set of observables, the wavefunction is not completely determined, and the particle can be entangled with other systems. Then it will be described by the reduced density matrix, and it will not be separable. Yet, something will still be localized, namely the observed properties. So they can be understood as the local beables. This also applies to the observation of composite systems, for example atoms, since even though they consist of entangled particles, the observed properties of the atom are localized. And the same happens in Everett's interpretation, where each branch of the wavefunction is exactly like the wavefunction that remains after the projection in standard quantum mechanics. Another kind of local beables is given by the reduced density matrix of the quantum fields at every point of space (Wallace and Timpson, 2010). All these kinds of local beables are just as good as the point-particles and the flashes.

But is it possible that the entire information contained in a generic wavefunction can be contained in a set of local beables? Even if the answer were negative, the same problem exists in all interpretations of quantum mechanics, including the pilot-wave theory and the GRW interpretation, since they all include the wavefunction.

Fortunately for all these interpretations, we have seen in this article that the wavefunction as-it-is is indeed an object on space, in the same sense in which geometric objects are objects on space, both in Euclidean geometry and in Klein geometry, and that this is consistent with the empirical data. Even so, the wavefunction is not explicitly a field on space, but why should we expect this? Nevertheless, if this is important, it was previously shown that it can be faithfully represented as an infinite-dimensional field on space (Stoica, 2019). But even without such a representation, here we made it clear that it already is an object on space, like classical objects or objects in Euclidean geometry or in Klein's Erlangen program are.

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