In Defense of a Constructive Truth Concept

First version

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Abstract

I address some major critical arguments against a constructive truth concept and intuitionist logic. I put the notions of in principle possibilities and valid constructions (mathematical proofs) under scrutiny. I argue that the objections against a constructive account of truth miss target, thus they are not decisive. Eventually, constructivism is at least as cogent and natural a stance as realism.

Keywords— intuitionism, constructivism, truth, possibilism, computability, proof

1 Intuitionism and Truth

While mathematical intuitionism brings along a remarkable truth concept and laudable care for proofs, it is still burdened by its original philosophical groundings after a hundred years gone by. Indeed, the philosophy of the founding father, Jan Brouwer can rightfully be frowned upon in several respects.[1] But once someone is ready to overlook the idealist and subjectivist or even soliptic sentiments (as they are detachable, for sure), in the wake of intuitionism a quite sober constructivist account of truth can be articulated for mathematics—and possibly beyond.[2]

1. On Brouwer's philosophy, see van Stigt 1998, Placek 1999 and Detlefsen 1990.
2. Semantical approaches to intuitionism come in different flavors and do not form any monolitic endeavor. Here I will not deal with formal semantics, where there is surely a lot to tell (see Dalen 1986 and 2001). I will focus only on the philosophical interpretations and the problem of truth.

My general claim is that such a truth concept can be the most modest, natural and reasonable rival to realism.

I cannot delve into a comprehensive analysis of realism here, maybe it is enough to note that endorsing it comes with an immense and unnecessary metaphysical burden. In particular, realism about the objects of mathematics, a view often tagged as Platonism, faces serious critics in many respects.[3] It is widely held that practicing mathematicians are also prone to deny their Platonism "on Sundays", i.e., when they reflect on their activities. However, on weekdays they talk about the objects of their subject, as if they were flesh and blood existents.[4] Though, by all means interwoven, here I am after the semantic side of the story rather than the ontological one. So let us begin with the language we are in, and its (philosophical) interpretations.

One of the most characteristic feature of intuitionistic logic[5] against the classical one is that it abandons the syntactic law of excluded middle:

(LEM) $\vdash A \vee \neg A$.

To explore the possible interpretations of this law, I will consider different forms of the corresponding semantic rule, the Principle of Bivalence. For now, I state it like this:

(PB1) Every statement of mathematics is true or false independently of our knowing which.

A common ground for these wordings will be that while a devotee for classical mathematics (a realist) holds that every intelligible mathematical statement is true or false, the intuitionist says that a statement is true if we have a proof for it. Something that is not proved, constructed or exhibited in a proper way, i.e. , we have no way to know it, cannot be regarded as true. Thus she will reject (PB1), and duly the corresponding rule of syntax, (LEM).

An informal interpretation for the logical constants of intuitionistic logic[6] was given by Kolmogorov (1932) in terms of *solutions* (for mathematical problems). Heyting (1956) features the term *constructions* (of proofs) and below I follow this latter vernacular:

(i) A mathematical statement p always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out;

3. See e.g. Benacerraf 1965, Benacerraf 1973, Field 1989.

4. "Most writers on the subject seem to agree that the typical working mathematician is a Platonist on weekdays and a formalist on Sundays." Davis and Hersh 1981, p.321

5. For different axiomatic formulations, see Heyting 1930, Kleene 1952, Heyting 1956, Dalen 2001.

6. Brouwer himself was not keen on logic, his disciple, Heyting was the first to present a formal system (1930).

(ii) $p \wedge r$ can be asserted if both can be;

(iii) $p \vee r$ can be asserted if both can be or one of them can be;

(iv) $\neg p$ can be asserted if and only if we possess a construction which from the supposition that a construction of p were carried out, leads to absurdity;

(v) $p \rightarrow q$ can be asserted if and only if we possess a construction r, which, amended by any construction proving p, result in a construction of a proof for q;

(vi) $\forall x A(x)$ means that $p(x)$ holds for every x in D (over which p ranges); in other words, we possess a general method of construction which, if any element a of D is chosen, yields $p(a)$;

(vii) $\exists x A(x)$ holds if and only if an element a of Q for which $p(a)$ holds has actually been constructed.

This interpretation is illuminating in that it suggests that truth is not a static property of formulas, sentences or assertions, but a terminus of certain interactive affairs. It may worth highlighting some peculiar points, though. Observe, e.g., that the condition for negation in (iv) is not defined in terms of a contradiction, but rather of absurdity. The reason is that a contradiction is generally conceived as $p \wedge \neg p$, which would make the interpretation circular. Instead, a certain statement which must be absurd in our intended system is dedicated to play the role of falsum, usually this is $0 = 1$. Thus $\neg p$ can be written as $p \rightarrow 0 = 1$.

And this is so because (v) describes more than a mere biconditional: it requires from p, r and q to be relevant of each other. In other words the operation $p \rightarrow q$ is supposed to turn one proof into another. Michael Dummett, however, makes and additional requirement (in all conscience after Kreisel 1962): we have to effectively realize a proof whenever we are presented with one (Dummett 1977, p. 13). This leads us straight to the question: what exactly counts as a proof? But let us leave it here and come back to this issue in section 5.

For now, here is another problem: should we actually have a proof in every case whenever we regard something as true? Consider an extremely large natural number. Should not we say that it is either prime or not without actually devoting the resources to prove? Panu Raatikainen intends to show that in this regard, contrary to the oversimplification characterizing the literature, intuitionsts have never been on common grounds, there are a bunch of different concerning views, and it is more than troublesome to give a cogent account anyway (Raatikainen 2004). Right or not in his conclusion, I take his historical analysis as a fine starting point (and I will also address his critical observations in the upcoming sections). To begin with, he makes an important distinction by dividing the intuitionist views on truth into two classes: *actualist* and *possibilist* accounts. By this, we can say true is:

(a) what is proved or constructed;

(b) what is *in principle* can be proven or constructed.

Though the account of the logical constants as given above clearly suggests an actualist reading, Raatikainen observes that it is not only hard to find a consequent representative of either kind of approach among the noted intuitionists, but we are also in trouble with reconstructing their respective views. Brouwer, to begin with, seems to hold that there is no non-experienced truth, and he expressed this thought at several places. For instance:

> [T]ruth is only in reality i.e. in the present and past experience of the consciousness. Amongst these are things, qualities of things, emotions, rules[...] and deeds [...]. But expected experiences and experience attributed to others are true only as anticipations and hypotheses; in their content there is no truth.
>
> (Brouwer 1948)

Or simply:

> ...in mathematics no truths could be recognized which had not been experienced.
>
> (Brouwer 1955)

The second quote by every means suggests an actualist reading: direct experience is needed for truth. The first, however, witnesses some of the highly puzzling points in Brouwer's philosophy. Tomasz Placek observes, that Brouwer suggests a redundant account on truth: whatever is, it's true (1999, p. 67). I am not so sure about Brouwer's ontological claims, but if Placek is right, then one may venture that there are different types of existence for which different types of truth apply. First there is the truth of direct individual experience as they are presented in Brouwer's consciousness. Then there are the experiences of others which are "true only as anticipations."[7]

Meanwhile, other places give hints on what is possible. Brouwer devoted considerable efforts to creating so called weak counterexapmles, mathematical problems often based on some unknown but possible properties in the unexplored parts of the decimal expansion of π (Dalen 1986, p. 230). By these experiments he intended to exhibit clear cases where (LEM) cannot be asserted, hence the name *counterexamples*. They are called weak, because they do not show that there are

7. Let me draw attention here to an interesting consonance with Bridgeman's operationalism (1927). I am not aware of any elaborated exposition of this analogy, though it might worth a paper.

absolutely undecidable problems. After all, someone in the future might prove that there are nine consecutive 9 digits somewhere in the decimal expansion of π.

Not least, holding that absolutely undecidable statements exist can be associated with the negation of an instance of (LEM): $\neg(p \vee \neg p)$. But it is uncomfortably stronger than just saying that there is no proof that p and no proof that $\neg p$. So much so that it is contradictory. To hold the assertion, there must be a proof that holding p leads to absurdity and holding $\neg p$ also leads to absurdity. However, showing that p leads to absurdity amounts to saying that $\neg p$ is true. A contradiction, since by accepting the formula, the falsity of $\neg p$ is assumed.[8]

Anyhow, it seems that Brouwer finally settled (if he ever did) on a view flirting with possibilism. Raatikainen summarizes his latter comments on truth, based on different sources, similarly to the following:

(c) A mathematical statement p is true when it is proved;

(d) p is false when it is proved to be absurd;

(e) at the moment we do not know the truth value of p but we have an algorithm to decide;

(f) at the moment we do not know the truth value of p and we have no algorithm to decide.[9]

After a comprehensive survey of his texts, Raatikainen concludes that Brouwer somewhat oscillated between the actualist and the possibilst accout of truth throughout his life. I have no reason to doubt. However, this latter summary is clearly and actualist one: true is what is actually proved. Still, what is really interesting in it, indeed concerns with possibilities. Note that there are two essentially different ways for mathematical statements not to be true or false. By (c), we have effective ways to decide. Given an integer, however large we have means to decide whether it is prime or not in finite time. By (d), we cannot tell whether the Goldbach-conjecture is true *at the moment*. We may, however, have the effective means to decide tomorrow. (If fact, even today we have an algorithm which may refute it in finite time, but, of course, it well may be that it never stops.) So we are dealing with different possibilities: if I have the proper tools, I might use them, but is is not excluded that only tomorrow will I have the needed algorithm for a given problem at hand. Thus in a certain sense we can say that it is in principle not impossible to decide the Goldbach-conjecture. By this token, this account remains wisely silent on absolutely undecidable statements.

Brouwer's most notable disciple was Arend Heyting, who, compared to his master, remained low-key in philosophical issues. In contrast to Brouwer's lofty

8. See e.g. Raatikainen 2013, p. 119.

9. In this coverage, it is interesting to see a Brouwer, who otherwise insisted that all mathematical is happening in the ideal mathematician's mind, finally letting in the thought of mechanical derivations.

idealism, he stressed the metaphysical neutrality of mathematics. When it comes to truth, he seems to be nihilistic, as he confessedly holds that truth has no meaning for the intuitionist, because, in order to speak about truth there should be a Platonic mathematical reality to which truth relates (Heyting 1958). Of course, here truth is understood in the classical sense of correspondence, not as constituted by proofs. Anyhow, with some minor exceptions, Heyting was faithfult to actualism, Raatikainen concludes (2004). So much so that instead of using the suspicious concept of truth, he expressed his actualist views straightly relying on the (arguably even more problematic) notion of existence. In particular, he maintained that no mathematical object exists unless it is construed.

> A formula of the form $\exists x A(x)$ can have no other meaning than: "A mathematical object x satisfying the condition has been constructed."
>
> (Heyting 1959, p. 69)

Without doubt, Michael Dummett gave the most thorough analysis of the problem of truth in intuitionist veins. He wrote extensively on the the topic over several decades, and meanwhile his claims evolved and changed considerably. In his early writings he tacitly endorse an actualist view (Dummett 1959), but later he explicitly addressed the problem and showed readiness for departure towards some kind of possibilism (Dummett 1975). He distinguishes direct (canonical) and indirect proofs, in case of the latter we in principle have the means for obtaining a direct proof. Interpreting this *in principle* clause is a key for a cogent possibilism, as I will argue below. By in principle available proofs, Dummett eventually arrives to what Raatikainen calls "liberalized actualism": a statement is true, if we either have a proof or we are in possession of a means for constructing one, independently of whether we are aware of this fact or not (Dummett 1998, Raatikainen 2004).

One may raise brows on this last amendment, as it seems to steal back the spirit of the realist's credo (PB1): there are truths out there, independently of our awareness. Another one of Dummett's distinctions may give a hint: some mathematical assertion are *determinately* true or false, without actually proving either. My reading is this: the formal verification procedure of mathematics is internally and inherently determined, independently of whether we are looking at it or not. The same does not go for empirical statements: the assertion that "the number of hairs on Professor Künne's head is even or odd" cannot be build on the counterfactual "if we had the means to count the number of hairs, we would find that it is even or we would find that it is odd" (Dummett 2007, pp. 349-350). May the belief in the efficiency of the formal derivative machinery however justified, in general, I would challenge the thought that in principle possibilities set apart mathematically and empirically valid procedures for constructing truth.

2 What Bob Wonders About

Above I have not given account of every twist and turn that Raatikainen nicely and meticulously covered in his historical survey. After all, in itself it is not so surprising that verdicts on a given issue by one and the same philosopher uttered at different occasions often diverge. He might, for instance, change his mind meanwhile.[10] What I find really important is the conceptual analysis along the lines of actualism and possibilism.

First it is to be seen that differentiating between actualist and possibilist accounts does not make a dichotomy. It is clear that a possibilist is always an actualist at the same time, since an actually exhibited proof is of course a proof for a possibilist. Rather, the intuitionist accounts of truth can be seen as grades on a scale running from the strictest actualism to the most liberal possibilist view. The place of a view is determined by the interpretation we give to the *in principle* clause when talking about in principle provability or constructibility. (I will address *in principles* in the next section.)

Second, it seems that the classical interpretations of intuitionist mathematics and the spirit of the different branches of constructive mathematics should all either lean toward some possibilism or relax on the notion of proof. At least I venture that no constructivist will deny, e.g., that given an integer however large we can in principle always decide whether it is prime or not, without actually devoting the needed resources.[11]

Someone who still wants to keep an actualist stance, i.e. insists that for a statement to be true (or false) we must actually possess a proof, has to face further difficulties. Citing Dag Prawitz (1987) Raatikainen (2004) indicates that equating truth with actual proofs, may lead to awkward consequences. For instance, probably everyone agrees with a statement like this one:

(1) If Bob has a proof for the Goldbach-conjecture, then Bob knows a great deal about the Goldbach-conjecture.

However, this implies the following, uncomfortable statement:

(2) If the Goldbach-conjecture is true, then Bob knows a great deal about it.

It indeed seems puzzling. For it is probably natural to suppose that if Bob has such a proof, then he is a mathematician well acquainted with the problem at issue. But observe that Bob's having a proof and the conjecture being true is not equivalent: the first implies the latter but not vice versa. What is more, the thought that Bob's proof alone is enough for the conjecture to be true can also be

10. One version of the famous bonmot credited to John Maynard Keynes goes like this: "When someone persuades me that I am wrong, I change my mind. What do you do?"

11. Although I can imagine objections based on some theoretical limits of physical possibilities.

challenged on the grounds that a lonely and isolated derivation is not enough, a proof must be canonized. Anyhow, Bob's having a proof cannot exhaust the meaning of a statement being true. Still, maintaining that Bob's having a proof for p implies p is true implies Bob knows a great deal about p, we have weird inference. At the very least it shows that there is a conflict between a constructive account of truth and everyday language use. But one may even insist that the conflict affects our common intuition. Now somebody else can argue – indicating that intuition is not always our best guide – that everyday language is messy, incorporates naive realist sentiments and every reflected and elaborated account should come into some kind of conflict with it. And I will argue accordingly below.

Raatikainen goes a bit further than Prawitz, showing that a possibilist view is also affected by similar nuisances. His point is that every theory of truth of a constructive nature, whether an actualist or a possibilist one, must admit that truth is in some sense temporal. For if the constructivist ties the truth of a propsition p to a construction of a proof, he cannot say that p was true before the construction.[12] A possiblist view cannot provide defense: if we have in priciple means to prove p, it may not have always been the case. Let we have:

(3) Bob wonders whether the Goldbach-conjecture is true.

Then, we must conclude along a possibilist account of truth that:

(4) Bob wonders whether the Goldbach-conjecture is provable by presently available methods.

Raatikainen observes that the (3) and (4) expressess quite different thoughts, they can by no means regarded as equivalent. I can only agree with him. Unfortunately the same is true for whatever reflected account of truth. Suppose Bob is a coherentist:

(5) Bob wonders whether the Goldbach-conjecture is coherent with his other beliefs.

Clearly (3) and (5) express quite distinct thoughts. But let Bob be a correspondace theorist, widely held to be the most "natural" stance:

(6) Bob wonders whether the proposition g expressing the Goldbach-conjecture corresponds to the fact G that every even number can be given as the sum of two primes.

12. An interesting question whether something may cease to be true after a point in time, but I will not address it here.

Do (3) and (6) express the same thoughts? By no means. A constuctivist account of truth may well be incriminated by comparing (3) and (4), but every other account is likewise suspicious as the other examples show above. I think, the moral is this: no reflected account of truth fares well in an intensional, natural language environment, should whatever be the case with temporality. But I would not regard it as an utter surprise, we have a good deal of examples bringing along similar cruxes like wondering Bob.[13] The reason is simple: natural languages incorporate their own metalanguages, and this often brings along mess and troubles.

Nevertheless, the temporal nature of constructive truth has often been causing headaches even to their advocates. Raatikainen intends to show that this uncomfortable feature even leads to absurdities (2004, p. 139). Let us assume that Bob strongly believes that some hypothesis *h* is true, and the following is the case:

(7) *h* is undecidable at Bob's time by the available means.

(8) *h* is provable by methods available well after Bob's time.

The point is, according to Raatikainen, that we cannot say that Bob's conviction was incorrect, but we should do this following the constructivist principles. And this is absurd.

As for me, I cannot really see why we should not say that Bob was *right*. If I say today that there will be a sea battle tomorrow, and indeed there is, you must admit that I was right. Even if a hypothesis is undecidable today, someone can endorse it and may provide to be right tomorrow. These natural languages peculiarities do not impair the constructive truth concept, since the latter has nothing to with an agent's convictions. All it asserts that truth is to be established through an actual or possible construction.

And after all, we can by no means say that a classical account is free of temporality issues. We know since Aristotle that a future possible sea battle may pose a sticky problem even to the most sober realist. But consider this:

(9) The proposition that the Eiffel tower is 330 meters high has always been true.

Trivially, the sight had not existed before the second half of the nineteenth century. But even if it did, there had not been meters around before the end of the eighteenth, as they were defined in 1791 as a certain fraction of the equator. Sure, metric scale is just a convention and one can say that the property itself has been just there naked out there in the world. However, we may venture that at some point in time there will be no such thing as Eiffel tower (or even references to it) anymore. A correspondence theorist can easily say that, of course, facts are changing and we can cope with this situation with a little time-indexing. So far so

13. Just consider the Cretan who says that every Cretan is a liar.

good, but now this theory is deeply engaged with temporality also. Note further, that holding (9) awakes a quite awkward version of the principle of bivalence:

(PB2) Every statement of a measurement result is true or false relative to an arbitrary convention on scales independently of our knowing which and in our outright inability for knowing it ever.

One could object at this point that referring contingent facts is one thing, asserting law-like ones is another. Let us see:

(10) It has always been true that the Higgs boson has the mass 125 GeV/c^2.

The mass of this particle (the existence of which is *established* enough today by a common consent of most experts) is not an entirely contingent fact of the world, but follows from an intricate and far-reaching theoretical machinery called the Standard Model. To gain empirical evidence, extremely complex equipment is needed, and also a so called *phenomenology*, i.e. established system of conventions on the interpretation of instrument readings. In this light holding (10) is at least uncomfortable, since the thing itself at issue and its properties are the result of a long, sophisticated and costly human endeavor, i.e. particle physics, which had been initiated in the twentieth century only. The realist, of course, insist that the the value of the mass of the Higgs boson has always been out there. By this token there are frightfully many other things out there with exact values on their measurable properties. Eventually, it could have been the case that this particle had never come along, but this does not prevent the mass in question to be a basic feature of nature itself. Still, this invokes a version of the principle of bivalence which is, I think, not so pleasing to hold and defend:

(PB3) Every well-formed statement of a physical hypothesis is true or false independently of our knowing which and our inability for knowing it ever.[14]

According to a widespread and longstanding tradition, mathematical truths are different, they have some internal and eternal, conceptual necessity in contrast to other kind of truths. Thus we may have:

(11) The hypothesis that every planar map is four-colorable has always been true.

Clearly, this hypothesis had not been worded before the eighteenth century and has been proved only in the seventies of the twentieth.[15] A Platonist, i.e. a

14. Interestingly, there are indeed physical theories reflecting such an attitude, see Baggott 2013.

15. This event is also remarkable due to the fact that this was the first theorem proven by the aid of a computer (Appel, Haken, and Koch 1977 and Appel and Haken 1977).

realist about a mathematical universe, would say that the truth had always been there waiting to be exposed, just as the element oxygen had been existed before its discovery. What is more, the existence of mathematical objects are even less arbitrary than that of plebeian elements, the are necessarily existing in an eternal and ideal immaterial world.

But if (11) is not awkward enough, consider this:

(12) It has always been true or false that there is a cardinality between those of the natural numbers and the reals numbers, independently of our knowing which and having no means whatsoever for knowing it.

A hardliner Platonist, like Gödel (1964) may still insist that there is a set theoretical universe out there, which is imperfectly described by all of our axiom systems. But many philosophers and practicing mathematicians are much more cautious than that. For one, Scott Aaronson (2023) suggests, somewhat in the spirit of Quine and Putnam, that there is an objective mathematical reality (at least) up to arithmetic, but set theoretical speculations are beyond that. They are quite interesting tinkering with formal systems on their own right, but outside the realm of reality and truth.

Still others, however, seem to outright deny the legitimacy of any weekday or weekend Platonism. Andrew Granville speaks in this vein. In his (2023) he provides a vivid snapshot on the current mathematical practice and presents his own reflections to the situation. According to him we should abandon ”the naive notion that formal proofs will improve objectivity,” but we should keep ”the community based approach to proof that has long served us so well,”[16] even when we seek for the assistance of computers during the proving procedure. By proofs, we are not in an approach for ever improving understanding of some objective truths – just as a Popperian realist would insist for empirical research –, but we are constituting mathematical truth with the kind help of machines in a cooperation of an expert community.[17] It does not at all mean that anything goes, the game has definite rules. I will come back to these points below.

3 What Do the *in Principle* Clauses Say?

As it seems, a constructivist must lean toward some kind of possibilism if she does not want to be too strict and deny assertions which are otherwise look evident. We know, for instance, that there are infinitely many primes[18] and also, trivially, that every natural number is either a prime or a composite. We also have an

16. Granville 2023, p. 11

17. It is, of course, does not mean that Granville is an intuitionist or constructive mathematician, only that he is an antirealist about mathematical objects.

18. The first proof which is attributed to Euclid used *reduction ad absurdum*, a non-constructive step. A very simple constructive proof is given by Saidak 2006.

algorithmic method for deciding for each number whether it is prime or not, however we may have trouble with applying it for very large numbers. And of course, we do not have means to decide it for all natural numbers at the same time. But it should be okay to say that *in principle* we can decide whether a number n is a prime or not.

Now the strength and the accountability of a possibilist constructive account of truth strongly depends on how one interprets these *in principle* clauses. According to Parsons (1997), we usually rely on what we know about the usual, observed abilities of human actors and characteristics of machines. But in itself it cannot be a strict guide. Suppose that x stands for the distance a rubber boot was ever thrown at by a human, e.g., in meters. Should not we say that it is in principle possible to throw a rubber boot at $x + 1$ meters by a human? Probably we should. $x + 2$? And any $x + y$? By every measure, one can insist that no one ever will throw a rubber boot to the orbit, or not even at $10x$ meters. But is it in principle impossible?

When we talk of *in principle*s, there is large scale of interpretations from *available*, through *physical* to *logical* possibilities. Having a pair of rubber boots in my shed I could, in principle, throw, say, the left one at some distance, even if I actually choose not to. I may not have a rubber boot at hand, but I might still be able to throw it at some distance in principle. Now give me one, surely I cannot cast it for a mile and I do not know anyone who can (I suppose I would know it if so), but logically it is possible, after all: it needs only proper muscular buildup and the right technique. But is it physically possible? To be sure, for human abilities we have vague terms.

It can be argued that in a mathematical context it is more important what a machine can do, or more exactly, what a human-computer collaboration can do. Thus, *in principles* should be characterized through physical[19] and computational (mathematical) limits. For what can be algorithmically solved, i.e., what is in principle computable, we have a delicate theory: that of computability. Its most important restrictive result, the Church-Turing thesis, roughly says that nothing can be computed beyond the capabilities of a Turing machine.

There are voices, however, that in the light of the recent development of large language model AIs, these old paradigms are no longer valid. With deep learning, new capabilities arouse spontaneously and we are no longer in the position to assess where the progress ends. I think otherwise. Pattern-based computation does not supersede the good old rule-based one, even it endows machines with new abilities. Now they are better and better at sensual recognition and imitating human linguistic behavior (by default with all of the flaws in character and with no regard to established facts). No doubt, AIs can be of great help also when we put problems in algorithmic setting, but I do not see how could they broaden the

19. For a die-hard Brouwerian, connecting mathematical truths to contingent physical facts would be unacceptable. Here I pursue a much more liberal approach.

set of computable functions. What we can in principle calculate is left intact by AI.

Perhaps another challenge for the received computation paradigm is the possibility of supertasks and hyper-computation. At least, some researchers argue that computation can go beyond the Turing-barrier in the sense that the laws of physics make it possible.[20] In particular, our prevailing theory on the structure of space-time allows for situations where some supposed observers can read their clocks so to realize that infinite time elapsed. And this time is obviously "enough" for a computer to prove the Goldbach-conjecture or even the consistency of set theory (say **ZFC**), simply by brute force: by probing each even integer or deriving all of the consequences of the axiom system.

All computations are, it can be argued, running on physical systems, humans included.[21] So the in principle clause should be highly affected by what is physically possible. Now one can say that even if the above scenarios are allowed by theory we are very far from harnessing them. But this is not the point. Somewhere in the past, e.g. in Leibniz's time, ordinary computers had also been a distant possibility. But observing the phenomena closer some day, auxiliary theories may come into the picture, warning for more restricted views on our *genuine* possibilities. By Newtonian mechanics we can give all the pool balls scattered across the billiard table exactly the right momentum so to form the initial pyramid rack. Still, due to imperfections in the setup, the *theoretical* imprecision in the measurement of the initial conditions and our robotic gadgets (with which we are by all probabilities would like to make such experiment), I suppose that no one regards it as a real, practical possibility. But can someone say that in principle it is possible? Maybe, but I would not join. I am inclined to put on the in principle clauses a narrower scope. In particular, I would not tag thought experiments as in principle possible, where the possibilities of one physical theory run against those of another one. In the case for hyper-computability, general relativity allows supertask scenarios. At the same time thermodynamics makes infinitely running machines impossible or at the very least infinitely improbable. So at the end of the day, in my view, hyper-computability is not something that we should consider as an in principle possibility in theorem proving.

On the other hand, some more mundane and immediate state-of-affairs of the physical world do pose challenges even for calculations well within the Church-Turing limits. Suppose we give a computer the relatively easy task of calculating

20. See Németi and Dávid 2006 and Andréka, Németi, and Németi 2009.

21. For the sake of simplicity I adopt this physicalist stance here, not distinguishing between different actors. However, it is not at all unproblematic. For instance, we are not in a good position when we are seeking a theory of meaning for the symbols which constitute a proof. Indeed, even when talking about different tokens and types of symbols, and thus about recognizing a proof when seeing one. I will not address these difficulties here, though I have argued against a radical physicalism elsewhere (Csatári 2012). My reason is that from the point of view of computation theory, it is quite apt to handle a machine and a human computer on the same page (Granville 2023, p. 3).

257^{729}, and then printing out the result in the unary system, i.e., by tally marks. The number of the digits needed for this simple output far exceeds the supposed number of atoms in the known universe.[22] Perhaps *actual* printing is not a genuine computation tasks after all, and we can surely choose a form and method so to fit a normal screen.[23] Let it, but as some results of computations theory warn us, only problems with a restricted complexity can have a solution factually viable on a computer. And problems, where the output is not *too large* related to the algorithmic size of the task and can be delivered in due time constitute only a fraction of all decidable problems. Since the volume of this practical set is not clear cut, however, as what we consider as reasonable time is up to technological advancements, one may claim that we are legitimate to use the concept of in principle computability up to decidability, i.e. up to the Church-Turing barrier. In a mathematical context we may well do with it as a characterization of *in principles*.

4 Incompleteness and Circularity

Even if we suppose that the scope of *in principles* is more or less clear, there may remain some further worries when ordering truth under proofs. An important one is that at this point the methods of the construction of proofs are not at all characterized, so far any procedure goes as a proof when it is in principle available. I will address this problem in the next section. Here I discuss two other difficulties. The first concerns with incompleteness, the second with a supposed circularity of a constructivist truth concept. As we will see, the these objections are not independent of each other, in fact, they are utterly intertwined.

Raatikainen (2004) observes that that there are most probably a finite number of available methods of proof at every point in time, giving room for a denumerable set of derivations. Naturally supposing that the methods do not contradict each other, they may be added up to a formal system. But if so, then by Gödel's first incompleteness theorem (1931) it follows that there are true sentences which cannot be derived. But this cannot be the case, since according to the intuitionist, true sentences are exactly those which are or can in principle be derived—a contradiction. Hence equating truth with a possible proof is ill-taken.

I take it that "methods" can be thought of as step by step rules, while axioms do not enforce definite procedures. It is thus at least questionable whether there is a *method* by which we can construct an axiom system encoding those and only those steps available in the existing methods. We know that first order intuitionist logic is undecidable (as a matter of fact so is classical predicate logic), meaning that there is no algorithmic way to decide for each and every sentence whether it follows from the system or not. So it does not seem to be a viable task to grab even

22. It is estimated somewhere at 10^{82}.

23. In fact, I typed the problem into Python, and it immediately gave the result in decimal, a 1757-digit number.

the underlying logic of a derivative system by adding up the available algorithmic rules, i.e. actual or potential derivations.

Each derivation ends with a proposition p, the very one supposed to be proved. (Of course, p may happen to be $\neg q$.) However, nothing ensures that for every well-formed statement r in the language at issue either r or $\neg r$ will end one of the derivations. In other words, it is still possible, that the rules for our methods of proof constitute a subsystem with the sufficient arithmetic, but I do not see why anyone should suppose that something like negation completeness holds for it, the very property denied by Gödel's first incompleteness theorem in case of sufficiently rich, consistent systems. In fact, it should not hold, since intuitionist logic explicitly denies it own completeness by not endorsing (LEM).

I would suggest more cautiousness when it comes to identification of syntactic and semantic layers, by making clear difference between the language of a formal system, its interpretations and the language (and metalanguage) of metalogic. The "truth" of the indeducible sentence is a truth in metalogic: an assessment of the formula's formal properties. Gödel's non-derivable formula does not speak about its own non-derivability, rather it makes a complex statement of arithmetic, which may belong to the formally defined set tagged as the true sentences. But this set could just as well been called as the correct sentences, nice sentences, tame sentences, whatever. Strictly speaking, we do not need any truth concept in order tell the incompleteness story. The incompleteness results, of course, deeply affect the endeavors of a formalist too, who happen to deprive the object language formulas of any meaning or truth values. And intuitionist arithmetic is of course affected by Gödel's theorems too: if consistent, it cannot prove its own consistency. Just like its classical counterpart.

When it comes to circularity, it worth noting that it is an ubiquitous suspicion whenever foundational theories are on trial. Founding mathematics by the means of mathematics, an endeavor also known as mathematical logic, is flirting with circularity in itself. Some say that the notion of a successor function already contains the concept of numbers, the very one it is supposed to build a foundation for. And elsewhere I showed that concatenation, as a basic empirical procedure for measurement is also burdened by circularity (Csatári 2020, pp. 30-33).[24]

Now let see the case against intuitionism. The construtivist truth concept relies on procedures of proof. But only those derivations are constructively valid, for with axioms and rules are set up in constructive (in this case: intuitionist) manner, i.e., they reflect the very truth concept themselves, which is to be defined by them. "If the explication of the notion of provability in turn presupposes intuitionistic interpretation of logical constants, the whole account appears to be viciously circular, or to lead to an infinite regress" (Raatikainen 2004, p. 140).

It is clear that traditionally we have the most interest in those formal systems, which reflect some of our ontological or alethic convictions. True, we may well indulge in tinkering with Hofstadter's **MIU**-system (1999, pp. 33-35), where

24. As a matter of fact, I did this to argue for a constructive approach.

downright meaningless strings of a tiny alphabet are manipulated by a couple of contingent rules. And this activity definitely has takeaways for formal systems in general. Typically, however, the focus is on systems which had been created with some meaning in head. First order logic (**FOL**) is intended to grab the essence of valid argumentation and the concept of consequence. Peano Arithmetic (**PA**) is vehicle for natural numbers and their manipulation on the top of **FOL**. Just like its intuitionist counterpart, Heyting Arithmetic (**HA**) on intuitionistic first order logic. By no means it comes as a surprise that the intended semantics deeply affects syntax, and the interpretations of logical and non-logical constants are not independent of the accepted truth concept.

Again, in a classical setup we say the the proposition p is true, iff it is the case that P, i.e. the state of affairs is indeed so as described by p. One can say that this truth concept is circular or at the very best *flat*, since the clause *true* adds nothing to the sentence p. I would not follow suit, because I put great importance on the observation that p and 'p is true' belong to different languages, or at least different layers of a language. Anyhow, classically $\neg\neg p \to p$ is a valid formula exactly because it is supposed that if it is not true that 'p is not true' then nothing else should be the case than P as described by p, exactly by the intended account of truth above. Intuitionistically, however, $\neg\neg p \to p$ should not be true, since the absurdity of 'p is absurd' will not constitute a positive proof that can be turned into a proof of p. I do not see genuine circularity here, however it is quite clear that both in the classical and the intuitionist cases, formal systems, logical constants and their interpretations are vehicles of higher devotions.

5 Proofs as Valid Procedures

I still owe with with an exposition on what proofs are, or in general, what can be regarded as a valid procedure for a construction exhibiting truth in mathematics (and perhaps beyond). While examining the nature of procedural activities resulting in proofs, I will leave intuitionistic logic behind. Or to be more precise, I will not be concerned here with the question whether there is any specific logic forced upon us if we base truth on these procedures.

By a valid procedure I mean canonized ways for establishing truth in a certain domain of knowledge. There is no scientific endeavor where everything goes. To measure the mass of the Higgs boson, to *construct* it, a complex and delicate theory, the hugest and most expensive accelerating machinery and large computers for the data handling are needed—by the consent of particle physicists. It is clearly not enough to tell fortune from coffee grounds, say.

When concerned with the nature of mathematical proofs, we can choose an easy-looking way to go on by "adopting the basic intuitionostic idealization that we recognize a proof when we see one" (Kreisel 1962, p. 201). Indeed, the road is not so smooth, though may lead in the right direction, since realization is an important point when the mathematical community reflects on proofs.[25] A recent study, intended as a collection of different views of contemporary practitioners, indicates two conceptual focus points when it comes to proofs, similar to the following (Bayer et al. 2024, p. 79):

(Def1) A mathematical proof is a deduction of a formula from the initial formulas (axioms) following the rules of deduction.

(Def2) A mathematical proof is a sequence of arguments that convinces the educated reader (i.e. the experts of the topic).

(Def1) is tagged as the formal definition (even called as *idealistic*), whereas (Def2) as the practical one, more faithfully reflecting the nature of weekday activities by mathematicians.

Mathematics is traditionally proud to be the strictest of all sciences, as all of its assertions are deemed to be based on some conceptual necessity. In this light it is all the more surprising that there is such a huge gap between an ideal proof definition, seemingly for the drawer, and a much more relaxed one for everyday use. However, may the first definition be sterile, the second one is for sure not sufficient. Since, after all, I may convince the educated colleagues with my arguments while bribing or blackmailing them, which by no means amounts to a mathematical proof. But do not split hairs, the intended message is clear.

If we look at papers in journals for mathematics, we see formal assertions as well as natural language arguments, invoking of former results and appeal to insight and intuition. Logical connectives are often also given as plain text arguments. It is a common wisdom among mathematicians, though, that all of those informal leaps and jumps could be translated into strict, formal derivations—*in principle*. But on what is at stake with such translations, opinions diverge.

Many hold that the essence of mathematics lies in understanding, heuristic thinking, mastering intuition and creativity (see e.g. Thurston 1994 and Bayer et al. 2024, pp. 81-82). Indeed, informal means constitute a large part and arguably the sunny side of the profession from journal argumentation through teaching to conference discussions both in the room and at the coffee counter. At the same time, never has the imperative that all mathematics ought to be based on strict, full-fledged formal proofs more articulated. And this is to do with the fact that proof assistants gained currency. Much of mathematics is now fully formalized and stored in the databases associated with proof assistants like Metamath, Lean or Isabelle. There are projects ongoing to translate not yet covered, highly complex proofs, such as the one of Fermat's Last Theorem, and also initiatives to require machine translation and check for every journal proof in the future (ibid.). So even if they are time-consuming, tiresome and bring no credit, full formalization and machine verification are slowly becoming the norm.

25. "[I]n practice a proof is what is considered to be a proof by all mathematicians (Bayer et al. 2024, p. 82)."

Having said this, one must admit that the validity of procedures for constructing truths is, to a large part, a normative concept. Acceptable methods are marked off by the scholarly community. At the same time, as the moral of formal *reconstruction* of former proofs clearly shows, truths of mathematics could be reached in several ways: by fully formal flawless derivations, and also by the good old, more sloppy, half-formal, intuitive means. All in all, recent changes seem to amount to this: while there is more emphasis on strict, mechanical verification, the playground for the intuitive and heuristic mathematics remains.[26]

Where more than one procedures are available, identical (or similar enough) results can be regarded as an indication for validity. Here I sketch three other principles, all of them may sound quite natural. They go beyond the scope of mathematics, and may worth some more elaboration elsewhere. It is important to note that these principles are not simple norms established by scholar communities, rather factors stemming from a common sense view of ”how things are,” thus regulating the received community norms.

The first requirement is *relevancy*. It is not surprising that we would like to see that the procedure we are pursuing towards truth is relevant to the subject matter. When aiming for some truth, say, on the behavior of certain bacteria, it is relevant to look into the microscope and record the observations of experiments with the given species. It is less adequate to ask a medicine man. (However, he might be aware of some of the effects of this behavior.) It is worth emphasizing that in practice, relevancy is often not transparent at all, it may be hidden in delicate theories and complex machinery. Looking at a collider with its many-mile long pipeline, extent supplier equipment and staff, it is not self-evident that crafted experiments with these are the relevant way to make true assertions about the masses of certain subatomic particles. Likewise, in complex mathematical proofs there may be trains of thoughts that seem to be completely irrelevant to the goal at the first sight.

Secondly, valid procedures must show *effectivity*. Practically this means that a method is more successful than others, and it is successful in several similar but distinct cases. Measuring distances by laser beams are effective in contrast to bare sight estimations. In the context of mathematics, Granville calls this feature *robustness* (2023, p.5). Well-used technical tools make verifying smooth, ”[a]nd even if there is a mistake, experience shows that a simple modification should be enough to make the argument work.”

Last but not least there is a need for *transparency*. The procedure must be open, each and every element of it must be built so that it could be scrutinized by everyone, but most importantly the expert peers. Given the complexity of many scientific projects, this principle may seem to be beyond the pale. A collider consists of highly sophisticated elements each built on intricate theories and running million lines of proprietary software. It is clear that even a relatively small part

26. However, with the advance of large language model AIs, machines may gain more and more grounds here as well.

is so complex that cannot be digested by one and the same human. Similarly, machine assisted proofs may require an extremely wide range of specific skills and may be so extent laid out in full, that it cannot be *surveyed* by any individual, mortal mathematician (Tymoczko 1979). And this kind of surveyability cannot even be the purpose. All we can require is that each process has to be legally and technically open for scrutiny, every part of the machinery must be open for a review by the community of specialist peers. This leads to interesting policy-related problems such as proper documentation, accessible literature and open source software, a realm which I will not explore here. For now, I it enough to declare that if a proof is to be recognized as such, transparency principle is there to help the scientific community in this recognition.

6 Conclusions

With my defense of a constructivist approach to truth I had no intention to suggest that intuitionistic logic is superior to classical one, or anything like this. Of course, no one wants to exile anyone from the set theoretical paradise.[27] My focus was on the different truth concepts, and all I wanted to show here that maintaining a constructive one is just as natural and at least as defendable as betting on a correspondence account. In doing so, I gave myself the freedom to loosely diverge from a strictly mathematical context every now and then.

I admitted the distinction of actualist and possibilists account of truth as a fruitful one, and concluded that a certain degree of possibilism is inevitable for a cogent constructive truth concept. At this point *in principle* clauses come into the picture, and I concluded that, at least in a mathematical context, what we can in principle do is by and large clear.

I examined some of the most important objections against a truth concept paired with intuitionist logic. I found that they can easily turned against correspondence theory and classical logic also, a fact showing that these difficulties go beyond the scope of a specific truth concept. And with the natural principles of relevancy, effectivity and transparency we can account for proofs as valid procedures, giving constructivism a good base to build its truth concept on.

All in all, attacking constructivism along the lines of circularity, and its truth definition going astray in natural language examples are not enough for settling on realism, as all of these critics likewise affect any other reflected viewpoint. If we want to choose a massive metaphysical stance, arguments must be searched for elsewhere.

27. See Hilbert 1926.

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