The Notorious Man-in-the-Street: Hermann Weyl and the Problem of Knowledge

Noah Stemeroff

Forthcoming in Studies in History and Philosophy of Science

Abstract

Hermann Weyl's philosophical reflections remain a topic of considerable interest in the history and philosophy of science. In particular, Weyl's commitment to a form of idealism, as it pertains to his reading of Husserl and Fichte, has garnered much discussion. However, much less attention has been given to Weyl's later, and at that only partial, turn towards a form of empiricism (i.e. from the late 1920s onward). This lack of focus on Weyl's later philosophy has tended to obscure some of the most significant lessons that Weyl sought to draw from his decades of research in the foundations of mathematics and physics. In this paper, I develop some aspects of what I will term as Weyl's 'modest' empiricism. I will argue that Weyl's turn toward empiricism can be read in the context of a development of Helmholtz's epistemological program and his unique form of 'Kantianism'. The hope is that this reading will not only provide a better understanding of Weyl's later thought, especially his (1954) criticism of Cassirer, but that it may also provide the basis for a novel 'Weylian' account of the mathematization of nature underwriting the group-theoretic methodology of parts of modern physics.

Keywords: Hermann Weyl; Hermann von Helmholtz; Ernst Cassirer; Empiricism; 'Kantian' and neo-Kantian Philosophy of Science

. . . .

> It is the common fate of man and his science that we do not begin at the beginning; we find ourselves somewhere on a road the origin and end of which are shrouded in fog (Weyl, 1948 [2017], p. 156).

1 Introduction

The history of twentieth century physics is filled with a colourful cast of characters, but even within this rarefied group, Hermann Weyl cuts a fine figure.

Weyl is best known, at least within the physics community, for the insightful and prescient nature of his work—e.g. on the early development of general relativity, the pursuit of unified field theory, and the application of group theory in quantum mechanics. Weyl's initial foray into physics was primarily concerned with its appropriate mathematical formulation. This was natural, given that Weyl was a mathematician by trade rather than a physicist. But the influential nature of Weyl's research served to bolster a broader trend within the mathematical community (at least in the Göttingen tradition of Klein, Minkowski, and Hilbert) towards a deeper engagement with the foundations of physics.

Weyl's work as a physicist tended to blur the boundaries between mathematics and nature, and his philosophical thought has often been interpreted in this context. This reading is certainly natural, but the problem with placing too much focus on any particular aspect of Weyl's work is that we run the risk of obscuring one of the most interesting features of his thought: namely, its eclectic nature (e.g. see Sigurdsson, 1991). Like many of his contemporaries, Weyl's interests were both wide-ranging and idiosyncratic, reaching far beyond his specific areas of expertise in mathematics and physics into the broader realm of philosophical discourse. However, unlike many of his contemporaries, Weyl's work, as a whole, was motivated by a fundamental belief in the unity of thought. Throughout his life, Weyl continually strove to interweave his vast and ever evolving studies (in particular those at the intersection of mathematics, physics, and philosophy) into a unified framework for physical enquiry. To Weyl, mathematics, physics, and philosophy did not represent independent fields of study, but interdependent means of tackling the fundamental problem of knowledge—i.e. to determine what is true and objective in our thought (e.g. Weyl, 1954).

Weyl not only derived inspiration from his philosophical studies, but also explored the philosophical implications of his work in mathematics and physics. However, at no point could Weyl's thought be said to neatly align with any particular philosophical school. In fact, Weyl's philosophical thought could hardly be said to be entirely coherent—e.g. one can find concurrent threads of idealism (including Leibnizian, Kantian, and later German idealism), Husserlian phenomenology, realism, empiricism, panpsychism, and forms of theism (including traditional theism, mysticism, and possibly even fideism—though that may be a bit of a stretch).[1] Nonetheless, Weyl's loosely 'Platonic' belief in a fundamental unity in mathematics, physics, and philosophy, as part of the general problem of knowledge, remained the essential insight motivating his philosophical thought. Indeed, it is the true depth of Weyl's belief in a profound harmony between mathematics, physics, and philosophy that makes him so inimitable from a present-day perspective.

[1]This is not to say that all of these "isms" are strictly incoherent with one another, but rather the collection as a whole is somewhat untenable. For the Leibnizian, Kantian, German idealist, and Husserlian aspects, see Weyl (1918, 1932 [2009], 1934 [2009], 1949a), Ryckman (2005), and Sieroka (2007). For the empiricist, see Weyl (1934 [2009]) and Scholz (2004, 2018). For the realist, see Weyl (1949a) and Sigurdsson (1991). For the panpsychism, see Weyl (1934 [2009]) and Sigurdsson (1991). For the theism, see Weyl (1932 [2009]), including possibly Fideism (i.e. in the sense that faith stands above reason) in Weyl (1932 [2009], p. 46).

In light of Weyl's significant contributions in both mathematics and physics, and his wide-ranging interest in philosophy, it should come as no surprise that his philosophical thought has been the subject of perennial discussion in the history and philosophy of science. In particular, Weyl's ever evolving idealism (and his commitment to a form of Husserlian phenomenology) has been the subject of much debate. In this context, it has rightly been noted that Weyl's reading of both Husserl and Fichte served as a major motivation for his work in mathematics and physics from the late 1910s to the middle of the 1920s (e.g. see Sigurdsson, 1991; Ryckman, 2005; and Sieroka, 2007). However, much less attention has been given to Weyl's later, and at that only partial, turn towards a form of empiricism (i.e. from approximately 1925 onward). This shift in philosophical focus was motivated by a crisis in Weyl's thought, which was brought on by his engagement with quantum theory and the need to account for its apparent empirical foundation (e.g. see Weyl 1927/1949a, 1928, 1929, 1948, 1949b, 1954; and Scholz, 2004, 2018, 2019). In Weyl's view, the development of quantum theory had made it clear that physical theory could no longer be led by considerations of mathematical harmony or theoretical unity alone, quantum theory had shown that we must now take our lead more directly from nature herself.

Weyl's later empiricism might loosely be called a form of 'modest' empiricism, in contrast to a 'strict' empiricism which seeks, or demands, an empirical ground for all knowledge. In his turn toward empiricism, Weyl sought to not only follow nature's lead, but to precisely identify the sense in which nature can serve to guide the theoretical construction of a picture of reality. However, Weyl never lost sight of the limitations of empiricism, and the need to account for the *a priori* basis of scientific thought. Weyl remained sensitive to the perils of both naive empiricism and unconstrained idealism. In his search for a viable middle ground between these extremes, Weyl traced out the foundations for a unique form of empiricism within what one might call a broadly Helmholtzian tradition—a tradition that sought to harmonize some of the central tenets of empiricism and transcendental idealism.

The lack of focus on Weyl's later turn toward a form of empiricism has tended to obscure some of the most significant lessons that Weyl sought to draw from his decades of research in the foundations of mathematics and physics. In this paper, I will look to develop some of the central tenets of Weyl's 'modest' empiricism, as one thread within the unique tapestry that makes up his later philosophy of science. In this context, I will place a particular focus on Weyl's shifting views concerning the relationship between mathematics, physics, and philosophy. Throughout, I will consider the extent to which Weyl's later empiricism can be read as a development of Helmholtz's 'Kantian' empiricism.[2] This development should not be read as a strict historical development of Helmholtz's thought, in its entirety, but rather as a looser philosophical development—one which looked to draw on certain key aspects of Helmholtz's philosophy as the

[2] Here, and in what follows, the label 'Kantian' should not be understood to indicate a direct development of Kant's thought, but rather a development of the broadly Kantian tradition in late 19th and early 20th century philosophy of science.

basis for future development, and provide a novel reformulation of the basic intuition or insight underwriting the Helmholtzian philosophical tradition. My aim is not to argue that this is the only suitable interpretation of Weyl's later thought, as there are many threads, but to highlight one way of understanding certain aspects of Weyl's later empiricism. The hope is that this reading may provide not only a better understanding of a few major themes in Weyl's later thought, particularly his (1954) criticism of Cassirer, but that it may also form the basis for a modern structural-empiricist philosophy in the Helmholtzian tradition, a 'Weylian' philosophy that could offer a novel understanding of the group-theoretic methodology underlying parts of modern physics.

In an attempt to make this paper somewhat self-contained, I will begin with a brief account of Helmholtz's 'Kantian' empiricism, as it emerges through his work on the problem of space (which concerns which geometrical structures can serve as a viable ground for the description of physical space) and the problem of knowledge (which concerns which aspects of our thought can be taken to be true and/or objective). I will then turn to a discussion of Weyl's work, and look to develop some aspects of his thought on Helmholtz, and related topics, as it pertains to his broader turn toward empiricism. I will then take a short detour to present a few basic aspects of Cassirer's neo-Kantian position. This detour will serve to provide important context for a more detailed engagement with Weyl's 1954 essay on Cassirer and the unity of knowledge. In analysing this essay, I will outline, in general terms, what I have labelled as Weyl's 'modest' empiricism.

2 Helmholtz's 'Kantian' Empiricism

Helmholtz's philosophical reflections were guided by both a deep appreciation for the Kantian tradition and a firmly held commitment to scientific empiricism. Following Kant, Helmholtz held that an appeal to *a priori* 'laws of thought' was necessary to account for the problem of knowledge, as they constituted the very possibility of experience.[3] However, as an adherent to the principles of scientific empiricism, Helmholtz also sought to provide a naturalist account of all physical and mental phenomena. The attempt to resolve the apparent conflict between these two positions was at the centre of Helmholtz's epistemological reflections.

With respect to Helmholtz's commitment to a form of empiricism, Hatfield (1990, p. 11, 166-168) draws an important distinction between traditional empiricism (i.e. a 'strict' empiricism), which holds that all knowledge is grounded on experience (pace Locke and Hume), with what he terms as Helmholtz's "empirism", which holds that all knowledge is gained, or learned, through experience, but not necessarily grounded on it (given Helmholtz's commitment to a form of critical idealism). Helmholtz's unique blend of empiricism and critical idealism remains a topic of long-standing discussion in the history and philoso-

[3] Here, by the use of the term 'laws of thought', I intend to indicate Helmholtz's loose commitment to some aspects of critical idealism, whether that includes the classic Kantian system or variations in its historical development.

phy of science.[4] However, I will argue that it was Helmholtz's 'empirism' that provided the essential insight and motivation for Weyl's 'modest' empiricism.[5]

In this section, I will provide a brief account of some aspects of Helmholtz's thought on the problem of space and the nature of perception. Here, I will place a particular focus on Helmholtz's development of a form of 'Kantian' empiricism (or 'empirism'). Throughout, I will highlight the points that will become most important for Weyl. As such, the subsequent discussion will focus on two of Helmholtz's most influential epistemological essays—i.e. "On the Origin and Significance of the Axioms of Geometry" (1876) and the "The Facts in Perception" (1878).

The basic problem underlying Helmholtz's essay on the axioms of geometry (1876) is whether the axioms of Euclidean geometry serve as the *a priori* form of intuition and a necessary presupposition of scientific thought, as Kant suggested. Helmholtz's essential insight is that the problem of space is not separate from, or unrelated to, the problem of knowledge in general. Thus, the question of the origin of the axioms of geometry becomes part of the general question of the relation between our means and ways of knowing (i.e. between our thought and our experience of nature). In particular, Helmholtz argued that in tackling the problem of space, the problem of perceptual knowledge is of paramount importance. It is not a question solely of how we think about nature (i.e. through geometry or mechanics), but also of how we see, hear, and touch it.

Helmholtz sets out his discussion from what he takes to be the basic Kantian position: namely, that spatial geometry serves as "a form, given *a priori*, of all outer intuition" (1876 [1977], p. 1), but he notes that this does not mean that spatial geometry is a formal scheme into which content is fit, rather only the content that is constrained in a particular 'lawlike' way can become intuitable for us. In order to study the *a priori* nature of this constraint, Helmholtz considers what limitations can be imposed on the structure of space under the most general assumptions concerning our spatial form of intuition. These assumptions, according to Helmholtz, are grounded on the very possibility of experience (understood within the context of the nature of our perceptual faculties). They entail that there exist fixed physical relations, i.e. rigid bodies, and that space is both homogeneous and isotropic. From these assumptions, Helmholtz derives the set of geometrical structures that can serve to underwrite any possible experience. It is here that Helmholtz points out that Euclidean geometry is not the only spatial structure that is consistent with these assumptions—in fact, any spatial structure with a constant curvature will do. The important conclusion being that Euclidean geometry is not a necessity of thought, at least in Helmholtz's sense. All that remains is to show that such spaces are imaginable, as a possible experience, as, for Helmholtz, this would "refute the claim that the axioms of geometry are in Kant's sense necessary consequences of a

[4]For example, see Turner 1977; Hatfield 1990; Cahan 1993; Schiemann, 1997; Heidelberger 1998; Lenoir, 2006; Patton 2009, 2018; De Kock 2014; and Biagioli, 2016.

[5]In fact, it is Weyl's unique account of how knowledge is gained, or learned, through experience that is the characteristic feature of his 'modest' empiricism.

transcendental form, given *a prior*, of our intuitions" (1876 [1977], p. 18).[6]

To demonstrate that non-Euclidean spaces are imaginable, Helmholtz presents his famous 'mirror world' thought experiment. Here, one imagines beings (with similar faculties as our own), who are the inhabitants of a world with spherical or pseudo-spherical geometry—Helmholtz suggests that we can imagine such a world in which the happenings of our world are mapped, e.g. as in a concave or convex mirror. The simple point being, to quote Helmholtz (1876 [1977], p. 23), is "to show how one can deduce from the known laws of our sense perceptions [...] the series of sense perceptions which a spherical or pseudo-spherical world would give us if it existed." He continues: "In this respect too we nowhere meet an impossibility or deductive fault". Indeed, suitably translated, both worlds would accord with all known facts, and Helmholtz concludes: "For this reason, we also cannot admit [that] the axioms of our geometry are based upon the given form of our faculty of intuition, or are connected with such a form in any way".[7]

However, at this point, the problem appears to remain only partially solved, as a lot still depends on Helmholtz's unique account of the relation between imagination and intuition, and his account of the nature of our senses and the laws governing perception. For this reason, Helmholtz's essay on the axioms of geometry is naturally complemented by a reading of his essay on the nature of perception (1878 [1977]). After some preamble, Helmholtz begins this essay with what he takes to be the fundamental question. Citing Fichte, he asks: "What is true in our intuition and thought?" and "In what sense do our representations correspond to actuality?" (1878 [1977], p. 117).

Drawing on his studies on the physiology of perception (e.g. 1867 [1925] and 1868 [1873]), Helmholtz sets out his basic position. He (1878 [1977], p. 121-122) notes that

> Inasmuch as the quality of our sensation gives us a report of what is peculiar to the external influence by which it is excited, it may count as a symbol of it, but not as an *image*. For from an image one requires some kind of likeness with the object of which it is an image—from a statue alikeness of form, from a drawing alikeness of perspective projection in the visual field, from a painting alikeness of colours as well. But a sign need not have any kind of similarity at all with what it is the sign of. The relation between the two of them is restricted to the fact that like objects exerting an influence under like circumstances evoke like signs, and that therefore unlike

[6] Here, it is important to note that Helmholtz interprets the notion of a possible experience along psychological lines, as the experience of a cognizing subject, and not along the typical Kantian line of a possible experience in general. I would like to thank an anonymous referee for highlighting this point.

[7] Of course, as Helmholtz notes, this all rests on a presumed mechanical account of the rigidity of bodies. It is only the union of geometry and mechanics that possesses any empirical significance, thus one could always hold fast to the Kantian position with a suitable change in one's mechanics.

> signs always correspond to unlike influences.[8]

For Helmholtz (1878 [1977], p. 122), this is enough to form

> an image of lawfulness in the process of the actual world. Every law of nature asserts that upon preconditions alike in a certain respect, there always follow consequences which are alike in a certain other respect.

However, he notes that in the very act of perception, one presupposes a law-like connection between the symbols of sensation and the objects of which they are a sign.[9] In our representations of objects, we assume that the laws of connection, whereby the symbols given by sensations are related to constitute a definite object, correspond to the relations characterizing the object of our experience.[10] In this sense, the very act of perception involves an act of thought—one presupposes that the laws of thought, whereby symbols are related, correspond to the laws of connection characterizing the objects of which they are representation.

Furthermore, Helmholtz suggests that when we hold such laws of connection to have an existence, independent of our representations, we call them a cause (1878 [1977], p. 139), and this lawfulness becomes "the essential presupposition for the character of the actual" (1878 [1977], p. 140).[11] On Helmholtz's account, this means that we have no access to, nor can we represent, the thing-in-itself (any attempt at such a 'representation' is a contradiction of terms for Helmholtz). All we can attain "is an acquaintance with the lawlike order in the realm of the actual, admittedly only as portrayed in the sign system of our sense impressions" (1878 [1977], p. 141). This 'modest' view should be kept in

[8]In one of his earlier popular lectures, Helmholtz (1853 [1873], p. 54) presents the basic idea a little more succinctly. He suggests that "Perhaps the relations between our sense and the external world may be best enunciated as follows: our sensations are for us only *symbols* of the external world, and correspond to them only in some such way as written characters of articulate words to the things they denote. They give us, it is true, information respecting the properties of things without us, but no better information than we give a blind man about colour by verbal descriptions".

[9]There is a possible tension here between the descriptive or naturalistic aspects of the Helmholtzian epistemological program and the manner in which Helmholtz could account for the normativity introduced through a priori elements (e.g. see Hatfield, 1990, ch. 5). It remains an open question whether Weyl's later empiricism inherits this tension from Helmholtz, given his weaker commitment to a methodological 'naturalism'. I want to thank an anonymous referee for bringing this issue to my attention.

[10]Here, we must not read this correspondence too strongly, and keep in mind, as Helmholtz notes in his Physiological Optics (1867 [1925] p. 20), that an "idea and the thing conceived [...] belong to two entirely different worlds."

[11]In his physiological optics, Helmholtz (1867 [1925] p. 32) notes that we can never come to know of an external world "except by inferring from the changing sensation that external objects are the causes of this change." Thus, "the law of causation, by virtue of which we infer the cause from the effect, has to be considered also as being a law of our thinking which is prior to all experience." In this sense, certain 'laws of nature' derive not from nature herself, but from our urge to understand.

mind in any reading of Helmholtz's empiricism.[12]

In this context, Helmholtz (1878 [1977], p. 141) is able to precisely define progress in science. He holds that "every correctly formed hypothesis sets forth, as regards its factual sense, a more general law of the appearances than we have until now directly observed — it is an attempt to ascend to something more and more generally and inclusively lawlike." Thus, lawfulness becomes the condition of cognition and comprehensibility. The belief in the complete comprehensibility of nature serves as an ideal, it is expressed through the law of causality, and the law of causality becomes the truly "*a priori* given [...] transcendental law " (1878 [1977], p. 142).

With this in mind, we are now able to fill in a few more details concerning Helmholtz's work on the problem of space. Helmholtz takes something to be imaginable if it can be shown to be a possible object of experience. Thus, the 'laws of sense perception' correspond to the laws of thought that demarcate the domain of possible experience. In this context, we immediately see that the problem with Kant is that the domain of possible experience that he outlines is too constrictive—the laws of thought (and of sense perception) can now be seen to be far more general in light of Helmholtz's account of scientific cognition.

To sum up, Helmholtz (1878 [1977], p. 162-163) notes that

> Kant's doctrine of the *a priori* forms of intuition is a very fortunate and clear expression of the state of affairs; but these forms must be devoid of content and free to an extent sufficient for absorbing any content whatsoever that can enter the relevant form of perception. But the axioms of geometry limit the form of intuition of space in such a way that it can no longer absorb every thinkable content, if geometry is at all supposed to be applicable to the actual world. If we drop them, the doctrine of the transcendentality of the form of intuition of space is without any taint. Here, Kant was not critical enough in his critique [...].

On Helmholtz's account, the Kantian line between the *a priori* and *a posteriori* has shifted. The *a priori* is now limited to the general form of intuition, which constrains the class of allowable geometries to those of constant curvature. This shift in the Kantian line between the *a priori* and *a posteriori* marks the entrance for a new form of empiricism. The *a priori* is now taken to constitute the general form of intuition, which itself must be general enough to encompass all conceivable content. But it is the content, that determines which of the possible forms are taken to be applicable, or actual. Thus, the *a priori* outlines the field of possibilities, while we must learn from nature which forms can serve as the basis for a scientific account of reality. It is here that the axioms of Euclidean geometry win the day, but Helmholtz notes that this remains a contingent fact.

[12] In one of Helmholtz's popular lectures (1868 [1873], p. 274-275), he characterizes the empirical nature of his theory, which he terms the 'Empirical Theory', as follows: "The Empirical Theory regards the local signs (whatever they really may be) as signs, the significations of which must be learnt, in order to arrive at a knowledge of the external world." This is characteristics of what Hatfield (1990, p. 11, 166-168) terms as Helmholtz's 'empirism'.

3 Weyl on the Problem of Space, Helmholtz, and Related Themes

In the early 1920s, Weyl famously pursued a deep and influential study of the problem of space (e.g. 1921, 1922, 1923a, 1923b) building on the tradition of Helmholtz, Klein, and Lie. In the late 1920s, he presented an expanded philosophical reflection on this study, and its implications for Helmholtz's thought on the relation between thought and reality, more broadly. In this section, I will provide a brief summary of some of the ideas behind Weyl's work on the problem of space, before turning to his later reflections on Helmholtz, the nature of perception, and the foundations of scientific knowledge.[13]

Weyl's interest in the problem of space developed out of prior work on the foundations of differential geometry (e.g. 1918b) and the geometrical structure of Einstein's theory of general relativity (1918a). In 1918a, Weyl sought to develop what he termed a 'purely local' version of Einstein's theory, which no longer assumed that length relations could be compared at distant points—i.e. a theory with no global standard of length or 'gauge'. Within this generalized geometrical structure, Weyl was able to provide a surprising formal unification of the theories of gravity and electromagnetism. This important result, led Weyl to consider, in even more general terms, the basic constraints on the allowable form of spacetime within any possible physical theory. In addition, in 1919, Weyl published a new edition of Riemann's habilitation lecture, together with his commentary, and through this commentary, Weyl pursued an even more general examination of the class of allowable geometries that could serve as a viable foundation for any conceivable physical geometry.

By the beginning of the 1920s, Weyl had already started to somewhat temper his hopes for a 'geometrical' unification of electromagnetism and gravitation, along the lines of his early gauge theory (1918a), but he still maintained that the generalized geometry at the basis of the theory contained an essential insight (Scholz, 2004, p. 174). It is this insight that Weyl sought to spell out through his work on the problem of space. Weyl's approach to the problem was firmly within what one might call the Helmholtzian tradition, though suitably re-interpreted in light of its group-theoretic refinement through the work of Sophus Lie (e.g. 1886/87, 1890a, 1890b, 1893).

Helmholtz had sought to determine the most general geometrical structures that would allow for the free mobility of a rigid body, which he took to be a principle grounded on the possibility of experience. However, Helmholtz simply assumed that the requirement of free mobility would delimit not only the allowable global structure of space, but its infinitesimal structure as well. But as Lie (1886/87, 1890a, 1890b) pointed out, this assumption was not entirely justified in Helmholtz's construction (e.g. see Scholz, 2016, p. 4; Biagioli, 2016, p. 159; Bernard, 2018, p. 48-50). As a result, Lie reformulated Helmholtz's requirement of free mobility in a modern group theoretic perspective—i.e. as a

[13] For a more detailed discussion of Weyl's work on the problem of space, and its historical context, see the edited volume on the subject by Bernard, et al., (2019).

group of transformations that preserve congruence—and simply stipulated that this group structure holds at the infinitesimal scale. The problem, at least from a philosophical perspective (e.g. see Bernard, 2018, p. 50-52), is that this stipulation seemed to undermine the motivation for Helmholtz's empiricist program in the context of the problem of space, as it was no longer based directly on the 'forms' underwriting the very possibility of experience.

Setting the problem of empirical support aside for the time being, Weyl's aim was to readdress the problem of space from the perspective of modern group theory (à la Lie) in light of recent developments in mathematics and physics. In the process, as Scholz (2019) notes, Weyl initially stripped Helmholtz's analysis of his intention to ground the choice of geometrical axioms directly on the 'facts of experience', and sought to readdress the question of the homogeneity of space (or rather now spacetime) in a manner that was appropriate in the context of general relativity, and its subsequent generalization in Weyl's gauge theory. But in the process, Weyl still pursued the problem from what one might call a broadly 'Kantian' perspective (e.g. see Bernard, et al., 2019, vi-x).

In an attempt to develop the 'Kantian' aspects of Helmholtz's work on the problem of space, Weyl sought to identify the *a priori* constraints on the 'essence of space' (e.g. see Scholz, 2004, p. 178-179). For Weyl, these constraints were no longer grounded on the possibility of experience, but on a formal study of the nature of congruence and similarity in the most general characterization of a geometrical structure. In this context, as Scholz (2019, p. 216-217) notes, Weyl "wanted to dig deeper and motivate, or even derive, a generalized metrical structure from congruence and similarity concepts". To accomplish this, Weyl looked to define an abstract group structure that could be taken to characterize these concepts.

In the fourth edition of *Space Time Matter* (1921 [1952]), Weyl provides a short outline of his initial plan (i.e. circa 1920-1921) for a group-theoretic solution to the problem of space. Here, Weyl looks to define a local metrical structure through a group of congruent transformations. He suggests that the "metrical constitution of the manifold at a point is known if, among the linear transformations of [a] vector body (i.e. the totality of vectors [at a point]), those are known that are congruent transformations of themselves" (1921 [1952], p. 138).[14] Weyl terms these infinitesimal congruent transformations "rotations" of the vector body, and notes that "since a rotation is "not to alter" the vector body it must obviously be a transformation that leaves the infinitesimal elements of volume unaffected" (1921 [1952], p. 139).[15] Through this terminology, Weyl clearly wants to suggest that the group of congruent point transformations can be taken to serve as an abstract generalization of the rotations of a body at a point in classical geometry (see Scholz, 2004, p. 176).

With a point congruence relation in hand, Weyl turns to the characterization

[14]He continues that "there are just as many different kinds of measure-determinations as there are essentially different groups of linear transformations".

[15]In the context of what Weyl terms a Pythagorean metrical space (i.e. a 'locally Euclidean' space), he notes that the "rotations" would correspond to the point transformations under which the Pythagorean-Euclidean metric is invariant.

of the "metrical relationship" between two separated points in an infinitesimal neighbourhood. He suggests that such a relationship can be defined though a notion of infinitesimal "congruent transference" (1921 [1952], p. 140), which determines the relationship between the congruence groups at each point in an infinitesimal neighbourhood. Weyl held that any viable notion of congruent transference must require that the congruent relations of a vector body at a given point are preserved when the vector body is transferred to a point in an infinitesimal neighbourhood—though the specific congruence group at each point may differ. Thus, any transference may be labelled as a congruent transference so long as the relevant congruence relations are maintained.[16] In this case, Weyl notes the infinitesimal congruence group at every point can be said to be of the same "kind", differing only in terms of what Weyl labels as their "orientation"—an abstract generalization of the congruence relations between bodies at different points in classical geometry (see Scholz, 2004, p. 176).[17] And while the congruence group, or group of rotations, at each point may differ, they are "similar" in that they define similar congruence relations; "thus there is a homogeneity in this respect" (1921 [1952], p. 140).

Yet, up to this point, Weyl had only presented a highly abstract characterization of the notions of similarity and congruence, and their role in the delineation of a geometrical structure. Weyl had yet to clarify the sense in which these general notions can be applied to delimit the form, or essence, of "real" space. To do this, Weyl (1921 [1952], p. 141) sets out to determine the category of metrical spaces to which "real space belongs", at least "according to Pythagoras' and Riemann's ideas". He notes that the existence of a group of rotations, or point-congruences, at every point defines "a property that belongs to space as a form of phenomena; it characterizes the metrical nature of space." In contrast, the "metrical relationship" between neighbouring points "is not determined by the nature of space, nor by the mutual orientation of the groups of rotation at the various points of the manifold." Rather, on what Weyl takes as Riemann's view, the metrical relationship is determined by the distribution of the "material content" of space.[18] Thus, the "metrical relationship", must be general enough to encompass any conceivable disposition of material content. This is what Weyl takes as his first axiom for the characterization of "real" space—i.e. that the metrical relationship between neighbouring points must be as flexible as possible to adapt to any material distribution.[19]

However, one problem remained. Weyl's account of the structure of "real"

[16]More precisely, Weyl defined such a notion of a "congruent transference" through a linear connection with arbitrary numerical coefficients.

[17]For example, in Weyl's conformal geometry (1918a), different "orientations" might correspond to different choices of length scales, or gauges, at each point, and the congruent transference would define the length connection (in addition to the usual metric connection) between neighbouring points in any infinitesimal neighbourhood.

[18]Weyl often associates this essential insight with Riemann and not Einstein.

[19]This first axiom could be taken as an expression of Helmholtz's belief that the *a priori* forms must be "devoid of content and free to an extent sufficient for absorbing any content", however the relevant constraint no longer emerges from the form of perception, but from the form of a dynamical spacetime theory.

space is not yet sufficient to account for what he took to be the essential insight contained in Riemannian geometry, and by extension Einstein's theory of relativity. Weyl had come to the realization, through the work of Levi-Civita (1917) and through his own studies (e.g. 1918a, 1918b, 1919), that the essence of Riemannian geometry is contained in the fact that its metrical structure defines a unique notion of parallel transport (e.g. see Dewar and Eisenthal, 2020). To maintain this conceptual insight, Weyl required, as a second axiom for the characterization of "real" space, that his "metrical relationship" define a unique affine connection (1921 [1952] p. 142).[20]

To conclude, Weyl (1921 [1952] p. 146-147) suggested that these two axioms, together with the specified congruence and similarity structure, may be sufficient to single out the class of geometries with a non-degenerate quadratic form. In his subsequent work, Weyl (e.g. 1923a, 1923b) was able to formally show that these constraints do indeed pick out such a class of metrical structures—which he later termed as the class of geometries of Euclidean-Pythagorean form (i.e. the generalized class of pseudo-Riemannian geometries at the heart of Weyl's early gauge theory) (e.g. see Scholz, 2004, p. 183).[21] This result stood as the culmination of Weyl's efforts on the problem of space, and served to capture the fundamental insight contained within the development of the theory of relativity and its subsequent generalization in Weyl's conformal geometry (1918a). For Weyl, at least in the early 1920s, this 'Euclidean-Pythagorean' form served as the true Kantian *a priori* form of space (or rather spacetime), constituting the underlying geometrical structure for any conceivable spacetime theory.

In the late 1920s, Weyl presented a short reflection on his study of the problem of space in an book on the philosophy of mathematics and natural science (1927 [1949a]).[22] Here, he notes that his study has shown, just as Helmholtz had done earlier, that 'the '*a priori* field of possibilities' is far more general than previously thought. However, in contrast to Helmholtz, Weyl had shown that the allowable class of metrical structures is not given *a priori*, but rather their more general "Euclidean-Pythagorean" form (Weyl, 1927 [1949a], p. 134). It is now the material content that determines the local metrical structure of any given spacetime region, *a posteriori.* Thus, once again, the Kantian line between the *a priori* and *a posteriori* has shifted. And while Helmholtz argued that Kant was not critical enough in his critique, in Weyl's opinion, neither was Helmholtz—though due to no fault of his own.

At this point, it is important to reiterate that, for Weyl, the motivation for the axioms underwriting his solution to the problem of space did not derive from empirical 'facts' relating to the nature of perception and cognition (following

[20]This second axiom can be taken as a revised expression of Helmholtz's principle of congruent motion—i.e. free mobility of a rigid body.

[21]Some of the relevant details of this story can be found in Coleman and Korté, 2001; Scholz, 2004, 2016; Bernard, et al., 2019; and Dewar and Eisenthal, 2020.

[22]This text will actually serve a double purpose in this paper. It was originally published in 1927, and subsequently revised and translated into English in 1949. However, the revisions in the translated text were limited to only certain sections, and thus some parts (e.g. those under discussion here) were written in 1927, and others (e.g. those discussed later on in this section) in 1949. The result is a fascinating mixture of Weyl's early and later thought.

Helmholtz), but from purely theoretical considerations. They were based on Weyl's detailed study of infinitesimal geometry and the role that the affine connection plays in general relativity, as the "guiding field" characterizing the inertial structure of the world (Weyl, 1927 [1949a], p. 106). Thus, Weyl's characterization of "real" space emerged not from our direct contact with nature, but from our broader theoretical understanding of it.[23] And though Weyl took himself to be working within the Helmholtzian tradition, he never sought to identify anything like an empirical ground for his work.

Weyl's early thought was heavily influenced by a belief in a pre-established harmony between mathematics and 'nature' (i.e. phenomenal reality), a belief that was prevalent in the Göttingen school of mathematics in which he was reared (Sigurdsson, 1991). Weyl's early approach to the philosophy of science, and the problem of knowledge more generally, was centred on a study of the nature and justification of this apparent harmony. But by the mid-1920s, Weyl's belief that theoretical construction could be guided solely by considerations of mathematical harmony, or theoretical unity, had been shattered. With the advent of quantum theory, Weyl quickly realized that he needed to reorient his thought toward a greater emphasis on the empirical basis of scientific thought.[24]

However, even when Weyl abandoned his broadly idealist 'geometrical program' in the foundations of physics, he still sought to maintain some of the essential insights that he had gained from his studies on the problem of space, particularly those concerning the essential role that mathematics plays in scientific cognition. Yet, by the late 1920s, the question of the empirical support for Weyl's work on the problem of space had remained unanswered, and it is natural to wonder how Weyl's thought could be reinterpreted along the lines of his developing empiricism. Given that Weyl abandoned Helmholtz's appeal to 'empirical facts' to ground the form of space (or spacetime), on what empirical basis, if any, could he characterize 'real' space out of the more general field of possibilities? To answer this question, and to understand how Weyl sought to reformulate central aspects of his earlier thought along empiricist lines, we will have to take a journey through Weyl's later reflections on Helmholtz, the nature of thought and perception, and the empirical grounds for symbolic construction in theoretical physics.

In an essay entitled "Mind and Nature" (1934 [2009]), Weyl presents a detailed discussion of Helmholtz's thought on the nature of perception and the relation between thought and 'reality'. Weyl begins with a detailed study of the sensations of sight and hearing. Here, he mainly summarizes Helmholtz's work,

[23]Note that in the preceding characterization of the empirical ground of the Helmholtzian program the relevant 'facts' are defined as preconditions for the possibility of experience (and scientific measurement). Helmholtz presupposes that the laws of thought, whereby the symbols given through sensation are related, correspond to the laws of connection characterizing the objects of which they are representation. The 'facts' underwriting the appeal to the free mobility of rigid bodies are 'laws of sense perception', which correspond to the laws of thought that demarcate the domain of possible experience.

[24]This crisis was motivated, in part, by the apparent fundamental length scale imposed by the quantum of action, in contradiction to speculative mathematical considerations underwriting Weyl's early gauge theory.

but he also provides an interesting, and novel, account of the relation between incident light and the sense of colour perception. Weyl appeals to his studies of projective geometry to suggest that, physiologically, our colour perception is limited to those features of incident light that are invariant under a projective mapping to the two-dimensional retinal plane. And this idea serves as the guiding metaphor for Weyl's subsequent account of the epistemology of science.

After a discussion of the physiology of the senses, and the physiological grounding of the psycho-physical relation, Weyl turns to a general discussion of Helmholtz and the relation between sensation and thought. In this discussion, Weyl appeals to Helmholtz's distinction between images and signs, and defends Helmholtz's account of the necessity of an assumed correspondence between the law-like ordering of signs and the law-like ordering in nature. Furthermore, he defends Helmholtz's empiricism (or empirism), whereby Weyl (1934 [2009], p. 94-95) notes that the signs given by sensation, are taken to stand initially without meaning, and that it is left to us to learn to "read", or better relate, these signs, according to our laws of thought, such that they can serve as a ground for action.

On the relation between signs and objects, Weyl suggests (1934 [2009], p. 95), drawing on his account of colour perception, that mathematics "has introduced the name isomorphic representation for the relation which according to Helmholtz exists between objects and their signs."[25] Thus, perceptual knowledge is limited to those features of the world that are invariant under a mapping to our perceptual faculties. He takes this to provide a clear and precise formulation of Helmholtz's view, specifically defining the relation whereby signs are related to their objects.

In a similar vein as Helmholtz, Weyl also sought to extend this account to characterize the structural relation that can be taken to hold between thought and reality, more generally—particularly in the case of scientific knowledge. Weyl suggests (1934 [2009], p. 95-96) that

> science can never determine its subject-matter except up to an isomorphic representation. The idea of isomorphism indicates the self-understood, insurmountable barrier of knowledge. It follows that toward the "nature" of its objects science maintains complete indifference.

Thus, on Weyl's view, scientific knowledge is limited to a group-theoretic description of certain properties in nature that are invariant under an isomorphism. However, Weyl will only partially clarify why this is the case, at least to the extent that scientific knowledge can be said to be grounded on the basic acts of sense perception (à la Helmholtz). It is only in a later essay on the relation between mathematics and nature, that he will address, at a more abstract level,

[25] Weyl then clarifies this thought through a discussion of the correspondence between the points of the projective plane and colour sensations. He highlights the sense in which these two domains are isomorphic to one another, and that this relation constitutes the fundamental limit of perceptual knowledge.

why this constraint (which merely derives only from a limitation on perceptual knowledge) is taken to apply to all forms of knowledge.

Towards the end of the essay, Weyl turns to the sceptical implications of this view concerning the form of knowledge that can be obtained through the senses. He begins with a brief summary of the history of scepticism concerning the veridicality of our senses (i.e. from the pre-Socratics onward). To Weyl, the most radical consideration is not the sceptical attack on secondary qualities, but Leibniz's attack on the primary qualities of shape and extension.[26] He notes (1934 [2009], p. 99) that Leibniz's view is given its classical expression in Kant, and that now "not even space and time may be attributed to the objective world", they "are instead intuitive forms of our consciousness".

Weyl then reflects on the implication of these thoughts as regards the construction of an objective picture of reality. Weyl (1934 [2009], p. 104) asks,

> How can this be accomplished, how can one get rid of space and time, if one is concerned with the objective world? At first glance, it seems quite impossible. But it can be done.

Weyl suggests that this is only possible through theoretical construction in mathematical physics. However, the 'reality' that is depicted in physics is only a symbolic construction, nothing more. But, Weyl notes (1934 [2009], p. 105) that the symbolic construction of nature is not arbitrary, it "is built up in several steps from what is immediately given; the transition from step to step is made necessary by the fact that the objects given at one step reveal themselves as manifestations of a higher reality, the reality of the next step."[27]

Weyl (1934 [2009], p. 109) summarizes his view as follows:

> Science proceeds realistically when it builds up an objective world in accordance with the demand which we previously expressed with Helmholtz that the objective configuration is to contain all the factors necessary for the subjective appearances: no diversity in experience that is not founded on a corresponding objective diversity. On the other hand, science concedes to idealism that this objective world is not given, but only propounded (like a problem to be solved) and that it can be constructed only by symbols. But the fundamental thought of idealism gains prevalence most explicitly by the [...] maxim: the objective picture of the world may not admit any diversity that cannot become manifest in some diversity of perception.

It is in this last maxim, that Weyl takes the structural limits of perception to constitute a limit on knowledge. However, it is important to note that Weyl

[26] Weyl's endorsement of Leibniz's thought is a common theme throughout his philosophical reflections.

[27] This echoes Helmholtz's views on the progress of science. However, in this context, Weyl does not cite Helmholtz directly, but his student Hertz, and his closely related views concerning the symbolic nature of scientific thought.

takes this notion of 'perception' to apply broadly.[28]

At this point, the sense in which Weyl initially sought to generalize the Helmholtzian program should be fairly clear. Both Helmholtz and Weyl looked to ground what Weyl would call the theoretical construction of reality (e.g. through a given set of axioms of geometry) on that which serves as a necessary and factual condition of the possibility of experience. Weyl holds that the 'facts of perception' must serve as the ultimate ground of scientific cognition, but he looks to re-construe these 'facts', and the notion of experience that they underwrite, in a more general sense. Furthermore, he precisely defines the formal manner in which perception can be related to 'reality', and how a scientific picture of 'reality' can be built up on this basis through the progress of science.

But still, we are left with a fundamental problem. In Weyl's work on the problem of space, he held that the 'essence of space' derives not from the direct facts of experience, but rather from the most general constraints on a geometrical structure in a dynamical spacetime theory. In his account of perception and cognition, he suggests that it is the nature of our perceptual faculties, very broadly construed, that constitute a fundamental limit on scientific knowledge. For Helmholtz, the two were coextensive, as the laws of sense perception, and laws of thought more generally, constituted the very possibility of experience, and set a constraint on the allowable form of spatial intuition in scientific cognition. This is what grounded Helmholtz's 'Kantian' empiricism.[29] However, for Weyl, there is a vast gulf between the group-theoretic constraints emerging from our forms of perception and the 'laws of connection' emerging from physical theory (e.g. a dynamical spacetime theory). How is this gulf to be overcome?

In order to answer this question, we must return to Weyl's (1949a) expanded and revised edition of his 1927 book on the philosophy of mathematics and natural science. This revision offered Weyl the opportunity to not only comment on his earlier philosophical thought, but to further elaborate on developments in his thought concerning the relation between mathematics and nature—particularly those which he had outlined in a recent lecture (1948). As Scholz (2018, p. 57) notes, Weyl's 1948 lecture can be read as a preliminary, but also a somewhat deeper, investigation of the central problem of the relation between mathematics and nature, as it appears in Weyl's later published writings (e.g. 1949a and 1952). In what follows, I will draw on the 1948 and 1949a texts somewhat interchangeably, as entire sections of the 1948 text are copied directly into the 1949a publication.

Weyl's later reflections on the problem of space, and the role of the *a priori* in

[28]This becomes evident in his discussion of the constraints that must be imposed on theoretical construction. The first is a demand for concordance, by which Weyl means a requirement of consistency (both an internal form of formal consistency and an external form of empirical consistency—i.e. consistent with experience, vaguely construed). The second is a demand for parsimony, by which he means that a theory should not have any superfluous parts.

[29]Recall that for Helmholtz, one must presuppose that the laws of thought, whereby symbols are related, correspond to the laws of connection characterizing the objects of which they are images—this is where Helmholtz appeals to an *a priori* principle of causality in physical theory.

delimiting scientific thought more generally, begin with a study of the problem of objectivity in both the mathematical and natural sciences. Following Felix Klein, Weyl (1948 [2017], p. 155), notes

> By whatever difficulties an epistemological analysis of objectivity is beset, we can say today in quite a definite manner what the adequate mathematical instrument is for the formulation of this idea. It is the notion of a group.

Within mathematics, Weyl held that the only relations that can be said to have any objective significance are those that are invariant under an automorphism group, as only once "the group is given [do] we know what like-ness or similarity means—namely two figures are similar (or alike, or equivalent) that arise from each other by a transformation of [the automorphism group]—and also under what condition a relation is objective, namely if it is invariant with respect to all transformations of [the automorphism group]" (1949a, p. 73-74). Thus, for Weyl, objectivity is a relational notion, as it is the choice of group that serves to define the very sense in which two structures, or sets of relations, can be compared.

In the context of modern mathematics, Weyl holds that any group can be taken to characterize an objective structure. In turn, the collection of all possible groups demarcates the collection of possible structures, up to an isomorphism. In his view, the same basic idea is true in physics, but here we are no longer free in the selection of a group structure, rather we are constrained by nature.

In the investigation of "real" space, for example, Weyl notes (1948 [2007], p. 156) that "neither the axioms nor the basic relations are given"—i.e. by nature. Rather, we select, at the outset, a set of basic relations that we hold to possess objective significance. But to do this, we must start with a group. The problem is that 'nature' does not wear her group structure on her sleeve. To even begin the process of understanding nature, we must come with a group structure in hand—as empty-handed, no science would be possible. Yet, there is no way to know at the outset which among the infinite possibilities is the 'right' choice. Thus, in science, Weyl notes (as in Dante's Divine Comedy), we must start in "mezzo del camin" (in the middle of the journey). He suggests that it is "the common fate of man and his science that we do not begin at the beginning; we find ourselves somewhere on a road the origin and end of which are shrouded in fog". But although we may find ourselves somewhat lost on a path, this does not mean that our journey is aimless. We must continually "question Nature to reveal to [us] her true group of automorphisms."

In this context, Weyl returns to the notion of congruence, as it pertains to the problem of space, and looks to draw a broader philosophical lesson from his earlier studies. In contrast to the notion of similarity, Weyl notes that congruence is a purely geometrical concept, which he takes to be based on our common-sense notion of a practically rigid body. He (1948 [2017], p. 158) notes that while the notion of congruence is, at first, relative to such a practically rigid body, its "factual independence of it is one of our most fundamental experiences." Classically, the congruent mappings of a space form a group, which Weyl terms the congruence group, or group of Euclidean motions. And classically, he suggests that the facts suggested an "interpretation according to which the congruence group [...] expressed an intrinsic structure of space itself; a structure stamped by space upon all the inhabitants of space."

Weyl then defines the formal sense in which a congruence group can be said to have objective significance. On his account, this can only be the case if the congruence group can be shown to be an invariant subgroup of the group of similarities, which Weyl defines as the group of automorphisms of Euclidean space (or what we would now call the symmetry group of Euclidean space). As an example, if, following Weyl (1948 [2017], p. 159-161; 1949a, p. 79-83), we associate the group Γ with the general group of similarities, or automorphisms, of Euclidean space, then we can show that Δ^+, the group of Euclidean motions (i.e. under translation and rotation), is an invariant subgroup of the group of similarities. Thus, congruent structures are necessarily similar (but not vice versa), and the group of congruent motions can be said to have objective significance in Euclidean space. This same relation holds in the case of the group of orthogonal transformations, Δ, and the group of parallel displacements.

From a mathematical perspective, a group (e.g. Δ) constitutes an objective relation provided that it is an invariant subgroup of the similarity transformations of a Euclidean space. However, from a physical perspective, e.g. the perspective of classical theory, Weyl (1948 [2017], p. 161; 1949a, p. 82) notes that a "far deeper aspect of the group Δ than that of describing the mobility of rigid bodies is revealed by its role as the group of *automorphisms of the physical world*." The idea is that from a mathematical perspective, the group of similarity transformations of a Euclidean space delimits the groups of transformations that can be considered to be invariant subgroups. Out of these subgroups, some have the added property of defining the group of "*automorphisms of the physical world*" in the sense that all the laws of nature are invariant under this group (1948 [2017], p. 160; 1949a, p. 83).

Thus, Weyl held that physical symmetries are a subset of a broader class of mathematical symmetries, which serve to define their objective significance. It is in this formal sense that Weyl held that mathematics, in particular group theory, serves a necessary presupposition of the possibility of experience, as one must have a mathematical structure in hand to define which physical structures posses objective significance. And from this perspective, the problem of space can only be solved by identifying the group structure that defines the broader mathematical conception of space, and then defining which invariant subgroups can be taken to have a physical significance. The aim is to identify the basic group of transformations under which the laws of nature, as they pertain to the motion of bodies, are invariant.

In classical physics, he (1948 [2017], p. 161) notes that

> Parts of space that arise from each other by a transformation of the group Δ [i.e. the group of orthogonal transformations] are *physically equivalent*. This is the way in which Helmholtz defines congruence. It is precisely the group Δ that plays this role.

In the context of relativistic physics, Weyl suggests that one would have to appeal to the generalization of the orthogonal group in spacetime.[30] But Weyl notes (1948 [2017], p. 161-162) that the development of general relativity has taught "us that the group of physical automorphisms is much larger than we had assumed so far." In general relativity, Weyl notes (1948 [2017], p. 162) that the congruence structure of special relativity has now been generalized to consist of "all transformations (satisfying certain continuity or differentiability conditions)." The essential point is that the generalization, or rather extension, of the mathematical symmetries of a physical theory would correspond to a potential extension of its physical symmetries—the extension would serve to ground a new 'field of *a priori* existing possibilities'.

Now, we can start to get a sense of the manner in which Weyl's later reflections on the problem of space, and the problem of knowledge more generally, lead to a profound generalization (or even reformulation) of the Helmholtzian program. Whereas Helmholtz sought to ground the constraints on the axioms of geometry on the 'facts' underwriting the possibility of experience, Weyl understood, better than almost anyone else, the problems that the advent of general relativity posed to such a program. In its stead, Weyl looked to ground his account of the 'essence of space' on a set of very general considerations concerning the structure of knowledge, but he also sought to connect these considerations back to our most basic forms of understanding. It is here that Weyl would argue that while the constraints on scientific knowledge are not directly derivable from the form of our perception, they are not entirely separate from, or prior to, experience, either.

If, following Helmholtz, we begin our study of the problem of space with our common-sense notion of a practically rigid body, Weyl would suggest that, in doing this, we are no longer taken to start at the beginning of knowledge, as Helmholtz argued. Our common-sense notions are simply based on a natural choice of underlying group structure—i.e. one that is grounded on the general form of our thought and motivated by the nature of our perceptual faculties and our initial assumptions concerning the law-like, and causal, ordering of nature. However, in the progress of science, this structure has been shown to be merely a local manifestation of a deeper, or more fundamental, structure. The Helmholtzian program is simply one step in the journey of thought, not a beginning but an initial foray, a key stage in our questioning of nature in the search for an extension of the group structure underlying our thought. The quest for knowledge requires a point of departure, but this point is not the fulcrum of an Archimedean level on which one can raise an objective reality. The search for such a point remains the problem of knowledge, it is the task of scientific enquiry, not its beginning.

The true *a priori* is not, as Helmholtz suggested, a principle of causality which defines the law-like ordering of the signs given in sense perception, but a more general structural orientation of thought toward reality based on a given

[30]Weyl had already accepted that quantum theory provides an absolute measure of length, which precludes the larger group of similarities from having direct physical significance.

group structure (a generalization of Helmholtz's law-like ordering) which characterizes the concept of objectivity.[31] This is true in both our understanding of nature and our own sense perception, which is the ground for its applicability. The specific laws of thought, i.e. the specific group structures that define our basic notions of similarity and congruence, which we must presuppose to even begin to process of understanding 'nature', are contingent. But Weyl would suggest that we must learn from nature which laws possess physical significance. To Weyl, the very process that sits at the foundation of the laws of thought, which underwrites our understanding of the forms of perception, is the very same process that underwrites modern physical theory. This process, whereby one builds up the latter from the former, is far more complex and convoluted, as it was for Helmholtz, but the basic lesson remains the same. The objective world is to be constructed from experience, "nicht gegeben, sondern aufgegeben", as the classic neo-Kantian dictum states, not given but to be propounded (Weyl, 1949a, p. 117).[32]

However, this reading of Weyl remains somewhat implicit in these later reflections (i.e. in Weyl 1948 and 1949a). Weyl will only make this strain of thought explicit in a later essay on the unity of knowledge (1954), where he presents a brief critique of Ernst Cassirer's neo-Kantian philosophy of science. Given Helmholtz's influence on both Weyl and Cassirer, it is natural to wonder whether Weyl's thought may be amenable to a more traditional neo-Kantian reading, à la Cassirer (1910 [1923], 1921 [1923], and 1936 [1956]). However, Weyl would explicitly deny such an association. Weyl's basic disagreement with the neo-Kantian position concerns the fundamental grounding of scientific thought. For Cassirer, this ground rests on the symbolic forms underlying scientific cognition, as he argues that it is on this basis that we fashion an objective reality. In Cassirer's thought, the symbolic forms serve as, what Weyl (1954) terms, "the luminous center" of our thought. Weyl seeks a different, and explicitly more empirical, ground. In understanding why, we can not only gain a better picture of Weyl's thought, but also his specific commitment to a form of Helmholtzian 'empirism'. To place this discussion in the appropriate context, I will first introduce some aspects of Cassirer thought on Helmholtz, group theory, and the progress of science.

[31] Here, Weyl would be in harmony with Poincaré, though this connection is not addressed directly by Weyl in this context.

[32] However, this distinction raises the question of whether Weyl's thought should be read along the lines of a 'modest' or Helmholtzian empiricism, or a more general neo-Kantianism (e.g. following the Marburg neo-Kantian tradition). This concern will be addressed in the following two sections.

4 A Brief Interlude: Cassirer's Neo-Kantian Philosophy of Science

Given the vast literature on Cassirer's philosophy of science, I will only touch on a few themes in his work that will help to better contextualize Weyl's criticism.[33] In particular, I will start with a brief discussion of Cassirer's (1938 [1944]) paper on group theory and the theory of perception. I will then turn to some of his earlier thought to expound upon a few of the themes that emerge in this later work.

In his group theory paper (1938 [1944]), Cassirer presents a reflection on Helmholtz's thought, but places a much greater emphasis on the constitutive role that group theory plays as an "organizing and unifying principle" in the foundations of science (1938 [1944], p. 1). Given the subject of the essay, Cassirer naturally begins with a discussion of Helmholtz and the problem of space. He suggests (1938 [1944], p. 2) that Helmholtz was

> Kantian in so far as he endorsed the thesis of space as a 'transcendental form of intuition', and he persistently clung to this thesis. But this thesis was to him the beginning, and not the solution, of the problem.

While the general form of space is given *a priori*, its specific structure (i.e. Euclidean or non-Euclidean) is not. This all depends on the axioms, which are determined by one's account of the manner in which figures can be displaced in space (a determination which itself requires certain presuppositions). Thus, for Helmholtz, "the axioms at the basis of every geometry may then be interpreted as statements concerning determinate groups of movements." The objective validity of these axioms depends not merely on the *a priori* 'form' of space, but upon one's account of experiments with 'rigid bodies'.

Following Klein, Cassirer (along with Weyl) holds that every system of geometry is characterized by its group. He notes that after Poincaré's pioneering work, the concept of a group becomes the true fundamental concept *a priori*. As for Poincaré, the concept of a group precedes and underwrites all experience. Poincaré (1902 [1905], p. 90) notes that "is imposed on us not as a form of sensibility, but as a form of understanding". In this context, Cassirer (1938 [1944], p. 4) points out that "all that experience can do is lead the mind in a certain direction, as a result of which it may construct such a system of geometrical concepts as yields the simplest and most convenient instrument for the description of physical phenomena." In our mind is the latent idea of a group, and experience merely guides us in selecting one such group for the construction of a physical geometry.

However, in contrast to Helmholtz (and Weyl), Cassirer (1938 [1944], p. 5) argues that experience is not "the source of concepts, but merely the occasional cause of their formation." He looks to show that the type of concept (i.e. that

[33] For a discussion of some of the current debates concerning Cassirer's philosophy of science, see Biagioli, 2016.

of a group and the invariant theory of objectivity it entails) to be a general form that extends far deeper and further than the domain of geometry. He suggests (1938 [1944], p. 19) that, “Metaphorically speaking, it extends down to the very roots of perception itself.”—i.e. the concept of group and the concept of invariance are necessary conditions of the constitution of both the perceptual world and that of geometrical thought.

To Cassirer (1938 [1944], p. 20), perception

> is a process of objectification, the characteristic nature and tendency of which finds expression in the formation of invariants. It is within this process that the distinction between “reality” and “appearances” emerges.

Cassirer suggests that the search for truth is the search for constancy. It is the process by which thought seeks out invariants to constitute the basis of our orientation towards an ‘objective existence’. He (1938 [1944], p. 21) then notes that this “function is as much a condition of perception of objective existence as it is a condition of objective knowledge.”

Cassirer holds (1938 [1944], p. 22) that these “reflections on the concept of *group* permit us to define more precisely what is involved in, and meant by, that “rule” which renders both geometrical and perceptual concepts universal. The rule may, in simple and exact terms, be defined as that *group of transformations* with regard to which variation of the particular image is considered.” On his view, one simply extends this basic idea to all forms of knowledge—this conception operates as “the constitutive principle of the construction of the mathematical universe”.

The difference in emphasis in Cassirer and Weyl should be apparent at this point. While Weyl continually sought to defend the empirical ground of the Helmholtzian tradition, Cassirer, following Poincaré places a much stronger emphasis on the constitutive role that group theory plays in our thought. Nature does not inform us of the appropriate group structure, it is rather us that dictates to nature the form that she must adopt, at least to be an object of our experience. To better understand this latter point, I will briefly present some of Cassirer’s earlier thought on the nature of scientific cognition and the development of general relativity.

In two famous texts, Substance and Function (1910 [1923]) and Einstein’s Theory of Relativity (1921 [1923]), Cassirer presents an insightful and influential neo-Kantian philosophy of science. In Substance and Function, he argues that concept formation in modern physics is a natural extension of the mode of concept formation in mathematics, where such reasoning gains its fullest clarity. He suggests that the essential character of all mathematical constructions, is that they gain their meaning by their connections within the system of relations defined by a mathematical formalism. Cassirer holds that the same basic idea is true for a physical theory. He (1910 [1923], p. 165) notes that the concepts of “mathematical physics have no other meaning and function than to serve as a complete intellectual survey of the relations of empirical being.” The basic idea being that we "inscribe the data of experience in our constructive schema, and thus gain a picture of physical reality; but this picture always remains a *plan*, not a *copy*" (1910 [1923], p. 186). Science aims at truth, but this truth concerns nothing other than the "unity and completeness in the systematic construction of experience" (1910 [1923], p. 187).[34]

On the neo-Kantian 'genetic' view of knowledge, scientific thought is taken to be both historically contingent and in a state of continuous development. Cassirer suggests that the systematic construction of experience is extended through the development of science. This extension does not take place according to some arbitrary caprice, but through a law of progress. As Cassirer (1910 [1923], p. 187) notes, this "law is the ultimate criterion of 'objectivity'." Thus, the progress of science is taken to be guided by fixed principles, and put simply the aim of critical philosophy (i.e. neo-Kantian philosophy) is to determine the principles that serve as the condition for the possibility of any conceivable physical theory (e.g. see 1910 [1923], p. 269). It is these principles that lie at the basis of the concept of connection according to natural law, which are appropriately termed *a priori*, in the sense that they constitute, or make possible, judgments concerning the facts of natural science. It is these connections that we term objective (e.g. see Cassirer, 1910 [1923], p. 273).

In addition, Cassirer suggests that our thought in this regard is strictly regulated, it is directed by "the idea of a 'fixed and permanent' realm of objectively necessary relations" (1910 [1923], p. 315). He holds that knowledge is constituted in a series of acts, a series that must be run through to gain an understanding of the rules for its progress. But the key point for Cassirer is that to grasp the sense in which science, as a whole, concerns an objective reality, "we must conceive the series as converging toward an ideal limit." In fact, he suggests (1910 [1923], p. 321) that the "system and convergence of the series takes the place of an external standard of reality". Experience, as it is described by science, can only be taken to be objective in the sense that the principles underwriting the development of our constructive schema can be said to be part of the final theory of nature—i.e. the final theory to which our current theories are taken to eventually, though only ideally, converge. Thus, Cassirer would agree with Helmholtz, that we must presuppose that the laws of our thought correspond to the laws of nature. Otherwise, no objective experience would be possible. But in contrast to Helmholtz, the significance of the laws of thought is not something that we must learn through nature, rather it is only to the extent that nature conforms that she is knowable in the first place.

The problem remains to identify the systematic connection on which this convergence rests. In his text on the revolution in thought brought about by the theories of special and general relativity, Cassirer (1921 [1923], p. 365) notes, echoing Helmholtz, that thought "can only transcend an earlier construction by replacing it by a more general and more inclusive one". In this context, he holds that it "is the general form of natural law which we have to recognize as

[34]It is in this sense that Cassirer's argued above that the search for truth is a search for constancy, as such constancy is the basis for a unified picture of reality.

the real invariant and thus as the real logical framework of nature in general" (1921 [1923], p. 374).[35] On this account, in the transition from the theory of special relativity to the theory of general relativity, the same principle for the construction of the concepts of natural science is taken to hold, only in a more general form. Now, natural law is freed entirely from any connection to a preferred set of coordinate systems. In this transition, the pure formal concepts persist as relatively fixed despite the change of physical ideals.[36]

In fact, Cassirer holds that general relativity stands at the end of a methodological development that unifies all systematic principles into the "supreme postulate" of the invariance of all magnitudes and laws under arbitrary transformations of the frame of reference (1921 [1923], p. 404). This is an expression of the "true systematic form of nature and its laws" (1921 [1923], p. 407). However, he holds out that possibility that even the most remote constructions of pure mathematics may find such a general application within physical thought. Thought advances by means of its own determinations (i.e. through its conception of reality according to natural law). The history of physics has witnessed a profound shift to more abstract mathematical construction, and it is in these constructions that the physicists finds her reality.

5 Weyl's Modest Empiricism

In his essay on the unity of knowledge (1954 [2009]), Weyl offers a critique of Cassirer's neo-Kantianism, and develops a contrasting form of 'modest' empiricism. At the outset, Weyl makes it clear that he holds Cassirer in very high esteem. But in his critique, he notes that Cassirer's account of knowledge is susceptible to a charge of vacuity. One can always highlight a constitutive structure underwriting any form of thought, but Cassirer, in his later writings (particularly, 1944), fails to draw these forms together in anything like a unified picture of cognition. Weyl (1954 [2009], p. 195), in his usual poetic manner, suggests that all we are left with is a "suite of bourrées, sarabands, minuets, and gigues" (i.e. a series of dances) rather than variations on a single theme, and he asks, are we not left "with a promise unfulfilled".[37]

Weyl, in defending an earlier, and more traditional, form of critical idealism

[35]It is interesting to note that Cassirer seems to have based his formal reading of general relativity, at least as is indicated by his citations, on the first edition of Weyl's Space-Time-Matter, and their early thought on general relativity can appear very closely related as a result.

[36]Here our general notions of "space and time are distinguished as the ultimate, agreeing unities. They seem, in this sense, also, to constitute the real *a priori* for any physics and the presupposition of its possibility as a science" (1921 [1923], p. 394). However, in Cassirer's view, in the formulation of modern field theory, the distinction between 'space' and 'matter' is inextricably blurred—"All dynamics tends more and more to be resolved into pure metrics". Here, Cassirer directly cites Weyl (1918a), to suggest that the theory of general relativity has finally fulfilled the dream of Descartes for a purely geometrical physics.

[37]Here, each dance is taken to characterize the constitutive structure of a given form of knowledge, but we are left without a unified theme tying the dances together to give a structural account of knowledge itself.

(e.g. in 1921), would have been much more sympathetic to the development of Kant's thought underlying Cassirer's neo-Kantian position. It is only after the shock of quantum theory, that Weyl came to believe that we must now take our lead more directly from nature herself. At this stage, the consonances and dissonances between Weyl and Cassirer should now be apparent. The key point of disagreement concerns Cassirer's claim that it is the form of our thought that imposes a strict constraint on the scientific account of nature. In Weyl's view the arrow goes the other around—it is not the abstract group-theoretic structure that constitutes the form of our objective reality, rather it is the most general group theoretic notions that are built up through our interactions with nature, by the very act of knowing and perceiving. In the progress of science they are refined and generalized, by way of a structural analogy, as we come to learn how to form a picture of reality in modern theoretical physics. It is in this sense that Weyl looks to defend central aspects of the Helmholtzian 'empirist' tradition.

However, this is not to say that Weyl himself is able to present anything like a unified picture of cognition, and he acknowledges such. Leaving aside the general problem of thought, he quickly turns to modern science, and presents a series of puzzles concerning our understanding of reality and consciousness. But despite these puzzles, he notes that there is a unity in scientific thought, not of content but of method—i.e. the method of symbolic construction.

In each science, Weyl holds that we construct a picture of reality, and these pictures must be in concordance with the assumed empirical facts in each domain, but that is all. We can no longer pretend that this picture corresponds to nature. Weyl suggests (1954 [2009], p. 199) that

> the words "in reality" must be put between quotation marks; who could seriously pretend that the symbolic construct is the true real world? Objective Being, reality, becomes elusive; and science no longer claims to erect a sublime, truly objective world above the Slough of Despond in which our daily life moves.

He notes that all we are left with is mathematical symbols, i.e. free creations of the human mind (Weyl, 1954 [2009], p. 202).

Of course, Weyl (1954 [2009], p. 199) holds that a theory of nature must be confronted with experience, but is does so

> as a whole, while the individual laws of which it consists, when taken in isolation, have no verifiable content. This discords with the traditional idea of truth, which looks at the relation between Being and Knowing from the side of Being, and may perhaps be formulated as follows: "A statement points to a fact, and it is true if the fact to which it points is so as it states." The truth of physical theory is of a different brand.

We are thus faced with the fundamental dilemma: "the objective Being which we hoped to construct as one big piece of cloth each time tears off; what is left in our hands are—rags."

Weyl continues (1954 [2009], p. 199), in his typical fashion,

> The notorious man-in-the-street with his common sense will undoubtedly feel a little dizzy when he sees what thus becomes of that reality which seems to surround him in such firm, reliable and unquestionable shape in his daily life. But we must point out to him that the constructions of physics are only a natural prolongation of operations his own mind performs (though mainly unconsciously) in perception, when, e.g., the solid shape of a body constitutes itself as the common source of its various perspective views. These views are conceived as appearances, for a subject with its continuum of possible positions, of an entity on the next higher level of objectivity: the three-dimensional body. Carry on this "constitutive" process in which one rises from level to level, and one will land at the symbolic constructs of physics. Moreover, the whole edifice rests on a foundation which makes it binding for all reasonable thinking: of our complete experience it uses only that which is unmistakably aufweisbar.[38]

That which is 'aufweisbar' is that which is readily exhibited—i.e. the 'facts of experience' in a given domain. It is the part of experience that Weyl takes to ground scientific knowledge, it is the empirical support upon which theoretical construction rests.

Taking a step back, and putting everything together, Weyl (1954 [2009], p. 202) suggests that at the basis of all knowledge there lie a few acts. The first two are the acts of intuition and understanding. They constitute the mind's original attempt to grasp reality and understand it. These are the acts through which, in science, we identify the basic 'experimental facts', which we take to be given directly through experience (understood broadly), i.e. Weyl's aufweisbar. For instance, at a certain stage in the development of science, this would entail our basic common-sense notions concerning empirical reality. If, following Helmholtz, we take as an example our common-sense understanding of a practically rigid body, Weyl would suggest that these notions are merely based on a specific understanding of our perceptual faculties, mode of experience, and the law-like ordering of nature. This understanding is a point of departure, not the infallible core of knowledge, and is subject to revision through our interaction with 'reality'.[39]

The next act defines the field of possibilities, which Weyl (1954 [2009], p. 202) terms a "mathematical game" in which we build up the domain of possible structure. For Weyl, the domain of possible groups demarcates the domain of possible structures, up to an isomorphism. Thus, the generalization, or rather extension, of the group structure (or mathematical symmetries) of a physical

[38] The term aufweisbar is left untranslated in the original.

[39] This Helmholtzian 'empirist' reading of the basic facts underwriting scientific intuition and understanding marks a profound shift in Weyl thought away from his earlier phenomenological reading of such terms.

theory would correspond to a potential extension of its domain of application. The extension would serve to ground a new 'field of *a priori* existing possibilities'. Indeed, this is how one might read Weyl's famous dictum (1952) that all *a priori* statements have their origin in symmetry. Thus, in the progress of science, a given structure may be shown to be merely a local manifestation of something deeper, or more fundamental. In this act, one probes the field of possibilities in the search for an extension of the group structure underlying scientific thought.

The final act is that of construction. Here one builds, on the basis of the pre-established field of possibilities, a new physical theory, and thereby searches the field of possibilities for extensions that possess what Weyl would term as 'physical significance'. But this significance is not an ontological but rather an epistemological significance—i.e. one leading to a novel physical insight. In addition, it is important to note that this act of construction entails both the theoretical construction of a picture of reality and the facts of experience which it entails (i.e. the new experimental facts of the theory). It is only in the later step that one attempts to 'close the loop', as it were, and re-establish a new set of empirical facts, a new aufweisbar, in the context of a novel theory, to serve as a ground for the next stage in the development of science.

Weyl suggests (1954 [2009], p. 203) that in this account he feels that he is "closer to the unity of the luminous center than where Cassirer hoped to catch it: in the complex symbolic creations which this lumen built up in the history of mankind." In our attempt to comprehend 'reality', Weyl suggests that we must be guided by nature herself. Of course, we must come to 'reality' with our laws of thought in hand, otherwise she would remain incomprehensible. These laws are mutually constituted by the nature of our cognitive and perceptual faculties, but that is only a contingent fact. Through the progress of science, we learn the meaning of these structural forms and their application by way of experience. And it is here that the nature of Weyl's 'modest' empiricism and his generalization of the Helmholtzian program comes to the fore.

Through his account of the basic forms of symbolic construction that underwrite both the nature of our perceptual faculties and our understanding of objectivity in the natural sciences, Weyl was finally able to bridge the apparent gulf between the group-theoretic constraints emerging from our forms of perception and the 'laws of connection' emerging from physical theory. Weyl came to the realization that the two could be seen to be defined as different stages of development within the same general mathematical structure, thus the latter could be constituted by the former through the progress of scientific cognition. In this sense, it is the mathematical constitution of cognition that finally bridges the gap and serves to underwrite the apparent harmony between mathematics and nature through defining the field of both domains. The abstract mathematical constructions one finds in physics are seen to be no more detached from reality as our common-sense understanding of any object, such as the table in front of me. They are both constituted by the same basic series of acts.

In this sense, Weyl's 'modest' empiricism presents not only a philosophical development but also a profound reformulation of certain aspects of the

Helmholtzian account of scientific knowledge. Weyl held, along with Helmholtz, that in the initial acts of intuition and understanding, the mind must seek out invariants as the ground for the construction of a picture of reality. This group structure is mutually constituted by the nature of our faculties and the 'reality' which we inhabit. However, it is precisely in the subsequent acts through which we define the field of possibilities, and pursue abstract theoretical construction, that Weyl outlines the contours of a novel structural empiricism—one that looks to ground the flights of group theoretic prognostication, which one finds throughout modern physics, on the fundamental acts of human understanding. Indeed, when a modern theoretical physicist 'plays around with group theory to try to get physics out of it' (an expression of Howard Georgi's)[40], Weyl would suggest that physicists are only doing what we have always done in the basic acts of perception and cognition, just at a much more abstract, and explicit, level. This process remains as connected to 'reality' as our common-sense notions, as both are based on the same fundamental series of acts. The only difference is that the former is built up from the latter through the history of thought, but in terms of its connection to reality, this remains a difference of degree not of kind (the former presumably being closer to 'reality').

6 Conclusion

Weyl's later turn toward a 'modest' empiricism marked a significant shift in his philosophical thought. The hope is that this study of Weyl's later turn toward a form of empiricism may open up Weyl studies to a broader reading of his eclectic philosophy of science. But, once more, this later turn toward empiricism should not be read as a strict historical development of Helmholtz's thought. It was rather a looser philosophical development—one which picked up certain key aspects of Helmholtz's philosophy as the basis for future development, and looked to provide a novel reformulation of the fundamental intuition or insight underwriting the Helmholtzian philosophical tradition. Of course, this is not the only viable interpretation of Weyl's later thought, as there are many threads, but this reading serves to highlight a significant theme in Weyl's later thought, one which is worthy of further study.

Acknowledgement

I would like to thank Dennis Lehmkuhl, Erhard Scholz, Alexander Blum, and the members of the Lichtenberg Group for the History and Philosophy of Physics (University of Bonn) and the Final Theory Program Research Group at the Max Planck Institute for the History of Science for helpful comments. I also want to thank the Alexander von Humboldt Foundation for generously supporting this research.

[40]This expression is taken from a research seminar that Georgi gave at the University of Bonn, in 2021.

References

[1] Bernard, Julien. 2018. Riemann's and Helmholtz-Lie's problems of space from Weyl's relativistic perspective. Studies in History and Philosophy of Modern Physics, 61: 41-56.

[2] Bernard, Julien, and Carlos Lobo (eds.). 2019. Weyl and the Problem of Space: From Science to Philosophy. Springer.

[3] Biagioli, Francesca. 2016. Space, Number, and Geometry from Helmholtz to Cassirer. Springer.

[4] Cahan, David eds. 1993. Hermann von Helmholtz and the Foundations of Nineteenth-Century Science. University of California Press.

[5] Cassirer, Ernst. 1910 [1923]. Substance and Function. Dover.

[6] Cassirer, Ernst. 1921 [1923]. Einstein's Theory of Relativity. Dover.

[7] Cassirer, Ernst. 1938 [1944]. The concept of group and the theory of perception. Philosophy and Phenomenological Research, 5(1): 1-36.

[8] Cassirer, Ernst. 1944. An Essay on Man. Yale University Press.

[9] Coleman, Robert Allen and Korté, Herbert. (2001). Hermann Weyl: Mathematician, Physicist, Philosopher. In Scholz, E., editor, Hermann Weyl's Raum-Zeit-Materie and a General Introduction to His Scientific Work. p. 198–270. Springer.

[10] De Kock, Liesbet. 2014. Hermann von Helmholtz's empirico-transcendentalism reconsidered: construction and constitution in Helmholtz's psychology of the object. Science in Context, 27(4): 709–744.

[11] Dewar, Niel, and Eisenthal, Josh. 2020. A Raum with a View: Hermann Weyl and the Problem of Space. In: Beisbart, C., Sauer, T., Wüthrich, C. (eds) Thinking About Space and Time. Einstein Studies, vol 15. Birkhäuser.

[12] Hatfield, Gary, 1990, The Natural and the Normative: Theories of Spatial Perception from Kant to Helmholtz. MIT Press.

[13] Heidelberger, Michael. 1998, From Helmholtz's Philosophy of Science to Hertz's Picture-Theory. In Baird, Hughes, and Nordmann (eds.). 1998. Heinrich Hertz: Classical Physicist, Modern Philosopher, Boston Studies in the Philosophy of Science Vol. 198, Dordrecht.

[14] Helmholtz, Hermann. 1853 [1873]. On Goethe's Scientific Researches. Translated and Reprinted in Atkinson eds. 1873. Public Lectures on Scientific Subjects by H. Helmholtz. Appleton.

[15] Helmholtz, Hermann von, 1867 (1910) [1925]. Treatise on physiological optics, Bd. 3: The perceptions of vision. Optical Society of America.

[16] Helmholtz, Hermann. 1868 [1873]. The Recent Progress of the Theory of Vision. Translated and Reprinted in Atkinson, E., eds. 1873. Public Lectures on Scientific Subjects by H. Helmholtz. Appleton.

[17] Helmholtz, Hermann. 1876 [1977]. On the Origin and Significance of the Axioms of Geometry. Vorträge und Reden, 5th ed., II: 1–31. Reprinted in Hertz, P., and M. Schlick, eds. 1977. Herman von Helmholtz: Epistemological Writings. Springer.

[18] Helmholtz, Hermann. 1876 [1977]. The Facts in Perception. Vorträge und Reden, II: 215–247, 387–406. Reprinted in Hertz, P., and M. Schlick, eds. 1977. Herman von Helmholtz: Epistemological Writings. Springer.

[19] Lenoir, Timothy. 2006. Operationalizing Kant: Manifolds, Models, and Mathematics in Helmholtz's Theories of Perception. In Friedman and Nordmann (eds.) The Kantian Legacy in Nineteenth-Century Science. MIT Press.

[20] Levi-Civita, Tullio. 1917. Nozione di parallelismo in una variatà qualunque. Rend. del Circ. Math. di Palermo, 42.

[21] Lie, Sophus. 1886/87. Bemerkungen zu Helmholtzs Arbeit: Über die Tatsachen, die der Geometrie zu Grund liegen. Leipziger Berichte Supplement 21.02.1887:337–342. In Lie, Sophus. 1935. Gesammelte Abhandlungen, Band 2, Teil I. Hrsg. F Engel, P. Heegard. Teubner and Aschehoug.

[22] Lie, Sophus. 1890. Über die Grundlagen der Geometrie I. Leipz. Ber. Heft II, abgeliefert 5. 11. 1890, S. Vorgelegt in der Sitzung vom 7. 7.

[23] Lie, Sophus. 1890. Über die Grundlagen der Geometrie II. Leipz. Ber. Heft III, abgeliefert 26. 2. 1891, S. Vorgelegt in der Sitzung vom 20. 10.

[24] Lie, Sophus. 1893. Theorie der Transformationsgruppen. Dritter und letzter Abschnitt. Teubner.

[25] Patton, Lydia. 2009. Signs, Toy Models, and the A Priori: from Helmholtz to Wittgenstein. Studies in the History and Philosophy of Science, 40(3): 281–289.

[26] Patton, Lydia. 2018. Helmholtz's Physiological Psychology, in S. Lapointe (ed.) Philosophy of Mind in the Nineteenth Century. London: Routledge.

[27] Poincaré, Henri. 1902 [1905]. Science and Hypothesis. Walter Scott.

[28] Ryckman, Thomas. 2005. The Reign of Relativity: Philosophy in Physics, 1915-1925. Oxford University Press.

[29] Sieroka, Norman. 2007. Weyl's 'agens theory' of matter and the Zurich Fichte. Studies in History and Philosophy of Science Part A, 38(1): 84-107. Schiemann2009 Schiemann, Gregor. 2009. Hermann von Helmholtz's Mechanism: The Loss of Certainty. A Study on the Transition from Classical to Modern Philosophy of Nature, trans. Cynthia Klohr. Springer.

[30] Scholz, Erhard. 2004. Hermann Weyl's analysis of the "problem of space" and the origin of gauge structures. Science in Context, 17: 165–197.

[31] Scholz, Erhard. 2016. The problem of space in the light of relativity: The views of Hermann Weyl and Elie Cartan. In Bioesmat-Martagon, L., editor, Eléments d'une Biographie de l'espace Géométrique, pages 255–312. Presses Universitaires de Nancy. References in text to arXiv draft (https://arxiv.org/abs/1310.7334).

[32] Scholz, Erhard. 2018. Weyl's search for a difference between 'physical' and 'mathematical' automorphisms. Studies in History and Philosophy of Modern Physics, 61: 57-67.

[33] Scholz, Erhard. 2019. The Changing Faces of the Problem of Space in the Work of Hermann Weyl. In Bernard, Julien, and Carlos Lobo (eds.). 2019. Weyl and the Problem of Space. From Science to Philosophy. Springer. 213-230.

[34] Sigurdsson, Skuli. 1991. Hermann Weyl, mathematics and physics, 1900–1927. PhD Dissertation, Harvard University.

[35] Turner, Steven, 1977. Hermann von Helmholtz and the empiricist vision. Journal of the History of the Behavioural Sciences, 13: 48-68.

[36] Weyl, Hermann. 1918. Gravitation und Elektrizität. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin. 465–480. GA II, 29–42.

[37] Weyl, Hermann. 1918. Reine Infinitesimalgeometrie. Mathematische Zeitschrift, 2:384–411. GA II, 1–28.

[38] Weyl, Hermann, 1919. Riemann, Bernhard. Über die Hypothesen, welche der Geometrie zu Grunde liegen. Neu herausgegeben und eingeleitet von H. Weyl. Springer.

[39] Weyl, Hermann. 1921 [1952]. Space, Time, Matter. Dover.

[40] Weyl, Hermann. 1922. Das Raumproblem. Jahresbericht DMV 30:92ff., GA II, 212–228.

[41] Weyl, Hermann. 1923. Mathematische Analyse des Raumproblems. Vorlesungen gehalten in Barcelona und Madrid. Berlin: Springer. Reprint 1963, Darmstadt: Wissenschaftliche Buchgesellschaft.

[42] Weyl, Hermann. 1923. Raum, Zeit, Materie (5th edn). Springer.

[43] Weyl, Hermann. 1927. Philosophie der Mathematik und Naturwissenschaft. Oldenbourg.

[44] Weyl, Hermann. 1928 [1949]. The Theory of groups and quantum mechanics. Dover.

[45] Weyl, Hermann. 1929. Elektron und Gravitation. Zeitschrift für Physik, 56:330–352. GA III, 245–267.

[46] Weyl, Hermann. 1932 [2009]. The Open World: Three Lectures on the Metaphysical Implications of Science. Yale University Press. Reprinted in: Pesic, Peter. 2009. Mind and Nature: Selected Writings on Philosophy, Mathematics and Physics. Princeton University Press.

[47] Weyl, Hermann. 1934. Mind and Nature. University of Pennsylvania Press. Reprinted in: Pesic, Peter. 2009. Mind and Nature: Selected Writings on Philosophy, Mathematics and Physics. Princeton University Press.

[48] Weyl, Hermann. 1948. Similarity and congruence: A chapter in the epistemology of science. In Weyl, Hermann. 1955 [2017]. Symmetrie, 3rd edition. Springer.

[49] Weyl, Hermann. 1949a [2009]. Philosophy of mathematics and natural science. Princeton University Press.

[50] Weyl, Hermann. 1949b. Relativity Theory as a Stimulus in Mathematical Research. Proceedings of the American Philosophical Society, 93(7): 535-541.

[51] Weyl, Hermann. 1954. The unity of knowledge. In Weyl, H. 2009. Mind and Nature: Selected Writings on Philosophy, Mathematics, and Physics. Princeton University Press.