Do arbitrary constants exist? A logical objection

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Abstract

In classical first-order logic (FOL), let T be a theory with an unspecified (arbitrary) constant c, where the symbol c does not occur in any of the axioms of T. Let psi(x) be a formula in the language of T that does not contain the symbol c. In a well-known result due to Shoenfield (the "theorem on constants"), it is proven that if psi(c) is provable in T, then so is psi(x), where x is the only free variable in psi(x). In the proof of this result, Shoenfield starts with the hypothesis that P is a valid proof of psi(c) in T, and then replaces each occurrence of c in P by a variable to obtain a valid proof of psi(x) in T, the argument being that no axiom of T is violated by this replacement. In this paper, we demonstrate that the theorem on constants leads to a meta-inconsistency in FOL (i.e., a logical inconsistency in the metatheory of T in which Shoenfield's proof is executed), the root cause of which is the existence of arbitrary constants. In previous papers, the author has proposed a finitistic paraconsistent logic (NAFL) in which it is provable that arbitrary constants do not exist. The nonclassical reasons for this nonexistence are briefly examined and shown to be relevant to the above example.

1 Shoenfield's theorem on constants

Let T^0 be a theory in classical first-order logic (FOL). Let T be a theory obtained by adding a new constant symbol c to the language of T^0. Note that the axioms of T do not contain the symbol c, and c is an unspecified (arbitrary) constant. Let $\vdash_T$ denote provability in T and let $\psi(x)$ be any formula in the language of T^0 (i.e., a formula of T that does not contain the symbol c), where x is the only free variable in $\psi(x)$. From Shoenfield [1] (see his "Theorem on Constants"), the following result is provable:

$$(\vdash_T \psi(c)) \rightarrow \vdash_T \psi(x). \tag{1}$$

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Shoenfield's proof of (1) is reproduced below:

Proof. Start with $\vdash_T \psi(c)$ as a hypothesis, i.e., suppose we have a proof in T of $\psi(c)$. Choose a variable y not occurring in the proof or in ψ, and replace c throughout the proof by y. This does not affect the nonlogical axioms, which do not contain the symbol c. It is easy to see that every other axiom becomes an axiom of the same type, and each application of a rule becomes a new application of the same rule. It follows that we obtain a proof in T^0 (which is also a proof in T) of $\psi(y)$. Hence, by the substitution rule, $\vdash_T \psi(x)$ follows. □

2 The meta-inconsistency

Note that a metatheory ($T_\vdash$) in which the above proof of (1) can be executed is simply the theory T together with a notion of provability. If T is already capable of formalizing provability (which is possible in FOL), then $T_\vdash$ would be an FOL theory, but it is not necessary for us to make this assumption. For our purposes, it is suffices to treat provability as a metatheoretical notion with respect to FOL theories, such as, T. We assume that $T_\vdash$ is specified by adding appropriate axioms to T that define provability in theories.

Let T^1 be the extension of T obtained by adding the axiom $c = b$ to T, where b is a specified constant and the specification (construction) for b does not contain the symbol c. Here we assume that $T^0 \vdash \exists x\ x = b$. Let $T^1_\vdash$ be the corresponding metatheory of T^1. Let $\mathcal{Q}(\mathcal{P}(c), \psi(c), T)$ denote "$\mathcal{P}(c)$ is a proof of $\psi(c)$ in the theory T". Further, let y be a variable that does not occur in $\mathcal{P}(c)$ or in $\psi(x)$.

From the given proof of (1), we conclude:

$$T_\vdash \vdash (\mathcal{Q}(\mathcal{P}(c), \psi(c), T) \to \mathcal{Q}(\mathcal{P}(y), \psi(y), T)), \tag{2}$$

where $\mathcal{P}(y)$ is obtained by replacing each occurrence of c in $\mathcal{P}(c)$ by the variable y. Given that c is a constant symbol, T^1 and $T^1_\vdash$ are legitimate extensions of T and $T_\vdash$ respectively. Indeed, T^1 ($T^1_\vdash$) is consistent if and only if T ($T_\vdash$) is consistent. Hence we obtain from (2):

$$T^1_\vdash \vdash (\mathcal{Q}(\mathcal{P}(c), \psi(c), T) \to \mathcal{Q}(\mathcal{P}(y), \psi(y), T^1)), \tag{3}$$

But the conclusion of (3) does not follow from the premise in the theory $T^1_\vdash$, because the replacement of each occurrence of c in $\mathcal{P}(c)$ by the variable y contradicts the axiom $c = b$ of $T^1_\vdash$. This axiom only permits replacing c by b, and not by a variable. Hence we obtain:

$$T^1_\vdash \nvdash (\mathcal{Q}(\mathcal{P}(c), \psi(c), T) \to \mathcal{Q}(\mathcal{P}(y), \psi(y), T^1)). \tag{4}$$

Obviously, (3) and (4) are contradictory. Therefore $T^1_\vdash$ is inconsistent. But if $T_\vdash$ is consistent, then so is $T^1_\vdash$. It follows that $T_\vdash$ is inconsistent. Clearly, the source of this inconsistency is the fact that T admits the existence of arbitrary constants.

The problem here is that an axiom of $T^1_\vdash$ (namely, $c = b$) is not consistent with a proof executed in its subtheory $T_\vdash$, which makes $T^1_\vdash$ inconsistent. This does not happen when we consider the reverse implication in (2). From Shoenfield [1], we obtain

$$T_\vdash \vdash (\mathcal{Q}(\mathcal{P}(y), \psi(y), T) \to \mathcal{Q}(\mathcal{P}(c), \psi(c), T)), \tag{5}$$

where it is assumed that there are no bound occurrences of y in $\psi(y)$ and $\mathcal{P}(y)$. Here a free variable y is substituted by an arbitrary constant c, which is a perfectly legitimate operation. Note that the proof of (5) does not make essential use of the fact that c is arbitrary, i.e., this proof will work even when c is replaced by a specified constant b. Hence it follows from (5) that

$$T^1_\vdash \vdash (\mathcal{Q}(\mathcal{P}(y), \psi(y), T) \to \mathcal{Q}(\mathcal{P}(c), \psi(c), T^1)). \tag{6}$$

Eq. (6) is unproblematic and does not lead to an inconsistency. On the other hand, the proof of (2), wherein c is replaced by a variable, does make essential use of the arbitrariness of c, and this leads to the contradictory conclusions (3) and (4). Thus (5) and (6), unlike (2) and (3), support the notion that "c is arbitrary, but fixed".

3 Resolution in the logic NAFL

In previous papers, the author has proposed a finitistic paraconsistent logic NAFL (non-Aristotelian finitary logic) [2] which refutes the existence of arbitrary constants. Here we will briefly examine the reasons for this nonexistence. Metatheorem 2 of [2] is reproduced below as:

Metatheorem 1 (Pairwise consistency implies consistency)**.** *Suppose the axioms of an* NAFL *theory NT are pairwise consistent, in the sense that every pair of axioms constitutes a consistent* NAFL *theory. Then NT is consistent.*

In the language of an NAFL theory NT with equality, let c_1, c_2 and c_3 be constant symbols, and let NT include the following three axioms:

$$c_1 = c_2, \;\; c_2 = c_3, \;\; c_3 \neq c_1. \tag{7}$$

Clearly these three axioms are inconsistent, but pairwise consistent in the classical sense. This is so because classical logic permits c_1, c_2 and c_3 to be specified nonconstructively, *i.e.*, they are arbitrary constants, and hence at least some of these constants are allowed to take multiple values in order to preserve the pairwise consistency of (7). However, we know from Metatheorem 1 that an inconsistent set of axioms of an NAFL theory cannot be pairwise consistent. In other words, multiple values are not allowed for constant symbols within the same NAFL theory. It follows that consistent NAFL theories do not permit the (nonconstructive) existence of arbitrary constants, as stated in Corollary 2 (Sec. 2.3) of [2].

A more direct derivation of this result is given in Metatheorem 6 (Sec. 3.2.3) of [2], as reproduced below:

Metatheorem 2. *Consistent* NFOL *theories (where* NFOL *is the* NAFL *version of* FOL*) do not admit arbitrary constants. If NT is a consistent* NFOL *theory and if $NT \vdash \exists x\, x = c$, where c is a constant symbol, then $NT \vdash c = a$, where $a \in \mathbb{U}$ and $\mathbb{U}$ is the universal class whose existence is provable in NT.*

Proof. Suppose, to obtain a contradiction, $NT \vdash \exists x\, x = c$ and c is an arbitrary constant. Then NT admits classical models in which $c = a_1$ and $c = a_2$, where $a_1 \in \mathbb{U}$, $a_2 \in \mathbb{U}$ and $a_1 \neq a_2$. The main postulate of NAFL semantics (see Sec. 2 of [2]) implies that there must exist a nonclassical model of NT in which $(c = a_1 \wedge c = a_2)$ holds. But such a nonclassical model cannot exist, because $(c = a_1 \wedge c = a_2)$ violates the axiom of equality corresponding to substitution for functions. □

The point of the above proof is that NAFL semantics requires constants to be uniquely determined by NAFL theories, which leaves no room for arbitrary constants. The key difference between variables and constants is seen from Metatheorem 9 of [2] (in Sec. 3.2.3), reproduced below as:

Metatheorem 3. *Let x be a free variable in a formula of an* NFOL *theory NT. Let $\mathbb{U} = \{a_j\}_{j \in \mathbb{N}}$ be the universal class of objects whose existence is provable in NT, i.e., when universally quantified, x ranges over the values $a_j, j \in \mathbb{N}$ (here $\mathbb{N} = \{0, 1, 2, \ldots\}$). Then x (when free) must assume an infinite superposition of all possible values, i.e.,*

$$(x = a_0) \wedge (x = a_1) \wedge (x = a_2) \ldots .$$

Remark 1 (Variables versus constants)**.** *A variable is allowed to assume multiple values within a theory, unlike a constant, whose value is fixed. This is why in the proof of Metatheorem 3 [2], a nonclassical model $\mathcal{M}$ of NT is required to exist in which a variable assumes a superposition of all possible values. The axioms for equality prohibit the existence of such a nonclassical model for constants, as seen from the proof of Metatheorem 2. What Metatheorem 3 shows is that variables are infinitary objects in* NAFL*, and hence the appropriate codes for variables are infinite classes rather than Gödel numbers [2]. It follows that only quantified formulas are meaningful in* NFOL *theories.*

4 Concluding remarks

Shoenfield's theorem on constants (from which (1) follows) leads to a meta-inconsistency in FOL, which raises serious questions on the existence of arbitrary constants. This meta-inconsistency does not arise in the logic NAFL [2], which refutes the existence of arbitrary constants, as seen in Sec. 3. Hence (1) is not a valid theorem in NFOL (the NAFL version of FOL). The role of NAFL in the foundations of mathematics and physics [2] deserves to be considered seriously.

References

[1] J. R. Shoenfield, *Mathematical Logic*, (Addison-Wesley Publishing Company, 1967), Sec. 3.3, pp, 33–34.

[2] R. Srinivasan, "Logical foundations of physics. Resolution of classical and quantum paradoxes in the finitistic paraconsistent logic NAFL", Preprint ID 22860, PhilSci Archive.