Are Maxwell gravitation and Newton-Cartan theory theoretically equivalent?

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Abstract

A recent flurry of work has addressed the question whether Maxwell gravitation and Newton-Cartan theory are theoretically equivalent. This paper defends the view that there are plausible interpretations of Newton-Cartan theory on which the answer to the above question is 'yes'. Along the way, I seek to clarify what is at issue in this debate. In particular, I argue that whether Maxwell gravitation and Newton-Cartan theory are equivalent has nothing to do with counterfactuals about unactualised matter, *contra* the appearance of previous discussions in the literature. Nor does it have anything to do with spacetime and dynamical symmetries, *contra* recent claims by Jacobs (2023). Instead, it depends on some rather subtle questions concerning how facts about the geodesics of a connection acquire physical significance, and the distinction between dynamical and kinematic possibility.

1 Introduction

It is well known that Newtonian gravitation admits, in addition to its usual static and kinematic shift symmetries, a symmetry known as Trautman gauge symmetry, in which the connection and gravitational potential are altered. Moreover, it is often claimed that just as kinematic shift symmetry motivates the transition from Newtonian to Galilean spacetime, so does Trautman gauge symmetry motivate the transition to a geometrised formulation of Newtonian gravitation, known as Newton-Cartan theory (NCT).[1]

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1. See e.g. Stachel (2007), Knox (2014), and Read and Møller-Nielsen (2020).

Recently however, Saunders (2013) and Dewar (2018) have challenged this orthodoxy—arguing that Maxwellian spacetime is the appropriate setting which encapsulates the lessons of Trautman gauge symmetry. But whilst the relationship between NCT and Galilean gravitation has been widely discussed, aspects of the relationship between NCT and Maxwell gravitation (MG) remain unclear. In particular, there is little consensus on the extent to which MG has less structure than NCT, or whether the two should be viewed as competitors at all.[2] Moreover, such questions have important implications for wider debates about theoretical equivalence, theoretical underdetermination, and how symmetries bear on the interpretation of theories.

Here, I aim to address these issues. First, I review some details of MG and NCT, as well as some preliminary results concerning the relationship between them. I then turn to the interpretation of these results. In §3, I discuss the fact that the models of these two theories are not in one-to-one correspondence, and clarify how this relates to the issue of test particles and counterfactuals about unactualised matter. §4 aims to diffuse Jacobs' (2023) recent argument that MG and NCT have different spacetime and dynamical symmetry groups. Finally, in §5 and §6, I use the resources of category theory to discuss how this relates to the question of theoretical equivalence. §7 concludes.

2 MG and NCT

Let M be a differentiable four-manifold (assumed connected, Hausdorff, and paracompact). A temporal metric t_a on M is a smooth, closed, non-vanishing 1-form;[3] a spatial metric h^{ab} on M is a smooth, symmetric, rank-$(2,0)$ tensor field which admits, at each point in M, a set of four non-vanishing covectors $\overset{i}{\sigma}_a$, $i = 0, 1, 2, 3$, which form a basis for the cotangent space and satisfy $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ for $i = j = 1, 2, 3$ and 0 otherwise. A spatial and temporal metric are compatible iff $h^{an}t_n = 0$. A vector field σ^a is spacelike iff $t_n\sigma^n = 0$, and timelike otherwise. Given the structure defined here, t_a induces a foliation of M into spacelike hypersurfaces, and relative to any such hypersurface, h^{ab} induces a unique spatial derivative operator D such that $D_a h^{bc} = 0$.[4] h^{ab} is flat just in case for any

2. A recent selection of competing views on the subject: Saunders (2013), Knox (2014), Weatherall (2016b), Dewar (2018), Wallace (2020), and Jacobs (2023).

3. Here and throughout, abstract indices are written in Latin script; component indices are written in Greek script; and the Einstein summation convention is used. Round brackets denote symmetrisation, square brackets antisymmetrisation.

4. See Weatherall (2018, 37–38), Malament (2012, §4.1) for details.

such spacelike hypersurface, D commutes on spacelike vector fields i.e. $D_{[a}D_{b]}\sigma^c = 0$ for all spacelike vector fields σ^a. Finally, let ∇ be a connection on M. ∇ is compatible with the metrics just in case $\nabla_a t_b = 0$ and $\nabla_a h^{bc} = 0$.

The first theory of Newtonian gravitation we will consider is Galilean gravitation. This theory has kinematically possible models (KPMs) of the form $\langle M, t_a, h^{ab}, \nabla, T^{ab}, \phi\rangle$, where ∇ is a flat, compatible connection, T^{ab} is the mass-momentum tensor for the matter fields F, and ϕ is a scalar field (which represents the gravitational potential). $\langle M, t_a, h^{ab}, \nabla, T^{ab}, \phi\rangle$ is a dynamically possible model (DPM) of Galilean gravitation just in case

$$\nabla_n T^{na} = -\rho \nabla^a \phi \tag{1a}$$

$$\nabla_n \nabla^n \phi = 4\pi\rho \tag{1b}$$

where $\rho := T^{nm}t_n t_m$ is the scalar mass density field. In what follows, we will be interested in the following transformation one can make on models of Galilean gravitation, known as *Trautman gauge symmetry*:

$$\nabla \to (\nabla, t_b t_c \nabla^a \psi) \tag{2a}$$

$$\phi \to \phi + \psi \tag{2b}$$

where $\nabla^a \nabla^b \psi = 0$.[5] This is a symmetry of Galilean gravitation, in the sense that $\mathfrak{M}$ is a model of Galilean gravitation just in case all its Trautman gauge symmetry-related cousins are. Trautman gauge symmetry-related models agree on T^{ab}, so at least appear to be empirically indistinguishable.[6] One might therefore wonder if there are theories which collapse the distinction between Trautman gauge symmetry-related models of Galilean gravitation. As is well known, the answer to this question is 'yes', and there are in fact two such theories—NCT and MG.

I will begin by introducing NCT. KPMs of this theory have the form $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$, where ∇ is a compatible connection, and T^{ab} is the mass-momentum tensor for F.

5. For details, see Malament (2012, §4). The notation here follows Malament (2012, proposition 1.7.3): $\nabla' = (\nabla, C^a{}_{bc})$ iff for all smooth tensor fields $\alpha^{a_1\dots a_r}{}_{b_1\dots b_s}$ on M,

$$\begin{aligned}(\nabla'_n - \nabla_n)\alpha^{a_1\dots a_r}{}_{b_1\dots b_s} = \alpha^{a_1\dots a_r}{}_{mb_2\dots b_s}C^m{}_{nb_1} + \dots + \alpha^{a_1\dots a_r}{}_{b_1\dots b_{s-1}m}C^m{}_{nb_s} \\ - \alpha^{ma_2\dots a_r}{}_{b_1\dots b_s}C^{a_1}{}_{nm} - \dots - \alpha^{a_1\dots a_{r-1}m}{}_{b_1\dots b_s}C^{a_r}{}_{nm}.\end{aligned}$$

6. As such, Trautman gauge symmetry is at least an epistemic symmetry in Dasgupta's (2016) sense.

$\langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$ is a DPM of NCT just in case

$$\nabla_n T^{na} = 0 \tag{NCT1}$$

$$R_{ab} = 4\pi \rho t_a t_b \tag{NCT2}$$

$$R^{a\ c}_{\ b\ d} = R^{c\ a}_{\ d\ b} \tag{NCT3}$$

$$R^{ab}{}_{cd} = 0. \tag{NCT4}$$

MG requires some further groundwork. This theory is set on Maxwellian spacetime, which is supposed to be equipped with a standard of rotation, but *not* a standard of absolute acceleration. But whilst the metrics and connection are by now standard notions, the rotation standard is not, and stands in need of further comment. This was introduced by Weatherall (2018): if t_a, h^{ab} are compatible temporal and spatial metrics on M, a standard of rotation $\circlearrowright$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, antisymmetric rank-$(2,0)$ tensor fields $\circlearrowright^b \xi^a$ on M, such that

1. $\circlearrowright$ commutes with addition of smooth vector fields;
2. Given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowright^a (\alpha \xi^b) = \alpha \circlearrowright^a \xi^b + \xi^{[b} d^{a]} \alpha$;
3. $\circlearrowright$ commutes with index substitution;
4. Given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = 0$ then $\circlearrowright^a \xi^b$ is spacelike in both indices; and
5. Given any smooth spacelike vector field σ^a, $\circlearrowright^a \sigma^b = D^{[a} \sigma^{b]}$.

One can then define a Maxwellian spacetime as a structure $\langle M, t_a, h^{ab}, \circlearrowright \rangle$, where $\circlearrowright$ is compatible with t_a and h^{ab}.

Now fix a spacetime $\langle M, t_a, h^{ab} \rangle$, and let ∇ and $\circlearrowright$ be a connection and standard of rotation on M, both compatible with the metrics. Following March (2023), I will say that a standard of rotation and connection are *compatible* just in case they agree with one another in the following sense: for any vector field η^a on M, $\nabla^{[a} \eta^{b]} = \circlearrowright^a \eta^b$.[7] Likewise, a connection ∇ is compatible with a spacetime $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ just in case it is compatible

7. See Weatherall (2018) for details; the basic fact is that any connection determines a unique compatible standard of rotation, but a standard of rotation does not similarly determine a unique compatible connection.

with the metrics and $\circlearrowright$. Finally, a spacetime $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ is *rotationally flat* just in case h^{ab} is flat and there exists a unit timelike vector field ξ^a on M such that $\circlearrowright^a \xi^b = 0$ and $\pounds_\xi h^{ab} = 0$,[8] or equivalently, just in case some flat derivative operator is compatible with $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ (Weatherall 2018, proposition 1).[9]

We need to say something about the Newtonian mass-momentum tensor T^{ab}. In both Galilean gravitation and NCT, we used the connection to extract vector fields from T^{ab}. In MG, we will likewise want to extract vector fields from T^{ab}, but without a connection. To do this, we impose the 'Newtonian mass condition': whenever $T^{ab} \neq 0$, $T^{nm}t_n t_m > 0$. This captures the idea that the matter fields we are interested in are massive, in the sense that there can only be non-zero mass-momentum in spacetime regions where the mass density is strictly positive. Since T^{ab} is symmetric, the Newtonian mass condition guarantees that whenever $T^{ab} \neq 0$, we can uniquely decompose T^{ab} as $T^{ab} = \rho\xi^a\xi^b + \sigma^{ab}$, where $\xi^a = \rho^{-1}t_n T^{na}$ is a smooth unit timelike future-directed vector field (interpretable as the net four-velocity of F), and σ^{ab} is a smooth symmetric rank-$(2,0)$ tensor field which is spacelike in both indices (interpretable as the stress tensor for F).

We can now introduce MG. This theory has KPMs $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab} \rangle$, where $\circlearrowright$ is compatible with the metrics and T^{ab} is the mass-momentum tensor for the matter fields F. $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab} \rangle$ is a DPM of MG just in case

(i) $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ is rotationally flat; and

(ii) For all points $p \in M$ such that $\rho \neq 0$, the following equations hold at p:

$$\pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} = 0 \tag{MG1}$$

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n(\xi^m \Delta_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3}D_m(\rho^{-1}D_n\sigma^{nm}) \tag{MG2}$$

$$\pounds_\xi(\circlearrowright^c \xi^a) + 2(\circlearrowright^n \xi^{[c})\hat{h}_{nm}\pounds_\xi h^{a]m} + \circlearrowright^c(\rho^{-1}D_n\sigma^{na}) = 0, \tag{MG3}$$

where $\hat{h}_{ab}$ is the spatial metric relative to ξ^a,[10] the $\overset{i}{\lambda}{}^a$ are three orthonormal connecting fields for ξ^a, and Δ is the "restricted derivative operator" defined in Weatherall (2018).

8. Here and throughout, $\pounds$ denotes the Lie derivative.

9. My justification for this terminology comes from (Malament 2012, propositions 4.2.4, 4.3.1), see also equation (5), and is analogous to the usual notion of flatness for a connection. A standard of rotation induces a standard of parallel transport for spacelike vectors along arbitrary (spacelike or timelike) curves, and this standard of parallel transport is path-independent just in case the spacetime is rotationally flat.

10. That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = 0$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b\xi^a$.

This acts on arbitrary spacelike vector fields σ^a at a point p according to

$$\eta^n \Delta_n \sigma^a := \pounds_\eta \sigma^a + \sigma_n \circlearrowright^n \eta^a - \frac{1}{2}\sigma_n \pounds_\eta h^{an} \tag{5}$$

where η^a is a unit timelike vector at p (the Lie derivative is taken with respect to any extension of η^a off of p). It also has the property that $\eta^n \Delta_n \sigma^a = \eta^n \nabla_n \sigma^a$ for any derivative operator ∇ compatible with $\circlearrowright$ (Weatherall 2018, 37).[11]

The relationship between MG and NCT is summarised by the following two propositions (March 2023; Chen 2023):

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be a model of NCT. Then there exists a unique standard of rotation $\circlearrowright$ such that ∇ is compatible with $\circlearrowright$ and $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ is a model of MG.*

Proposition 2. *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of MG. Then there exists a derivative operator ∇ compatible with $\circlearrowright$ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of NCT. Moreover, this derivative operator is not unique. If ∇ is such a derivative operator, then $\nabla' = (\nabla, t_b t_c \sigma^a)$ also satisfies the above conditions, where σ^a is any spacelike, twist-free, and divergence-free vector field such that $\rho\sigma^a = 0$.*

Corollary 2.1 (Chen, 2023)**.** *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of MG such that $\rho \neq 0$ on some open set $O \subset M$. Then there exists a unique derivative operator ∇ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of NCT.*

This concludes my presentation of MG and NCT. With this in hand, I will now consider two features of the relationship between these theories which have become focal points in the literature on MG and NCT. The first has to do with the fact that the models of MG and NCT are not in one-to-one correspondence as *per* proposition 2; the second has to do with the spacetime and dynamical symmetry groups of the two theories.

11. For further details on this way of presenting MG, including the interpretation of the equations (MG) and its relation to Dewar's (2018) theory by the same name, see March (2023) and Chen (2023).

3 On geodesics, counterfactuals, and distinctions without differences

We begin with the fact that typically, a model of MG does not carry enough information to fix a unique model of NCT. Now, it is worth emphasising that corollary 2.1 substantially restricts the space of matter distributions for which this failure of uniqueness occurs. For example, Chen (2023, 9) claims that the models of MG and NCT are in one-to-one correspondence over "all but the vaccuum sector." This is technically correct, though I think that it does not quite do justice to the fact that there are models of NCT which do not fall under the scope of corollary 2.1 but which are standardly taken *not* to represent solutions of NCT in which no matter is present. The obvious case of this is point-particle matter distributions: if e.g. $T^{\mu\nu} = m\delta^3(\mathbf{x})\xi^\mu\xi^\nu$ in some Maxwellian coordinate system x^μ on $M \cong \mathbb{R}^4 \backslash \mathbb{R}^1$, it is straightforward to show that if ∇ is a Newton-Cartan connection for this matter distribution then so is $(\nabla, t_b t_c \nabla^a \phi)$, where $\phi = 2z^2 - x^2 - y^2$.[12] A similar construction works for matter distributions involving a pair of point-particles with vanishing angular momentum,[13] though it does not generalise much beyond this.[14]

So the scope of this failure of uniqueness is limited. Nevertheless it has been taken to show that NCT draws its distinctions finer than MG, and hence that the two theories are inequivalent. Saunders puts the concern as follows:

> What of possible worlds, and distinctions among them drawn in NCT, invisible to ours? Take possible worlds each with only a single, structureless particle. Depending on the connection, there will be infinitely-many distinct trajectories, infinitely-many distinct worlds of this kind. But in [Maxwellian terms], as in Barbour-Bertotti theory, there is only one such world—a trivial one, in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn. From the point of view of the latter theories, the fault lies

12. ϕ is harmonic, and $(-2x, -2y, 4z)\delta^3(\mathbf{x}) = 0$ in the sense of distributions.

13. We can always find a Maxwellian coordinate system in which such particles are confined to the line $x = y$, $z = 0$, so can take $\phi = (x^2 - y^2)z$.

14. The basic mathematical fact is that if ϕ is harmonic on some spacelike hypersurface S, then either $\nabla^a\phi = 0$ on S or the zeros of $\nabla^a\phi$ form a subspace of S with dimension at most one. This is sufficient to exclude almost all point-particle distributions with three or more particles, as well as (providing there are no missing points in our spacetime, other than perhaps the particle worldlines themselves) two-particle distributions with non-vanishing angular momentum.

> with introducing a non-trivial connection—curvature—without any source, unrelated to the matter distribution. At a deeper level, it is with introducing machinery—a standard of parallelism for time-like vectors, defined even for a single particle—that from the point of view of a relationalist conception of particle motions is unintelligible. (Saunders 2013, 46–47)

Let's count the steps here. There is the mathematical fact that there are matter distributions for which the correspondence between models of NCT and MG is many-to-one. These models of NCT disagree only on the value of the Newton-Cartan connection in empty spacetime regions. Then there is the claim that these disagreements about the Newton-Cartan connection are distinctions without differences. But just what are the distinctions in question? Since the geodesics of the Newton-Cartan connection encode facts about the trajectories of test particles, a number of authors have suggested that they have to do with counterfactuals about unactualised matter. Here is Dewar (2018):

> Consider a pair of such materially identical models M, M' of [NCT]. [...] On the one hand, M and M' agree with respect to all material structure: thus, the full collection of every piece of observational data regarding M is identical to that regarding M'. On the other, it is not straightforwardly the case that M and M' agree on the content of all possible observations. For although there is not (in fact) any matter in the empty regions, there could have been, and were such matter to have been introduced, the motions that it would have made would suffice to empirically discriminate between M and M' (or to rule them both out in favor of some third alternative). More generally, the distinction at issue is whether unactualized dispositions may properly be considered as empirically respectable properties. (265–266)

In a similar vein, Wallace (2020) attempts to diffuse Saunders' concern by arguing that:

> [Insofar] as these counterfactuals [about the behaviour of unactualised matter] are indeterminate (perhaps because a Humean view of laws [...] is assumed) so is the Newton-Cartan connection. (29)

However, this cannot be the whole story. To see this, it is helpful to lay out explicitly the kind of reasoning which Dewar and Wallace appear to be engaging with here:

1. Let M, M' be a pair of distinct (non-isomorphic) but materially identical models of NCT, which represent worlds W, W'.

2. M and M' disagree about the geodesics of the Newton-Cartan connection in empty spacetime regions.

3. The geodesics of the Newton-Cartan connection in empty spacetime regions of M, M' correspond to the trajectories of test particles in empty spacetime regions of W, W', by the geodesic principle.

4. Facts about test particle trajectories in empty spacetime regions of W, W' represent counterfactuals about the behaviour of unactualised matter in W, W'.

5. Counterfactuals about unactualised matter are physical facts.

C. Therefore, W and W' disagree about some physical fact, so $W \neq W'$. But this means that M and M' represent distinct possibilities, and therefore NCT draws its distinctions finer than MG.

So counterfactuals are invoked to bridge the gap between the mathematical facts 1-3 about materially identical models of NCT, and C, which is a claim about how these models represent possible worlds. The crucial premise here is 4, which takes us from a claim about test particle trajectories to a claim about the behaviour of unactualised matter. 4 certainly seems plausible, and is very much in line with physics practice.

But 4 should also give us pause. After all, if we introduce unactualised matter into some empty spacetime region, then this matter will also act as a source field for the equations (NCT). This will bring with it a new Newton-Cartan connection, which will then give determinate predictions for the behaviour of the matter in question. So if we wish to evaluate counterfactuals about the behaviour of unactualised matter, there is another obvious strategy—which is to modify T^{ab} to include the unactualised matter as well as the original background matter distribution, and then examine how this new mass-momentum tensor evolves under the dynamics of the theory.[15] Indeed, this is precisely

15. Note that the resulting model won't be a DPM of NCT (or MG), since modifying the mass-momentum tensor to insert matter into empty spacetime regions will violate (NCT1) (or (MG1)). The models I have in mind are those KPMs of NCT where (a) the equations (NCT) all hold until some time t, (b) the equations (NCT) all hold after t, but (c) we take $T^{ab} \to T^{ab} + T'^{ab}$ at t, where $\mathrm{supp}(T^{ab}) \cap \mathrm{supp}(T'^{ab}) = \varnothing$, so that (NCT1) is violated, and analogously for MG. I discuss this in more detail in what follows.

the strategy one must use to evaluate counterfactuals about unactualised matter in MG, since the theory lacks a connection.

Note that this strategy for evaluating counterfactuals about unactualised matter is strikingly similar to possible worlds analyses of counterfactuals familiar from metaphysics. Consider e.g. Lewis's (1973; 1973; 1979) account. According to Lewis, the counterfactual 'If it were the case that A, then it would be the case that C.' is true at some world W just in case some world where both A and C are true is more similar to W than any world where A is true but C is false. Similarity amongst worlds, for Lewis, is to be evaluated using the following criteria, in order of most to least importance (Lewis 1979):

- Avoid widespread, diverse violations of law.[16]
- Maximise the region of perfect match of particular fact.
- Minimise small, simple violations of law.

In practice, then, Lewis's prescription for evaluating counterfactuals about the behaviour of unactualised matter is as follows: take a world which is a perfect duplicate of W before some time t,[17] insert a small violation of law at t to introduce unactualised matter into the region of interest, and then evolve the laws forward.

But now compare this to the strategy outlined above. We take the model that we are using to represent some world of interest W. We discontinuously modify T^{ab} at some time t to insert unactualised matter into the region of interest—thereby violating at least (NCT1) (or (MG1), in the case of MG). And then we evolve the laws forward to examine how it behaves. If Lewis's account is adequate, it is this method—and not the use of test particles—which is the correct way to evaluate counterfactuals about unactualised matter.[18] In this case, 4 is false. Counterfactuals about the behaviour of unactualised

16. 'Law' here refers to the laws of the world W we are considering.

17. Lewis (1979) claims that his similarity ordering ensures that worlds which are perfect duplicates before time t but diverge thereafter will be more similar than worlds which differ before t but are perfect duplicates after t. Whilst this is controversial (see e.g. Elga (2001)), I am assuming that it works as intended.

18. Doesn't using test particles avoid the need to either violate the laws or change the matters of particular fact? Not if the matter we are interested is massive (and note that, given the Newtonian mass condition, this is the only matter we can meaningfully talk about in NCT or MG). If we use test particles to evaluate counterfactuals about the behaviour of some massive body, this amounts to neglecting its role as source matter in the equations (NCT). So in those worlds where (a) there is unactualised matter in the region of interest, and (b) the matter in question behaves as a test particle, the equations (NCT1) and (NCT2) are violated *at all times*. I take this to be a greater violation of law than is involved in modifying the mass-momentum tensor on only one spacelike hypersurface. Plausibly, using test particles

matter are not represented by test particle trajectories, but elsewhere in the theory (I have argued, among some subset of the KPMs—recall footnote 15).

This gives us a better handle on what is at issue in claims that NCT draws its distinctions finer than MG. To say that NCT draws distinctions without differences is to say that there are models of NCT which represent distinct possible worlds, but which correspond to the same model of MG. Proposition 2 takes us some way towards that—but it does not take us the whole way. After all, it might still be the case that materially identical models of NCT represent the same possible world. One way to motivate the idea that these models *do not* represent the same possible world is to appeal to the geodesic principle. This tells us that any pair of such models disagree as to the behaviour of test particles in empty spacetime reasons.

But in virtue of what do facts about test particles in empty spacetime regions count as physical facts? One answer would be to go via 4, and say that they represent counterfactuals about unactualised matter. If we accept 4, then we might be led to think that MG is unable to make sense of counterfactuals about unactualised matter (since the theory lacks a connection). We are also forced to look elsewhere for the source of disagreement about whether NCT draws its distinctions finer than MG. Hence, Wallace suggests that it has to do with 2, arguing that on Knox's or Brown's view, the Newton-Cartan connection is indeterminate in empty spacetime regions. Dewar, on the other hand, appears to accept 1-5, but goes on to suggest that the disagreement has to do with a close cousin of 5—whether counterfactuals about unactualised matter are *empirical* facts.

I have argued that this was a mistake: counterfactuals about unactualised matter are represented among the KPMs of the theory, and show up in precisely the same way in both NCT and MG. As such, the relevant question for whether NCT draws its distinctions finer than MG is not whether counterfactuals about unactualised matter are indeterminate in NCT or MG, *pace* Wallace. Nor is it whether unactualised dispositions constitute empirical content, *pace* Dewar. Rather, it is how (if at all) we are to justify the inference from 1-3 to C.

Now, one might wonder if we can do this without 4. We might, instead, try replacing 4 and 5 with

also involves changing some matters of particular fact: there is unactualised matter in the region of interest, but this fact is just not being explicitly represented in the theory's formalism.

4′. Facts about test particle trajectories in empty spacetime regions of W, W' are physical facts.

The business of arguing that NCT draws its distinctions finer than MG then comes down to finding some justification for 4′, without appealing to 4 and 5.

How else might we justify 4′? Here is a thought. Model the behaviour of a particle plus some background matter distribution, and consider what happens when we ignore the role which that particle plays as source matter. Facts about test particles count as physical facts in virtue of the fact that making this idealisation preserves some (or all) of the salient features that we get from an exact treatment—the approximate trajectory of the particle, perhaps.

The thought is tempting. We should see whether this way of thinking holds up to scrutiny. In NCT, the central result concerning the behaviour of test particles is Weatherall's (2011) Newtonian geodesic theorem (where I have modified his statement of the theorem slightly to match the terminology used here):

Proposition 3 (Weatherall, 2011). *Let $\langle M, t_a, h^{ab}\rangle$ be a non-relativistic spacetime, ∇ a compatible derivative operator on M and suppose that M is oriented and simply connected. Suppose also that $R^{ab}{}_{cd} = 0$. Let $\gamma : I \to M$ be a smooth curve. Suppose that given any open subset O of M containing $\gamma[I]$, there exists a smooth symmetric field T^{ab} on M such that:*

- *T^{ab} satisfies the Newtonian mass condition;*
- *$\nabla_n T^{na} = 0$;*
- *$\mathrm{supp}(T^{ab}) \subset O$; and*
- *There is at least one point in O at which $T^{ab} \neq 0$.*

Then γ is a timelike curve that can be reparametrised as a geodesic.

The interpretation of proposition 3 is as follows. Fix a Newton-Cartan spacetime which satisfies (NCT4). Then the only curves in that spacetime which are apposite to represent the worldlines of test particles, in the sense that they may be traversed by an arbitrarily small, non-interacting matter distribution, are timelike geodesics.

Note that proposition 3 is exactly the right sort of construction for modelling a body when we neglect its role as source matter. Ignoring the role of some matter T^{ab} as a source in the equations (NCT) amounts to neglecting T^{ab} in (NCT1) and (NCT2) when we fix the Newton-Cartan connection, and then allowing T^{ab} to evolve according to (NCT1) in the resulting spacetime. One could also interpret proposition 3 as saying that, if we consider a sequence of such matter distributions which become arbitrarily small about some curve, then that curve is a geodesic.

All this is well and good when the Newton-Cartan spacetime thus determined is unique. But when it is not unique, propositions 2 and 3 tell us that neglecting the particle's role as source matter in this way renders the theory viciously indeterministic. Meanwhile, a realistic treatment of the target system *does* give deterministic predictions for the behaviour of such particles. In these cases, the idealisation of bodies as test particles fails to preserve even such basic features as the existence of unique predictions for the motion of the body. So $4'$ is not obviously true of matter distributions which admit distinct (non-isomorphic) Newton-Cartan connections.

(There is one option which I have not discussed, but is worth mentioning. One way of fixing a unique Newton-Cartan connection is via a choice of boundary conditions. If we expect these to come endowed with a physical interpretation—perhaps because we are modelling a subsystem of a larger universe—then at least in practice, this might explain why it is sometimes appropriate to interpret models which differ only as to the Newton-Cartan connection as physically distinct. However, it is not then clear what we are supposed to say about the fact that models of NCT can also be used to represent complete physical histories. In view of these difficulties, I won't consider this option further.)

This concludes my discussion of the positive case for materially identical models of NCT representing distinct physical states of affairs. What can be said in favour of the opposite view—that these models represent the very same physical state of affairs?

For this, I will make use of a result due to March (2023, 24). March shows that the equations (NCT) are equivalent to the conjunction of the equations (MG), the rotational flatness condition, and (the geometrised version of) Newton's second law

$$\rho \xi^n \nabla_n \xi^a = -\nabla_n \sigma^{na}, \tag{NII}$$

with $\circlearrowright$ now interpreted as the unique standard of rotation compatible with ∇. This makes it apparent that only the standard of rotation, rather than the connection, is needed for the internal dynamics of the matter distribution. Moreover, the degrees of freedom of ∇ not fixed by $\circlearrowright$ now figure only in the equation (NII). This suggests that we should think (NII) as providing a (partial) fixing of these remaining degrees of freedom, rather than as a constraint on T^{ab} itself. Whenever $\rho \neq 0$ throughout some open region O, (NII) defines the connection uniquely, and moreover furnishes it with a physical interpretaion—as the unique connection relative to which fluid elements obey (NII).

But now consider what happens when there are no such regions. Given the Newtonian mass condition, (NII) now provides non-trivial constraints on the connection, if at all, on a set of measure zero. At those points where $\rho \neq 0$ it still makes sense to interpret the connection as the unique one (at those points) relative to which Newton's second law holds—but this will no longer be sufficient to specify ∇ throughout all spacetime. So we cannot give an analogous physical interpretation to ∇ in regions where $\rho = 0$.

This suggests a view on which materially identical models of NCT represent the same physical state of affairs. If the Newton-Cartan connection has its physical significance in virtue of (NII), is not clear that the irrotational degrees of freedom of ∇ represent anything at all in those regions where they are underdetermined by (NII). Under this interpretation, NCT might exhibit representational redundancy, but would draw its distinctions no finer than MG.

4 On spacetime and dynamical symmetries

I will now turn to the second area of focus in the literature on MG and NCT, due to Jacobs (2023), which has to do with spacetime and dynamical symmetries. Jacobs begins his analysis of MG and NCT by defining an 'active' version of the dynamic shift—analogous to the standard kinematic and static shifts—which produces a linear time-dependent acceleration of the matter content of the original solution. Since active dynamic shifts are a dynamical symmetry but not a spacetime symmetry of Galilean gravitation, the theory violates Earman's (1989, 46) 'adequacy conditions', which demand that there be a match between the spacetime and dynamical symmetries of a theory, in the following sense:

SP1: Any dynamical symmetry of T is a spacetime symmetry of T.

SP2: Any spacetime symmetry of T is a dynamical symmetry of T.

Jacobs then argues that, although both NCT and MG restore SP1, they do so in different ways. In moving to MG, we enlarge the spacetime symmetries from the Galilei to the Maxwell group. Meanwhile, in moving to NCT, we employ the opposite strategy—"[removing] the dynamical symmetries themselves from the theory" (Jacobs 2023, 12). For Jacobs, this means that MG and NCT are inequivalent, since they have different spacetime and dynamical symmetry groups.

Never mind whether theories with different spacetime and dynamical symmetry groups can be equivalent. Instead, I want to focus on Jacobs' technical claim viz. the spacetime and dynamical symmetries of NCT. I claim that this rests on a mistake. Earman (1989, 45) defines the spacetime symmetries of a theory as the automorphism group of its absolute objects, where the absolute objects "are supposed to be the same in each dynamically possible model."[19] In arguing that MG and NCT have different spacetime and dynamical symmetry groups, Jacobs (2023, proposition 3) assumes that the Newton-Cartan connection is an absolute object. But the Newton-Cartan connection is *not* an absolute object: it is dynamical, and depends on the matter distribution we are considering.[20]

What, then, are the absolute objects of NCT? The metrics are invariant across the entire space of DPMs. But the rotation standard associated with the Newton-Cartan connection is *also* invariant—up to isomorphism—across the entire space of DPMs.[21] Now we have a choice. The rotation standard does not appear explicitly in the models of NCT, so we could say that the absolute objects of NCT are just the metrics. But the reason that the rotation standard does not appear explicitly in the models of NCT is that it is *definable* from the Newton-Cartan connection. In other words: since we are always free to (harmlessly) rewrite models of NCT as $\langle M, t_a, h^{ab}, \circlearrowright, \nabla, T^{ab} \rangle$ (where $\circlearrowright$ is the unique standard of rotation associated with ∇), we should take the absolute objects of NCT to be the metrics and the standard of rotation.

Now, the reader may be concerned whether the above rewriting is harmless, and a full

19. Here, 'same' is in the sense that they are isomorphic, see Earman (1989, 45).

20. As is obvious from e.g. (NCT2). To put the point pithily, taking the Newton-Cartan connection to be an absolute object would mean taking there to be only one nomically possible mass density field according to the theory.

21. This is an immediate consequence of the fact that DPMs of NCT are rotationally flat, which follows from (NCT4) and (Malament 2012, proposition 4.2.4).

discussion of this point is beyond the scope of this paper. But in brief: it is absolutely standard to take a theory to be committed to structures which do not explicitly appear between the angle brackets of its models, when those structures are definable from other structures in the theory (the canonical example of this is the Levi-Civita connection of general relativity (GR), which is—I take it uncontroversially—part of the structure of the theory even though models of GR are standardly presented in the form $\langle M, g_{ab}, \phi \rangle$). And if we accept this, the worry that the rotation standard doesn't show up explicitly in the models of NCT seems to me uncompelling. That is: we shouldn't be concerned that we have chosen to present NCT in a way that makes the fact that the standard of rotation is an absolute object 'less visible' than it is in e.g. MG.[22]

This presents a serious problem for Jacobs' argument that MG and NCT are inequivalent, and likewise for his claim that the two theories represent different ways of restoring SP1. If the absolute objects of NCT are the metrics and rotation standard, then MG and NCT share the same spacetime and dynamical symmetry groups. That the spacetime symmetries of NCT are the Maxwell group is immediate. And if $h : M \to M$ is a diffeomorphism generated by an arbitrary Maxwell transformation, then the induced map $\langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle \to \langle M, t_a, h^{ab}, h^*\nabla, h^*T^{ab} \rangle$ preserves both solutionhood of the equations (NCT), and all the absolute objects.

Of course, this requires that we follow Earman in saying that it is preservation of the absolute objects which is relevant for the definition of spacetime symmetries. It also requires that we allow dynamical symmetry transformations to act on the connection—a piece of spacetime structure—as well as the matter distribution. I will merely point out that this is completely standard; it is precisely the notion of spacetime and dynamical symmetries implicit in the claim that the spacetime and dynamical symmetries of GR are the full diffeomorphism group.

Still, there is one part of Jacobs' analysis which carries over intact. Maxwell transformations of the mass-momentum tensor preserve solutionhood of the equations (MG). But they do not preserve solutionhood of the equations (NCT). In general, if $\langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$ is a solution of (NCT), then $\langle M, t_a, h^{ab}, \nabla, h^*T^{ab} \rangle$ will violate at least (NII), where $h : M \to M$ is a diffeomorphism generated by an arbitrary Maxwell transformation. *Prima facie*, this reveals an important difference between MG and NCT: once we move

22. Dewar (2016, 152) and Wallace (2020, 28) make similar points.

to consider the entire space of KPMs, there will be non-solutions of NCT which correspond to solutions of MG. I return to this issue in §6.

5 MG, NCT, and categorical equivalence

In §3 and §4, we considered previous discussions of MG and NCT, and isolated two questions which bear on whether MG and NCT are equivalent: whether materially identical models of NCT represent the same physical state of affairs, and whether the Newton-Cartan connection is an absolute object.

It turns out that these debates can be usefully represented in terms of categories of models associated to MG and NCT. In recent years, there has been a movement to represent the collection of models of the theory as a category, and analyse the relationships between theories using category-theoretic methods. One of the advantages of the theories-as-categories programme is that it is flexible enough to distinguish between different interpretations of the same formalism, insofar as these interpretations can be realised by different choices of arrows for the category of models of the theory. Moreover, the category-theoretic tools developed for analysing the relationships between theories then naturally take account of these interpretative differences when making comparisons such as whether two theories are equivalent, or whether one has more structure than another.

My approach will to be to associate a category to a theory by taking its objects to be the theory's models, and its arrows to be inter-model relationships which preserve physical content.[23] For our purposes, we can assume that there are two such relationships:

- Isomorphisms of the theory's models; and
- Physical content-preserving transformations which do not fall under the above.

For MG, this gives us the following category:

MG: Objects are models of MG. Arrows are diffeomorphisms which preserve the metrics, standard of rotation, and mass-momentum tensor.[24] Composition of arrows is composition of diffeomorphisms.

23. This is in the spirit of e.g. Weatherall (2016a), Barrett (2019), and Nguyen, Teh, and Wells (2020), though note that this is not completely standard. For example, when Barrett and Halvorson (2022) talk about theories as categories they have in mind specifically first-order theories, and categories whose objects are the theory's models and whose arrows are elementary embeddings.

24. If $\mathfrak{M} = \langle M, X_i \rangle$ for some collection of X_i on M and $\chi : M \to M'$ is a diffomorphism, $\chi_* \mathfrak{M} = \langle M', \chi_*(X_i) \rangle$. $\chi : M \to M'$ preserves some structure X on M, X' on M' iff $\chi_*(X) = X'$.

For NCT, however, there are *prima facie* (at least) two plausible options for the arrows of our category of models. On the one hand, we might say that the only relationships of physical equivalence between models are isomorphisms. On the other hand, I have argued that there is good reason, from the perspective of NCT, to say that gauge transformations of the Newton-Cartan connection $\nabla \to (\nabla, t_b t_c \sigma^a)$, where σ^a is spacelike, twist-free, divergence-free and satisfies $\rho\sigma^a = 0$, also relate physically equivalent models. This gives us the following two categories:

NCT$_1$: Objects are models of NCT. Arrows are diffeomorphisms which preserve the metrics, Newton-Cartan connection, and mass-momentum tensor. Composition of arrows is composition of diffeomorphisms.

NCT$_2$: Objects are models of NCT; arrows are maps $(\chi, \sigma^a) : \mathfrak{M} \to \chi_* \mathfrak{M}'$, $\mathfrak{M}' = \langle M, t_a, h^{ab}, (\nabla, t_b t_c \sigma^a), T^{ab} \rangle$, where σ^a is spacelike, twist-free, divergence-free, and satisfies $\rho\sigma^a = 0$, and $\chi : M \to M'$ is a diffeomorphism which preserves the metrics, (gauge-transformed) Newton-Cartan connection $(\nabla, t_b t_c \sigma^a)$, and mass-momentum tensor. The composition operation on arrows is given by $(\chi, \sigma^a) \circ (\psi, \tau^a) = (\chi \circ \psi, \psi^* \sigma^a + \tau^a)$.

I will now show that these categories allow one to capture, in a precise sense, how the debates about the interpretation of NCT in §3 and §4 lead to different judgements on whether MG and NCT are theoretically equivalent. For this, I will make use of the following criterion of theoretical equivalence, which has been brought to bear on a number of debates in recent years:[25]

Categorical equivalence: T_1 and T_2 are equivalent just in case there exists an equivalence functor between the categories of models of T_1 and T_2 which preserves empirical content.

An equivalence functor $F : \mathbf{T}_1 \to \mathbf{T}_2$ is one which is full, faithful, and essentially surjective.[26] Such a functor exists just in case there are functors $F : \mathbf{T}_1 \to \mathbf{T}_2$, $G : \mathbf{T}_2 \to \mathbf{T}_1$ such that FG is isomorphic to $\mathrm{id}_{\mathbf{T}_2}$, and likewise GF is isomorphic to $\mathrm{id}_{\mathbf{T}_1}$. As such, categorical equivalence appears to capture the idea that we can translate between T_1 and

25. See e.g. Rosenstock, Barrett, and Weatherall (2015), Weatherall (2016a), Barrett (2019), and Nguyen, Teh, and Wells (2020).

26. See, e.g. Mac Lane (1998) for details, and Weatherall (2017) for a philosophically-oriented presentation.

T_2, in a way that preserves empirical content, and that these translations are—up to isomorphism—inverses of each other.

We begin with $\mathbf{NCT}_1$:

Proposition 4. *Let* $F : \boldsymbol{NCT}_1 \to \boldsymbol{MG}$ *be the functor which takes each model of NCT to its corresponding Maxwell model, as given in proposition 1, and each arrow to an arrow generated by the same diffeomorphism. Then* F *is not an equivalence functor; it is not full.*

Proof. There are two ways to see this. First, consider the objects $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, 0\rangle$, $\mathfrak{M}' = \langle M, t_a, h^{ab}, (\nabla, t_b t_c \nabla^a \phi), 0\rangle$ in $\mathbf{NCT}_1$, where $\phi = e^x e^y \sin(\sqrt{2}z)$ in some Maxwellian coordinate system x^μ on M and ∇ is flat.[27] Now consider the arrow $\text{id} : F(\mathfrak{M}) \to F(\mathfrak{M}')$ in $\mathbf{MG}$. I claim that this is not the image of any arrow $\chi : \mathfrak{M} \to \mathfrak{M}'$ in $\mathbf{NCT}_1$. For this, note that ∇ transforms as $\nabla \to (\nabla, t_b t_c \sigma^a)$ under the action of any Maxwell transformation on ∇, where σ^a is a spacelike vector field which is twist-free and rigid ($\nabla^a \sigma^b = 0$). $\nabla^a \phi$ is not rigid, so there are no arrows $\mathfrak{M} \to \mathfrak{M}'$ in $\mathbf{NCT}_1$.

Alternatively, let $\mathfrak{M}$ be as above, and let $\chi : M \to M$ be a diffeomorphism generated by an arbitrary Maxwell transformation (which is not also a Galilean transformation). Consider the arrow $\text{id} : F(\mathfrak{M}) \to F(\chi_* \mathfrak{M})$ in $\mathbf{MG}$. Since $\chi_* \nabla \neq \nabla$, this arrow is not the image of any arrow $\mathfrak{M} \to \chi_* \mathfrak{M}$ in $\mathbf{NCT}_1$ under F.[28] □

Beginning with the first argument given in proposition 4, this tells us that F is not an equivalence functor because NCT admits distinct (non-isomorphic) models which correspond to the same model of MG. Since there are no arrows between these models in $\mathbf{NCT}_1$, this means that F is not full, and so in the terminology of Baez et al. (2006) forgets structure.[29] One might take this to capture Saunders' idea that MG and NCT are inequivalent because NCT draws its distinctions finer than MG, and so has 'surplus structure' over MG.

Proposition 4 also allows us to make sense of Jacobs' argument that MG and NCT are inequivalent because they have different spacetime and dynamical symmetries and hence "different structures" (Jacobs 2023, 13). For this, we need to look at the second argument given in proposition 4. What goes wrong here is not that the models $\mathfrak{M}$

27. I take this example from Dewar (2018, 265).
28. Many thanks to an anonymous referee for suggesting this to me.
29. For more on the connection between fullness and (amount of) structure, see Barrett (2022).

and $\chi_*\mathfrak{M}$ have no arrows between them in $\mathbf{NCT}_1$—indeed, they are isomorphic—but that the diffeomorphism χ is not an automorphism of the Newton-Cartan spacetime $\langle M, t_a, h^{ab}, \nabla\rangle$, whereas it is an automorphism of $\langle M, t_a, h^{ab}, \circlearrowright\rangle$. Again, this means that F is not full, and so forgets structure.

How does this relate to Jacobs' argument about spacetime and dynamical symmetries? Notice that the models we are considering here are vaccuum models. When $T^{ab} = 0$, all we are left with is a Maxwellian spacetime and a Newton-Cartan spacetime, and the problem arises because the two have different automorphism groups. If one then assumes, with Jacobs, that the Newton-Cartan connection is an absolute object, these automorphism groups correspond to the spacetime symmetry groups of MG and NCT, and the second argument given in proposition 4 can be understood as saying that it is precisely because of these different symmetry groups that the two theories are inequivalent.

Now on the one hand, given that the Newton-Cartan connection is not (in fact) an absolute object, this cannot be the correct way to cash out the significance of proposition 4 (one might, instead, cash it out in terms of the non-uniqueness we see in proposition 2). On the other hand, this seems to me a plausible way of making sense of Jacobs' argument, given that Jacobs also wants to say that isomorphic pairs of models of NCT (related by a Maxwell transformation) represent the same physical state of affairs (Jacobs 2023, 9-10). To reiterate, Jacobs' point is not to do with relationships of physical equivalence between distinct but isomorphic models of NCT, nor is it (primarily) to do with the fact that models of MG sometimes induce multiple non-isomorphic models of NCT. But it is very much to do with the fact that the automorphisms of Maxwellian and Newton-Cartan spacetimes do not coincide—in Jacobs' words, that "[Maxwellian] spacetime *by itself* [i.e. when the mass-momentum tensor vanishes] has no standard of acceleration" (13, emphasis in original)—which is precisely what is going on in the second part of proposition 4.

Moving on now to $\mathbf{NCT}_2$, this category *is* equivalent to $\mathbf{MG}$:

Proposition 5. *There exists an equivalence of categories between* $\boldsymbol{NCT}_2$ *and* $\boldsymbol{MG}$ *which preserves empirical content.*

Proof. Let $F : \mathbf{NCT}_2 \to \mathbf{MG}$ be the functor which takes each model of NCT to its corresponding Maxwell model, as given in proposition 1, and each arrow $(\chi, \sigma^a) \to \chi$. F preserves empirical content since it preserves T^{ab}, and by proposition 2 is essentially surjective. It remains to show that F is full and faithful. First, let $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$, $\mathfrak{M}' = \langle M', t'_a, h'^{ab}, \nabla', T'^{ab} \rangle$ be two objects in $\mathbf{NCT}_2$. Suppose that there exist distinct arrows (χ, σ^a), $(\chi', \sigma'^a) : \mathfrak{M} \to \mathfrak{M}'$, and suppose for contradiction that $\chi = \chi'$. Then $\sigma^a \neq \sigma'^a$, since the arrows were assumed distinct. But then $(\nabla, t_b t_c \sigma^a) \neq (\nabla, t_b t_c \sigma'^a)$, so that $\chi_*(\nabla, t_b t_c \sigma^a) \neq \chi'_*(\nabla, t_b t_c \sigma'^a)$. But by assumption (χ, σ^a), (χ', σ'^a) are both arrows from $\mathfrak{M}$ to $\mathfrak{M}'$, i.e. $\chi_*(\nabla, t_b t_c \sigma^a) = \chi'_*(\nabla, t_b t_c \sigma'^a)$, so by contradiction $\chi \neq \chi'$ and F is faithful. Finally, let $\chi : F(\mathfrak{M}) \to F(\mathfrak{M}')$ be an arrow in $\mathbf{MG}$. Since $\chi_* \circlearrowright = \circlearrowright'$, we know that $\chi_* \nabla$ and ∇' are rotationally equivalent, so that $\chi_* \nabla = (\nabla', t'_b t'_c \sigma'^a)$, where σ'^a is a spacelike vector field on M' (Weatherall 2018, proposition 1). It follows that $\nabla' = \chi_*(\nabla, -t_b t_c \chi^* \sigma'^a)$, where we have used the fact that χ preserves the metrics. Now consider the tuple $\langle M, t_a, h^{ab}, (\nabla, -t_b t_c \chi^* \sigma'^a), T^{ab} \rangle$. This is an object in $\mathbf{NCT}_2$, since it maps to $\mathfrak{M}'$ under χ. Moreover, it agrees with $\mathfrak{M}$ on the metrics and mass-momentum tensor. It follows that $\chi^* \sigma'^a$ is spacelike, twist-free, and satisfies $\rho \chi^* \sigma'^a = 0$ and $\nabla_n \chi^* \sigma'^n = 0$ (see the proof of proposition 2). So $(\chi, -\chi^* \sigma'^a) : \mathfrak{M} \to \mathfrak{M}'$ is an arrow in $\mathbf{NCT}_2$ which maps to χ under F. Hence F is full. □

6 Categorical equivalence and theoretical equivalence

We have seen how proposition 4 can be used to capture Jacobs' and Saunders' arguments that NCT—interpreted after $\mathbf{NCT}_1$—is inequivalent to MG. But the interpretation of NCT which I have advocated for here is $\mathbf{NCT}_2$, which *is* categorically equivalent to MG. *If* categorical equivalence is sufficient for theoretical equivalence—and a full discussion of that is beyond the scope of this paper—then I take proposition 5 to have established that there is a plausible and natural interpretation of NCT on which it is equivalent to MG. But for those unconvinced by the antecedent of this claim (myself included!), my final aim is to say something in support of the verdict which categorical equivalence gives us in proposition 5.

For this, I want to return to the discussion in §3, where we considered the fact that the equations (NCT) are equivalent to the conjunction of rotational flatness, (NII), and the equations (MG) (March 2023). On this basis, I urged that we should think of (NII) as providing an implicit definition of the irrotational degrees of freedom of the Newton-Cartan connection—so that the Newton-Cartan connection has its physical significance, if at all, insofar as it is fixed uniquely by (NII).

Should we say that NCT is equivalent to MG, in this case? I will approach this question roundaboutly, beginning with a remark made by Dewar (2018). Dewar notes that a model of NCT where $\rho \neq 0$ "carries a [...] form of redundancy: provided we know the standard of rotation associated to ∇, and provided we know the character of T^{ab}, we can "fill in the blanks" to reconstruct ∇ itself" (264). He likens this feature of NCT to comments made by Pooley (2013, §4.5) about the redundancy of standard presentations of Newtonian spacetime: given a Newtonian spacetime $\langle M, t_a, h^{ab}, \nabla, \xi^a \rangle$, we are always free to define ∇ from the remaining structure in the theory.

However, I would like to suggest that the 'redundancy' we see in NCT is much more akin to the fact that Newtonian gravitation—restricted to the island universe sector, and coupled with the assumption that the centre of mass of the universe is at absolute rest—also has a certain redundancy to it. Given a Galilean spacetime and the mass-momentum tensor, we can always define ξ^a as the unique vector field which results from parallel transporting the centre of mass velocity field throughout all spacetime. ξ^a is irrelevant to the internal dynamics of the matter distribution, just as the irrotational degrees of freedom of ∇ are in NCT. Notice also that in both cases, this definition sometimes results in a failure of unique recovery. Just as (NII) does not fix a unique connection when $\rho = 0$, so does the demand that ξ^a is the centre of mass velocity field fail to fix a unique vector field outside of the island universe sector, where the centre of mass is not well-defined. And there is also an obvious parallel to Jacobs' discussion of MG and NCT. Kinematic shift symmetry in Newtonian gravitation is—via Earman's SP1—standardly taken as motivation for the move from Newtonian to Galilean spacetime. But we can also restore SP1 by restricting the dynamical symmetries to the Newtonian group. Now, it might *appear* that we can accomplish this by demanding that the centre of mass of the universe is at absolute rest. But by tying the standard of rest to facts about the matter distribution in this way, it is no longer an absolute object.[30] As a result, the spacetime (and dynamical) symmetries of the theory remain the Galilei group.

Now, compare this version of Newtonian gravity theory to Galilean gravitation. The only difference between the two is that in the former theory, we have promoted a particularly convenient choice of gauge—the practice of taking the centre of mass of the

30. cf. Earman (1989, 39–40).

universe as a reference frame—to a dynamical law. Clearly this is harmless, providing that we do not then interpret the centre of mass velocity field as ontologically subsistent spacetime structure. Moreover, the fact that the 'standard of rest' so-defined is not an absolute object guards against precisely this mistake. Rather, it suggests an interpretation on which the vector field ξ^a is simply an additional piece of structure introduced to represent (somewhat redundantly) the centre of mass velocity of the universe.

The analogy to NCT and MG is immediate. From the perspective of MG, the decision to work with a connection with respect to which (NII) holds amounts simply to a choice of gauge. But in moving to NCT, we promote (NII) to a dynamical law. My claim is just that to the extent that one thinks that this modified version of Newtonian gravitation is equivalent to Galilean gravitation, one should also think that NCT, interpreted after $\mathbf{NCT}_4$, is equivalent to MG.

Finally, the view developed here also suggests a response to Jacobs' concern at the end of §4 about there being non-solutions of NCT which correspond to solutions of MG. Thus far, I have described the move from MG to NCT as a matter of fixing the Newton-Cartan connection by imposing (NII) as a dynamical constraint. But we could go further, and interpret (NII) as a *kinematical* constraint. This would avoid the problem of non-solutions of NCT in which the centre of mass of the universe is accelerated mapping to solutions of MG. It would be consistent with the idea that the move from MG to NCT simply involves a choice of gauge, this time imposed equally across the KPMs. And it fits naturally with the suggestion that the Newton-Cartan connection is not ontologically subsistent spacetime structure, but rather has its physical significance in virtue of (NII). If (NII) is a dynamical constraint, this makes it unclear how to interpret ∇ outside of the DPMs. But if (NII) is a kinematical constraint, then ∇ can be given a consistent physical interpretation throughout the entire space of KPMs.

If this is right, then the suggestion that (NII) should be interpreted as an implicit definition of the Newton-Cartan connection is more radical than it first appears. It also requires a discussion of the distinction between kinematical and dynamical possibility, which I do not have space to attempt here. A proper treatment of these issues will have to wait for another time.

7 Conclusion: respecting corollary VI?

In a recent paper, Chen (2023) has raised a challenge for the idea that MG and NCT are equivalent. Chen argues that corollary VI to the Laws of Motion in Newton's *Principia* presents a *prima facie* problem for NCT. Corollary VI, recall, expresses the fact that dynamic shifts (in Jacobs' (2023) sense) are dynamical symmetries of Newtonian physics. So in choosing ∇ to satisfy (NII), we are making "precisely the sort of physical judgment we had sought to refrain from making" (Chen 2023, 11). Chen goes on to argue that NCT can nevertheless be thought to respect corollary VI on the basis that putative alternatives to (NII) fail to produce a connection with a different Riemann curvature, so that there is always "a canonical distinction between that part of the gravitational potential of any given model that is interpretable as curvature and that which is not" (Chen 2023, 15).

Of course, Chen is absolutely correct in his technical claim here—but his substantive claim should give us pause. It is worth taking the time to unravel carefully why this is the case. Chen is not explicit about what he thinks it would mean for a theory to "respect" corollary VI, but it seems clear from his comments that this would involve collapsing the distinction between models of Galilean gravitation related by dynamic shifts. In fact, there are a few ways of glossing this requirement; here are two plausible options

- A theory respects corollary VI just in case its spacetime and dynamical symmetry groups are the Maxwell group.
- A theory respects corollary VI just in case any pair of models related by a Maxwell transformation of the mass-momentum tensor are isomorphic.

On the first option, we have already seen that the spacetime and dynamical symmetries of NCT are the Maxwell group. On the second, if (NII) is a kinematical constraint (as I have urged), then pairs of models related by Maxwell transformations of the mass-momentum tensor are also related by Galilean transformations of the mass-momentum tensor, and hence are isomorphic. This is because Maxwell transformations of the mass-momentum tensor which are not Galilean transformations take us outside the space of KPMs.

This brings out the reason why NCT respects corollary VI—Newton's second law notwithstanding. If the role of (NII) is to *define* the connection, then it cannot permit the drawing of unphysical distinctions. In Newton-Cartan terms, we cannot give sense to the idea that the centre of mass of the universe might have had a different acceleration, any more than we can do in MG. We can give sense to the idea that the acceleration of some subsystem might have been different—but only insofar as this produces a corresponding difference in the relative positions, velocities, and accelerations of material bodies.

Other choices for the Newton-Cartan connection are possible. The existence of other choices does not mean that they correspond to distinct physical judgements. Just as a choice of inertial frame does not amount to making a physical judgement about absolute velocities, neither does the decision to work with a connection with respect to which (NII) holds amount to making a physical judgement about absolute accelerations. In the end, it is not the naturalness of (NII) as a definition, but the fact that it is a definition, which means that MG and NCT are theoretically equivalent.

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