Review of "Quantum Uncertainty as an Intrinsic Clock", by Etera K. Livine (*Journal of Physics A: Mathematical and Theoretical*, 56, 485301, 2023)

James Read*

For *Mathematical Reviews*

It's well known that time is 'the great simplifier' [6]. The article under review here exploits the time reparameterisaton symmetry of classical and quantum particles in time-dependent potentials in order to identify a 'maximally simplifying' temporal parameter and in turn to derive some interesting results about how this parameter might be operationalised using quantum uncertainty. In a little more detail, the article does the following:

(a) uses the time-reparameterisartion symmetry of the dynamics of a classical particle in a time-dependent potential to map an arbitrary time-dependent frequency to a constant frequency—this amounts to a choice of simplifying 'clock';

(b) derives a non-linear differential equation which must be satisfied by this simplifying time parameter;

(c) identifies the conserved charges associated with this time-translation symmetry;

(d) turns to the quantum context, and in particular to the evolution of a Gaussian wavepacket in a time-dependent potential, and notices that the evolution of the width of the wavepacket in position space satisfies exactly the above-mentioned non-linear equation, meaning that this width operationalises the dynamics-simplifying 'clock' from the classical case;

(e) shows that this 'quantum uncertainty' is associated with a conserved charge.

In the remainder of this review, I'll go over the key points of the article, before closing with some (positive) reflections on the significance of this work.

*james.read@philosophy.ox.ac.uk

1 The classical case

In this article, Livine first considers a classical particle evolving in the following time-dependent potential:

$$s_\omega[t, q(t)] = \int dt \left[\frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega(t)^2 q^2 \right]; \tag{1}$$

the following time reparameterisations are symmetries of the dynamical equations associated with the above action after varying in accordance with Hamilton's principle:

$$\begin{cases} t \mapsto \tilde{t} := f(t), \\ q(t) \mapsto \tilde{q}(\tilde{t}) := h(t)^{\frac{1}{2}} q(t), \end{cases} \tag{2}$$

where $h = \dot{f}$. If one wants to exploit these symmetries in order to map a time-dependent frequency $\omega(t)$ to a constant frequency Ω, one can derive, defining $\eta := h^{-\frac{1}{2}}$, that these two frequencies must be related by the following non-linear differential equation:

$$\ddot{\eta} + \omega^2 \eta = \frac{\Omega^2}{\eta^3}. \tag{3}$$

The solution of this equation defines a 'simplifying clock',

$$\tau(t) := \int^t \frac{dt}{\eta^2}, \tag{4}$$

which maximally simplifies the dynamics of the system by mapping it to a simple harmonic oscillator with constant frequency Ω.

2 The quantum case

In the quantum case, the relevant system is a one-dimensional wavefunction $\psi(t, x)$ with dynamics given by the TDSE:

$$i\hbar \partial_t \psi = -\frac{\hbar}{2m} \partial_x^2 \psi + \frac{1}{2} m\omega(t)^2 x^2 \psi. \tag{5}$$

Livine then consider a Gaussian ansatz for the wavefunction,

$$\psi_{\text{Gaussian}}(t, x) = N e^{i\gamma} e^{i\frac{xp}{\hbar}} e^{-A(x-q)^2}, \tag{6}$$

which is well-known to be a 'maximally classical' state. For such a Gaussian wavefunction, the position uncertainty is then shown to satisfy exactly the non-linear differential equation (4). In this way, one sees that the quantum uncertainty can be used as an intrinsic clock, and that the 'beats' of the clock maximally simplify the description of the system.

One upshot of this—as Livine identifies—is that the quantum uncertainty can operationalise a solution to a classical mechanics problem. Livine then goes on to show that the invariant associated with the quantum uncertainty—the Ermakov-Lewis invariant—can be obtained by taking the classical limit of the Noether charge associated with the conformal invariance of the classical oscillator in a time-dependent potential.

3 Discussion

The paper raises several interesting questions. Here are three:

1. One can ask about the extent to which the approach can be generalised: can other moments in a quantum mechanics problem be associated with other physically significant parameters in a classical problem? (These questions are also raised in the original article.)

2. One wonders about the extent to which the approach can be extended to relativistic and curved spacetime settings, and about the extent to which it might be applicable to the long-standing and vexed issue of operationalising clocks in curved spacetimes (see e.g. the introduction to [5] for some sceptical remarks; for discussion on this point, see [2]).

3. In response to the well-known 'problem of time' in canonical approaches to quantum gravity (also mentioned briefly by Livine in the article's concluding section)—according to which it follows from the Wheeler–DeWitt equation $\hat{H} |\psi\rangle = 0$ that no observables can have any time-dependence—the Page–Wootters (PW) formalism [7] proposes to identify a time parameter with the readings of some subsystem of the universe which can be regarded as a clock. (For detailed assessment of the PW approach, see [1].) One might think that, given their natural 'adaptation' to the unfolding of physical processes, the 'clocks' considered in the article under review here could feature as obvious candidates for such clocks on the PW approach. Moreover, one might wonder how these 'clocks' relate to other 'intrinsic' clocks which have been introduced in order to address the problem of time—see e.g. [3, 4, 8].

One could surely multiply possible applications: in general, this article and potential extensions have the potential to be of great value to a number of important issues in the foundations of physics.

Acknowledgements

I'm grateful to Emily Adlam, Niels Linnemann, and Etera Livine for helpful discussions.

References

[1] Emily Adlam, “Watching the Clocks: Interpreting the Page–Wootters Formalism and the Internal Quantum Reference Frame Programme”, Foundations of Physics 52(99), 2022.

[2] Emily Adlam, Niels Linnemann and James Read, *Constructive Axiomatics for Spacetime Physics*, Oxford: Oxford University Press, 2024.

[3] Romeo Brunetti, Klaus Fredenhagen and Marc Hoge, “Time in Quantum Physics: From an External Parameter to an Intrinsic Observable”, Foundations of Physics 40, pp. 1368–1378, 2010.

[4] Paul Busch, Marian Grabowski and Pekka J. Lahti, “Time Observables in Quantum Theory”, Physics Letters A 191, pp. 357–361, 1994.

[5] Jürgen Ehlers, Felix AE Pirani and Alfred Schild, “Republication of: The Geometry of Free Fall and Light Propagation”, General Relativity and Gravitation 44(6): pp. 1587–1609, 2012.

[6] Charles Misner, Kip Thorne, and John Wheeler, *Gravitation*, San Francisco: Freeman & Co., 1973.

[7] D. N. Page and W. K. Wootters, “Evolution Without Evolution: Dynamics Described by Stationary Observables”, Physical Review D 27, pp. 2885–2892, 1983.

[8] Alexander R. H. Smith and Mehdi Ahmadi, “Quantizing Time: Interacting Clocks and Systems”, Quantum 3, 2019.