Some remarks on recent approaches to torsionful non-relativistic gravity

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Abstract

Over the past decade, the physics literature on torsionful non-relativistic gravity has burgeoned; more recently, philosophers have also begin to explore this topic. As of yet, however, the connections between the writings of physicists and philosophers on torsionful non-relativistic gravity remain unclear. In this article, we seek to bridge the gap, in particular by situating within the context of the existing physics literature a recent theory of non-relativistic torsionful gravity developed by philosophers Meskhidze and Weatherall (2023); we also discuss the philosophical significance of that theory.

Contents

1 Introduction

Newton-Cartan theory (NCT) was developed initially by Cartan (1925) and Friedrichs (1928) as a curved spacetime model of Newtonian gravity. The theory went through a classical phase of investigation in the 1960s and 70s (see

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in particular Dautcourt 1964; Dixon 1975; Havas 1964; Künzle 1972, 1976; Trautman 1965), two of the many fruits of which were the Trautman (1965) geometrisation and recovery theorems, which together establish a precise sense in which NCT is 'equivalent' to standard Newtonian gravity. More recently, NCT has undergone a renaissance in which it and its torsionful generalizations have been applied to non-relativistic holography (see e.g. Christensen et al. 2014a,b) and condensed matter physics, especially the fractional quantum Hall effect (see e.g. Geracie, Prabhu, et al. 2016; Son 2013; Wolf, Read, and Teh 2023).

More recently, philosophers have also begin to study torsion in the classical spacetime context. Motivated by questions raised by Knox (2011), Read and Teh (2018) explore the extent to which the mappings between NCT and ungeometrised, potential-based Newtonian gravity (henceforth NGT)—made precise in Trautman geometrisation/recovery—can be understood as a case of 'teleparallelisation'—i.e., the map relating general relativity to its torsionful equivalent, teleparallel gravity (TPG)—arguing for an affirmative answer: NGT is in a certain sense the teleparallel equivalent of NCT, and the gravitational potential of that former theory can be understood as a (gauge-fixed) 'mass torsion', associated with the mass gauge field which arises once one gauges the Bargmann algebra (Andringa et al. 2011; Read and Teh 2018; Teh 2018; Wolf, Read, and Teh 2023). Read and Teh (2018) also show that this NCT–NGT correspondence is the non-relativistic limit of the GR–TPG correspondence, when said 'limit' is implemented via null reduction; an alternative non-relativistic limit (now implemented via a $1/c$ expansion) is undertaken by Schwartz (2023), from which the NCT–NGT correspondence is also obtained. Building on this, Read and Teh (2022) exploit these connections in order to explore the status of 'Newtonian equivalence principles'; Wolf, Read, and Vigneron (2023) use these results to motivate the construction of a purely non-metric equivalent to NCT, thereby completing a 'non-relativistic geometric trinity';[1] and March et al. (2023) identify that the 'common core' of this non-relativistic trinity is Maxwell gravitation (on which see Chen 2023; Dewar 2018; March 2023).[2]

To this by-now quite mature physics literature, and still-blossoming philosophical literature, Meskhidze and Weatherall (2023) have recently added their own contribution.[3] In their article, they seek to construct a non-relativistic theory of gravitation which is (in some sense) equivalent to NCT, yet the gravitational effects in which are manifestations only of (spatiotemporal) torsion. This theory is certainly interesting and worthy of study; however, there remains much to be said about it, especially with respect to the following questions:

[1]This yields the non-relativistic analogue of the relativistic 'geometric trinity' of gravity (Jiménez et al. 2019), which is a trio of three empirically equivalent gravitational theories which are formulated using different geometric degrees of freedom: curvature for GR, torsion for TPG, and non-metricity for 'symmetric teleparallel gravity' (STGR). See Wolf and Read (2023) and Wolf, Sanchioni, et al. (2023) for further philosophical discussion on issues concerning theory equivalence and underdetermination in this context.

[2]Here, 'common core' is meant in the sense of Le Bihan and Read (2018).

[3]To be perfectly clear on the chronology: (Meskhidze and Weatherall 2023) appeared as an online preprint a couple of months before (March et al. 2023; Wolf, Read, and Vigneron 2023).

1. How is the Meskhidze-Weatherall theory best situated with respect to the existing physics literature on non-relativistic torsionful theories of gravitation?
2. What is the best interpretation of the Meskhidze-Weatherall theory *qua* theory, and how best to extract its philosophical significance?

Our goal in this article is to undertake a systematic exploration of the above two questions. Accordingly, the structure of the note is as follows. In §2, we introduce the technical details of the Meskhidze-Weatherall theory; in §3 we answer question (1) by situating this theory with respect to the existing physics literature; in §4, we answer question (2) by engaging in a thoroughgoing interpretation of this theory.

2 The Meskhidze-Weatherall theory

Let's first recall the details of the Meskhidze-Weatherall theory of non-relativistic torsionful gravitation (henceforth MWT). Kinematical possibilities of this theory are tuples $\langle M, t_a, h^{ab}, \nabla, \rho \rangle$, where the first four elements denote a classical (i.e., non-relativistic) spacetime (assumed to be temporally orientable) in the sense of Malament (2012, ch. 4), and ρ is a scalar field denoting the matter density content. In this theory, ∇ is a derivative operator with torsion (which, recall, encodes the antisymmetry of the connection—see Wald 1984, p. 53) generically decomposable as

$$T^a{}_{bc} = 2F^a{}_{[b}\, t_{c]}; \tag{1}$$

where $F^a{}_b$ is spacelike in the upper index; one can treat this as a kinematical restriction on the content of this theory. Dynamical possibilities of MWT are picked out by the field equation

$$\delta^n{}_a \nabla_{[n} F^a{}_{b]} = 2\pi\rho t_b; \tag{2}$$

gravitating but otherwise force-free test bodies with velocity vectors ξ^a are subject to

$$\xi^n \nabla_n \xi^a = -F^a{}_n\, \xi^n; \tag{3}$$

hence, such bodies experience torsion-dependent forces and thereby exhibit non-geodesic motion. Meskhidze and Weatherall (2023) prove a 'recovery' theorem *à la* Trautman (1965) relating the models of Newton-Cartan theory (NCT) to models of MWT—we return to this in §4.

3 Situating the Meskhidze-Weatherall theory in the literature

With the details of MWT on the table, we now consider and assess some claims made by Meskhidze and Weatherall (2023) with respect to the existing physics literature on torsion in non-relativistic gravity—claims which they use to motivate the construction of MWT. There are, in particular, three claims made with respect to that literature upon which we here focus: (i) claims regarding the relationship between the existence of torsion and the closed nature of t_a in a(n orientable) classical spacetime model (§3.1), (ii) claims regarding the literature's supposedly restricted attention to a specific form of the connection (§3.2), and (iii) a particular terminological choice regarding 'spatial torsion' and 'temporal torsion' which it is worth reconciling with the existing literature (§3.3).

3.1 Does a closed clock form imply vanishing torsion?

Meskhidze and Weatherall (2023) assert that physicists often claim "that taking $\partial_\mu t_\nu = 0$, where ∂ is a (torsion-free) coordinate derivative operator *will always result in a torsion-free spacetime*" (p. 9, emphasis in original). Presumably, Meskhidze and Weatherall (2023) mean to have antisymmerised their equation in the above passage, for the condition of interest is that the clock form be closed, so $dt = 0$. Since all of the important action with respect to the above quote concerns the relationship between (a) the existence of torsion and (b) the closed nature of t_a, we'll focus our considerations upon said relationship in the remainder of this subsection.

On this relationship, earlier in their article, Meskhidze and Weatherall (2023) write that "it is widely claimed that a classical spacetime with torsion cannot have a temporal metric that is closed[, but] this is not true" (p. 2). Denoting schematically all torsion by T, Meskhidze and Weatherall (2023), in other words, impute to the literature the claim that

$$T \neq 0 \quad \overset{?}{\Longrightarrow} \quad dt \neq 0, \tag{4}$$

which of course by contraposition is equivalent to the claim that

$$dt = 0 \quad \overset{?}{\Longrightarrow} \quad T = 0. \tag{5}$$

Now, on the one hand, Meskhidze and Weatherall (2023) are completely correct that this claim is false (hence our oversetting with '?' above)—one need only look to the expressions for torsion in terms of exterior derivatives of gauge fields given by e.g. Bergshoeff, Hartong, et al. (2014), Geracie, Prabhu, et al. (2015), Hartong and Obers (2015), and Read and Teh (2018) to see that $dt = 0$ does not imply that all components of the torsion vanish (see also §3.3 where these formulae are reproduced). In addition, physicists have mentioned explicitly that one can consider spacetimes with both torsion and absolute time. Consider Geracie, Prabhu, et al. (2015, p. 14), who mention in a specific context that one can still have a closed temporal metric and a spacetime with torsion (between their equations (2.48) and (2.49)): "In case we have a torsionful causal spacetime ($\boldsymbol{n} \wedge d\boldsymbol{n} = 0$) or a torsionful spacetime with absolute time ($d\boldsymbol{n} = 0$), we propose an invariant spatial Newtonian condition of the form [...]". One can, indeed, also make this point by deriving the relationship—see e.g. Bekaert and Morand

(2014, proposition 3.2)—

$$t_a T^a{}_{bc} = (dt)_{bc}, \tag{6}$$

from which we see that, although the torsion need not vanish when $dt = 0$, it is nonetheless constrained by this condition: in particular, the upper (vector) index of T must lie in the kernel of t, and is thus 'spacelike'.

All of the above is well-known. Thus, although Meskhidze and Weatherall (2023) are correct that the implication (5) fails, they are not entirely fair in imputing to the physics literature a widespread failure to recognise this.

3.2 Justifying the form of the connection considered in the literature

In any case, moving on from the above, Meskhidze and Weatherall (2023) further target the form of the connection used in the 'TTNC' literature ('twistless torsionful Newton-Cartan theory'—i.e., most recent physics work on this topic)—i.e., one with coefficients

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu t_\nu + \frac{1}{2} h^{\lambda\rho} \left(\partial_\rho h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right) \tag{7}$$

(Geracie, Prabhu, et al. 2015, eq. 2.45)—claiming that the only reason the torsion of their connection vanishes when the clock form t_a is closed is because "*they have adopted such a strict definition for their connection*" (p. 10, emphasis in original).

Taken at face value, this claim is true—but it fails to give due credit to the full reasons underlying *why* physicists who work on non-relativistic gravity have used a connection of this form. Motivations for the particular form of the connection used in non-relativistic gravity research include both exploring the non-relativistic limits of relativistic theories and understanding the full spectrum of non-relativistic structures that can emerge when one expands Lorentzian geometries in powers of $1/c$, as well as specific modelling concerns unique to non-relativistic physics.

To focus first on the former: one can expand the standard Levi-Civita connection of GR in powers of $1/c$ (Künzle 1972), which results in a similar, but slightly different connection of the form (up to a choice of the Newton-Coriolis two-form Ω_{ab}, defined in §4.2 below):

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_{(\mu} t_{\nu)} + \frac{1}{2} h^{\lambda\rho} \left(\partial_\rho h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right). \tag{8}$$

One can immediately see that this connection does not have torsion as $\Gamma^\lambda{}_{[\mu\nu]} = v^\lambda \partial_{[(\mu} t_{\nu)]} = 0$. This is due to a technical fact that it is only the zeroth order in the expansion of the Levi-Civita connection that transforms as a connection. One must then *impose* that $dt = 0$ so that the minus-first order vanishes; the zeroth order of the expansion then becomes the leading order and can serve as a proper connection for the Galilean theory (see e.g. (Künzle 1972; Van den

Bleeken 2017) for discussion on this point). Indeed, historically much of the non-relativistic literature has assumed this condition for this technical reason and the choice of $dt = 0$ seems forced in the most straightforward way of taking the non-relativistic limit of GR.

However, the choice of vanishing temporal torsion is in fact quite restrictive in the context of modern non-relativistic physics. There are a few considerations:

- As explained by Hansen et al. (2019a) and Hartong, Obers, and Oling (2022, Sect. 5.5), the presence of mass in non-relativistic theories necessarily implies the non-vanishing of temporal torsion.

- Similarly, conservation of energy requires the introduction of non-zero temporal torsion in non-relativistic gravity. As emphasized by Geracie, Son, et al. (2015, Sect. II.C), relativistic physics has a single stress-energy tensor that captures both energy current and stress. This does not hold in non-relativistic physics as the energy current is independent of stress. One in general finds that t_a can be understood as a source for energy current and that properly defining an energy current requires that $dt \neq 0$. This has been exploited to great effect in condensed matter systems and non-relativistic field theory (see e.g. Geracie, Son, et al. 2015; Gromov and Abanov 2015; Jensen 2018).

- The vanishing temporal torsion condition is not conformally invariant, and is thus unsuitable for non-relativistic holography studying the relationship between non-relativistic gravitational theories and their dual conformal field theories (Bergshoeff, Chatzistavrakidis, et al. 2017, §1).

The inclusion of non-vanishing temporal torsion ($dt \neq 0$) is not a purely arbitrary choice—rather, it is essential if one is interested in investigating many non-relativistic systems of physical interest. Furthermore, as explained by Hartong, Obers, and Oling (2022, Sect. 5.2.1), a connection of the form (7) is the "closest analogue of the Levi-Civita connection for Newton–Cartan geometry" that allows one to incorporate non-zero torsion. One then normally restricts to the case that $t \wedge dt = 0$ so as to maintain causality (Bekaert and Morand 2016), resulting in the standard TTNC version of Newton-Cartan theory considered in the literature.

Hansen et al. (2019a, 2020) then show that such a non-relativistic theory can be obtained by taking the non-relativistic limit of GR. They do so by first rewriting the metric and connection of standard GR in terms of what they call 'pre-non-relativistic' variables (resulting in a slightly generalized 'pre-non-relativistic' connection), which they argue are natural for the non-relativistic limit. They then implement a $1/c$ expansion in order to find the non-relativistic limit of GR in this parameterization and arrive at what the authors have dubbed 'type II' Newton-Cartan geometry.

One then has a connection of the form (7) and the freedom to consider a number of choices regarding restrictions placed on t_a. When one imposes the condition that $dt = 0$ on this more general connection, one arrives at 'type

I' NCT (i.e., NCT *à la* Malament (2012)); when one relaxes this condition in favour of the condition $t \wedge dt = 0$, one arrives at the type II NCT found in Hansen et al. (2019a, 2020). Type II NCT particularly interesting as it is a novel theory with both curvature and torsion which exhibits a remarkable overlap with GR in terms of its empirical content, as it can also account for the strong field gravitational physics of perihelion precession, gravitational redshift, and the bending of light that was previously thought to be the exclusive purview of relativistic physics. As was the case in the examples above, temporal torsion plays a crucial role in encoding these strong gravitational effects (see e.g. Hansen et al. 2019a,b; Van den Bleeken 2017; Wolf, Sanchioni, et al. 2023).

So yes: the particular form of the TTNC connection "ensures that the only way to allow torsion is to sacrifice having a closed temporal metric" (Meskhidze and Weatherall 2023, p. 10), but this choice is motivated by the fact that many of the non-relativistic systems physicists are interested in investigating require that we sacrifice having a closed temporal metric. That is, $dt \neq 0$ is a crucial physical requirement for these systems! These physicists are aware that it is possible to write down connections that are more general and manifest all different kinds of geometric qualities (see e.g. Geracie, Prabhu, et al. (2015, eq. 2.27)). Within the physics literature, however, the choices we have seen all have particular physical motivations. When one works with a non-relativistic connection with $dt = 0$, one thereby restricts oneself to traditionally understood notions of Newtonian gravity and absolute time. When one works with a non-relativistic connection with $dt \neq 0$, one can then explore a broader spectrum of interesting non-relativistic systems for which the role of temporal torsion plays a crucial role, including condensed matter systems, non-relativistic holography, and non-relativistic (but non-Newtonian) gravity.

3.3 The meaning of 'temporal torsion' and 'spatial torsion'

Our next point pertains to a discrepancy between the use of the terms 'temporal torsion' and 'spatial torsion' in the hands of Meskhidze and Weatherall (2023) when compared with the rest of the existing literature. Typically in the contemporary physics literature, Newtonian theories are treated as gauge theories of the Bargmann algebra, with generators $\{M, H, P, G, J\}$, and associated torsions and curvatures given by the Cartan equations:

$$(\boldsymbol{f})_{\mu\nu} := T_{\mu\nu}(M) = 2\partial_{[\mu} m_{\nu]} - 2\omega_{[\mu}{}^{\mathsf{a}} e_{\nu]\mathsf{a}}, \tag{9}$$

$$T_{\mu\nu}(H) = 2\partial_{[\mu} t_{\nu]}, \tag{10}$$

$$T_{\mu\nu}{}^{\mathsf{a}}(P) = 2\partial_{[\mu} e_{\nu]}{}^{\mathsf{a}} - 2\omega_{[\mu}{}^{\mathsf{ab}} e_{\nu]\mathsf{b}} - 2\omega_{[\mu}{}^{\mathsf{a}} t_{\nu]}, \tag{11}$$

$$R_{\mu\nu}{}^{\mathsf{a}}(G) = 2\partial_{[\mu} \omega_{\nu]}{}^{\mathsf{a}} - 2\omega_{[\mu}{}^{\mathsf{ab}} \omega_{\nu]\mathsf{b}}, \tag{12}$$

$$R_{\mu\nu}{}^{\mathsf{ab}}(J) = 2\partial_{[\mu} \omega_{\nu]}{}^{\mathsf{ab}}. \tag{13}$$

(A thorough review of this material is given by Andringa et al. (2011).) One then defines (now suppressing indices) $T(H)$ (i.e., the torsion associated with time translations) as the 'temporal torsion', and $T(P)$ (i.e., the torsion associated with spatial translations) as 'spatial torsion'. Together with the 'mass torsion' $\boldsymbol{f}$, one then defines the 'extended torsion' $(T(H), T(P), \boldsymbol{f})$.

This terminology is different from that of Meskhidze and Weatherall (2023), who use 'vanishing spatial torsion' to refer to the condition $T^{abc} = 0$ (Meskhidze and Weatherall 2023, p. 6). This is *not* equivalent to the requirement that $T(P) = 0$, but rather that $T(P)|_S = 0$ for any spacelike hypersurface S. Meskhidze and Weatherall also assume that t_a is closed, from which it follows (along with metric compatibility) that $T(H) = 0$. So in the more usual Cartan terminology, (a) MWT has no temporal torsion, and (b) MWT has spatial torsion, but such that it vanishes when restricted to any spacelike hypersurface.

4 Analysing the Meskhidze-Weatherall theory

In this section, we explore the structure of MWT *per se*. Our focus here will be twofold: (i) on clarifying the relationship between MWT and NCT, and (ii) on the orbit structure of MWT, and the extent to which it can be viewed as analogous to that of NGT.

4.1 Geometrisation and recovery

Meskhidze and Weatherall (2023) prove a 'recovery' theorem to the effect that any (compatible, torsion-free) non-relativistic spacetime $\langle M, t_a, h^{ab}, \tilde{\nabla}, \rho\rangle$ which satisfies $\tilde{R}^{ab}{}_{cd} = 0$ and $\tilde{R}_{ab} = 4\pi\rho t_a t_b$ (i.e., NCT without explict commitment to the 'Newtonian' curvature condition $\tilde{R}^{a}{}_{b}{}^{c}{}_{d} = \tilde{R}^{c}{}_{d}{}^{a}{}_{b}$) gives rise, non-uniquely, to a model of MWT (Meskhidze and Weatherall 2023, theorem 1). However, they do not similarly prove a 'geometrisation' theorem linking models of MWT to models of (this version of) NCT. In the absence of such a theorem, the relationship between MWT and NCT remains somewhat unclear, so we begin by filling in this gap on Meskhidze and Weatherall's behalf:

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla, \rho\rangle$ be a model of MWT such that $F^n{}_m F^m{}_n = 0$. Then there exists a unique torsion-free derivative operator $\tilde{\nabla}$ compatible with the metrics such that $\tilde{R}^{ab}{}_{cd} = 0$, $\tilde{R}_{ab} = 4\pi\rho t_a t_b$, and for all unit timelike vector fields on M, $\xi^n\tilde{\nabla}_n\xi^a = 0 \Leftrightarrow \xi^n\nabla_n\xi^a = -F^a{}_n\xi^n$.*

Proof. Let $\tilde{\nabla} = (\nabla, -F^a{}_b t_c)$. We claim that it satisfies the required conditions. First, note that $\tilde{\nabla}$ is compatible with the metrics since $\tilde{\nabla}_a h^{bc} = \nabla_a h^{bc} + F^b{}_a t_n h^{nc} + F^c{}_a t_n h^{bn} = 0$ and $\tilde{\nabla}_a t_b = \nabla_a t_b - F^n{}_a t_b t_n = 0$, where we have used that ∇ is compatible and $F^a{}_b$ is spacelike in the a index. $\tilde{\nabla}$ is also torsion-free since $-2F^a{}_{[b}t_{c]} = \tilde{T}^a{}_{bc} - T^a{}_{bc} = \tilde{T}^a{}_{bc} - 2F^a{}_{[b}t_{c]} \Leftrightarrow \tilde{T}^a{}_{bc} = 0$. Furthermore, if ξ^a is a unit timelike vector field on M such that $\xi^n\nabla_n\xi^a = -F^a{}_n\xi^n$ then $\xi^n\tilde{\nabla}_n\xi^a = \xi^n\nabla_n\xi^a + F^a{}_n t_m\xi^n\xi^m = 0$ (conversely, if ξ^a is geodesic with respect to $\tilde{\nabla}$ then $\xi^n\nabla_n\xi^a = -F^a{}_n\xi^n$). $\tilde{\nabla}$ is clearly unique in this regard, since an arbitrary derivative operator $(\nabla, C^a{}_{bc})$ will satisfy the above condition just in case $C^a{}_{nm}\xi^n\xi^m = -F^a{}_n\xi^n$ for *any* unit timelike vector field ξ^a, from which it follows that $C^a{}_{bc} = -F^a{}_b t_c$.

It remains to verify that $\tilde{R}^{ab}{}_{cd} = 0$ and $\tilde{R}_{ab} = 4\pi\rho t_a t_b$. First, using the expression relating two Riemann tensors

$$\begin{aligned}\tilde{R}^a{}_{bcd} &= R^a{}_{bcd} - 2\nabla_{[c}F^a{}_{d]}t_b + 2t_b F^n{}_{[c}F^a{}_{d]}t_n + 2F^n{}_{[c}t_{d]}F^a{}_n t_b\\ &= -2\nabla_{[c}F^a{}_{d]}t_b + 2F^n{}_{[c}t_{d]}F^a{}_n t_b,\end{aligned}$$

where we have used that ∇ is flat and again that $F^a{}_b$ is spacelike in the a index. It follows immediately that $\tilde{R}^{ab}{}_{cd} = 0$ since ∇ is compatible. Meanwhile, using (2) and that $F^a{}_b$ is spacelike in the a index we have

$$\begin{aligned}\tilde{R}_{ab} &= -2\delta^n{}_m\nabla_{[b}F^m{}_{n]}t_a + 2\delta^n{}_m F^r{}_{[b}t_{n]}F^m{}_r t_a\\ &= 2\delta^n{}_m\nabla_{[n}F^m{}_{b]}t_a - F^r{}_n F^n{}_r t_a t_b\\ &= 4\pi\rho t_a t_b - F^r{}_n F^n{}_r t_a t_b\\ &= 4\pi\rho t_a t_b,\end{aligned}$$

where we have used that the last equality holds just in case $F^n{}_m F^m{}_n = 0$. □

So MWT, as presented by Meskhidze and Weatherall (2023), does not quite admit a 'geometrisation' theorem analogous to Meshdiske and Weatherall's 'recovery' result; however, this may straightforwardly be rectified with the additional assumption that $F^n{}_m F^m{}_n = 0$. Since this condition also holds with respect to the recovered models of MWT considered in Meskhidze and Weatherall's theorem 1, we will assume it in what follows.

4.2 Gauge orbits and torsion in the Meskhidze-Weatherall theory

We turn now to the gauge orbit structure of MWT. We begin by recalling some basic facts about the orbit structure of NGT. Recall that if ξ^a is a unit timelike vector field on $\langle M, t_a, h^{ab}\rangle$, then any spatiotemporally torsion-free compatible connection ∇ can be represented by a pair $(\nabla_\xi, t_{(b}h^{an}\Omega_{c)n})$ for some 2-form $\Omega_{ab} = -2\hat{h}_{n[a}\nabla_{b]}\xi^n$, where ∇_ξ is the special connection for ξ^a.[4] This representation manifestly carries some redundancy, though, since it's clear that there are multiple pairs $(\nabla_\xi, t_{(b}h^{an}\Omega_{c)n})$ which can be used to represent ∇. This is encoded in the notion of Milne symmetry. Let σ^a be a spacelike vector field on M. Then under the transformation $\xi^a \to \xi^a + \sigma^a$, the 2-form Ω_{ab} in the representation of ∇ transforms as

$$\Omega_{ab} \to \Omega_{ab} + d_a\Phi_b \tag{14}$$

where $\Phi_a = \hat{h}_{an}\sigma^n - 1/2\hat{h}_{nm}\sigma^n\sigma^m t_a$. We then have the following result (Teh 2018, Proposition 1):

[4] i.e. the unique spatiotemporally torsion-free connection such that ξ^a is twist-free and geodesic with respect to ∇_ξ.

Proposition 2. *The affine space of Milne orbits is canonically isomorphic to the affine space of Newton-Cartan connections.*

This orbit structure is not quite the orbit structure of NGT, however. As articulated by Teh (2018), the orbit structure of NGT can be obtained by 'gauge-fixing' this space of representations of ∇ so that the reference connections ∇_ξ are the special connections for unit timelike vector fields which are all twist-free and rigid with respect to ∇_ξ (we are guaranteed that such exist by the condition $R^{ab}{}_{cd} = 0$). In this case, we have $\Omega_{ab} = -2\hat{h}_{n[a}t_{b]}\xi^m\nabla_m\xi^n$. The Milne symmetry action on representations of ∇ becomes

$$\begin{aligned} \xi^a &\to \xi^a + \sigma^a \quad (\nabla^a\sigma^b = 0), \\ \Omega_{ab} &\to \Omega_{ab} + d_a\Phi_b. \end{aligned} \tag{15}$$

Note that the reference connections ∇_ξ in this restricted space of representations are all flat, since ξ^a is constant with respect to ∇_ξ.

How does this change when we move to the torsionful case? Here, we can begin as before: any unit timelike vector field ξ^a has a unique compatible special connection ∇_ξ. Crucially for our purposes, when t_a is closed, this special connection is torsion-free (Bekaert and Morand 2014). The first disanalogy comes when we look at the form of the difference tensor, which now depends on both the 2-form Ω_{ab} and the spatiotemporal torsion tensor transverse to ξ^a, $U^a{}_{bc} = T^a{}_{bc} - \xi^a d_b t_c$. In particular, ∇ is now represented by a pair $(\nabla_\xi, t_{(b}h^{an}\Omega_{c)n} + U^a{}_{bc} + 2h^{an}U^m{}_{n(b})\hat{h}_{c)m})$. Specialising to the MWT case where ∇ has no spatial or temporal torsion, $U^a{}_{bc} = F^a{}_{[b}t_{c]}$, where $F^a{}_b$ is spacelike in the upper index and $F^{ab} \neq 0$. Once again, under a Milne boost with parameter σ^a, we have

$$\begin{aligned} \Omega_{ab} &\to \Omega_{ab} + 2\nabla_{[a}\Phi_{b]}, \\ U^a{}_{bc} &\to U^a{}_{bc} - 1/2\sigma^a d_b t_c, \end{aligned} \tag{16}$$

so that again specialising to the MWT case with no spatial or temporal torsion

$$F^a{}_{[b}t_{c]} \to F^a{}_{[b}t_{c]}. \tag{17}$$

This allows us to clarify the extent to which the orbit structure of MWT can be viewed as analogous to that of NGT. Unlike the connections of NGT, the connections of MWT cannot in general by constructed by considering Milne boosts between different representations of the same Newton-Cartan connection (since the temporal torsion vanishes, so that any special connection is spatiotemporally torsion-free). The MWT orbits do have the NGT orbits as a substructure, though: as can be seen by noting that once we have constructed an NGT orbit, we are always free to replace each reference connection ∇_ξ in the orbit with $(\nabla_\xi, F^a{}_{[b}t_{c]})$.

We now move on to consider a second point of disanalogy between the orbit structure of MWT and NGT, which is that the connections of MWT do not form an affine space. Indeed, this can be seen straightforwardly by noting the extra condition $F^n{}_m F^m{}_n = 0$ which the recovered models of MWT must satisfy, as established in Proposition 1: since this constraint is non-linear, it will not be preserved under arbitrary affine transformations. Again though, the orbits of MWT do have affine subspaces—which are just the substructures corresponding to the NGT orbits, as explained above.

The final point of disanalogy between MWT and NGT concerns the status of mass torsion in MWT. On this, contrasting their theory with other theories of torsional Newtonian gravity considered in the literature (in particular those by Read and Teh (2018) and Schwartz (2023)), Meskhidze and Weatherall write that

> [MWT], insofar as it features *spacetime* torsion instead of mass torsion, is a stronger analog to a classical TPG. (Meskhidze and Weatherall 2023, p. 10, emphasis in original)

So Meskhidze and Weatherall claim *inter alia* that MWT has no mass torsion. To assess this claim, let's remind ourselves how mass torsion arises in NGT. As noted in §3.3, NGT and NCT can be understood as gauge theories of the Bargmann algebra. In this approach, one begins by fixing a Bargmann structure, which is locally represented by a one-form m_μ on M which encodes the $U(1)$ connection of a Bargmann spacetime. Under a Milne boost, this one-form transforms as

$$m_\mu \to m_\mu + \sigma_a e^a{}_\mu - 1/2\sigma^2 t_\mu \tag{18}$$

and the associated mass torsion as

$$f_{\mu\nu} \to f_{\mu\nu} + \sigma_a T_{\mu\nu}{}^a - 1/2\sigma^2 d_\mu t_\nu. \tag{19}$$

In the gauge-theoretic approach to NCT, one then notes that given a choice of Bargmann structure the two-form Ω_{ab} is uniquely determined by the mass torsion via $\Omega_{\mu\nu} = f_{\mu\nu} - d_\mu a_\nu$ and *vice versa*. In particular, for each Bargmann structure, there exists a unique extended torsion-free connection, which is Newtonian and is identified with the Newton-Cartan connection. The connections of NGT (for this Bargmann structure) then have an associated mass torsion, as follows from the discussion of orbits in NGT above (where we note that with the Newtonian condition in place, we now have that the 2-form Ω_{ab} is closed—see Malament 2012, prop. 4.3.5).

How does the situation differ in MWT? We can see immediately from (19) that for models of MWT with non-vanishing spatial torsion, vanishing of the mass torsion is not preserved under arbitrary Milne boosts. So Meskhidze and Weatherall cannot insist that (a) the mass torsion always vanishes, and (b) their theory is Milne invariant. Moreover, since the orbits of MWT are not Milne orbits, and unlike the situation in (non-Newtonian) NGT, one cannot then restore Milne invariance by identifying the Newton-Cartan connection with its orbit of recovered models. Note also that in standard Newtonian NGT, the connections and 2-forms in the orbits are built from Milne invariant objects, and so are Milne-invariant by construction; however, this construction no longer works when one does not impose the Newtonian condition as in MWT.

How might Meskidze and Weatherall respond to these disanalogies? We can see at least three (not all mutually exclusive) options:

- They could relax the condition that t_a is closed. This would mean that models of MWT have a non-trivial Milne orbit for the spatial torsion, though there would still remain substructures of the MWT orbits which are not Milne orbits.
- They could drop the demand that the mass torsion always vanishes. Then Milne boosts would generate an orbit for the mass torsion, as in NGT. Again though, this Milne orbit structure still wouldn't encompass the full orbit structure of MWT.
- They could resist the language of mass torsion and the gauge-theoretic approach to NGT/NCT. However, it remains the case that for those models of NCT which satisfy the Newtonian condition, the two-form Ω_{ab} (which is related to the antisymmetric part of MW's torsional force via ${\Omega^a}_b = h^{an} F_{[nb]}$) is coextensive with the mass torsion. In this sense, it is debatable the extent to which MWT can be said to eliminate mass torsion, since their theory generically contains terms which can always consistently be reinterpreted as such.

To end this section, we will explore one analogy between MWT and other torsional non-relativistic theories considered in the literature, such as those from Read and Teh (2018) and Schwartz (2023). In their discussion, Meskhidze and Weatherall appear to suggest that these theories do not feature spacetime torsion. However, as Meskhidze and Weatherall (2023, theorem 1) show, we are always free to 'gauge fix' ${F^a}_b$ so that the spatiotemporal torsion vanishes in the recovered models of MWT. In this respect, MWT is precisely analogous to the theories considered by Read and Teh (2018) and Schwartz (2023), in which the spatiotemporal torsion is generically non-vanishing, but nevertheless can always consistently be chosen to vanish via a suitable fixing of the torsion and frame.

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