Reconstructions of Quantum Theory: Methodology and the Role of Axiomatization

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Abstract

Reconstructions of quantum theory are a novel research program in theoretical physics which aims to uncover the unique physical features of quantum theory via axiomatization. I focus on Hardy's "Quantum Theory from Five Reasonable Axioms" (2001), arguing that reconstructions represent a modern usage of axiomatization with significant points of continuity to von Neumann's axiomatizations in quantum mechanics. In particular, I show that Hardy and von Neumann share similar methodological ordering, have a common operational framing, and insist on the empirical basis of axioms. In the reconstruction programme, interesting points of discontinuity with historical axiomatizations include the stipulation of a generalized space of theories represented by a framework and the stipulation of analytic machinery at two levels of generality (first by establishing a generalized mathematical framework and then by positing specific formulations of axioms). In light of the reconstruction programme, I show that we should understand axiomatization attempts as being context–dependent, context which is contingent upon the goals of inquiry and the maturity of both mathematical formalism and theoretical underpinnings within the area of inquiry. Drawing on Mitsch (2022)'s account of axiomatization, I conclude that reconstructions should best be understood as provisional, practical, representations of quantum theory that are well suited for theory development and exploration. However, I propose my context–dependent re–framing of axiomatization as a means of enriching Mitsch's account.

Keywords—axiomatization, quantum theory, reconstructions, methodology

Declarations

Competing Interests The author declares that there are no conflicts of interest, and that the work is the sole property of the submitting author.

1 Introduction

Quantum theory is one of the most successful scientific theories to date, underpinning nearly all of our best theories of physics, excepting gravity. It is a theory that is overwhelmingly experimentally accurate, has led to a variety of technologies, and has driven advances in mathematics and other areas of science. However, a singular interpretation of quantum theory which describes what the theory says about the nature of reality is greatly debated, with little consensus amongst physicists and philosophers as to what the answer may be. The standard interpretational approaches to quantum mechanics (exemplified by the Everett interpretation or de–Broglie–Bohm's theory) typically supplies an interpretation based on the accepted mathematical formalism for quantum theory. Interpretation tends to be the main focus of standard work in quantum foundations, focusing on providing solutions to the measurement problem and analyzing the problem of nonlocality. In spite of the insights discovered from the standard interpretational approach,[1] a unified picture of what the formalism of quantum theory says about what exists in the world has yet to emerge.

The reconstruction programme refers to "both a mathematical and a conceptual paradigm that allows one to derive the usual formulation of quantum theory from a set of primitive assumptions" (Selby, Scandolo, and Coecke 2018, p. 1). The aim of reconstructions is to reformulate quantum theory from a base set of physical principles which are used to motivate a set of mathematically–formulated axioms; this reformulation is done in order to derive the key postulates of quantum theory (the von Neumann postulates). The overarching goal of the reconstruction programme is to "find a compelling set of axioms that singles out quantum theory from among all possible theories" (Chiribella and Spekkens 2016, p. 4). One common ambition is to subvert the importance of interpretation and instead focus on developing an abstract generalization of the fundamental features of quantum mechanics, seeking to "remove the interpretative bottleneck" (Goyal 2022, p. 18) through the derivation of the mathematical formalism. The hope is that reformulation from primitive assumptions will reveal the core physical structures that underlie the mathematical formalism.

I argue that the reconstruction programme represents a modern usage of axiomatization with significant points of continuity to historical uses of axiomatization. The historical context of the method reveals how axiomatization developed, particularly as it was applied to the physical sciences. We see the axiomatic method in David Hilbert's formalization of mathematics and its application to a young quantum mechanics in 1928 by Hilbert, Nordheim, and von Neumann entitled *Über die Grundlagen der Quantenmechanik*. I will show that there are important continuous features of axiomatization in Hilbert et al. and von Neumann's axiomatizations which are also present in reconstructions of quantum theory. Here I explore Lucien Hardy (2001)'s "Quantum Theory from Five Reasonable Axioms", a reconstruction which was pivotal in the development of the research programme. Further, I argue that we should understand axiomatization attempts as being context–dependent, context which is relative to the goals of inquiry and the maturity of the mathematical formalism and theoretical concepts within an area of knowledge. Using Mitsch (2022) as a starting point, I show that not only should we understand axiomatizations as context–dependent, but we can also have a more robust understanding of axiomatizations as axiomatic completions. I take the context–dependency of axiomatization as a means of enriching Mitsch's account.

I begin in section 2 by identifying strands of development of the axiomatic method

[1] Including no-go theorems, the Hilbert space formalism, and new concepts such as Bohr's *complementarity*, to name a few (Goyal 2022, p. 15).

that are continuous with the use of axiomatization in reconstructions, starting with David Hilbert's axiomatization of geometry and his axiomatization of mechanics. I then discuss a paper by Hilbert, Nordheim, and von Neumann, which was an earlier axiomatization of a young quantum mechanics before moving on in section 3 to von Neumann's axiomatic derivation of the trace function. Throughout this historical overview I highlight how the maturity of the area of inquiry influenced the ease with which an axiomatization could proceed. Section 4 gives a sketch of the general methodology of reconstructions of quantum theory, then focuses on a discussion of Hardy (2001)'s 'Five Axioms'. Here I argue that there are several points of continuity between von Neumann's axiomatization and Hardy's reconstructions, including similar methodological ordering, a similarly operational framing, and the insistence on the empirical basis of axioms. I conclude that these points of continuity situate Hardy's reconstruction as an historical successor to von Neumann's axiomatization and thus to Hilbert's axiomatic project. In section 5 I show how reconstructions use features rooted in earlier axiomatizations but manifest differently in a matured quantum theory. I highlight these methodological features, which include the stipulation of a framework as a generalized space of theories and stipulation of analytic machinery at two levels of generality (once at the framework level and again at the level of axioms). Lastly in section 6 I discuss Mitsch's account of axiomatization as axiomatic completion attempts, arguing that this is a good way to understand the value of axiomatization in the case of reconstructions. I apply this understanding to the reconstruction program, showing that we should conceptualize reconstructions as axiomatic completions. Thus I show that reading axiomatizations as context–dependent is complementary to the notion of axiomatic completions.

2 A Brief History of Axiomatization

David Hilbert is widely considered to be the father of the axiomatic programme which called for the formalization of mathematics in an axiomatic form. This had resounding effects on foundations of mathematics and any scientific endeavour that used mathematics as a basis (Zach 2019). Hilbert's interest in applying axiomatization went well beyond foundations of mathematics, extending his attention to physics. This extension led to Hilbert's Sixth Problem, a 'programmatic call' for the axiomatization of physical theories. Hilbert took inspiration from the success of axiomatization in the foundations of geometry, hoping to apply the method to 'those physical sciences' where mathematics plays a major role.

Hilbert's axiomatization of geometry was an attempt to derive key propositions in Euclidean geometry (such as Desargue's theorems) from physical axioms. The question was *which* physical assumptions (for example, the parallel postulate) would result in the derivation of those key propositions. Hilbert succeeded in this endeavour by "translating geometric statements into a formalism whose necessary assumptions had already been elucidated (the arithmetic equations); with the formalism's structure already clear, the axiom candidates for Euclidean geometry could be translated into the formalism's language and their relationship precisely characterized" (Mitsch 2022, p. 5). Hence, in light of these physical assumptions Hilbert could identify a formalism into which the theory in question could be translated, the arithmetic equations. It could then be determined which axioms were necessary for the derivation of the key propositions by relying on the structure of the identified formalism.

However, the application of axiomatization to physical sciences which were closely connected to experimentation and empirical information challenged the plausibility of the expansion of the use of axiomatization beyond mathematics. Hilbert had seen that axiomatization had the benefit of clarifying starting assumptions and concepts, and the axiomatization of geometry exemplified the power of the methodology. So Hilbert attempts to axiomatize mechanics in 1905 (Corry 1997, p. 131). Though the basic facts of the discipline (e.g. vector addition) were well established, the arrangements of those basic concepts (e.g. nuanced readings of time, space, mass, microscopic mechanical descriptions, etc.) were still open to revision.[2] Hilbert's axiomatization of mechanics, however, "was not followed by an analysis of the independence of the axioms, based on the construction of partial models, such as Hilbert had carried out for geometry" (Corry 1997, p. 187). Mechanics, compared to geometry, had to consider a range of observed phenomena which made the axiomatization much more difficult. Hilbert himself maintained that there was yet much work to do in the axiomatic treatment of mechanics (Corry 1997, p. 131).

Just as axiomatization encountered roadblocks in mechanics, the problems faced in the axiomatization of quantum mechanics were magnified tenfold. Although Hilbert called for the full mathematization of physics, quantum mechanics was very much in its infancy when Hilbert presented the Göttingen lectures in 1905 (Lacki 2000, p. 280). By 1925 there were several formalisms vying to be *the* formalism of the discipline, including Heisenberg's matrix mechanics, Dirac's q–numbers, Schrödinger's wave mechanics, and the operator calculus of Born–Wiener (Lacki 2000, p. 281). The existence of multiple formulations posed significant problems for what we could reasonably conclude about the physical world—in light of multiple mathematical descriptions, which could we physically interpret? Likewise, was it possible that a more fundamental and undiscovered formulation underpinned the variant formulations? This latter question is what Hilbert et al. explore in 1928.

2.1 Hilbert, Nordheim, and von Neumann

Hilbert et al.'s 1928 paper was an attempt to find an underlying basis between the disparate formulations of quantum theory using axiomatization. This paper is significant because it is an explicit description of how the authors see axiomatization proceeding in the very 'young' quantum theory. Hilbert et al. state that "this way is thus that of an axiomatization, as it has been carried out, for example, with geometry" (1928, p. 2). Rédei understands Hilbert et al. to characterize axiomatization to include (Rédei 2005, p. 47):

1. Physical axioms
2. Analytic machinery (i.e. formalism)
3. Physical interpretation

The basis of physical axioms is meant to be empirical, gained by past experiences and trends (Hilbert, von Neumann, and Nordheim 1928, p. 2). These physical axioms will be "formulated for certain physical quantities and relations among them" (Rédei and Stöltzner 2006, p. 3). Thus the physical axioms must be related in certain ways—where certain 'physical *demands*' must be met—as indicated by our past experiences. The analytic machinery is a formalism constructed to reflect those relations. As such the analytic machinery is a "mathematical structure containing quantities that have the same relation among themselves as the relation between the physical quantities" (Rédei 2005, p. 47). The analytic machinery is

[2] In his discussion of mechanics, Hilbert discusses several differing approaches to its foundations, including Hertz's perspective that force was explained by rigid connections between bodies and Boltzmann's presentation that focused on the central forces between "any two mass points" (Corry 1997, p. 142).

then give a physical interpretation based on those physical demands (Hilbert, von Neumann, and Nordheim 1928, p. 2). In other words, our physical interpretation is 'read off' of the formalism in whatever way satisfies those physical demands given by the physical axioms. This is the 'suggested' method of axiomatization put forth by Hilbert et al. (1928, p. 3) which I will label 'optimal axiomatization'. Hilbert et al. assert that this method is inspired by the axiomatization of geometry:

> Through the axioms the relations between the elements of geometry, point, line, plane, are characterized, and then it is shown that these relations are exactly satisfied by an analytic apparatus, namely the linear equations. Through it one can again recover geometric propositions from the properties of the linear equations. (1928, p. 2)

Hilbert et al. maintain that *optimally* the three steps listed above follow in a 'natural' temporal order, where the physical axioms are robust enough to completely articulate the analytic machinery necessary to describe them (Rédei 2005, p. 48). In the context of quantum mechanics, the analytic machinery Hilbert et al. refer to are the general probability relations of the theory (1928, p. 1). However, the authors assert that "the above suggested procedure of axiomatization is not typically followed in physics now, but rather is the way to the erection of a new theory" (1928, p. 3). Thus axiomatization as it is used in the practice of science and particularly in the development of new theories will differ from the 'optimal' method above. Instead, Hilbert et al. claim that the analytic machinery is typically already formulated and the physical axioms are then determined based on the analytic machinery (Rédei 2005, p. 48). As such we gain insight into what the physical axioms might be based on an established formalism. Hilbert et al. maintain that in the practice of physics the physical axioms are determined via an interpretation of the formalism, rather than the physical interpretation being borne out of the physical axioms (Rédei and Stöltzner 2006, p. 4). I call this different characterization of axiomatization 'practical' axiomatization. Indeed, Lacki notes that Hilbert et al. maintained "the difficulty, even impossibility, of carrying out the axiomatization process in its most ambitious form, namely that going from the physical requirements to the finding of the formalism" (2000, p. 297). This was the last of Hilbert's work in physics, though the axiomatic program was carried forward notably by John von Neumann.

From the work by Hilbert et al., what we observe is that axiomatization as developed by Hilbert for geometry had to become more flexible (specifically as it relates to the order in which analytic machinery and physical axioms are posited) in light of the method's application to the physical sciences. It was the application of axiomatization to mechanics that highlighted the problem of the strictness of axiomatization, particularly in light of physical theories with greater demands on empirical information. Hilbert et al. (1928)'s paper was the realization and attempted execution of the 'loosening off' of axiomatization in order to contend with the formalism of an immature quantum mechanics. Rédei and Stöltzner maintain that axiomatization became more flexible ('soft'), less well-defined, and more intuitive (2006, p. 3). This increase in flexibility widened the scope of what axiomatization could be applied to and the order in which axioms, physical interpretation, and analytic machinery could be posited.

The Hilbert et al. (1928) paper is significant for my purposes because it illustrates how the way an axiomatization proceeds would change depending on the area of inquiry being axiomatized. Geometry offered an area of inquiry that could more easily be axiomatized as its experimental foundations were generally accepted and the theory could be "turned into a *pure mathematical* science" (Hilbert, 1898-9 *Mechanik* Göttingen Lecture). However, mechanics lacked a clear arrangement of basic concepts which proved difficult for a similarly precise axiomatic treatment that was enjoyed by geometry. The axiomatic treatment of quantum mechanics posed additional challenges due to its multiple formulations. The lack of a single well–established mathematical language hindered the articulation of key theorems and concepts. John von Neumann aimed to resolve this by utilizing axiomatization to ascertain the correct formulation of quantum mechanics.

3 Von Neumann on Axiomatization

Von Neumann published two important papers in 1927, the *Mathematische Begründung der Quantenmechanik* (1927a) and *Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik* (1927b). In these works von Neumann presented quantum mechanics in a mathematically rigourous manner that solidified the use of the abstract Hilbert space formalism (Lacki 2000, p. 300). It is in *Aufbau* (von Neumann 1927b) that the trace function is derived, which von Neumann had determined was a key proposition in any formulation of quantum mechanics (Mitsch 2022, p. 14). The goal was now to identify which basic assumptions would result in the derivation of the trace function.

In the introduction of this proof, von Neumann highlights the problem "of the determination of the probability for obtaining a determinate value of a given physical quantity" (Lacki 2000, p. 301). In quantum mechanics, due to the impossibility of certain simultaneous measurements[3] a measurement will not necessarily result in a definite value being attributed to a physical quantity (Lacki 2000, p. 301). Von Neumann's aim was to define the *statistics* of physical quantities over an ensemble of identical systems, which would yield an expectation value.[4] A trace function is the sum of the diagonal elements of a matrix (sum of the complex eigenvalues), essentially determining measured probabilities while maintaining basis invariance. Hence a trace is the same regardless of coordinates. In the proof, von Neumann begins by stipulating two physical axioms regarding the expectation values of physical quantities in a statistical ensemble (an ensemble being a collective of systems that are characterized by the operator U^5):

A. Expectation value assignments are linear (where 'a', 'b' represent a given physical quantity:

$$Exp(\alpha a + \beta b + ...) = \alpha Exp(a) + \beta Exp(b)... \tag{1}$$

B. Expectation value assignments are positive:

$$Exp(a) \geq 0 \tag{2}$$

Rédei and Stöltzner argue that both of these axioms are physical axioms in the sense that they are based in our experience and observations in the laboratory (empirical) (2006, p. 3). They also note that the physical quantities a and b are left unspecified while the two axioms represent the expectation values of the observed statistical behaviour of an ensemble of systems.[6] It is here that von Neumann draws insight from the 'analytic machinery'

[3]This is a result of the uncertainty principle, which rules out simultaneous measurements of incompatible properties or conjugated quantities (pairs of observables whose operators do not commute).

[4]An expectation value is the average value predicted from a collection of measurements.

[5]U here is some statistical operator that is both positive (real, non-negative valued) and linear (Rédei and Stöltzner 2006, p. 5).

[6]By unspecified I take Rédei and Stöltzner to be designating a general scope in terms of what those

i.e. formalism, which is the set of all selfadjoint operators on a Hilbert space (Hermitian operators representing physical quantites) (Rédei and Stöltzner 2006, p. 5). In other words, von Neumann assumes that the physical quantities were represented by a type of operator that would only result in real eigenvalues. He then stipulates:

C. If operators S, T...represent physical quantities $a, b...$ then the operator $\alpha S + \beta T + ...$ represents the physical quantity $\alpha a + \beta b...$

D. If operator S represents physical quantity a then the operator $f(S)$ represents the physical quantity $f(a)$.

These two requirements fill out the physical interpretation, linking the physical axioms to the analytic machinery. Von Neumann clearly stipulates that the physical quantities are represented by all linear operators on a Hilbert space, thus the properties of the physical quantities are postulated by considering the 'structural' properties of the Hilbert space operators (Rédei and Stöltzner 2006, p. 5). Hence C and D represent and capture "the central focus in quantum mechanics on the functional relationship among experimentally measurable quantities" (Mitsch 2022, p. 16).

The collection of $A + B + C + D$ resulted in the derived

$$Exp(a) = Tr(US) \tag{3}$$

with a representing a physical quantity with its associated operator S (Rédei and Stöltzner 2006, p. 5). This is the trace formula that is "the heart of the whole story" (Rédei and Stöltzner 2006, p. 5).

Von Neumann successfully derived the trace function and in doing so isolated the basic assumptions necessary for its derivation. This derivation was based on the determination of *physical* requirements which then enabled him to isolate the central proposition of quantum mechanics. As we can see from the consideration of *Aufbau* and the joint Hilbert et al. paper, von Neumann was driven by the 'spirit of axiomatization' imparted on him by his work with David Hilbert, and the trace function case illustrates his employment of the method. In the trace function case, von Neumann employed axiomatic methodology in order to identify the unique formalism for quantum mechanics. Von Neumann was working in an immature area of inquiry, both in the development of its mathematical formulation as well as its theoretical underpinnings. The derivation of the trace function was a step toward providing a solid mathematical foundation on which to build a unified representation of quantum mechanics.

Thus far, I have traced axiomatization from Hilbert's axiomatization of geometry to von Neumann's continuation of the method in the derivation of the trace function. I have highlighted how the maturity of an area of inquiry, in both its mathematical formalism as well as its theoretical underpinnings, influences the way in which an axiomatization could proceed. An area of inquiry with a mature mathematical formalism (such as geometry) and theoretical underpinnings could be axiomatized in a more straight–forward way wherein formal notions such as completeness, uniqueness, and independence, for example, could be surveyed. However, axiomatization of an immature area of inquiry requires more leniency in how an axiomatization is ordered. In particular, the maturity of an area of inquiry influences whether physical axioms are stipulated before or after looking at the analytic machinery. In von Neumann's trace function case we see how the axiomatic project continued as he

physical quantities could be, even if they can be specified at the time of measurement. This detail is not expanded on in Rédei and Stöltzner.

searches for an underlying formalism for quantum mechanics. Von Neumann's main goal was to discover a unique formalism for quantum mechanics, one that unambiguously derived the key proposition of the theory.[7] This derivation was made possible by his discovery of physical axioms which would give rise to the key proposition.

Next I introduce the reconstruction programme in the foundations of quantum theory. I demonstrate the continuity in axiomatic methodology from von Neumann's trace function case to Hardy's "Five Axioms," particularly concerning methodological ordering, an operational reading, and the empirical basis of axioms. I conclude that these points of continuity support the interpretation that the reconstruction programme is a continuation of Hilbert's axiomatic project.

4 Methodology of Reconstructions of Quantum Theory

This section details how axiomatization is used in Hardy (2001)'s reconstruction, highlighting significant similarities and differences to von Neumann's axiomatization. What we will see is the continuation of the historical story of axiomatization from von Neumann, but used in a very modern context.

Methodologically, reconstructions begin by establishing a framework which describes a landscape of possible theories including classical, quantum, and extra–quantum theories. This framework is used to stipulate foil theories which serve as nonquantum contrasts (Chiribella and Spekkens 2016, p. 5). These nonquantum foils enable us to pinpoint the uniquely quantum case, and it is the relations between theories within the landscape that are studied. These foils function as a means of answering Wheeler (1971)'s "Why the quantum?" question, which asks both why our world is described by quantum theory and why it is not described by something else. Thus if we can envision a landscape of sufficient variety, then we are tackling both the question "why this?" and "why not *that*?" (Chiribella and Spekkens 2016, p. 4). A framework establishes a broad theory space via a generalized mathematical formalism which best represents the core principles of an area of inquiry. For example, the Generalized Probability Theory (GPT) framework describes any theory that can be articulated in terms of experimental probabilities, and the generalized formalism represents the core principles in terms of preparation, transformation, and measurement procedures.

Under the general framework, individual axiomatizations pick out what researchers take to be the fundamental physical principles of quantum theory. An axiom is a particular mathematical representation within the broader framework, serving as a mathematical description of a physical principle. Axioms function as specific constraints on the framework in order to isolate what is uniquely quantum in the space of theories. Different axiomatizations will highlight different physical principles that researchers take to be fundamental to quantum theory. It is these physical principles which will be given divergent mathematical formulations which are not mathematically translatable. From these axioms, a successful reconstruction will derive the von Neumann postulates which are fundamental to quantum theory, i.e. the Hilbert space formalism in terms of linear operators in a Hilbert space. This process demonstrates that the postulated physical principles, initially considered as

[7] While this is true at this stage, it should be noted that von Neumann later abandons the Hilbert space formalism he was so integral in developing. He was aware of significant conceptual problems—including an infinite trace—in the derivation of the trace function (Rédei 1996, p. 495). Von Neumann moved to develop his type II operator algebra (now called a von Neumann algebra) in an attempt to solve the conceptual problems in the Hilbert space formalism (Rédei 1996, p. 495).

assumptions at the axiom level, along with their corresponding mathematical expressions, are compatible with the von Neumann postulates.

4.1 Hardy's 'Five Axioms'

Hardy (2001)'s seminal "Quantum Theory from Five Reasonable Axioms" kick–started the reconstruction programme and is an excellent example of how axiomatization is active in theoretical physics today. Hardy aims to "recover the basic structure of quantum theory along with the most general type of quantum evolution possible" (2001, p. 1). He seats his reconstruction in the Generalized Probability Theory framework based on his intuition that quantum theory is a new type of probability theory. Hardy's five posited axioms are based first on 'states' typified as integers K and N. K represents the number of degrees of freedom (the minimum number of probability measurements required to determine the state) and N represents the maximum number of states that are capable of being distinguished from one another in a single measurement (2001, p. 2). The five axioms are:

1. Probabilities. Relative frequencies are given for given measurements on an ensemble of systems via a specific preparation procedure.
2. Simplicity. Given N where $N = 1, 2, ...$, the minimum value is taken that is consistent with the axioms.
3. Subspaces. A system whose state belongs to an M subspace behaves like a system of dimension M.
4. Composite systems. A composite system comprised of A, B satisfies $N = N_A N_B$ and $K = K_A K_B$.
5. Continuity. There exists a continuous reversible transformation between pure states in a system.

All of the axioms are physically motivated and abstractly formulated. Hardy maintains that these axioms are less obscure than the axioms of traditional approaches (e.g. complex Hilbert spaces, Hermitean operators, etc.) and that these fundamental axioms can be articulated most clearly in the context of probability theory. The axioms are meant to be less obscure because they are based on clear, generalized, empirical physical principles. In contrast, the traditional formulation of quantum theory reveals very little about what physical insights we might derive from the formalism. Hardy discovers that it is the continuity axiom that is the feature that picks out the quantum case. By omitting the continuity axiom Hardy claims to obtain classical probability theory ($K = N$) instead of quantum theory ($K = N^2$), with the aid of the simplicity axiom (2001, p. 2).[8] Thus, Hardy's axiomatization begins with the acceptance of the formalism of probability theory situated within the GPT framework, postulates physical principles which are then encoded mathematically as axioms, and ends with the derivation of the von Neumann postulates. Below, I identify the methodological order that Hardy follows in his axiomatization, comparing it to earlier applications of axiomatization:

[8] Although Hardy claims to obtain classical probability theory with the removal of axiom 5, the issue of how and when we are situated in a classical region in the space of theories is a subtle one. Hardy uses the continuity axiom to rule out various theories that do not correspond to quantum theory, including classical probability theory (Hardy 2001, p. 15). It is sufficient to rule out alternatives to the Bloch sphere. However, omitting continuity could result in either classical or quantum theories in the absence of assumptions about the transformation properties of states. I am thankful to a reviewer for this insight.

1. Analytic machinery (probability theory, specifically Generalized Probability Theory)
2. Identification of physical principles (e.g. subspaces, composite systems)
3. Analytic machinery, physical principles formulated as mathematical axioms
4. Derivation (e.g. von Neumann postulates)

Hardy situates his reconstruction in the Generalized Probability Theory framework. The physical principles are stipulated as axioms via a mathematical formulation within the framework. The ordering of Hardy's reconstruction aligns with 'practical' axiomatization from Hilbert et al. insofar as some analytic machinery is accepted (GPT) before the physical axioms are articulated.

We can note that the 'physical interpretation' step noted in Hilbert et al.'s view is not present in Hardy's reconstruction. Hardy explicitly states that he is "principally interested in deriving the structure of quantum theory rather than solving the interpretational problems" (2001, p. 10). The focus on the structure of quantum theory is to be expected of the reconstruction programme, which does not aim to provide a full, conventional interpretation of quantum theory. However, there is a subtle point to be made as it relates to 'interpretation' in both Hilbert et al. and in Hardy's reconstruction. Hilbert et al.'s aim of physically interpreting the analytic apparatus (which is identified via empirical information) is to "fully formulate the physical demands so that the analytic apparatus is clearly defined" (1928, p. 2). This motivation conveys that what we really want are statements about real, physical things. I take 'physical interpretation' here to be closer to 'those real physical objects the analytic apparatus is meant to represent' rather than interpretations of quantum theory in a modern sense.[9] In its modern sense, interpretation refers to an attempt to provide a full ontic account of the connection between theory and world (for example, the Everettian interpretation). For Hilbert et al., we have a more minimal understanding of interpretation concerned mostly with how the analytic machinery connects to the physical facts of the domain of inquiry within an axiomatization. Von Neumann's trace function case is likewise not solving for interpretation in the modern sense, and better aligns with Hilbert et al.'s ideas about the mathematics–world connection. The trace function, as a necessary proposition in any formulation of quantum mechanics, is a piece of the analytic apparatus that is designated as physically representative. The way the physical principles are 'true' in von Neumann is similar to Hardy. Although the 'physical interpretation' step is absent from Hardy's reconstruction, there is a 'thin' physical interpretation given via the identification of the axioms. In reconstructions, the physical intuitions that inform the axioms are taken as 'true' in some sense, as researchers choose the physical features they take to be central to quantum mechanics (in contrast to classical mechanics or other foil theories). The physical principles are empirically well–verified and understood to be unlikely to be revised. This is because they are borne from clear empirical generalizations that are meant to be physically transparent. Hence, the basis of the physical principles are taken to be accurate representations of the world as true generalizations of phenomena.

A point of continuity between von Neumann and Hardy's 'Five Axioms' is a shared operationalist framing. In the trace function case von Neumann restricts his representation of requirements C and D to only those quantities that are experimentally measurable and

[9] Mitsch avoids the conflation of 'physical interpretation' by instead referring to 'physical facts' in Hilbert et al.'s work. Mitsch maintains that 'physical facts' is a more apt terminology as it relates to Hilbert's account (2022, p. 4).

based on experience and intuition. We can see that this bears a similarity to Hardy's reading of his axiomatization as operational. This is not to say that von Neumann's axioms are articulated in an operationalist fashion, but rather that Hardy's operationalism and von Neumann's 'intuitive, empirical basis' are similar in spirit. In both accounts, the basis of the axioms must be empirical. In Hardy's case, the GPT framework is operational, describing only that which is experimentally measurable. What this means is that Hardy's axioms, which are mathematically formulated based on an overarching physical principle, will be necessarily connected to empirical information. Hence, the formalism used to express operational axioms will necessarily have a thin physical interpretation.

As I have shown, there are points of continuity between von Neumann's historical axiomatization of the trace function and Hardy's modern quantum reconstruction. In particular, both axiomatizations are ordered similarly according to the 'practical' ordering, have a similar operationalist flair, and base axioms on empirical observations. As such, Hardy's reconstruction should be considered an historical successor to axiomatization as it was used by von Neumann.

5 Diverging Methodology in the Reconstruction Programme

There is historical precedence to the evolution of the method of axiomatization, beginning with Hilbert et al. and continuing with von Neumann. Reconstructions can be regarded as extensions of this method. However, what is perhaps more interesting are the ways in which certain features rooted in aspects of earlier applications of axiomatization diverge in the reconstruction programme. To this end, I will now demonstrate that axiomatizations are tailored to the different mathematical and theoretical context that the reconstruction programme finds itself in.

5.1 The Space of Theories

One important feature in von Neumann's work and reconstructions is the interest in capturing a general characterization of quantum theory. For von Neumann this feature arises from his interest in discovering a generalized formalism for quantum mechanics in his trace function example. The trace function was a fundamental feature of any formalism in quantum mechanics. Von Neumann was looking for the widest formal generalization of quantum theory in order to provide a unified mathematical representation of the theory, and this interest in generalization informed his goal to derive the unique formalism for quantum mechanics.[10] In reconstructions, the interest in generalization arises out of the stipulation of a broad landscape of theories. Within this generalized theoretical space, reconstructionists aim to discover the physical principles that isolate the quantum region in the space

[10]It is possible to read von Neumann's later shift from the Hilbert space formalism (HSF) to type II operator algebras as similar to the interest in different formulations in the reconstruction case. As Rédei notes, von Neumann abandons the idea that the HSF is the exclusive framework for quantum mechanics (1996, p. 495). Von Neumann looks to different mathematical formulations to solve conceptual problems in the HSF. It does appear that there is a potential similarity with the methodology of the reconstruction programme. However, the discussion in this paper is confined to von Neumann's work within the HSF in the period up to 1932. Von Neumann began investigating the Type II framework after this period. It would be worth investigating this possible similarity with von Neumann's later work in mind. I am thankful to a reviewer for this thought–provoking insight.

of theories. I proceed now to emphasize how this important feature is represented in the methodology of reconstructions.

The methodology of reconstructions establishes a framework that stipulates an expansive landscape of theories, allowing for the exploration of possible theories that are *not* quantum theories but especially those close in theory space to quantum theory. This framework is devised to highlight the distinctions between these regions in theory space. Individual reconstructions will pick out the fundamental features of quantum theory differently which means that a different set of theories within the fixed landscape will be highlighted. This also results in divergent physical insights insofar as the physical principles which inform axioms will confer different physical interpretations. For example, Hardy's 'Five Axioms' is formulated within the Convex Operational Theories framework[11] as one axiomatization. Other axiomatizations under the Convex Operational Theories framework include (Barnum, Barrett, et al. 2007), (Barnum and Wilce 2011), (Barnum, Barrett, et al. 2012), and (Barrett 2007). I suggest that establishing a broad space of theories is methodologically important as this framework is more likely to include crucial features that might have been overlooked in a narrower set of theories. The methodology of reconstructions provides the necessary structure to do this, both in the stipulation of the broad framework and the use of axiomatization as a constraint.

Axiomatic methodology in reconstructions is even more significant in the field of quantum foundations since one of the main advantages of the method is conceptual clarification. Here I take 'conceptual clarification' to be the simplification of the chosen fundamental physical principles with an accompanying mathematical formulation. Fundamental physical principles are intended to assume as little as possible while being firmly based on empirical information. Axiomatization functions well here because it allows us to 'toggle off' an axiom which will affect the space of theories. Toggling is akin to testing an electrical panel. If we switch off a breaker we are able to see the effect on the lighting in the room. In the context of reconstructions, this is methodologically advantageous because it is a means of clarifying which physical concepts and formalisms are ones we consider to be foundational. According to Rédei, Hilbert also supported the dropping of single axioms in order to study alternative theories and 'deepen the foundations' (2005, p. 4).[12]

Both the methodology of reconstructions and axiomatization provide the structure (a framework *and* axioms) to generate alternative theories while placing fundamental principles centrally. As Chiribella and Spekkens note:

> Given an axiomatic derivation of quantum theory, it suffices to modify a single axiom in order to get a consistent alternative. Furthermore, this approach can be used to avoid an important pitfall of more ad hoc approaches to developing alternatives to quantum theory, namely, that the latter may inadvertently violate fundamental principles that one would prefer not to abandon (2016, p. 5).

[11] The Convex Operational Theories framework is a further specification of the Generalized Probability Theory framework (Chiribella and Spekkens 2016, p. 8). Hence, Hardy's 'Five Axioms' is within the GPT framework, stipulated specifically as a convex operational variety.

[12] Stöltzner (2002) gives a much more detailed account of the different ways Hilbert deepens the foundations. There may be echoes of these different ways of deepening foundations in different implementations of the reconstruction project. For example, Stöltzner notes a type of deepening in the sense that deepening mathematical foundations might yield concepts that are physically more fundamental (2002, p. 257). This is similar in spirit to how Hardy uses a minimal Hilbert space formalism in order to highlight continuity as an important physical axiom of quantum theory. The potential connection between Stöltzner's types of deepening and the reconstruction project will be explored in future work. I am grateful to a reviewer for this valuable insight.

The conceptual clarification comes out of the derivation of quantum theory from a physically perspicuous set of axioms and those axioms are not likely to be subject to revision. By placing those physical principles centrally, we are more easily able to track the principles and their interactions with other posited features. Axiomatization as it was used first by von Neumann and then by the reconstructionists has the advantage of flexibility insofar as the adding, dropping, or modifying of a single axiom enables one to devise a large number of alternative theories. The construction of foil theories within the framework also allows us to prove the independence of a set of axioms: "if one axiom is independent from another, then one should be able to devise a foil theory that satisfies the former but violates the latter" (Chiribella and Spekkens 2016, p. 5). Axiomatization is particularly advantageous in the context of theory development, exploration, and 'deepening foundations' because it opens up the space of theories we are willing to consider.

5.2 Axiomatic Starting Points

One difference between Hardy's reconstruction and von Neumann's axiomatizations is the starting point of both researchers. Specifically, how mature the mathematical formalism and theoretical concepts is in both uses of axiomatization influences how each axiomatization proceeds. In the case of the trace function, the insight that was desired at the outset of the axiomatization was insight into the *formalism.* Von Neumann was looking to define the statistics of physical quantities over an ensemble of identical systems. The trace function was a key piece of new mathematical machinery developed via axiomatization, accomplished in part by leaning on trusted physical axioms (the linearity and positive value of expectation values). Von Neumann's starting point included intuitive physical axioms and a minimal formalism of probability theory used to articulate those axioms. This result was pivotal because the discipline was still trying to find its mathematical and conceptual footing (Lacki 2000, p. 281).

Von Neumann's use of axiomatization in the above cases diverges from Hardy's 'Five Axioms', given that Hardy has a mature formalism to work with (the Hilbert space formalism). This mature formalism acts not only as a solid mathematical foundation from which to begin his reconstruction but also as a constraint as the end point. That is, Hardy employs an operational description in order to mathematically define the physical principles which inform his axioms, and the derivation of the von Neumann postulates indicates that the reconstruction is successful. Hardy also takes the GPT framework to be fixed, even if axioms can be formulated in different ways. However, a different framework could be chosen, for example one that is device-independent (Popescu and Rohrlich 1998) or category-theoretic (Selby, Scandolo, and Coecke 2018). A change in framework results in a different generalized formalism and different allowable formulations of axioms. As such, the insight that is desired in Hardy's reconstruction (and in reconstructions in general) is predominantly *physical* insight, accomplished in part by leaning on a mature formalism. Hardy's starting point includes intuitive physical postulates, a minimal, mature formalism, and the specific results we need to reproduce at the end of the reconstruction. The end point is a proof that the von Neumann postulates are recovered, which also functions as a proof that the formulations of Hardy's axioms are well–formed. In this case, when an axiom is well–formed the mathematical formulation of the physical principle will have the right mathematical ingredients to derive the VNP.

What is highlighted is how the goals of inquiry shift relative to the maturity of the area of inquiry. Goyal (2022) offers a helpful distinction between the *developmental* and *reflective* phase of a theory. In the developmental phase, the workability of a theory is more important than, for example, mathematical rigour. The result is that mathematical formalism often lacks clear physical connection (2022, p. 1). The reflective phase establishes the connection between mathematical formalism and its physical content, occurring after a theory's formalism has satisfactorily captured "some basic regularities in nature's workings" (Goyal 2022, p. 1). This distinction allows us to highlight the difference in goals between von Neumann's derivation of the trace function and the overarching goal of the reconstruction programme. Von Neumann works in the developmental phase of quantum theory, whereas reconstructionists work in the reflective phase. Both von Neumann and the reconstructionists are searching for the proper formulation of quantum mechanics. While each share a similar goal, the maturity of the area of inquiry being axiomatized is quite different. For reconstructions, the more mature area of inquiry allows them to integrate the framework level in the methodology, which allows them to situate quantum mechanics in a generalized theory space. Due to the maturity of the formalism, reconstructionists know of the von Neumann postulates which function as an indicator of success. Von Neumann is working to establish a unique mathematical language, while the reconstructionists focus on discovering a set of axioms that picks out the quantum case in the landscape. They have a similar goal, but the field they are working in has different levels of development.

The above discussion leads us to a unique feature of reconstructions that distinguishes the programme from von Neumann's axiomatizations. Von Neumann's trace function example occurs in the context of probability theory. His focus was how to determine the probability of obtaining a determinate value for a physical quantity. However, the analytic machinery that is used in the axiomatization is stipulated once, which is at the level of axioms (the linearity and positivity of expectation value assignments). This is where the reconstruction programme diverges, as analytic machinery is stipulated at two points. Once at the framework level and once in the specific formulations of axioms. In Hardy's case, Generalized Probability Theory is stipulated as the broad, overarching framework that describes the space of theories, which includes a generalized mathematical formalism. The specific formulations of the axioms are the second instance where analytic machinery is stipulated, which in Hardy's case include, for example, the physical principle of composition represented mathematically as subsystems A and B satisfying $N = N_A N_B$ and $K = K_A K_B$. The key difference is the stipulation of a framework. Here two instances of analytic machinery are stipulated at different 'levels' of generality, a broad overarching framework and the specific formulations of axioms. Although von Neumann's derivation of the trace function occurs in the context of probability theory, a specific framework is not *stipulated*, particularly a framework that has the same degree of generality as the framework in reconstructions. It is partly the stipulation of a broad framework in reconstructions that is methodologically advantageous as it enables us to probe the boundaries of what is uniquely quantum. It might seem imprudent to stipulate such a generalized space of theories if what we are trying to accomplish is the identification of a set of axioms that isolates quantum theory. However, *because* we already know the formalism that typifies quantum theory (the von Neumann postulates) we can use it as a constraint within that space of theories. Thus, concerns about the scope of the space of theories are offset by the known formalism. In von Neumann's trace function and hidden variables cases, axiomatization was used to arrive at a broader formalism for quantum mechanics in the hopes of establishing a formalism on which to build.

5.3 Inference tracking

There are many advantages to axiomatization as a methodology, whether Hilbert's more formal notion of axiomatization or later von Neumann's more flexible version. Ultimately, axiomatization as a methodology is understood to lend epistemic credibility as a semi-formal means of parsing out fundamental and easily understandable constraints (axioms stipulated from physical postulates) that give rise to some solution or collection of solutions. Axiomatization provides a more directed means of inference tracking, particularly in the physical sciences. Part of the trustworthiness of axiomatization is being able to find mistakes within a derivation from a set of axioms. This derivation facilitates the finding of mistakes in our reasoning. If the stipulated axioms are resistant to revision, like Hilbert stressed, then it is more likely that the resulting derivations and conclusions of those derivations will preserve that resistance to revision as well.

The significance of how an axiomatization is ordered has been highlighted throughout the historical story. The key distinction between 'optimal' axiomatization and 'practical' axiomatization is whether the physical postulates help us to identify the analytic machinery, or if the analytic machinery gives us insights into what the physical postulates are. The optimal version begins with the stipulation of physical postulates which are then given a mathematical formulation. The practical version is the inverse. It was important for Hilbert et al. to separate formalism and physical interpretation because the mathematics was resistant to revision while physical interpretation was not. Hilbert et al., and particularly Hilbert, wanted to remove any specifics of meaning in order to de-empiricize a theory, thus transforming it into a pure mathematical exercise[13]. However, I argue that in the context of reconstructions the ordering of an axiomatization, though it was significant for Hilbert et al., is important *not* due to *when* analytic machinery and physical content is ordered but rather *that* the connection is stipulated at all. And so, the designation of optimal versus practical ordering is not the important insight we get from following such a procedure.

Rather, the insight we get by distinguishing between formalism and physical content is pinpointing where our mathematics applies to the physical world. What is significant is not that we stipulate mathematical formalism before physical content or the inverse. It is the specification that some mathematical feature x is representative of some physical insight y. The insight desired, however, is relative to the goals one has in performing an axiomatization. If we are, e.g., looking for physical insight, then as we see in reconstructions it becomes important to seek out the formalism–physical content connection. In other axiomatizations this connection may be less important such as those axiomatizations where understanding of the mathematical formalism is being sought. Von Neumann uses physical insights in his trace function case in order to search for the right mathematics. Axiomatization provides the methodological structure to facilitate the bridging of our mathematics to what it purports to represent. It is important that it is the physical axioms in reconstructions that are taken to be unlikely to be revised, rather than mathematical formalism, which is a feature that is distinctive in how axiomatization is used.[14] Though the physical principle is unlikely to be revised, this is not necessarily the case for how that physical principle is mathematically formulated as an axiom. How those physical principles are formulated as

[13]To de-empiricise a theory was to disentangle empirical content from the analytic machinery. This was successful when a formalism was identified whose structure represented the important relations of the theory without empirical content.

[14]This relates to how reconstructions are also distinct from the standard interpretational project in quantum mechanics: the standard interpretational approach accepts the formalism of quantum mechanics and instead aims to provide a physical interpretational of that formalism.

axioms is dependent on the choice of researchers. The methodology of reconstructions also cements the importance of physical intuitions that guide researchers in the choice of formalism, not only at the level of axioms but at the framework level. Axiomatization provides the semi-formal structure to aid researchers in specifying the connection between physical ideas and their mathematical representations.

Next I show how a liberalized axiomatic method solidifies the role axiomatization can play in reconstructions, relying in part on Mitsch (2022)'s recent work on 'Hilbert-Style Axiomatic Completion'.

6 Reconstructions as Axiomatic Completions

In "Hilbert–Style Completion: On von Neumann and Hidden Variables in Quantum Mechanics" (2022) Mitsch offers a nuanced interpretation of Hilbert's axiomatic method. Mitsch's elucidation of the role that axiomatization plays in the development of theories is directly applicable to the reconstruction programme. I apply this understanding to demonstrate that Mitsch's conception of axiomatization is the best means of understanding the value of the methodology. I also argue that a context–dependent understanding of axiomatization complements Mitsch's. In particular, Mitsch's description of axiomatization as a provisional, practical,[15] meta-mathematical procedure concerned with orienting and ordering an area of inquiry aligns with my conception of the methodology. With this understanding in mind, what we see is that reconstructions of quantum theory successfully use axiomatization in a way that has and will continue to clarify and develop quantum theory.

Mitsch argues that Hilbert's axiomatic method was both provisional and practical, rather than a strict formalization "in the service of radical epistemological or metaphysical goals" (2022, p. 2).[16] According to Mitsch, the goal of an axiomatization is a meta-mathematical one, wherein the relationship between mathematics and reality is addressed "insofar as an axiomatization will identify necessary physical assumptions based on the theorems central to an area of inquiry" (2022, p. 6). In contrast, the mathematician determines what propositions result from a given set of axioms.

This meta-mathematical project proceeds, in the Hilbertian tradition (Mitsch 2022, p. 6):

1. The identification of central theorems and concepts of an area of knowledge.
2. The identification of a formalism whose structure reflects the above area of knowledge.
3. The determination of the necessity of the candidate axioms based on the formalism.

Mitsch calls the relations between candidate axioms the "*uniqueness question*: are the other axioms sufficient for deciding the structure of the formalism w.r.t. the candidate axiom?" (2022, p. 6). Uniqueness questions determine if and when the collection of axioms is representative for the theory while admitting no other realizations (Mitsch 2022, p. 20). An axiomatic completion is achieved when all uniqueness questions are answered, in which case the axiomatic structure of the theory of the area of knowledge is completely determined

[15] Mitsch is not referring to 'practical' in the sense of the ordering of analytic machinery and physical concepts that I have outlined prior from Hilbert et al.. Rather, 'practical' refers more to the notion that axiomatization contributes to scientific progress (Mitsch 2022, p. 3).

[16] Mitsch disagrees with Lacki that Hilbert was interested in axiomatization strictly in terms of logical clarification and rational reconstruction (2022, p. 3). Rather, Mitsch interprets Hilbert as having a more pragmatic and liberal idea of what axiomatization could achieve.

(Mitsch 2022, p. 6). However, Mitsch maintains that axiomatic completions, even if they are successful, are provisional—they "generate provisional representations of reality insofar as axiomatic completions rely on fallible steps" (2022, p. 7). This is because the central theorems and concepts are liable to change within the theory of an area of knowledge. Axiomatic completions are also practical as "they are a tool meant to generate helpful representations of a field of knowledge" (Mitsch 2022, p. 7) while simultaneously orienting and ordering a theory (Mitsch 2022, p. 8). A theory is *oriented* when the independence of propositions is surveyed and an axiomatization orients an area of inquiry by directing our attention to certain physical, epistemological, or mathematical considerations (Mitsch 2022, p. 29). A theory is *ordered* when the lack of contradictions between propositions is guaranteed. An axiomatic completion of a theory is successful when it is able to both orient and order an area of knowledge alongside its mathematical investigation (Mitsch 2022, p. 8).

Mitsch's description of axiomatization is a nuanced reading of axiomatization as a liberalized methodology that is pragmatically useful, in contrast with the notion of axiomatization as a strict formalization. He concludes that "von Neumann effectively summarized and clarified where we *had been*—in physics as well as in mathematics—in an effort to identify where we *could go*" (2022, p. 30). I concur with this understanding of the methodological role that axiomatization has to play in the development and exploration of theories. For my purposes, I am primarily interested in Mitsch's characterization of axiomatization as a practical, provisional, meta–mathematical methodology which aims to order and orient an area of knowledge. This is precisely how we should understand the role that axiomatization plays in the reconstruction programme. Individual reconstructions should be taken as provisional, alternative formulations of quantum mechanics. The methodology of reconstructions should be understood as a predominantly meta-mathematical process which enriches both our mathematical and physical conceptions in foundations of quantum theory. The meta–mathematical project first identifies central theorems and concepts, then identifies a formalism whose structure reflects those theorems and concepts, and lastly determines the necessity of candidate axioms based on the formalism. It is here where we understand an axiomatization to be provisional, as the central theorems and concepts may change. Below we see how the meta–mathematical project plays out in the methodology of reconstructions:

1. The identification of central theorems and concepts of an area of knowledge (e.g. the von Neumann postulates, specific axioms like composite systems)
2. The identification of a formalism whose structure reflects the above area of knowledge (e.g. the GPT framework *a la* Hardy)
3. The determination of the necessity of candidate axioms based on the formalism (e.g. the continuity axiom in 'Five Axioms')

The central postulates to be derived in the reconstruction programme are the von Neumann postulates, the minimal theoretical and empirical result that must be recovered in any axiomatization of quantum theory. Specific axioms may differ depending on what researchers take to be fundamental to quantum theory. For example, both Hardy in 'Five Axioms' and Selby et al. (2018) include the physical concept of composition (composite *systems* in Hardy and the composition *of processes* in Selby et al.) as a central concept of quantum theory. The identification of an appropriate formalism is by framework choice, such as an operational formulation in Hardy's case or a process theory formulation in Selby et al.[17].

[17] A process theory formalism is meant to capture the intuition that "the conceptual bare-bones of quantum

In reconstructions, formalism stipulated at two points adds another level of specification to the analytic machinery. Lastly, we determine the necessity of candidate axioms in light of the axiomatic structure of the formalism.

This final step requires a more careful treatment in the reconstruction programme. Mitsch describes this final step as determining if the collection of axioms is representative for the theory while admitting no other realizations. An axiomatic completion is achieved if the axiomatic structure of the theory has been completely determined, i.e. if the necessity of candidate axioms is determined. Uniqueness questions help us decide if other axioms are needed. Von Neumann's goal is to show that transformation theory is a unique representation of quantum theory that did not allow for hidden variables. Hence, transformation theory is unique insofar as the collection of axioms was sufficient to describe quantum theory with no other realizations. The result is a single formulation of quantum theory which is a general characterization of the theory. In the reconstruction case, successful reconstructions will rederive the von Neumann postulates from specified sets of axioms and in doing so will highlight a collection of axioms necessary for its derivation. These sets of axioms will not be unique in the sense that there will be no other realizations. There will be many different sets of axioms which isolate the quantum space of theories within a framework. As such, reconstructions will not be unique in the sense that von Neumann meant.

Have we achieved an axiomatic completion in the context of reconstructions? The answer is not obvious. Reconstructions fail to be unique in the sense that there is a single realization that represents quantum theory. However, reconstructions *do* succeed in determining axiomatic structures which describe quantum theory when they isolate the quantum region in the space of theories. Reconstructions are able to reproduce the von Neumann postulates by considering which axioms are necessary to situate us in a quantum space of theories. If we are able to isolate the quantum region within the space of theories, then other candidate axioms are not required. What is unique in the reconstruction programme is the region in theory space that is quantum theory. Hence, I maintain that reconstructions successfully determine the axiomatic structure of quantum theory by isolating the quantum region in the space of theories. There will however be many realizations, in which case there will be no unique reconstruction which represents the theory. However the lack of a unique reconstruction makes sense in the context of the programme. The goal of reconstructionists is to discover a general characterization of quantum theory in a contrastive theoretical space. This contrastive space enables researchers to highlight the differences between the classical, the quantum, and other possible theories. The results of the programme are different candidates for the distinct features of quantum theory. Thus, the intention is not to determine a single formulation for quantum theory, but to highlight the distinctions between the classical, quantum, extra–quantum, and non–quantum in the space of theories. As such I argue that reconstructions successfully determine axiomatic structures for quantum theory, even if there are multiple realizations. The methodology is designed to give us the theoretical space (the framework) in order to investigate what the core physical principles of quantum theory might be.

But one could ask why, if von Neumann was successful in achieving an axiomatic completion of quantum mechanics in 1932, we should consider the reconstruction programme to be examples of another attempt at axiomatic completions of quantum theory. Why, in other words, is the project ongoing if von Neumann already succeeded? This is where the provisional nature of an axiomatic completion comes to the fore. If we understand axioma-

theory concerns the manner in which systems and processes compose" (Selby, Scandolo, and Coecke 2018, p. 1).

tization attempts as being context-dependent (relative to the goals and maturity of the area of inquiry), then it is unproblematic that the search for axiomatic completions continues. In von Neumann's hidden variables case, the goal was to show that transformation theory is a unique representation of quantum mechanics (*excluding* hidden variables). In the reconstruction programme, the goal is to identify a set of axioms which isolates the quantum case in the space of theories. The goals of inquiry differ between both cases. As I have argued above, the maturity of the mathematical formalism for quantum mechanics is quite different in the present day, as are the central theorems and concepts. The context of the area of knowledge has shifted, which is why axiomatic completion attempts continue. And this contextual reading is complementary to the provisional nature of axiomatic completions. As such, I argue that Mitsch's account of axiomatic completions is further enriched by framing axiomatizations as context dependent, relative to the goals and maturity of the area of inquiry. As we can see, both the mathematical and meta–mathematical project will be relative to the maturity of the area of inquiry. An immature area of inquiry will result in a greater degree of difficulty in identifying central theorems and concepts as well as an accompanying formalism whose structure reflects the relations between them. The central theorems, concepts and formalism are liable to change, and this will be relative to how mature the theory is. Thus, steps (1) and (2) in the meta–mathematical project will be influenced, which inevitably influences step (3). In the reconstruction case, because quantum theory is reasonably mature, reconstructionists have a clearer idea of what steps 1-3 should look like, even if they are still investigating the arrangement of candidate axioms. However, as long as the provisional nature of axiomatic completions is maintained, axiomatization functions as a means of structuring how we probe the mathematical, conceptual, and meta–mathematical structure of theories.

According to Mitsch, the meta–mathematical goal of an axiomatization is to address the connection between mathematics and the reality it purports to represent. This goal is achieved when the necessary physical assumptions are identified, based on the central theorems of an area of inquiry. The mathematical goal concerns the derivation of the propositions that result from a set of axioms. In the reconstruction case, the goal of rederiving the von Neumann postulates (the central concepts) is the meta–mathematical project. As I have described, reconstructionists aim to determine which physical assumptions are necessary to pick out the quantum case. As per Mitsch's distinction, this is a meta–mathematical procedure. Further, the provisional nature of an axiomatic completion is complementary to reconstructions. Though the chosen axioms stipulated at the outset are taken to be fundamental and *resistant* to revision, both the collection of those axioms and their mathematical formulation can be altered. Reconstructions are best understood as provisional representations of quantum theory, while having a decidedly pragmatic flair. Recall that an axiomatization is pragmatic insofar as it generates different representations of a field of knowledge. Here Mitsch refers to Hilbert's consideration of both Boltzmann and Hertz's axiomatizations of Lagrangian mechanics. According to Hilbert, both representations contributed by unearthing a deeper layer of mechanics (Mitsch 2022, p. 8). However, separate representations[18] will be evaluated via their ability to order and orient an area of inquiry (Mitsch 2022, p. 8). An axiomatization is also pragmatic insofar as it orients an area of inquiry by surveying the independence of propositions. In the reconstruction programme we see this in the construction of foil theories where we might test the independence of axioms. The methodology of reconstructions is well-suited to contend with questions of

[18]Which might even be *mutually inconsistent*, which is an unproblematic end in the interest of scraping out the mathematical core of a theory (Mitsch 2022, p. 8).

independence, aligning with Mitsch's reading of axiomatization.

As I have emphasized, the reconstruction programme results in multiple axiomatizations of quantum theory, based on different physical principles which are encoded as axioms. Though the common theoretical and empirical result (the von Neumann postulates) must be derived, separate reconstructions will act as different representations of quantum theory. Since researchers will carve up the essence of quantum theory in divergent ways, reconstructions will be distinct in their physical axioms, the formalism used within the axiomatization, and the associated physical implications of the axiomatization.[19] The generation of multiple representations of an area of inquiry is a result of axiomatization, a result that is useful in the exploration and development of theories. Although the reconstruction programme is relatively new, it has already enriched mathematics, physics, and philosophy. The focus on operational formulations has become an effective methodological tool, while also spurring research on how to understand such an approach (Adlam 2022). Recent work offers some potential general interpretative insights that can be drawn from several operational reconstructions including different notions of space, measurement, and time (Goyal 2022, p. 35). More practically, the reconstruction programme is tied intimately with the advancement of quantum technologies and quantum information. For example, device–independent cryptography such as Barrett, Hardy, and Kent's "No signaling and quantum key distribution" (2005) arose out of foundational work on information–processing (Chiribella and Spekkens 2016, p. 5). Though there is much work to be done in this area, there is great promise in the reconstruction programme.

I have shown, using Mitsch's articulation of axiomatization, that we should understand reconstructions of quantum theory as provisional, alternative formulations of quantum mechanics. They are best understood as a primarily meta–mathematical process which has benefited mathematics, physics, and philosophy, and is likely to continue to enrich foundational work in quantum theory.

7 Conclusion

Reconstructions of quantum theory employ a modern version of axiomatization with important points of continuity to axiomatization as it was practiced by von Neumann. Hardy's 'Five Axioms' shares a number of core features of axiomatization with the form of axiomatization that was used by von Neumann. These core features include a shared operational approach, similarly empirical axioms, and the same methodological ordering between analytic machinery and physical axioms.

However, though there are points of continuity between these cases, I have demonstrated that the ways in which historical features of axiomatization appear in reconstructions are methodologically distinct. Specifically, I show that the inclusion of a generalized framework is methodologically advantageous, primarily in its capacity to provide a comparative platform of alternative formulations of quantum mechanics within the framework. Further, the stipulation of analytic machinery at two distinct levels of generality allows for greater specification of the mathematical tools used in a reconstruction. This specification is significant because it allows us to both constrain the space of theories and to clearly identify the success condition of deriving the von Neumann postulates. These features are methodologically advantageous in the context of the reconstruction programme, which enables reconstructions

[19]For example, the physical implications of time–reversibility (Selby, Scandolo, and Coecke 2018, p. 4) is difficult to comprehend.

to be vehicles for theory exploration. In light of historical uses of axiomatization and its modern usage in reconstructions, I showed that axiomatization should be construed as context–dependent, where the relevant context is contingent on the goals of inquiry and the maturity of the mathematical formalism and theoretical concepts in an area of knowledge. A mature formalism and set of central theorems and concepts allow reconstructionists to integrate a generalized framework in the methodology, which aids researchers in identifying the quantum case in the space of theories.

Using Mitsch (2022)'s understanding of axiomatizations as axiomatic completions, I show that reconstructions are provisional, practical representations of quantum theory that order and orient the discipline. This characterization clarifies the role that axiomatization plays in the exploration and development of theories. The context–dependency of axiomatization further enriches Mitsch (2022)'s account of axiomatic completions, as both the mathematical and meta-mathematical project will be relative to the maturity and goals of the area of inquiry.

Thus, just as von Neumann took seriously the task of "spreading the spirit of axiomatization" (Lacki 2000, p. 281), so too do reconstructionists of quantum theory.

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