On Coordinates and Spacetime Structure*

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Abstract

Philosophers and physicists often claim that the 'privileged coordinates' of a physical theory provide a window into its structure. The purpose of this paper is to examine whether this is the case. We show that there are general relativistic spacetimes that admit the same privileged coordinates but have different structure, and we infer from this that privileged coordinates do not provide a perfect guide to underlying structure. We conclude by isolating the conditions under which privileged coordinates *do* perfectly reflect structure.

1 Introduction

The structure of the Euclidean plane is usually presented using the tools of differential geometry. One defines the smooth manifold $\mathbb{R}^2$ and then lays down on it the Euclidean metric tensor. This method of presenting a geometric space is considered standard by physicists and philosophers of physics, but it is not the only way to go. A less familiar method proceeds as follows:

> One begins with the set $\mathbb{R}^2$, but rather than directly defining manifold structure and then laying down a metric, one instead singles out a collection C of 'privileged coordinate systems' for the space. In the case of the Euclidean plane, C contains the standard 'x-y coordinate system', along with its rotations, translations, and reflections. One then stipulates that the structure of the space will be given by those features that are 'agreed upon by' the privileged coordinate systems in C. We often say that these features are 'shared in common by' or 'invariant under change of' privileged coordinates. One can show that in this case the privileged coordinates agree upon just the smooth manifold structure of $\mathbb{R}^2$ and the standard Euclidean metric (and metrics isometric to it).

We therefore have a second method of presenting the Euclidean plane. These two methods go by various names. Norton (1993, 1999, 2002) occasionally

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calls the first “additive” and the second “subtractive”; Dewar (2019) has called the first “internal” and the second “external”. In this paper, we will follow Wallace (2019) in calling the first the “Riemannian method” and the second the “Kleinian method”.

It is sometimes suggested that the Kleinian method has drawbacks. For example, North (2021, p. 23) remarks that it is “less direct”, Maudlin (2012, p. 31) suggests that it is “obscure”, and Arntzenius and Dorr (2012, p. 232) say that it is “spectacularly unsatisfying from a foundational point of view”. But the adequacy of the Kleinian method is rarely questioned further. In particular, it is often implicitly assumed that the Kleinian method can be used to present all of the same geometric spaces that are presentable using the Riemannian method. North (2021, p. 23), for example, writes the following:

> As we see in the case of the Euclidean plane, the kinds of coordinates we can use for a space can indicate its structure. [. . .] There are simply two ways of characterizing a given structure, and two corresponding routes to learning about it. A structure can be characterized more directly, as in the case of the Euclidean plane and the metric tensor. Or it can be characterized less directly, by means of the coordinate systems we can use for the space and the features that are invariant under transformations of them.

Wallace (2019, p. 135) has similarly remarked that, in addition to the Riemmanian presentation, “it is generally fine, and often actively useful, to characterize mathematical spaces via classes of preferred coordinatizations of these spaces”.

The aim of this paper is to carefully investigate the generality of the Kleinian method. Euclidean space can be presented using either the Riemannian or Kleinian method, but one wonders which other geometric spaces can be described in Kleinian terms. In particular, we will ask whether all of the spacetimes of general relativity can be presented using the Kleinian method:

Question. *Can every relativistic spacetime be presented using the Kleinian method, i.e. by singling out a collection of privileged coordinates?*

We will show that the answer is “no”. We do so by exhibiting non-isometric relativistic spacetimes that nonetheless have the same privileged coordinates. Despite their structural differences, these spacetimes ‘look the same’ from the Kleinian perspective, and hence one loses expressive power if one opts for the Kleinian method over the standard Riemannian method. The Kleinian method ‘washes out’ the structural differences between some spacetimes.

At issue is whether the ‘privileged coordinates’ of a geometric space provide a guide to its structure. Many philosophers and physicists have claimed that they do. North (2021, p. 26), for example, suggests that

> the features or quantities that are agreed upon by all the different coordinate systems we can use for the plane, the coordinate-independent, invariant features, correspond to the intrinsic nature of the plane, to aspects of the plane itself, apart from our descriptions of it — that is, to what I have been calling its *structure*.

Fock (1964, p. 374) expresses the same thought when he remarks that "the existence of a preferred set of coordinates [...] reflects intrinsic properties of space-time" and stresses the "fundamental significance" of such coordinates (Fock, 1964, p. 375). And even when it is not explicitly stated, the idea that privileged coordinates reflect a theory's structure is relied upon in common inferences about structure. North (2021, p. 9) remarks that philosophers and physicists often "take the mathematical form of the laws in different kinds of coordinate systems [...] to indicate the underlying nature of the world." When we do so, we are using a collection of privileged coordinates — in this case, often those in which the laws 'take a particularly nice form' — as a guide to the structure that the theory posits. In demonstrating that the answer to our question is "no", we will be showing that, at least in general relativity, privileged coordinates do not provide a perfect guide to spacetime structure. There are relativistic spacetimes that have different structure but the same privileged coordinates.

In order to carefully examine this question, we will employ a framework that was recently introduced into philosophy of physics by Wallace (2019): the framework of locally G-structured spaces. This framework provides one natural way of precisely stating the Kleinian method. In brief, a locally G-structured space is a set S with a collection C of maps from S to $\mathbb{R}^n$, which can be thought of as the collection of privileged coordinates of the space. Wallace (2019, p. 126) remarks that this framework provides "a legitimate and informative way to formulate" theories, and he offers evidence for this claim by working through a number of illustrative examples showing how to use locally G-structured spaces to present various geometric spaces. Our aim in what follows is to examine where the limits of this framework lie, and consequently the extent to which privileged coordinates provide a guide to structure.

2 How does the Kleinian method work?

We begin with some preliminaries about the Kleinian framework. In particular, we need to define a locally G-structured space and show how one uses it to present a geometric space. Our presentation in this section will follow that of Wallace (2019), but we will include additional details that will be important for our subsequent results.

It is worth making a few practical remarks at the outset. We assume familiarity with basic differential geometry; the reader is invited to consult Malament (2012) for details. We will divide our discussion throughout into the simpler case of 'global coordinates' (and G-structured spaces) followed by the more technically complicated case of 'local coordinates' (and locally G-structured spaces). The arguments provided in each case parallel one another. The global case is worth discussing because it is technically less complex. To streamline our presentation, we have placed lemmas and proofs of propositions in an appendix.

2.1 Global coordinates

We begin by defining a G-structured space. A G-structured space presents a geometric space by singling out a collection of privileged global coordinates for the space. Recall that a **transformation group** on a set X is a group G of bijections from X to itself. Let G be a transformation group on $\mathbb{R}^n$ that is contained in the group of diffeomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$. A **G-structured space** is then a pair (S, C), where S is a set and C is a non-empty set of bijections from S to $\mathbb{R}^n$ that has the following property:

Compatibility condition. If $f \in C$, then $f' \in C$ if and only if $f \circ f'^{-1} \in G$.

One can think of the maps in C as the 'privileged global coordinates' on the space S.

One can recover a geometric space — in the form of a smooth manifold with structure on it — from the data given by a G-structured space (S, C). We begin by building smooth manifold structure on S using the maps in C. In order to do so, one needs to define an atlas on S, i.e. a collection of coordinate charts that endow S with its 'smoothness structure'. This is easily done using the global coordinates in C. We consider the collection

$$C_0 = \{(c^{-1}[U], c|_{c^{-1}[U]}) : c \in C \text{ and } U \subset \mathbb{R}^n \text{ is open}\}$$

of n-charts on C. We let our atlas C^+ be the collection of n-charts on S that are compatible with all of the charts in C_0. And we then have the following result.

Proposition 2.1.1. *(S, C^+) is a smooth n-dimensional manifold.*

Now that we have recovered manifold structure on S, we have a simple guarantee about the maps in C.

Proposition 2.1.2. *Every $c \in C$ is a diffeomorphism from the smooth manifold (S, C^+) to $\mathbb{R}^n$.*

We have therefore recovered the structure of a smooth manifold from the G-structured space (S, C). And moreover, Proposition 2.1.2 implies that the recovered manifold (S, C^+) has the same smoothness structure as $\mathbb{R}^n$. Our next step is to recover structure on the manifold (S, C^+) in the form of tensor fields.

There is a particularly natural way to do this. In brief, we recover a transformation group Γ on S, and then use this transformation group to implicitly define various tensor fields on the manifold. One first notices that the maps in C suffice to determine the **coordinate transformation group** $\Gamma = \{c^{-1} \circ d : c, d \in C\}$ of (S, C) on S. This transformation group is the group of bijections 'generated by' the coordinate maps in C. Intuitively, it contains all of the bijections from S to itself that 'switch us' from one of our privileged coordinate systems in C to another one of them. One can easily verify that it is a transformation group on S.

Now that we have the coordinate transformation group Γ defined on S, we can consider those tensor fields that are 'implicitly defined' by Γ, i.e. those tensor fields that remain unchanged when pulled back via any of the maps in Γ. One might say that the tensor fields implicitly defined by Γ are those that are 'invariant under coordinate transformations'. More precisely, if M is a manifold with G a transformation group on M that is a subset of the group of diffeomorphisms of M, we will say that a smooth tensor field α (of arbitrary index structure) on M is **implicitly defined** by G just in case $h^*(\alpha) = \alpha$ for each $h \in G$. This is how to recover tensor fields from the data given by our G-structured space. We equip (S, C^+) with those smooth tensor fields that are implicitly defined by the coordinate transformation group Γ.

We have therefore recovered from a G-structured space (S, C) a manifold with tensor fields on it. This captures a strong sense in which one can use 'privileged coordinates' to present a geometric space. At heart, the idea behind this Kleinian framework of G-structured spaces is simple. The privileged coordinates in C immediately endow S with smooth manifold structure and a coordinate transformation group Γ. Γ then gives rise to further structure on our geometric space by implicitly defining tensor fields on the manifold (S, C^+).

An example illustrates this method of recovering a manifold with structure from (S, C). We show how one presents Minkowski spacetime using the apparatus of G-structured spaces. Recall that a **relativistic spacetime** is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth Lorentzian metric on M. **Minkowski spacetime** is the pair $(\mathbb{R}^4, \eta_{ab})$, where η_{ab} is flat and geodesically complete.

Example 1. Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ can be presented as a G-structured space in the following manner. Let S be the set $\mathbb{R}^4$, let C be the set of diffeomorphisms $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that $f^*(\eta_{ab}) = \eta_{ab}$, and let $G = C$. One can verify that this defines a G-structured space (S, C). ⌟

We can now straightforwardly see how to recover the structure of Minkowski spacetime from (S, C). First, we begin with the manifold structure. Proposition 2.1.2 implies that the recovery procedure described above equips S with the manifold structure of $\mathbb{R}^4$, the same as that of Minkowski spacetime. So we have recovered from (S, C) exactly the manifold structure underlying Minkowski spacetime. And second, we turn to the metric. We can easily see that $h^*(\eta_{ab}) = \eta_{ab}$ for each $h \in \Gamma$. Since $h \in \Gamma$, it must be that $h = f^{-1} \circ g$ for $f, g \in C$. We then compute that

$$h^*(\eta_{ab}) = (f^{-1} \circ g)^*(\eta_{ab}) = g^* \circ f_*(\eta_{ab}) = \eta_{ab}$$

The first equality follows since $h = f^{-1} \circ g$, the second follows from properties of the pullback and pushforward, and the third follows from the defining condition of the maps in C. So η_{ab} is implicitly defined by Γ, and hence (S, C) recovers the Minkowski metric.

One can therefore present the structure of Minkowski spacetime using the Kleinian framework of G-structured spaces. We note, however, that nothing said here implies that η_{ab} is the *only* metric recoverable on (S, C) using our recovery procedure. There may, for example, be more than one metric implicitly defined by Γ. We will return to this point later on.

2.2 Local coordinates

We now turn to the case of local coordinates. Proposition 2.1.2 implies that the manifold structure recovered by a G-structured space is that of the smooth manifold $\mathbb{R}^n$. Since relativistic spacetimes can have underlying manifolds that are not diffeomorphic to $\mathbb{R}^n$, the framework of G-structured spaces clearly does not have the expressive capabilities required to present all of general relativity. It is for this reason that Wallace (2019) considers a more flexible formalism in which spaces are given their structure by specifying their privileged *local* coordinates. This is the framework of locally G-structured spaces. It will take a bit more work to set up the results in this section, but the basic ideas are exactly the same as in the case of G-structured spaces.

One begins with the definition of a pseudogroup (Kobayashi and Nomizu, 1996, p. 1). While a transformation group is collection of bijective structure-preserving maps from a space to itself, a **pseudogroup** is a collection of bijective structure-preserving maps between *open subsets* of a topological space. It is, in essence, the 'local analogue' of a transformation group. A precise definition is contained in the appendix (Definition 2.2.1), but examples serve well to illustrate the core idea. The simplest example of a pseudogroup is the **diffeomorphism pseudogroup** of a smooth manifold M, i.e. the class of diffeomorphisms $f : U \to V$ between open sets U and V of M. If (M, g_{ab}) is a relativistic spacetime, then the **isometry pseudogroup** of (M, g_{ab}) is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(g_{ab}) = g_{ab}$. (These are pseudeogroups by Lemma 2.2.1, which is stated and proved in the appendix.)

We can now define a locally G-structured space. Let G be a pseudogroup on $\mathbb{R}^n$ that is contained in the diffeomorphism pseudogroup of $\mathbb{R}^n$. A **locally G-structured space** is then a pair (S, C), where S is a set, C is a collection of injective partial functions $c : S \to \mathbb{R}^n$, and the following three conditions hold:

Cover condition. For every point $p \in S$ there is a map $c \in C$ such that $p \in \text{dom}(c)$.

Range condition. For every map $c \in C$ there is a map $g \in G$ such that $\text{ran}(c) = \text{dom}(g)$.

Compatibility condition. For any partial function $f : S \to \mathbb{R}^n$ whose range is the domain of an element of G, $f \in C$ if and only if for every $f' \in C$ such that $\text{dom}(f) \cap \text{dom}(f')$ is non-empty, $f \circ f'^{-1} \in G$.

We can think of the maps in C as the 'privileged local coordinates' on our space S. The cover condition guarantees that every point in our space S lies in some or other privileged coordinate chart. The range condition guarantees that the ranges of elements of C are open subsets of $\mathbb{R}^n$. And the compatibility condition is a generalization of the earlier compatibility condition for G-structured spaces; it guarantees that the maps in C 'play nicely' with the maps in G.

Let (S, C) be a locally G-structured space. We need to show how one recovers a geometric space from (S, C). As in the global case, we begin by showing how (S, C) inherits smooth manifold structure, and we then show how to recover the structure of tensors on that manifold. We first need to build an atlas on S. This is simpler in the local case because for each $f \in C$, $(\text{dom}(f), f)$ is an n-chart on S. Let C^+ be the collection of all n-charts on S that are compatible with all these n-charts in C. We then have the following result.

Proposition 2.2.1. *(S, C^+) is a smooth n-dimensional manifold.*

We have therefore recovered smooth manifold structure from the data given by the locally G-structured space (S, C). Unlike in the global case, the manifold (S, C^+) is not necessarily diffeomorphic to $\mathbb{R}^n$. We do, however, have the following analogue to Proposition 2.1.2.

Proposition 2.2.2. *Every $c \in C$ is a smooth map $c : dom(c) \to \mathbb{R}^n$. And moreover, $c : dom(c) \to ran(c)$ is a diffeomorphism.*

We now show how to recover various levels of geometric structure on the manifold (S, C^+). As in the global case, one notices that the maps in C suffice to induce a pseudogroup on S. Intuitively, this **coordinate transformation pseudogroup** Γ again contains all of the maps between open subsets of (S, C^+) that 'switch us' from one of our privileged coordinate systems in C to another one of them. One can picture Γ as containing those diffeomorphisms between open sets of (S, C^+) generated by functions of the form $f^{-1} \circ g$, where f and g are in C. See Definition 2.2.2 in the appendix for a precise account.

We have hence recovered a manifold (S, C^+) with a coordinate transformation pseudogroup Γ defined on it. As in the global case, this provides us a way to recover the geometric structures on (S, C^+). We will say that a smooth tensor field α (of arbitrary index structure) on a smooth manifold M is **implicitly defined** by a pseudogroup G on M just in case $h^*(\alpha) = \alpha|_U$ for all $h : U \to V$ in G. We now simply equip (S, C^+) with those smooth tensor fields α that are implicitly defined by the coordinate transformation pseudogroup Γ.

Overall, locally G-structured spaces operate in much the same manner as G-structured spaces did. There are, however, two important differences. First, there is an increase in expressive power that one buys when moving from G-structured spaces to locally G-structured spaces. Because the latter do not require the maps in C to be bijections from S to $\mathbb{R}^n$, the underlying manifold of the recovered geometric space is not necessarily diffeomorphic to $\mathbb{R}^n$. Second, the coordinate transformation pseudogroup contains 'more maps' than the coordinate transformation group does. The latter requires its maps to be bijections from the entirety of S to itself, while the former admits maps that are bijections merely between open subsets of S. This means that the coordinate transformation pseudogroup provides us with more data than the mere coordinate transformation group, and therefore the former has a better chance of encoding the entire variety of geometric structures one needs for general relativity.

3 Does the Kleinian method always work?

We have so far shown how to use a (locally) G-structured space to present a geometric space. One wonders whether *all* relativistic spacetimes can be presented using (locally) G-structured space. We now turn to this question.

3.1 Global coordinates

We begin with the global case. Now that we have a clear method of recovering a relativistic spacetime from a G-structured space, we can make our main question from the introduction precise. We restrict attention to relativistic spacetimes whose underlying manifold is diffeomorphic to $\mathbb{R}^n$. Without this restriction, the answer to our question would trivially be "no" because Proposition 2.1.2 implies that the manifold (S, C^+) recovered by a G-structured space is diffeomorphic to $\mathbb{R}^n$.

Question 1 (Global case)**.** *Can every relativistic spacetime whose underlying manifold is diffeomorphic to $\mathbb{R}^n$ be recovered from a G-structured space?*

We will begin by describing a sense in which the answer to Question 1 is "yes". We will then see that this answer is unsatisfactory, and the better answer is "no". We discuss the "yes" answer first to illustrate the best kind of answer that one might give to the question on behalf of those who claim that privileged coordinates do fully reflect to structure.

Answer 1: yes

In brief, we will show that every relativistic spacetime whose underlying manifold is diffemorphic to $\mathbb{R}^n$ determines a G-structured space, and this G-structured space is capable of recovering, in the sense described in section 2.1, the smooth manifold and the metric of the relativistic spacetime that we began with.

Suppose that we have a relativistic spacetime (M, g_{ab}) such that M is diffeomorphic to $\mathbb{R}^n$. We need to say what the 'privileged coordinates' of (M, g_{ab}) are. We do this by showing how one naturally builds a G-structured space (S, C) from (M, g_{ab}); the maps in C will be the privileged coordinates on (M, g_{ab}). We will then look to this G-structured space (S, C) to recover (M, g_{ab}).

We turn to our the definition now. There is, by assumption, a diffeomorphism $\phi : M \to \mathbb{R}^n$. This means that the pushforward $\phi_*(g_{ab})$ of g_{ab} to $\mathbb{R}^n$ is a metric on $\mathbb{R}^n$. We build a G-structured space (S, C) as follows.

- Let $S = M$.
- Let C be the collection of diffeomorphisms $f : M \to \mathbb{R}^n$ such that $f_*(g_{ab}) = \phi_*(g_{ab})$.

- Let G be the the set of diffeomorphisms $s : \mathbb{R}^n \to \mathbb{R}^n$ such that $s^*(\phi_*(g_{ab})) = \phi_*(g_{ab})$.

One shows (Lemma 3.1.1 in the appendix) that this (S, C) is indeed a G-structured space. We will call (S, C) **a G-structured space determined by** (M, g_{ab}). Note that for now we cannot call it *the* G-structured space determined by (M, g_{ab}), since we had to arbitrarily pick a diffeomorphism $\phi : M \to \mathbb{R}^n$ to complete our construction. We will show in a moment that this choice actually makes no difference. We first state an important result.

Proposition 3.1.1. *Let (S, C) be a G-structured space determined by (M, g_{ab}). Then both of the following hold:*

1. *The identity map 1_M is a diffeomorphism between the manifold (S, C^+) and M.*
2. *The coordinate transformation group Γ on S is the isometry group of (M, g_{ab}), i.e. the collection of diffeomorphisms $f : M \to M$ such that $f^*(g_{ab}) = g_{ab}$.*

(Note that 1 and 2 make sense since $S = M$.)

This result captures the close relationship that (S, C) bears to the relativistic spacetime that determines it. The first clause implies that (S, C) recovers the manifold structure of M. And the second clause implies that the coordinate transformation group Γ is closely related to the metric g_{ab}; it contains all and only the diffeomorphisms that preserve g_{ab}.

Recall that we made an arbitrary choice of diffeomorphism $\phi : M \to \mathbb{R}^n$ when defining (S, C). Fortunately, had we chosen a different diffeomorphism $\psi : M \to \mathbb{R}^n$ in our construction, the resulting G-structured space would be isomorphic to (S, C). In order to demonstrate this, we need to first discuss the conditions under which two G-structured spaces are isomorphic. Let (S, C) and (S', C') be G- and G'-structured spaces. A map $f : S \to S'$ is an **isomorphism** between them if the following two conditions obtain:

1. The map $f : S \to S'$ is a diffeomorphism between the manifolds (S, C^+) and (S', C'^+).
2. The map $s \mapsto f \circ s \circ f^{-1}$ is a bijection $\Gamma \to \Gamma'$ between the coordinate transformation groups on S and S'. (Note that this map is guaranteed to be a group isomorphism if it is a bijection, since one can show trivially that it preserves composition.)

It is worth taking a moment to discuss this notion of isomorphism. Isomorphic G-structured spaces recover the same manifold structure. This is what clause 1 of the definition guarantees. Clause 2 guarantees that they recover the same tensor fields in the following sense.

Proposition 3.1.2. *Suppose that $f : S \to S'$ is an isomorphism between (S, C) and (S', C') and α is a tensor field (of arbitrary index structure) on S. Then α is implicitly defined by Γ if and only if $f_*(\alpha)$ is implicitly defined by Γ'.*

Suppose that (S, C) and (S', C') are G- and G'-structured spaces and $f : S \to S'$ is an isomorphism between them. Now equip the manifolds (S, C^+) and (S', C'^+) with the smooth tensor fields that are implicitly defined by their coordinate transformation groups Γ and Γ'. Then $f : S \to S'$ is an isomorphism between these two geometric spaces in the sense that it is a diffeomorphism from (S, C^+) to (S', C'^+) and for any smooth tensor field α on (S', C'^+), α is one of the smooth tensor fields that we have equipped S' with if and only if $f^*(\alpha)$ is one of the smooth tensor fields we have equipped S with. Proposition 3.1.2 thus provides us with a sanity check on our definition of isomorphism between G-structured spaces. It shows us that isomorphisms do indeed preserve all of the structure that we 'care about' on G-structured spaces.

We now have the following result.

Proposition 3.1.3. *Let (S, C) and (S', C') be G- and G'-structured spaces determined by (M, g_{ab}). Then (S, C) and (S', C') are isomorphic.*

This proposition implies that it makes sense to talk about **the G-structured space determined by** (M, g_{ab}). All G-structured spaces determined by a particular relativistic spacetime are isomorphic. This means that we have a method of recovering a unique (up to isomorphism) G-structured space (S, C) from a relativistic spacetime (M, g_{ab}).

We finally have the resources to provide our affirmative answer to Question 1. Let (M, g_{ab}) be a relativistic spacetime with M diffeomorphic to $\mathbb{R}^n$, and consider the G-structured space (S, C) determined by (M, g_{ab}). Proposition 3.1.1 implies that (S, C^+) has the same manifold structure as M, and moreover, that the metric g_{ab} is implicitly defined by Γ, meaning that (S, C) recovers this metric structure. This captures a sense in which the G-structured space (S, C) determined by (M, g_{ab}) recovers all of the structures of (M, g_{ab}). It recovers its manifold structure and it recovers its metric structure.

Answer 2: no

We now turn to the negative answer to Question 1. In order to present it, we need some further preliminaries. Recall that the **isometry group** of a relativistic spacetime (M, g_{ab}) is the group of diffeomorphisms $f : M \to M$ such that $f^*(g_{ab}) = g_{ab}$. An object has a **trivial isometry group** if its only isometry is the identity map.

We have the following two simple results.

Proposition 3.1.4. *There are non-isometric relativistic spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ with trivial isometry groups.*

Proposition 3.1.5. *Suppose that (S, C) and (S', C') are G- and G'-structured space with trivial coordinate transformation groups Γ and Γ' (i.e. both contain only the identity map). Then (S, C) and (S', C') are isomorphic.*

The idea behind Proposition 3.1.4 is simple. It is similar to the existence results proven by Barrett et al. (2023), and it follows directly from the results of Manchak and Barrett (2024). It guarantees that there are different metrics g_{ab} and g'_{ab} that one can lay down on $\mathbb{R}^2$ that are so asymmetric that the only isometry they admit is the identity map. And the sense in which the metrics g_{ab} and g'_{ab} are 'different' is strong; they are not isometric. Proposition 3.1.5 provides a sufficient condition for two G- and G'-structured spaces to be isomorphic. They are isomorphic if their their coordinate transformation groups are trivial. Intuitively, this is because the two spaces agree in all the respects that we care about; they recover the same manifold (namely, $\mathbb{R}^n$), and since they have the same trivial coordinate transformation group they recover the same tensor fields too.

We now use these two propositions to prove the following result, which is the key ingredient in our negative answer to Question 1.

Theorem 3.1.1. *There are non-isometric relativistic spacetimes* $(\mathbb{R}^2, g_{ab})$ *and* $(\mathbb{R}^2, g'_{ab})$ *that determine isomorphic G-structured spaces.*

Proof. Proposition 3.1.4 implies that there are non-isometric relativistic spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ with trivial isometry groups. Let (S, C) be the G-structured space determined by $(\mathbb{R}^2, g_{ab})$ and (S', C') be the G'-structured space determined by $(\mathbb{R}^2, g'_{ab})$. Since $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ have trivial isometry groups, clause 2 of Proposition 3.1.1 implies that both (S, C) and (S', C') have trivial coordinate transformation groups Γ and Γ' (i.e. only contain the identity map). Proposition 3.1.5 then implies that (S, C) and (S', C') are isomorphic. $\square$

Theorem 3.1.1 implies that the answer to Question 1 is "no". In brief, it shows that the framework of G-structured spaces lacks the expressive resources of the standard differential geometric framework (even when we restrict the underlying manifolds to just $\mathbb{R}^n$). One cannot use it to present both of the spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ from the theorem. Since they determine isomorphic G-structured spaces, they 'look the same' from the perspective of the Kleinian framework. The structural differences between some non-isometric spacetimes are therefore 'washed out' when one moves to the framework of G-structured spaces. This means the G-structured space that a relativistic spacetime determines does not provide a perfect guide to the structure that the spacetime has. The information provided to us by a G-structured space — or in other words, the collection of 'privileged global coordinates' on a spacetime — simply does not tell us everything about the structure of the spacetime.

We can isolate exactly what went wrong with our initial affirmative answer. We showed that there is a sense in which every relativistic spacetime is recoverable from the G-structured space that it determines. All of the structures of (M, g_{ab}) — both the manifold and the metric — are recoverable from the G-structured space (S, C) that it determines. But it may be that these are not the only structures recoverable. Indeed, it may be that (S, C) is such that many different metrics are implicitly defined by its coordinate transformation group Γ. In cases where the relativistic spacetime has a trivial isometry group,

(S, C) implicitly defines every metric on M. In such a case, clause 2 of Proposition 3.1.1 implies that Γ only contains the identity map. And every metric is invariant under the identity map. The sense in which (S, C) actually recovers (M, g_{ab}) is therefore weak. If it recovers (M, g_{ab}), then in the same sense it recovers many other non-isometric spacetimes. This is the exact issue that Theorem 3.1.1 isolates. In essence, we are simply pointing out that the recovery procedure described in section 2.1 — despite how natural it appears — is not well-defined. There may be more than one metric implicitly defined by the coordinate transformation pseudogroup Γ on a G-structured space (S, C), and when that is the case, one does not recover a unique relativistic spacetime from (S, C).

3.2 Local coordinates

The same exact arguments from the global case go through in the local case as well. Our question now is whether all relativistic spacetimes, regardless of their underlying manifold, can be recovered from a *locally* G-structured space.

Question 1 (Local case)**.** *Can every relativistic spacetime be recovered from a locally G-structured space?*

It is natural to think that one has a better chance of providing a compelling affirmative answer to Question 1 in the local case than one did in the global case. As we mentioned at the end of section 2.2, the coordinate transformation pseudogroup contains more information than the coordinate transformation group does. It admits many more maps, and therefore has greater potential to encode all of the geometric structures that we might want to present.

We begin again with an affirmative answer to Question 1.

Answer 1: yes

We first need to define the locally G-structured space that a relativistic spacetime (M, g_{ab}) determines. As before, this is tantamount to saying what the 'privileged coordinates' of (M, g_{ab}) are. We will build a locally G-structured space (S, C) from (M, g_{ab}), and the maps in C will then be the privileged coordinates on (M, g_{ab}). We will then look to this locally G-structured space (S, C) to recover (M, g_{ab}).

In order to build (S, C), we need to rely on one crucial fact. We will say that a relativistic spacetime $(\mathbb{R}^n, g'_{ab})$ is a **representation of** (M, g_{ab}) if for every point $p \in M$, there are open sets $O \subset M$ and $O' \subset \mathbb{R}^n$ such that $p \in O$ and (O, g_{ab}) is isometric to (O', g'_{ab}). Intuitively, a representation of (M, g_{ab}) is just a spacetime with underlying manifold $\mathbb{R}^n$ that 'reflects' the structure of (M, g_{ab}) in the following precise sense. Around each point $p \in M$, there is an open set that is isometric to an open set in the representation. Note that if (M, g_{ab}) is a flat spacetime, then Minkowski spacetime $(\mathbb{R}^n, \eta_{ab})$ is a representation of it (O'Neill, 1983, p. 223). With some work, one can show (Lemma 3.2.2 in the appendix) that every relativistic spacetime has a representation.

This fact provides us with a method of constructing a locally G-structured space from a relativistic spacetime (M, g_{ab}). We will simply let the 'privileged coordinates' on (M, g_{ab}) be those partial functions from (M, g_{ab}) to some fixed representation $(\mathbb{R}^n, g'_{ab})$ that preserve the metrics. In particular, let (M, g_{ab}) be a relativistic spacetime with $(\mathbb{R}^n, g'_{ab})$ a representation of it. We then define the following.

- Let $S = M$.
- Let G be the isometry pseudogroup of $(\mathbb{R}^n, g'_{ab})$.
- Let C be the collection of isometries between open subsets of (M, g_{ab}) and open subsets of $(\mathbb{R}^n, g'_{ab})$, i.e. diffeomorphisms $c : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $c^*(g'_{ab}) = g_{ab}|_O$.

One shows (Lemma 3.2.3 in the appendix) that this (S, C) is indeed a locally G-structured space. We will call it a **locally G-structured space determined by** (M, g_{ab}). We cannot yet call it *the* locally G-structured space determined by (M, g_{ab}), since we had to arbitrarily pick a representation of (M, g_{ab}) to complete our construction and we have no guarantee that this representation is unique. We will show shortly that this choice makes no difference, but we first need to catalogue how this space is related to the relativistic spacetime (M, g_{ab}) that determined it.

Proposition 3.2.1. *Let (M, g_{ab}) be a relativistic spacetime. If (S, C) is a locally G-structured space determined by (M, g_{ab}), then*

1. *The identity map 1_M is a diffeomorphism between (S, C^+) and M.*
2. *The coordinate transformation pseudogroup Γ associated with (S, C) is the isometry pseudogroup of (M, g_{ab}).*

(Note that both 1 and 2 make sense since $S = M$.)

The content of Proposition 3.2.1 is precisely the same as its global analogue 3.1.1. The first clause guarantees that (S, C) recovers the manifold structure of M; the second clause guarantees that the coordinate transformation pseudogroup on (S, C) bears a close relationship to the metric g_{ab}.

One wonders whether by picking different representations of (M, g_{ab}), one might end up with different locally G-structured spaces (S, C). But while different representations can result in non-equal locally G-structured spaces, we will here show that different choices of $(\mathbb{R}^n, g'_{ab})$ will always result in *isomorphic* locally G-structured spaces. The definition of isomorphism between locally G-structured spaces parallels the global case. Let (S, C) and (S', C') be locally G- and G'-structured spaces, respectively. An **isomorphism** $f : (S, C) \to (S', C')$ is a bijection $f : S \to S'$ such that

1. f is a diffeomorphism between (S, C^+) and (S', C'^+) and

2. the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection between Γ and Γ', the pseudogroups associated with (S, C) and (S', C'). (Note that this map is guaranteed to preserve composition between the pseudogroups Γ and Γ', in the sense that the image of the composition of two elements of Γ is the composition of their images.)

An isomorphism preserves the smooth manifold structure (clause 1) and the pseudogroups (clause 2) that the spaces inherit, so that all structures invariant under the one correspond to structures invariant under the other. We have an exact analogue to Proposition 3.1.2.

Proposition 3.2.2. *Suppose that $f : S \to S'$ is an isomorphism between locally G- and G'-structured spaces (S, C) and (S', C') and let α be a tensor field (of arbitrary index structure) on S. Then α is implicitly defined by Γ if and only if $f_*(\alpha)$ is implicitly defined by Γ'.*

This result again provides us with a 'sanity check' on our definition of isomorphism between locally G-structured spaces. It shows us that isomorphisms, as we have defined them, do indeed preserve all of the structure that we 'care about' on locally G-structured spaces. They preserve both the smooth manifold structure and the tensors recovered on that manifold via implicit definability.

We can now show that our translation from relativistic spacetimes to the underlying locally G-structured spaces they determine is well-defined. The following proposition guarantees that we will end up with the same (up to isomorphism) locally G-structured space, regardless of which representation of (M, g_{ab}) we choose.

Proposition 3.2.3. *Let (M, g_{ab}) be a relativistic spacetime. Suppose that (S, C) and (S', C') are locally G-structured spaces determined by (M, g_{ab}). Then (S, C) and (S', C') are isomorphic.*

We can therefore call (S, C) **the locally G-structured space determined by** (M, g_{ab}).

It is now easy to see our affirmative answer to Question 1. There is a sense in which the locally G-structured space (S, C) determined by (M, g_{ab}) can recover the relativistic spacetime (M, g_{ab}). The sense of recovery is captured by Proposition 3.2.1, and is exactly analogous to the global case. The manifold structure (S, C^+) that (S, C) naturally inherits is the same as that of M; we know this since 1_M is a diffeomorphism between the two manifolds. And moreover, (S, C) naturally inherits the metric structure g_{ab}, in the sense its coordinate transformation pseudogroup Γ implicitly defines g_{ab}, i.e. all of the maps in Γ are isometries with respect to g_{ab}. This is because Proposition 3.2.1 guarantees Γ is the isometry pseudogroup of (M, g_{ab}). Since we have defined this 'translation' for every relativistic spacetime (M, g_{ab}), this captures a sense in which every relativistic spacetime can be recovered from the locally G-structured space that it determines.

Answer 2: no

As in the global case, however, there is a more compelling negative answer. There are non-isometric relativistic spacetimes that determine the same locally G-structured space. We show this exactly as before. We will say that a pseudogroup on a manifold M is **trivial** if it only contains identity maps. In addition, we will say that a relativistic spacetime (M, g_{ab}) is **Heraclitus** if, for any open subsets $U, V \subset M$ and any isometry $\psi : U \to V$, it follows that (i) $U = V$ and (ii) ψ is the identity map. Manchak and Barrett (2024) show that a Heraclitus spacetime exists. Such a spacetime is maximally asymmetric; no two regions of it look alike. One can easily verify that the isometry pseudogroup of (M, g_{ab}) is trivial if and only if (M, g_{ab}) is Heraclitus. And hence there are spacetimes with trivial isometry pseudogroups.

We have the following two simple results, both of which are exact analogues to Propositions 3.1.4 and 3.1.5.

Proposition 3.2.4. *There are non-isometric relativistic spacetimes* $(\mathbb{R}^2, g_{ab})$ *and* $(\mathbb{R}^2, g'_{ab})$ *with trivial isometry pseudogroups.*

Proposition 3.2.5. *Let* (S, C) *and* (S', C') *be locally* G*- and* G'*-structured spaces with trivial coordinate transformation pseudogroups* Γ *and* Γ'*. If the manifolds* (S, C^+) *and* (S', C'^+) *are diffeomorphic, then* (S, C) *and* (S', C') *are isomorphic.*

And we now have our main result.

Theorem 3.2.1. *There are non-isometric relativistic spacetimes* $(\mathbb{R}^2, g_{ab})$ *and* $(\mathbb{R}^2, g'_{ab})$ *that determine isomorphic locally* G*-structured spaces.*

Proof. Proposition 3.2.4 implies that there are non-isometric relativistic spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ with trivial isometry pseudogroups. Let (S, C) be the locally G-structured space determined by $(\mathbb{R}^2, g_{ab})$ and (S', C') be the locally G'-structured space determined by $(\mathbb{R}^2, g'_{ab})$. Since $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ have trivial isometry pseudogroups, clause 2 of Proposition 3.2.1 implies that both (S, C) and (S', C') have trivial coordinate transformation pseudogroups Γ and Γ'. Proposition 3.2.1 also implies that (S, C^+) and (S', C'^+) are diffeomorphic, since they are both diffeomorphic to $\mathbb{R}^2$. So Proposition 3.2.5 implies that (S, C) and (S', C') are isomorphic. □

This is exactly analogous to the global case, but it is worth briefly discussing. Recall that our affirmative answer to Question 1 showed that there is a sense in which every relativistic spacetime is recoverable from the locally G-structured space that it determines. The problem with that argument was again that the sense of 'recovery' at play was weak. As the results here demonstrate, it still may be that (S, C) is such that *many* different metrics are implicitly defined by its coordinate transformation pseudogroup Γ, even though now the coordinate transformation pseudogroup Γ admits more maps than the mere coordinate transformation group did. Indeed, in the examples we have been considering, where the relativistic spacetime has a trivial isometry pseudogroup, the locally G-structured space (S, C) determined by the relativistic spacetime (M, g_{ab}) implicitly defines *every* metric on M, since Γ only contains identity maps, and so every metric on M is implicitly defined by Γ.

We have provided an example of two non-isometric spacetimes with (up to isomorphism) the same underlying locally G-structured space. So the answer to Question 1 is "no". This means that the structural differences between non-isomorphic spacetimes are sometimes lost when we move to the framework of locally G-structured spaces. In showing this, we have shown that the 'privileged local coordinates' of a relativistic spacetime do not fully capture its structure. The information provided to us by a spacetime's underlying locally G-structured space — or in other words, its collection of 'privileged local coordinates' — does not tell us everything about the structure of the spacetime.

4 When does the Kleinian method work?

We conclude by inquiring into the conditions under which privileged coordinates *do* tell us everything about the structure of a relativistic spacetime. We will show that for a certain class of (highly symmetric) spacetimes, their structure can be perfectly presented using the Kleinian method.

4.1 Global coordinates

We consider the following question.

Question 2 (Global case)**.** *Under what conditions can one present a relativistic spacetime (whose underlying manifold is diffeomorphic to $\mathbb{R}^n$) by providing a G-structured space?*

We have seen that there are cases where one cannot do this. Theorem 3.1.1 shows that certain highly asymmetric spacetimes admit too few privileged coordinates for their full structure to be encoded in a G-structured space. This suggests, however, that if a spacetime admits 'enough' symmetries, in the sense that its isometry group is sufficiently large, the G-structured space it determines encodes its structure.

We will now show that a spacetime is presentable using this Kleinian apparatus just in case its symmetries 'determine' its structure. We need two definitions. First, we will say that a relativistic spacetime (M, g_{ab}) is **determined by isometry** if all relativistic spacetimes (M, g'_{ab}) with the same isometry group as (M, g_{ab}) are isometric to (M, g_{ab}). It only takes a moment to unravel the idea behind this definition. If a spacetime (M, g_{ab}) is determined by isometry, then its isometry group uniquely determines the spacetime. This is because if some spacetime admits the same isometries as (M, g_{ab}), then it must be isometric to (M, g_{ab}). We might say that if one knows the isometry group of such a spacetime, one can know the structure of the spacetime.

The second definition that we require is slightly more subtle. Let (M, g_{ab}) be a relativistic spacetime that determines the G-structured space (S, C). We will say that (M, g_{ab}) is **globally presentable** if for all relativistic spacetimes (M', g'_{ab}) (that determine the G-structured space (S', C')), if (S, C) and (S', C') are isomorphic, then (M, g_{ab}) and (M', g'_{ab}) are isometric. A spacetime (M, g_{ab}) that is globally presentable is one that can be genuinely recovered from its underlying G-structured space (S, C). If some relativistic spacetime has the same underlying G-structured space (up to isomorphism) as (M, g_{ab}), then that spacetime is guaranteed to be isometric to (M, g_{ab}). This means that the structure of (S, C) determines the structure of (M, g_{ab}) (up to isometry). One can also put this basic idea in the following manner. If one knows (S, C), one will know (M, g_{ab}) too, since it's the only relativistic spacetime that determines a G-structured space isomorphic to (S, C). Theorem 3.1.1 provides an example of a spacetime $(\mathbb{R}^2, g_{ab})$ that is not globally presentable, since there is a non-isometric $(\mathbb{R}^2, g'_{ab})$ that nonetheless determines an isomorphic G-structured space. Globally presentable spacetimes are thus those whose privileged global coordinates fully reflect their structure.

We now have the following result. G-structured spaces can be used present all and only those relativistic spacetimes that are determined by isometry. Since the proof of Theorem 4.2.1 is perfectly analogous, we skip the proof here.

Theorem 4.1.1. *Let (M, g_{ab}) be a relativistic spacetime whose underlying manifold M is diffeomorphic to $\mathbb{R}^n$. Then (M, g_{ab}) is globally presentable if and only if it is determined by isometry.*

One naturally wonders which spacetimes are determined by isometry. One can show that at least Minkowski spacetime is.

Proposition 4.1.1. *Minkowski spacetime is determined by isometry.*

In conjunction with Theorem 4.1.1 and Example 1, this result implies that Minkowski spacetime can be presented using the apparatus of G-structured spaces. In Example 1 we saw that there is a G-structured space (S, C) that recovers Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$. Proposition 4.1.1 and Theorem 4.1.1 imply that η_{ab} is (up to isometry) the only metric on (S, C^+) is implicitly defined by the coordinate transformation group Γ. For suppose that Γ implicitly defines some metric g_{ab}. One can easily verify that the G-structured space determined by $(\mathbb{R}^4, g_{ab})$ is isomorphic to (S, C). Since Minkowski spacetime is globally presentable, this implies that $(\mathbb{R}^4, g_{ab})$ is isometric to Minkowski spacetime.

It is also easy to see that an analogue of Proposition 4.1.1 holds for Euclidean space, and this substantiates the story that we told at the outset. Euclidean space can indeed be presented by singling out a collection of privileged coordinates. But Minkowski spacetime and Euclidean space have incredibly rich isometry groups, and one conjectures that there are few other geometric spaces that are determined by isometry, and hence few relativistic spacetimes — even with underlying manifold $\mathbb{R}^n$ — that can be presented in the framework of G-structured spaces.

4.2 Local coordinates

An analogous result holds in the local case. We begin with the local analogue of Question 2.

Question 2 (Local case)**.** *Under what conditions can one present a relativistic spacetime by providing a locally G-structured space?*

Theorem 3.1.1 implies that there are cases where one cannot do this. We now demonstrate that one can do this just in case the *local* symmetries of the relativistic spacetime 'determine' its structure. We need the following two definitions. First, we will say that a relativistic spacetime (M, g_{ab}) is **determined by local isometry** if all relativistic spacetimes (M, g'_{ab}) with the same isometry pseudogroup as (M, g_{ab}) are isometric to (M, g_{ab}). Second, let (M, g_{ab}) be a relativistic spacetime that determines the locally G-structured space (S, C). We will say that (M, g_{ab}) is **locally presentable** if for any relativistic spacetimes (M', g'_{ab}) (that determines the locally G'-structured space (S', C')), if (S, C) and (S', C') are isomorphic, then (M, g_{ab}) and (M', g'_{ab}) are isometric.

The idea behind these two definitions is exactly the same as in the global case. If a spacetime (M, g_{ab}) is determined by local isometry, it is possible to 'know what the spacetime is' just by looking at its isometry pseudogroup. And a spacetime (M, g_{ab}) that is locally presentable is one that can be genuinely 'recovered' from its underlying locally G-structured space (S, C). Since Minkowski spacetime is determined by isometry, it is also determined by local isometry. But our two Heraclitus spacetimes from the proof of Theorem 3.2.1 are not. They are non-isometric but have the same trivial isometry pseudogroup.

We now have the following analogue of Theorem 4.1.1. Locally G-structured spaces can be used present all and only those relativistic spacetimes that are determined by local isometry. We provide the proof in full.

Theorem 4.2.1. *Let (M, g_{ab}) be a relativistic spacetime. Then (M, g_{ab}) is locally presentable if and only if it is determined by local isometry.*

Proof. Suppose first that (M, g_{ab}) is locally presentable. Let (M, g'_{ab}) be a relativistic spacetime that has the same isometry pseudogroup as (M, g_{ab}). Let (S', C') be the locally G'-structured space determined by (M, g'_{ab}) and (S, C) the locally G-structured space determined by (M, g_{ab}). Proposition 3.2.1 implies that the identity maps $1_M : S' \to M$ and $1_M : S \to M$ (which make sense since $S = M = S'$) are diffeomorphisms from (S', C'^+) to M and from (S, C^+) to M, and that Γ' and Γ are the isometry pseudogroups of (M, g'_{ab}) and (M, g_{ab}). This means first that $1_M : S \to S'$ is a diffeomorphism, and hence satisfies clause 1 of the definition of isomorphism. And it means second that $\Gamma = \Gamma'$, since (M, g'_{ab}) and (M, g_{ab}) have the same isometry pseudogroup. Hence $1_M : S \to S'$ satisfies clause 2 of the definition of isomorphism too, and therefore 1_M is an isomorphism between (S, C) and (S', C'). Since (M, g_{ab}) is locally presentable, (M, g_{ab}) and (M, g'_{ab}) are isometric, and hence (M, g_{ab}) is determined by isometry.

Now suppose that (M, g_{ab}) is determined by isometry and let (M', g'_{ab}) be a relativistic spacetime. Suppose that $f : S \to S'$ is an isomorphism between

(S, C) and (S', C'), the locally G- and G'-structured spaces determined by (M, g_{ab}) and (M, g'_{ab}), respectively. We need to show that (M, g_{ab}) and (M', g'_{ab}) are isometric. We first show that $f : M \to M'$ is a diffeomorphism. (Note that f is a function $M \to M'$ since $S = M$ and $S' = M'$.) Since f is an isomorphism, we know that it is a diffeomorphism between the manifolds (S, C^+) and (S', C'^+). Proposition 3.2.1 implies that $1_M : M \to S$ is a diffeomorphism between M and (S, C^+) and $1_{M'} : S' \to M'$ is a diffeomorphism between (S', C'^+) and M', so the composition $f = 1_{M'} \circ f \circ 1_M : M \to M'$ is a diffeomorphism.

Consider the metric $f^*(g'_{ab})$ on M. We show that (M, g_{ab}) and $(M, f^*(g'_{ab}))$ have the same isometry pseudogroup. First, suppose that $h : U \to V$ is in the isometry pseudogroup of $(M, f^*(g'_{ab}))$, so $h^*(f^*(g'_{ab})) = f^*(g'_{ab})|_U$. This immediately implies that $f_* \circ h^* \circ f^*(g'_{ab}) = g'_{ab}$, so $(f \circ h \circ f^{-1})^*(g'_{ab}) = g'_{ab}$. Since Γ' is by Proposition 3.2.1 the isometry pseudogroup of (M', g'_{ab}), this means that $f \circ h \circ f^{-1} \in \Gamma'$. Clause 2 of the definition of isomorphism implies that $f^{-1} \circ f \circ h \circ f^{-1} \circ f \in \Gamma$, so $h \in \Gamma$. Proposition 3.2.1 then implies that $h : U \to V$ is in the isometry pseudogroup of (M, g_{ab}). Second, suppose that $h : U \to V$ is in the isometry pseudogroup of (M, g_{ab}), so $h^*(g_{ab}) = g_{ab}|_U$. Proposition 3.2.1 implies that $h \in \Gamma$. Clause 2 of the definition of isomorphism implies that $f \circ h \circ f^{-1} \in \Gamma'$. Proposition 3.2.1 then implies that $(f \circ h \circ f^{-1})^*(g'_{ab}) = g'_{ab}$, which means that $h^*(f^*(g'_{ab})) = f^*(g'_{ab})$, so h is in the isometry pseudogroup of $(M, f^*(g_{ab}))$. Since (M, g_{ab}) and $(M, f^*(g'_{ab}))$ have the same isometry pseudogroup, the fact that (M, g_{ab}) is determined by isometry implies that (M, g_{ab}) and $(M, f^*(g'_{ab}))$ are isometric. Since $f : M \to M'$ is an isometry between $(M, f^*(g'_{ab}))$ and (M', g'_{ab}), this means that (M, g_{ab}) and (M', g'_{ab}) are isometric, and hence (M, g_{ab}) is locally presentable. □

This Kleinian framework can therefore present all and only those relativistic spacetimes whose collection of symmetries are sufficiently rich. There will be many relativistic spacetimes that can be presented in the framework of locally G-structured spaces if and only if there are many relativistic spacetimes that are determined by local isometry. There are certainly more spacetimes determined by local isometry than there are determined by isometry, but our results here imply that not all are. And indeed, one conjectures that spacetimes determined by local isometry are much more the exception than the rule. (See, for example, Proposition 1 of Sunada (1985).) We leave a careful investigation of this issue to further work.

5 Conclusion

In sum, our results demonstrate that the Kleinian method — when made precise using the framework of (locally) G-structured spaces — does not allow one to present all relativistic spacetimes. There is no Kleinian formulation of general relativity. And moreover, we have shown that the spacetimes that *can* be presented in this framework are those that are highly symmetric, in the precise sense that they are determined by (local) isometry. We conjecture that such spacetimes are rare. At the very least, our results imply that it is not always the case that privileged coordinates provide a perfect guide to structure.

We will conclude with two final movements. We will first look to the past and draw two connections between our results and the surrounding philosophy of physics literature. And second, we will look to the future and make some suggestions about potential routes forward for the proponent of Kleinian methods.

5.1 The Past

There are two important connections between our discussion of coordinates and the existing philosophy of physics literature. First, our results allow one to appreciate the close relationship between recent discussions of coordinates and longstanding debates about symmetry. It is common for philosophers and physicists to use a theory's symmetries as a guide to its underlying structure. (See, for example, Barrett (2018), Earman (1989), Dasgupta (2016, 2015), North (2009, 2021) and the references therein.) Weyl (1952, 144–45) famously remarked that "whenever you have to do with a structure-endowed entity X, try to determine its group of [symmetries], the group of those element-wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of X in this way."

Our results here imply that one can present a relativistic spacetime by singling out a collection of privileged coordinates (i.e. it is (locally) presentable) if and only if that spacetime's structure can be read off from its collection of symmetries (i.e. it is determined by (local) isometry). This is the content of Theorems 4.1.1 and 4.2.1. This means that a relativistic spacetime's privileged coordinates provide a perfect guide to its structure if and only if its symmetries provide a perfect guide to its structure. The question that we have been investigating — the extent to which a theory's privileged coordinates are a guide to its structure — is therefore closely related to the question investigated in much of the literature on symmetry — the extent to which a theory's symmetries are a guide to its structure.

The second connection we would like to draw concerns a precedent for our results. In particular, it has been remarked before that the move to general relativity represents a move to a Riemannian conception of geometry and away from a Kleinian conception. Norton (1999, p. 129) suggests that "Klein's strategy was entirely appropriate within the context of special relativity" and that "special relativity provided a beautiful illustration of the power of Klein's approach." Norton attributes to Cartan (1927) the thought that moving to general relativity "threw into physics and philosophy the antagonism that existed between the two principle directors of geometry, Riemann and Klein. The spacetimes of classical mechanics and special relativity are of the type of Klein, those of general relativity are of the type of Riemann" (Norton, 1999, p. 128). Norton and Cartan's basic idea is echoed by Wallace (2019), who writes the following:

> Norton regards the Kleinian approach to geometry as essentially su-

> perseded in contemporary spacetime physics. Specifically, he draws a contrast between two rival programs for the characterisation of geometry: Klein's, in which geometry is characterised via the invariance groups of the geometry under transformations, and Riemann's, in which geometry is characterised via metric tensors and similar differential geometric spaces. As Norton sees it, the move from special to general relativity is really a move from a Kleinian to a Riemannian conception of spacetime geometry.

Our results here sharpen and substantiate these thoughts. All and only those spacetimes determined by (local) isometry are presentable in our Kleinian framework. This is what Theorems 4.1.1 and 4.2.1 tell us. Minkowski spacetime, the setting for special relativity, is one such spacetime (by Proposition 4.1.1). This makes precise the idea that the Kleinian method is appropriate within the context of special relativity. But not every relativistic spacetime is determined by (local) isometry, which concretely shows that the Kleinian method breaks down when confronted with general relativistic geometric structure.

More importantly, we can clearly see exactly *why* the Kleinian method does not work in general relativity. One might have suspected that it is because of the flexibility one gains in general relativity to formulate a spacetime on an arbitrary manifold. But this is not the problem. For suppose that (M, g_{ab}) is a relativistic spacetime that determines a locally G-structured space (S, C). Proposition 3.2.1 guarantees that the manifold (S, C^+) that (S, C) recovers is diffeomorphic to M. This means that the manifold M is presentable using the Kleinian method. Presenting arbitrary manifolds is not the issue. Rather, the Kleinian approach fails because arbitrary *metrics* are not presentable using this the Kleinian apparatus. They are too flexible, and some admit too few symmetries to be characterized in a Kleinian manner.

Moreover, we can also see that the failure of Kleinian methods is robust. One might not be surprised that Kleinian methods fail when one only considers privileged global coordinates, as in the case of G-structured spaces. It is natural to think that the prospects are better when one also takes into account the privileged local coordinates of our geometric space. We have seen here that even this 'local' Kleinian method does not succeed.

5.2 The Future

We conclude with three possible routes forward for the proponent of the Kleinian method and (locally) G-structured spaces: restriction, revision, and reservation. Along the way, we will catalogue some questions for future work.

Restriction

First, a proponent of Kleinian methods might first argue that (locally) G-structured spaces suffice to present geometric spaces within some restricted domain. We have shown that the Kleinian method works just in case a relativistic spacetime is determined by (local) isometry. One potential route forward is to investigate which spacetimes are determined by (local) isometry. After doing so, one might argue that all 'physically reasonable' spacetimes have this property or that all spacetimes are (in some sense) sufficiently 'close' to spacetimes that are determined by (local) isometry.

Restriction 1. Are all 'physically reasonable' spacetimes determined by (local) isometry?

Restriction 2. Are all spacetimes sufficiently 'close' to spacetimes that are determined by (local) isometry?

If either question were answered in the affirmative then one might say that Kleinian methods come close to successfully presenting general relativity. It would be surprising were the answer to Restriction 1 "yes" since, as we mentioned above, we conjecture that spacetimes determined by (local) isometry are rare. Restriction 2 is more interesting, and it is closely related to recent debates about 'approximate symmetries' in general relativity. (See Fletcher (2020, 2021), Linnemann et al. (2024), and Fletcher and Weatherall (2023a,b)).

Relatedly, one might argue that (locally) G-structured spaces can be used to capture the content of some physical theories other than general relativity. Perhaps the symplectic manifolds of Hamiltonian mechanics or the flat classical spacetimes of standard Newtonian gravitation theory, for example, can be presented using Kleinian methods.

Restriction 3. Can other physical theories be presented using (locally) G-structured spaces?

This suggestion is promising. Other geometric spaces do not have the same kind of flexibility admitted by relativistic spacetimes. It would be difficult for one to design, for example, a flat classical spacetime or symplectic manifold that admits a small enough collection of symmetries to generate problems for the Kleinian approach, so we conjecture that Kleinian methods will work in other contexts.

Revision

The second potential route forward for the proponent of the Kleinian method involves revising the recovery procedure that one utilizes to build a relativistic spacetime from a (locally) G-structured space. We have shown that one particularly natural way to recover a relativistic spacetime from a locally G-structured space — described in section 2 — does not work. In brief, the problem is that this recovery procedure is not well-defined. It can be that more than one metric is invariant under the coordinate transformation (pseudo)group of a (locally) G-structured space, so equipping the manifold $(S, C+)$ with the tensors invariant under this (pseudo)group does not always result in a well-defined relativistic spacetime. One might be equipping the manifold with more than one Lorentzian metric. This is nevertheless the most natural recovery procedure one can come up with. Indeed, it is standard in the literature; the idea of using the 'privileged coordinates' to induce a collection of symmetries, and then looking to the 'invariants' under these symmetries is, for example, pointed to by Norton (2002) and Wallace (2019).

One might nonetheless try to find a better method. Instead of looking to the (locally) G-structured space determined by a spacetime (M, g_{ab}), one might think that another (locally) G-structured space offers a better chance of recovering the structure of (M, g_{ab}).

Revision 1. Is there a better candidate than the (locally) G-structured space determined by (M, g_{ab}) for recovering the structure of (M, g_{ab})?

Revision 1 is asking whether there is a better account of the 'privileged coordinates' of (M, g_{ab}) than the one we gave in sections 3.1 and 3.2. There are certainly other possible accounts. For example, given a relativistic spacetime (M, g_{ab}) with underlying manifold diffeomorphic to $\mathbb{R}^n$, one can consider a diffeomorphism $\phi : M \to \mathbb{R}^n$, and let $S = M$, $C = \{\phi, \phi^{-1}\}$, and $G = \{1_S\}$. It is easy to verify that this defines a G-structured space. The problem with this trivial recipe is that one will not be able to use this G-structured space to recover the structure of (M, g_{ab}). This is because applying this recipe to any two relativistic spacetimes will yield the same (up to isomorphism) G-structured space. And hence this answer to Revision 1 makes it so that one cannot recover the structure of (M, g_{ab}) from (S, C).

Alternatively, one might consider the Lorentz normal coordinates of the spacetime (M, g_{ab}). Lorentz normal coordinates about a point $p \in M$ are coordinates (U, ϕ) with $p \in U$ such that $\phi(p) = (0, \ldots, 0) \in \mathbb{R}^n$ and both the metric g_{ab} and its associated derivative operator 'take a simple form' at p in (U, ϕ) coordinates, in the sense that the 'Christoffel symbols' vanish at p and the metric g_{ab} takes Minkowskian form at p in (U, ϕ) coordinates. If one takes the privileged coordinates of (M, g_{ab}) to be its Lorentz normal coordinates, then one is forced to dramatically change the procedure with which one recovers the structure of (M, g_{ab}) from these privileged coordinates. This is because the resulting coordinate transformation pseudogroup Γ will not in general implicitly define g_{ab}. Consider, for example, the case when (M, g_{ab}) is Heraclitus. Let $p, q \in M$ be distinct points and suppose that we have Lorentz normal coordinates (U, ϕ) about p and (V, ψ) about q. We can consider the 'coordinate transformation' map $\psi^{-1} \circ \phi$, which maps a neighborhood of p to a neighborhood of q. It will be in Γ, but since (M, g_{ab}) is Heraclitus, $\psi^{-1} \circ \phi$ will only preserve g_{ab} if it is the identity map. And it cannot be the identity map since $\psi^{-1} \circ \phi(p) = \psi^{-1}(0, \ldots, 0) = q$. It may be possible to answer Revision 1 and give a different account of privileged coordinates than we have given above. But when doing so, one has to make sure that the spacetime is recoverable from those privileged coordinates.

This brings us to our second variety of revision. Instead of revising our account of privileged coordinates, one might revise our method of recovery. In particular, one might notice that the problem with the recovery procedure at heart has to do with implicit definability. The property of 'being invariant under' a collection of maps — like those in the coordinate transformation (pseudo)group — is known to be a particularly weak kind of 'definability' (Winnie, 1986; Barrett, 2018). So one might instead try to recover a metric on $(S, C+)$ by using some other kind of 'explicit' definability. This would involve somehow using the maps in C to explicitly 'build' or 'construct' a metric on (S, C^+), rather than simply looking to the metrics remain unchanged under coordinate transformations.

Revision 2. Is there a variety of 'explicit definability' that allows one to recover (M, g_{ab}) from the (locally) G-structured space that it determines?

Implementing this idea would require a substantial change in spirit. When philosophers of physics discuss the significance of privileged coordinates, it is most common to speak of those structures 'invariant under changes of coordinates'. And this suggests that some kind of implicit definability is the operant method of recovering structure, rather than explicit definability. For example, Norton (2002, p. 259) writes that under the Kleinian method a "geometric theory would be associated with a class of admissible coordinate systems and a group of transformations that would carry us between them. The cardinal rule was that physical significance can be assigned just to those features that were invariants of this group". Similarly, North (2021, p. 48) writes that "Klein suggested that any geometry can be identified by means of the transformations that preserve the structure, likewise by the quantities that are invariant under the group of those transformations". Wallace (2019, p. 135) remarks that the Kleinian method involves characterizing spaces "via the invariance groups of the geometry under transformations".

Of course, one might comfortable with a substantial revision of the Kleinian method. But two other challenges are worth mentioning. First, it is difficult to talk about explicit definability when one is not working within the confines of a formal language. Our best physical theories, like general relativity, are not formulated within such confines. Steps have been taken both in the direction of formulating these theories within frameworks where we can easily talk about explicit definability (see, for example, the work of Andréka and Németi (2014)) and in the direction of generalizing concepts like explicit definability — see for example the 'maximally structured' spacetimes of Manchak (2024), the concept of 'covariant definability' of Glymour (1977), and the category theoretic methods discussed by Halvorson (2019), Barrett (2021), and Weatherall (2019). The second more general challenge is that when revising the method of recovery one must take care to ensure that it does not collapse back into the standard Riemannian method. We will turn to this issue shortly.

Reservation

Finally, a proponent of Kleinian methods might argue for a more reserved conclusion. One might think that although the privileged coordinates of a relativistic spacetime do not tell us *everything* about its structure, they still tell us something. In particular, it might be that some information about a relativistic spacetime can be recovered from the (locally) G-structured space it determines.

One possibility is that the privileged coordinates of a relativistic spacetime capture the 'amount of structure' that the spacetime has.

Reservation 1. Does the (locally) G-structured space determined by a geometric space encode the 'amount of structure' that the geometric space has?

An affirmative answer is suggested, for example, when (North, 2021, Ch. 4) argues that standard Newtonian mechanics posits more structure than Lagrangian mechanics. Her idea is that since the former privileges a 'smaller' collection of coordinates than the latter, it will posit more structure, since there will be more features that are agreed upon by all of the coordinate systems in this smaller class. And these "coordinate-independent, invariant features, correspond to the intrinsic nature" of the space described by the theory (North, 2021, p. 26). North's argument has been questioned (Barrett, 2022), but the basic idea that privileged coordinates provide a window into *amounts* of structure is worth investigating carefully. We conjecture, however, that there can be spacetimes that posit different amounts of structure but nonetheless determine the same (locally) G-structured space. If so, then privileged coordinates would also not be a perfect guide to amounts of structure.

There is another variety of reservation that one might pursue.

Reservation 2. Can *some* of the structure of a geometric space be recovered from the (locally) G-structured space it determines?

Wallace (2019, p. 135) has noted a sense in which the answer to Reservation 2 is "yes". Part of every relativistic spacetime (M, g_{ab}) can be presented using a locally G-structured space. Indeed, the manifold M just *is* a locally G-structured space (M, C), where C is the standard atlas on M and G is the diffeomorphism pseudogroup of $\mathbb{R}^n$. So the underlying manifold structure of (M, g_{ab}) can trivially be presented using this Kleinian apparatus.

There are two points worth making about this affirmative answer to Reservation 2. First, one wonders whether this gives rise to a genuinely Kleinian presentation of general relativity. One might be tempted to say that we can take a kind of 'hybrid' approach to presenting the theory. In effect, one is changing the method of recovering the structure of (M, g_{ab}) from (S, C). One first uses Kleinian methods and the apparatus of locally G-structured spaces to present M, and one then defines the metric g_{ab} on M in the standard Riemannian manner (Wallace, 2019, p. 134–135). The issue here is that this is not a genuinely hybrid approach. Or at the very least, it is at heart no different from the standard differential geometric presentation of of relativistic spacetime (M, g_{ab}). As we have seen, a smooth manifold M in the standard differential geometric presentation is just a locally G-structured space. So while this approach does represent a legitimate and informative way of formulating general relativity, it does not represent one that is any different from the standard way of formulating general relativity.

One might be inclined to say that this means that geometry in modern physics is partially Kleinian in character (Wallace, 2019, p. 135). It does seem that there is an echo of these Kleinian methods lurking below the surface in general relativity. But bolder claims about coordinates and structure have been made, and this affirmative answer to Reservation 2 — the mere claim that the manifold structure of (M, g_{ab}) can be presented by appeal to privileged coordinates — does not suffice to substantiate them. For example, North (2021, p. 26) writes that "the coordinate-independent, invariant features, correspond to the intrinsic nature" of the space. If one includes as privileged coordinates the entire atlas C of M, as this variety of reservation suggests we do, then we will have 'too many' privileged coordinates. Indeed, the metric g_{ab} will not count as part of the 'intrinsic nature' of the spacetime because the resulting maps in Γ will not necessarily preserve g_{ab}. This variety of reservation is therefore too modest for proponents of the idea that privileged coordinates determine the structure of a geometric space.

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Appendix

The purpose of this appendix is to provide proofs of the propositions that appear in the paper. We organize the proofs by which section of the paper they appear in.

Proofs in section 2.1

Lemma 2.1.1. *Let M be a non-empty set with (M, c) an n-chart. If (U, ϕ) and (V, ψ) are n-charts on M that are both compatible with (M, c), then they are compatible with each other.*

Proof. Suppose that $U \cap V$ is non-empty. Since both of the charts (U, ϕ) and (V, ψ) are compatible with (M, c), we know that $c[U]$, $c[V]$, $\phi[U]$, and $\psi[V]$ are all open, and $c \circ \phi^{-1}$,$\phi \circ c^{-1}$, $c \circ \psi^{-1}$, and $\psi \circ c^{-1}$ are all smooth. First, we need to show that $\phi[U \cap V]$ and $\psi[U \cap V]$ are open. Since $c[U]$ and $c[V]$ are both open, $c[U] \cap c[V] = c[U \cap V]$ is open. Both $\phi \circ c^{-1}$ and its inverse $c \circ \phi^{-1}$ are smooth, which implies that $\phi \circ c^{-1} : c[U] \to \phi[U]$ is a homeomorphism and hence an open map. So since $c[U \cap V] \subset c[U]$ is open, $\phi \circ c^{-1}[c[U \cap V]] = \phi \circ 1_M[U \cap V] = \phi[U \cap V]$ is open. The second equality follows since U and V are subsets of M. One reasons in precisely the same manner to show that $\psi[U \cap V]$ is open. Second, we need to show that both $\phi \circ \psi^{-1} : \psi[U \cap V] \to \mathbb{R}^n$ and $\psi \circ \phi^{-1} : \phi[U \cap V] \to \mathbb{R}^n$ are smooth. This follows since each is equal to the composition of two smooth functions. For example, in the former case, the fact that U and V are subsets of M implies that $\phi \circ \psi^{-1} = \phi \circ 1_M \circ \psi^{-1} = (\phi \circ c^{-1}) \circ (c \circ \psi^{-1})$. Hence (U, ϕ) and (V, ψ) are compatible. $\square$

Proposition 2.1.1. *(S, C^+) is a smooth n-dimensional manifold.*

Proof. Each member $(c^{-1}[U], c|_{c^{-1}[U]})$ of C_0 is indeed an n-chart since $c|_{c^{-1}[U]} : c^{-1}[U] \to U$ is injective and U is open. Now by Proposition 1.1.1 of Malament (2012), it will suffice to show that C_0 satisfies conditions M1, M2, and M3. We take them in order.

(M1) Let $c \in C$ and consider the chart (S, c). We know that this chart is in C_0 since $S = c^{-1}[\mathbb{R}^n]$ and $\mathbb{R}^n$ is an open subset of itself. If we can show that every chart in C_0 is compatible with (S, c), then Lemma 2.1.1 will imply that any two charts in C_0 are compatible. So let $(d^{-1}[U], d|_{d^{-1}[U]})$ be a chart in C_0. First, we need to show that $d[d^{-1}[U] \cap S]$ and $c[d^{-1}[U] \cap S]$ are open subsets of $\mathbb{R}^n$. The former is just U, which we know to be open because $(d^{-1}[U], d|_{d^{-1}[U]})$ is in C_0. Since $d \in C$, the compatibility condition implies that both $c \circ d^{-1}$ and $d \circ c^{-1}$ are in G and hence diffeomorphisms, since G is a subset of the group of diffeomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$. So $c \circ d^{-1}$ and $d \circ c^{-1}$ are homeomorphisms and thus open maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. We know that U is open, so $c \circ d^{-1}[U] = c[d^{-1}[U] \cap S]$ is open too. Second, we need to show that both $c \circ d^{-1}$ and $d \circ c^{-1}$ are smooth. But we have already shown that they are diffeomorphisms, which immediately implies that they are smooth.

(M2) The domains of the charts in C_0 trivially cover S since (S, c) is a chart in C_0 for any $c \in C$.

(M3) Let p and q be distinct points in S and let $c \in C$. Since c is injective, $c(p)$ and $c(q)$ are distinct points in $\mathbb{R}^n$, and hence there are disjoint open sets U and V such that $c(p) \in U$ and $c(q) \in V$. Then the n-charts $(c^{-1}[U], c|_{c^{-1}[U]})$ and $(c^{-1}[V], c|_{c^{-1}[V]})$ are in C_0, they have disjoint domains, and $p \in c^{-1}[U]$ and $q \in c^{-1}[V]$. $\square$

Proposition 2.1.2. *Every $c \in C$ is a diffeomorphism from the smooth manifold (S, C^+) to $\mathbb{R}^n$.*

Proof. We know that c is a bijection. Since (S, c) is a global chart on (S, C^+) and $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ is a global chart on $\mathbb{R}^n$, the fact that $1_{\mathbb{R}^n} \circ c \circ c^{-1} = 1_{\mathbb{R}^n}$ is smooth implies (via (Lee, 2012, Prop 2.5)) that c is smooth. Similarly, since $c \circ c^{-1} \circ 1_{\mathbb{R}^n} = 1_{\mathbb{R}^n}$ is smooth, (Lee, 2012, Prop. 2.5) implies that c^{-1} is smooth. And hence $c : S \to \mathbb{R}^n$ is a diffeomorphism. $\square$

Lemma 2.1.2. $\Gamma = \{c^{-1} \circ s \circ c : c \in C, s \in G\}$.

Proof. ($\subset$) Let $c^{-1} \circ d \in \Gamma$. We know that $d \circ c^{-1} \in G$ by the compatibility condition. And since $c^{-1} \circ d = d^{-1} \circ (d \circ c^{-1}) \circ d$, $c^{-1} \circ d \in \{c^{-1} \circ s \circ c : c \in C, s \in G\}$. ($\supset$) Consider $c^{-1} \circ s \circ c$, where $s \in G$ and $c \in C$. Since $c \in C$

and $s \in G$, the compatibility condition implies that $s \circ c \in C$. This means that $c^{-1} \circ s \circ c \in \Gamma$. □

Proofs in Section 2.2

Definition 2.2.1. Let (X, τ) be a topological space. A **pseudogroup** on (X, τ) is a class G of transformations that satisfies the following conditions:

PG1. Each $f \in G$ is a homeomorphism from an open set $\mathrm{dom}(f) \subset X$ onto another open set $\mathrm{ran}(f) \subset X$.

PG2. If $f \in G$, then $f|_U \in G$ for every open set $U \subset \mathrm{dom}(f)$.

PG3. Let $U = \bigcup_i U_i$ where each U_i is an open set of X. A homeomorphism f of U onto an open set of X belongs to G if $f|_{U_i} \in G$ for every i.

PG4. For every open set U of X, the identity map $1_U : U \to U$ is in G.

PG5. If $f \in G$, then $f^{-1} \in G$.

PG6. If $f \in G$ is a homemorphism of U onto V and $f' \in G$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \circ f$ of $f^{-1}[V \cap U']$ onto $f'[V \cap U']$ is in G.

Lemma 2.2.1. *Let M be a manifold with $\{\alpha_i\}_{i \in I}$ a collection of smooth tensor fields (of arbitrary index structure) on M. Then the collection G of diffeomorphisms $f : U \to V$ between open subsets U and V of M such that for all $i \in I$ $f^*(\alpha_i) = (\alpha_i)|_U$ forms a pseudogroup.*

Proof. One needs to show that PG1–PG6 hold of G. PG1, PG4, and PG5 are straightfoward, so we show the others.

(PG2) Suppose that $f : U \to V$ is in G and let O be an open subset of U. We know that $f|_O : O \to f[O]$ is a diffeomorphism since f is a diffeomorphism. We see that at each point $p \in O$ and for each $i \in I$,

$$f|_O^*(\alpha_i)|_p = f^*(\alpha_i)|_p = \alpha_i|_p$$

The first equality holds since $f|_O$ and f agree on the open set O, and the second follows since $f \in G$ and thus $f^*(\alpha_i) = \alpha_i|_U$. Hence $f|_O^*(\alpha_i) = \alpha_i|_U$ and it must be that $f|_O \in G$.

(PG3) Let $U = \bigcup_i U_i$ where each U_i is an open set of M, and suppose that $f : U \to V$ is a homeomorphism such that $f|_{U_i}$ is in G for every i. Since each $f|_{U_i}$ is a diffeomorphism, (Lee, 2012, Corollary 2.8) implies that f is a diffeomorphism. Let $p \in U$. Since $p \in U_i$ for some i, we see that

$$f^*(\alpha_j)|_p = f|_{U_i}^*(\alpha_j)|_p = \alpha_j|_p$$

The first equality follows since $f|_{U_i}$ and f agree on the open set U_i, and the second since $f|_{U_i} \in G$. So it must be that $f^*(\alpha_j) = \alpha_j|_U$ for every j, and hence $f \in G$.

(PG6) Suppose that $f : U \to V$ is in G and $f' : U' \to V'$ is in G and $V \cap U'$ is non-empty. Since U, V, U', V' are all open and f and f' are diffeomorphisms, it must be that the domain $f^{-1}[V \cap U']$ and range $f'[V \cap U']$ of $f' \circ f$ are both open and $f' \circ f$ is a diffeomorphism. It is also easy to see that for each $p \in f^{-1}[V \cap U']$,

$$(f' \circ f)^*(\alpha_i)|_p = f^* \circ f'^*(\alpha_i|_{f' \circ f(p)}) = f^*(\alpha_i|_{f(p)})|_p = \alpha_i|_p$$

where the first equality follows from properties of the pullback, and the second and third follow since f and f' are both in G. Hence $f' \circ f$ is in G. □

Proposition 2.2.1. *(S, C^+) is a smooth n-dimensional manifold.*

Proof. We first remark that the pair $(\text{dom}(f), f)$ is an n-chart on S for each $f \in C$. The map $f : \text{dom}(f) \to \mathbb{R}^n$ is injective and $f[\text{dom}(f)]$ is an open subset of $\mathbb{R}^n$, since the range condition guarantees that it is the domain of one of the partial functions in G, all of which have open sets as their domains. By Proposition 1.1.1 in Malament (2012), it will suffice to show that C satisfies conditions M1, M2, and M3 listed there.

(M1) Let $(\text{dom}(f), f)$ and $(\text{dom}(g), g)$ be two n-charts in C. We show that they are compatible. Suppose that $U = \text{dom}(f) \cap \text{dom}(g)$ is non-empty. The compatibility condition implies that $f \circ g^{-1} : g[U] \to f[U]$ and $g \circ f^{-1} : f[U] \to g[U]$ are in G. Since G is contained in the pseudogroup of local diffeomorphisms between open sets of $\mathbb{R}^n$, both f and g are smooth. Since $f \circ g^{-1} : g[U] \to f[U]$ is in G, PG1 implies that both $f[U]$ and $g[U]$ are open, and hence the two charts are compatible.

(M2) It follows immediately from the cover condition that the domains of the charts in C cover S.

(M3) Let p and q be distinct points in S. Suppose that these two points cannot be separated by charts in C with disjoint domains. The cover condition then implies that there must be a chart $(\text{dom}(f), f)$ such that both p and q are in $\text{dom}(f)$. Consider the points $f(p)$ and $f(q)$ in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, there are open subsets U and V of $\text{ran}(f)$ that separate $f(p)$ and $f(q)$. By PG4, we know that the identity maps 1_U and 1_V are in G. This implies that $1_U \circ f \in C$ and $1_V \circ f \in C$. The domain of $1_U \circ f$ is $f^{-1}[U]$ and the domain of $1_V \circ f$ is $f^{-1}[V]$. One easily sees that $f^{-1}[U]$ and $f^{-1}[V]$ separate p and q in S. □

Proposition 2.2.2. *Every $c \in C$ is a smooth map $c : dom(c) \to \mathbb{R}^n$. And moreover, $c : dom(c) \to ran(c)$ is a diffeomorphism.*

Proof. This follows immediately from the fact that $(\text{dom}(c), c)$ is a chart in C^+ for every $c \in C$. □

Definition 2.2.2. The set of domains of the charts in C^+ form a basis for a topology on the manifold (S, C^+). We define the **coordinate transformation pseudogroup** on this topological space. We begin with the following definition.

$$\Gamma_0 = \{f^{-1} \circ g : g^{-1}[\text{ran}(f) \cap \text{ran}(g)] \to f^{-1}[\text{ran}(f) \cap \text{ran}(g)]$$

such that $f, g \in C$ and $\text{ran}(f) \cap \text{ran}(g)$ is non-empty$\}$

We can think of the maps in Γ_0 as the 'coordinate transformations' on S that are determined by the coordinate charts in C. We now define the pseudogroup Γ by 'closing' Γ_0 under unions of functions. A bijection $h : U \to V$ between open subsets U and V of S is in Γ if either of the following conditions hold:

(i) $h \in \Gamma_0$ or

(ii) there is a family of open sets U_i of S such that $U = \bigcup_i U_i$ and $h|_{U_i} \in \Gamma_0$ for every i.

We will call Γ the **coordinate transformation pseudogroup** on (S, C). We note for future reference that each $h : U \to V$ in Γ is a diffeomorphism from its domain to its range.

Lemma 2.2.2. *The coordinate transformation pseudogroup Γ is a pseudogroup on the topology inherited by (S, C^+).*

Proof. One shows that PG1–PG6 hold of Γ. We here show PG2, PG3, and PG6 and leave the others to the reader.

(PG2) Suppose first that $h \in \Gamma_0$, so $h = f^{-1} \circ g$ for maps $f, g \in C$. Let $O \subset \text{dom}(h)$ be open. Since $\text{dom}(h) \subset \text{dom}(g)$ and g is a diffeomorphism, it must be that $g[O]$ is open. PG4 implies that $1_{g[O]}$ is in G, and this means that $1_{g[O]} \circ g = g|_O$ is in C. Since $f^{-1} \circ g|_O = (f^{-1} \circ g)|_O$, this means that $(f^{-1} \circ g)|_O \in \Gamma_0$, and hence $h|_O \in \Gamma$. Now suppose that $h \in \Gamma - \Gamma_0$, so $h : \cup U_i \to V$ is a diffeomorphism and each $h|_{U_i}$ is in Γ_0. Let $O \subset \cup U_i$ be open. We know that each $(h|_{U_i})|_O = h|_{U_i \cap O}$ is in Γ_0 by the result earlier in this paragraph. And $h|_O : O \to h[O]$ is such that $O \cap U_i$ is open for each i, $O = \cup_i (O \cap U_i)$, and $(h|_O)|_{O \cap U_i} = h|_{O \cap U_i}$ is in Γ_0, which implies that $h|_O \in \Gamma$ by clause (ii) of our definition of Γ.

(PG3) Let $U = \cup_i U_i$ where each U_i is an open subset of S. Suppose that $h : U \to V$ is a homeomorphism and $h|_{U_i} \in \Gamma$ for each i. Since $h|_{U_i} : U_i \to h[U_i]$ is in Γ, there is a family of open sets O_{ij} such that $U_i = \cup_j O_{ij}$ and $(h|_{U_i})|_{O_{ij}} = h|_{O_{ij}}$ is in Γ_0 for every j. (Note that if $h|_{U_i}$ is itself in Γ_0, then this is trivial.) So $\cup_i U_i = \cup_{i,j} O_{ij}$, each O_{ij} is open, and $h : \cup_{i,j} O_{ij} \to V$ is such that $h|_{O_{ij}}$ is in Γ_0 for each pair i, j. This means that $h \in \Gamma$ by clause (ii) of the definition of Γ.

(PG6) Let $f : U \to V$ and $f' : U' \to V'$ be homeomorphisms in Γ. Suppose that $V \cap U'$ is non-empty. We aim to show that $f' \circ f$ is in Γ. First suppose that both f and f' are in Γ_0. So $f = g^{-1} \circ h$ and $f' = g'^{-1} \circ h'$, so $f' \circ f = g'^{-1} \circ h' \circ g^{-1} \circ h$. We know that $h' \circ g^{-1}$ is an element of G by the compatibility condition, so $h' \circ g^{-1} \circ h \in C$. This immediately implies that $g'^{-1} \circ (h' \circ g^{-1} \circ h) = f' \circ f \in \Gamma_0$. Now we proceed to the general case. We know that $f : \cup_i U_i \to V$ is a bijection, where each U_i is open and $f|_{U_i}$ is in Γ_0 for each i. And $f' : \cup_j U'_j \to V'$ is a bijection, where each U'_j is open and $f'|_{U'_j}$ is in Γ_0 for each j. (Note that this is trivially the case if f or f' is itself in Γ_0.) We know that $f^{-1}[V \cap U'_j] \cap U_i$ is open, since f is a homeomorphism and V, U'_j and U_i are all open. Now one can compute that $(f' \circ f)|_{f^{-1}[V \cap U'_j] \cap U_i} = f'|_{U'_j} \circ f|_{U_i}$. Since $f'|_{U'_j}$ and $f|_{U_i}$ are in Γ_0 by assumption, we have that $(f' \circ f)|_{f^{-1}[V \cap U'_j] \cap U_i}$ is in Γ_0 by the result earlier in this paragraph. Now $f' \circ f : f^{-1}[V \cap U'] \to f'[V \cap U']$ is such that $f^{-1}[V \cap U'] = \cup_{i,j} f^{-1}[V \cap U'_j] \cap U_i$, a union of open sets, and each $(f' \circ f)|_{f^{-1}[V \cap U'_j] \cap U_i}$ is in Γ_0, and therefore clause (ii) of the definition of Γ implies that $f' \circ f \in \Gamma$. □

Proofs in Section 3.1

Lemma 3.1.1. *Let (S, C) be a G-structured space determined by (M, g_{ab}). Then (S, C) is a G-structured space.*

Proof. We immediately see that G is a transformation group that is a subset of the group of diffeomorphisms of $\mathbb{R}^n$. So all that needs to be shown is that the compatibility condition is satisfied. Let $f \in C$. Suppose that $f' \in C$. We know that $f \circ f'^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism since both f and f'^{-1} are diffeomorphisms. We also see that

$$\phi_*(g_{ab}) = f'_* \circ f^*(\phi_*(g_{ab})) = (f \circ f'^{-1})^*(\phi_*(g_{ab}))$$

The first equality follows since f and f' are in C, and the second follows from basic facts about pullbacks. This implies that $f \circ f'^{-1} \in G$. The other direction of the compatibility condition follows in an analogous manner. □

Proposition 3.1.1. *Let (S, C) be a G-structured space determined by (M, g_{ab}). Then both of the following hold:*

1. *The identity map 1_M is a diffeomorphism between the manifold (S, C^+) and M.*

2. *The coordinate transformation group Γ on S is the isometry group of (M, g_{ab}), i.e. the collection of diffeomorphisms $f : M \to M$ such that $f^*(g_{ab}) = g_{ab}$.*

Proof. We start with 1. Let $\phi : M \to \mathbb{R}^n$ be the diffeomorphism that we chose in the above construction of (S, C). We know that $\phi \in C$, so $\phi : S \to \mathbb{R}^n$ is a diffeomorphism from (S, C^+) to $\mathbb{R}^n$ by Proposition 2.1.2. Hence $\phi^{-1} \circ \phi = 1_M$ is the composition of two diffeomorphisms — one from (S, C^+) to $\mathbb{R}^n$, the other from $\mathbb{R}^n$ to M — and thus itself a diffeomorphism from (S, C^+) to M.

For 2, Let $h \in \Gamma$. This means that $h = c^{-1} \circ d$ for $c, d \in C$. Since c and d are diffeomorphisms from (S, C^+) to $\mathbb{R}^n$, h is a diffeomorphism from (S, C^+) to itself. The result in 1 above then implies that $h = 1_M \circ h \circ 1_M$ is a diffeomorphism from M to itself. We compute that

$$h^*(g_{ab}) = d^* \circ c^{-1*}(g_{ab}) = d^*(\phi_*(g_{ab})) = g_{ab}$$

The first equality follows from properties of the pullback, the second and third by the definition of C and since $c, d \in C$. So h is in the group of isometries of

(M, g_{ab}). Now let $f : M \to M$ be a diffeomorphism with $f^*(g_{ab}) = g_{ab}$. And let $c \in C$. We see that $(c \circ f)_*(g_{ab}) = \phi_*(g_{ab})$ since f is an isometry and $c \in C$. Hence $c \circ f \in C$. This means that $f \in \Gamma$ since $c^{-1} \circ (c \circ f) = f$ and both c and $c \circ f$ are in C. □

Proposition 3.1.2. *Suppose that $f : S \to S'$ is an isomorphism between (S, C) and (S', C') and α is a tensor field (of arbitrary index structure) on S. Then α is implicitly defined by Γ if and only if $f_*(\alpha)$ is implicitly defined by Γ'.*

Proof. We show the 'left-to-right' implication. The other follows similarly. Let $h' \in \Gamma'$. We know then that $f^{-1} \circ h' \circ f \in \Gamma$ by clause 2 of the definition of isomorphism. Since α is implicitly defined by Γ we see that $(f^{-1} \circ h' \circ f)^*(\alpha) = \alpha$. Simplifying implies that

$$\alpha = (f^{-1} \circ h' \circ f)^*(\alpha) = f^* \circ h'^* \circ f^{-1*}(\alpha)$$

And this means that $f_*(\alpha) = h'^*(f_*(\alpha))$. Since $h' \in \Gamma'$ was arbitrary, Γ' implicitly defines $f_*(\alpha)$. □

Proposition 3.1.3. *Let (S, C) and (S', C') be G- and G'-structured spaces determined by (M, g_{ab}). Then (S, C) and (S', C') are isomorphic.*

Proof. Consider the identity map $1_M : S \to S'$. (Note that it is well-defined since $S = M = S'$.) By Proposition 3.1.1, it is a diffeomorphism. That same proposition implies that $\Gamma = \Gamma'$, and hence 1_M also satisfies condition 2 of the definition of isomorphism. □

Proposition 3.1.4. *There are non-isometric relativistic spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ with trivial isometry groups.*

Proof. Let (M, g_{ab}) be a 2-dimensional Heraclitus spacetime, the existence of which is guaranteed by Manchak and Barrett (2024). Let (U, ϕ) be a coordinate chart on M with $O \subset \phi[U]$ be an open ball, and consider the open set $\phi^{-1}[O] \subset M$. Since $\phi^{-1}[O]$ is a manifold in its own right, we can consider the relativistic spacetime $(\phi^{-1}[O], g_{ab}|_{\phi^{-1}[O]})$. It immediately follows from the fact that (M, g_{ab}) is Heraclitus that $(\phi^{-1}[O], g_{ab}|_{\phi^{-1}[O]})$ has a trivial isometry group. And moreover, $\phi^{-1}[O]$ is diffeomorphic to $\mathbb{R}^n$ since O is diffeomorphic to $\mathbb{R}^n$ (see Lee, Example 2.14a) and $\phi : \phi^{-1}[O] \to O$ is a diffeomorphism. We have therefore built one relativistic spacetime with trivial isometry group whose underlying manifold is diffeomorphic to $\mathbb{R}^n$. Now let $O' \subset \phi[U]$ be an open ball not equal to O. The exact same argument as above demonstrates that the spacetime $(\phi^{-1}[O'], g_{ab}|_{\phi^{-1}[O']})$ has a trivial isometry group and has an underlying manifold diffeomorphic to $\mathbb{R}^n$. And $(\phi^{-1}[O], g_{ab}|_{\phi^{-1}[O]})$ and $(\phi^{-1}[O'], g_{ab}|_{\phi^{-1}[O']})$ are not isometric. For if they were, there would be an isometry $\psi : \phi^{-1}[O] \to \phi^{-1}[O']$ where $\phi^{-1}[O] \neq \phi^{-1}[O']$, contradicting the fact that (M, g_{ab}) is Heraclitus. □

Proposition 3.1.5. *Suppose that (S, C) and (S', C') are G- and G'-structured space with trivial coordinate transformation groups Γ and Γ' (i.e. both contain only the identity map). Then (S, C) and (S', C') are isomorphic.*

Proof. We already know that the manifolds (S, C^+) and (S', C'^+) are diffeomorphic, since they are both diffeomorphic to $\mathbb{R}^n$. So let $f : S \to S'$ be a diffeomorphism. Then we immediately see that f also satisfies clause 2 of the definition of isomorphism, since $f \circ 1_S \circ f^{-1} = 1_{S'}$. □

Proofs in Section 3.2

Lemma 3.2.1. *Let (M, g_{ab}) be an n-dimensional spacetime and let $p \in M$. Let $(\mathbb{R}^n, \eta_{ab})$ be n-dimensional Minkowski spacetime. Let $B \subset \mathbb{R}^n$ be an open unit ball. There is a neighborhood U of p, a spacetime $(\mathbb{R}^n, g'_{ab})$, and an open set $U' \subset B$ such that (i) $g'_{ab} = \eta_{ab}$ outside of B and (ii) (U, g_{ab}) is isometric to (U', g'_{ab}).*

Proof. We work in two dimensions to simplify the presentation; one generalizes in a straightforward way. Let (M, g_{ab}) be any two-dimensional spacetime and let $p \in M$. Let $(\mathbb{R}^2, \eta'_{ab})$ be Minkowski spacetime given in standard (t', x') coordinates: $\eta'_{ab} = \nabla_a t' \nabla_b t' - \nabla_a x' \nabla_b x'$. Let $B \subset \mathbb{R}^2$ be an open unit ball. Without loss of generality, we take B to be centered at the origin $o = (0, 0) \in \mathbb{R}^2$.

Consider a chart (O, φ) containing $p \in M$ such that $\varphi(p) = o$ and the coordinate maps $t : O \to \mathbb{R}^2$ and $x : O \to \mathbb{R}^2$ associated with (O, φ) are such that $\varphi^*(t') = t$, $\varphi^*(x') = x$, and g_{ab} at the point p is $\nabla_a t \nabla_b t - \nabla_a x \nabla_b x$. We can now express g_{ab} on O as $f_{tt} \nabla_a t \nabla_b t + f_{xx} \nabla_a x \nabla_b x + 2 f_{tx} \nabla_{(a} t \nabla_{b)} x$ for some smooth scalar fields $f_{tt} : O \to \mathbb{R}$, $f_{xx} : O \to \mathbb{R}$, and $f_{tx} : O \to \mathbb{R}$. Since $-1 = f_{xx}(p) < 0 < f_{tt}(p) = 1$, we can find an open ball $B_\epsilon(o) \subset \varphi[O]$ centered at o with radius $0 < \epsilon < 1$ small enough that $f_{xx} < 0 < f_{tt}$ on all of $\varphi^{-1}[B_\epsilon(o)]$. Let $N' = B_\epsilon(o) \subset B$ and let $N = \varphi^{-1}[N']$. We divide N into three disjoint regions: U, V, W. For convenience, let r be the scalar function on N defined by $r = \sqrt{t^2 + x^2}$. Let U be the region where $r < \epsilon/3$; let V be the region where $\epsilon/3 \leq r < 2\epsilon/3$; let W be the region where $2\epsilon/3 \leq r < \epsilon$.

We now define a smooth tensor field γ_{ab} on N. First, we use φ to pull back the the metric η'_{ab} on $\mathbb{R}^2$ to the define the metric $\eta_{ab} = \varphi^*(\eta'_{ab})$ on N. Since $\varphi^*(t') = t$ and $\varphi^*(x') = x$ we know that that $\eta_{ab} = \nabla_a t \nabla_b t - \nabla_a x \nabla_b x$. On the region W, let $\gamma_{ab} = \eta_{ab}$. On the region U, let $\gamma_{ab} = g_{ab}$. In order to define γ_{ab} on V, consider any smooth bump function $\theta : N \to \mathbb{R}$ such that $\theta = 1$ on U, $0 < \theta < 1$ on V, and $\theta = 0$ on W. On the region V, let $\gamma_{ab} = h_{tt} \nabla_a t \nabla_b t + h_{xx} \nabla_a x \nabla_b x + 2 h_{tx} \nabla_{(a} t \nabla_{b)} x$ where $h_{tt} = \theta f_{tt} - \theta + 1$, $h_{xx} = \theta f_{xx} + \theta - 1$, and $h_{tx} = \theta f_{tx}$. By construction, γ_{ab} is smooth on N.

We now show that γ_{ab} is a Lorentzian metric on N. Clearly, it is symmetric on N and is nondegenerate on U and W. We claim it is nondegenerate on V as well. Note first that since $f_{xx} < 0 < f_{tt}$ and $0 < \theta < 1$ on V, we have $h_{xx} < 0 < h_{tt}$ on V as well. Let q be any point in V and let ξ^a be any vector at q. We can express ξ^a as $\alpha(\partial/\partial t)^a + \beta(\partial/\partial x)^a$ for some $\alpha, \beta \in \mathbb{R}$. Consider $\gamma_{ab}\xi^a$. It must come out as $[\alpha h_{tt}(q) + \beta h_{tx}(q)] \nabla_b t + [\alpha h_{tx}(q) + \beta h_{xx}(q)] \nabla_b x$. Now suppose that $\gamma_{ab}\xi^a = \mathbf{0}$. This implies that $\alpha h_{tt}(q) + \beta h_{tx}(q) = 0$ and $\alpha h_{tx}(q) + \beta h_{xx}(q) = 0$. It follows that $\alpha[h_{tt}(q) h_{xx}(q) - h_{tx}(q)^2] = 0$ and $\beta[h_{tt}(q) h_{xx}(q) - h_{tx}(q)^2] = 0$. So either $\alpha = \beta = 0$ or $h_{tt}(q) h_{xx}(q) = h_{tx}(q)^2$. But the latter case cannot obtain since $h_{xx} < 0 < h_{tt}$. So $\alpha = \beta = 0$ and thus $\xi^a = \mathbf{0}$. So γ_{ab} is non-degenerate and hence a metric on V. So, γ_{ab} is a smooth metric on all of N. Since γ_{ab} is Lorentzian at p and N is connected, γ_{ab} is Lorentztian on all of N.

Now let $(\mathbb{R}^2, g'_{ab})$ be such that $g'_{ab} = \varphi_*(\gamma_{ab})$ on N' and $g'_{ab} = \eta'_{ab}$ on $\mathbb{R}^2 - N'$. By construction, $\varphi_*(\gamma_{ab}) = \varphi_*(\eta_{ab}) = \eta'_{ab}$ on the region $\varphi[W]$. It follows that g'_{ab} is smooth on the boundary of N' and hence smooth everywhere. If we let $U' = \varphi[U]$ and recall that $\gamma_{ab} = g_{ab}$ on U, we see that that $g'_{ab} = \varphi_*(\gamma_{ab}) = \varphi_*(g_{ab})$ on U'. So $\varphi_{|U}$ is an isometry between the spacetimes (U, g_{ab}) and (U', g'_{ab}). Since U is a neighborhood of p and $U' \subset N' \subset B$, we have our result. □

Lemma 3.2.2. *Every n-dimensional relativistic spacetime (M, g_{ab}) has a representation.*

Proof. We show that there is a relativistic spacetime $(\mathbb{R}^n, g'_{ab})$ such that for every point $p \in M$, there is an open neighborhood U of p such that (U, g_{ab}) is isometric to (U', g'_{ab}) for some $U' \subset \mathbb{R}^n$. We work in two dimensions to simplify the presentation; one generalizes in a straightforward way. Let (M, g_{ab}) be any two-dimensional spacetime. Let $(\mathbb{R}^2, \eta_{ab})$ be Minkowski spacetime in standard (t, x) coordinates where $\eta_{ab} = \nabla_a t \nabla_b t - \nabla_a x \nabla_b x$. Let $B \subset \mathbb{R}^2$ be an open unit ball. We know from Lemma 3.2.1 that for each $p \in M$, there is an open neighborhood U_p of p, a spacetime $(\mathbb{R}^2, \hat{g}_{ab})$, and an open set $\hat{U} \subset B$ such that (i) $\hat{g}_{ab} = \eta_{ab}$ outside of B and (ii) (U_p, g_{ab}) is isometric to $(\hat{U}, \hat{g}_{ab})$. Let $\{U_p\}$ be the collection of all such open neighborhoods of all $p \in M$. Since M is second countable and $\{U_p\}$ is an open cover of M, we know from Lindelöf's lemma that there is a countable open subcover $\{U_{p_i}\}$ of M where $\{p_i\}$ is some countable collection of points in M indexed by $i \in \mathbb{N}$.

For each p_i, let $(\mathbb{R}^2, g'_{ab}(i))$ be the spacetime constructed using Lemma 3.2.1 where $B \subset \mathbb{R}^2$ is taken to be the open unit ball $B(q_i)$ centered at the point $q_i = (2i, 0)$. So we have (i) $g'_{ab}(i) = \eta_{ab}$ outside of $B(q_i)$ and (ii) (U_{p_i}, g_{ab}) is isometric to $(U'_i, g'_{ab}(i))$ for some $U'_i \subset B(q_i)$. Now let $(\mathbb{R}^2, g'_{ab})$ be the spacetime defined by setting $g'_{ab} = \eta_{ab}$ in the region outside of $\bigcup B(q_i)$ and let $g'_{ab} = g'_{ab}(i)$ in the region $B(q_i)$ for each $i \in \mathbb{N}$. By construction, each U_{p_i} is such that (U_{p_i}, g_{ab}) is isometric to (U'_i, g'_{ab}) for some $U'_i \subset B(q_i)$. But since $\{U_{p_i}\}$ is an open cover of M, we know that each $r \in M$ is contained some U_{p_i}. So for each $r \in M$, there is some open neighborhood U_{p_i} of r such that (U_{p_i}, g_{ab}) is isometric to (U'_i, g'_{ab}) for some $U'_i \subset B(q_i)$. □

Lemma 3.2.3. *Let (S, C) be a locally G-structured space determined by (M, g_{ab}). Then (S, C) is a locally G-structured space.*

Proof. The cover condition holds of (S, C) because of Lemma 3.2.2. Each point in M is contained in the domain of some map in C. The range condition holds since the maps in C are maps from open sets of M to open sets of $\mathbb{R}^n$, and each open set in $\mathbb{R}^n$ is the domain of the identity function on that open set, which is contained in G. We show that the compatibility condition is satisfied. Let $f : O \to \mathbb{R}^n$ have an open set as its range (or in other words, its range is the domain of some element of G).

Suppose first that $f \in C$, and let $f' \in C$ be such that $\text{dom}(f) \cap \text{dom}(f')$ is non-empty. Then $f \circ f'^{-1}$ is a diffeomorphism, as it is the composition of two diffeomorphisms, and it preserves g'_{ab} since

$$(f \circ f'^{-1})^*(g'_{ab}) = f'^{-1*} \circ f^*(g'_{ab}) = f'^{-1*}(g_{ab}) = g'_{ab}$$

The first equality follows from properties of the pullback, while the second and third follow from the fact that f and f' are in C. This gives us the left-to-right direction of the compatibility condition.

Now suppose that for every $f' : O' \to \mathbb{R}^n$ that is in C with $\text{dom}(f) \cap \text{dom}(f')$ non-empty, $f \circ f'^{-1} \in G$. Note that this means that $f \circ f'^{-1}$ is a diffeomorphism. In order to show that $f \in C$, we first need to show that f is a diffeomorphism between an open $O \subset M$ and open $f[O] \subset \mathbb{R}^n$. Since $f|_{O \cap O'} = (f \circ f'^{-1}) \circ f'$, $f|_{O \cap O'} : O \cap O' \to f[O \cap O']$ is the composition of two diffeomorphisms and is therefore one itself. The cover condition then implies that for each point $p \in O$ there is some $f'_i : O'_i \to \mathbb{R}^n$ in C with $p \in O'_i$. And the preceding argument guarantees that $f|_{O \cap O'_i}$ is a diffeomorphism for each i. Since $f : O \to \mathbb{R}^n$ is such that $O = \cup_i O \cap O'_i$ where each $O \cap O'_i$ is open and $f|_{O \cap O'_i}$ is a diffeomorphism, the gluing lemma for smooth maps (Lee, 2012, Corollary 2.8) implies that $f : O \to f[O]$ is a diffeomorphism. Since by assumption $f[O]$ is open, it then must be that O is open too. We now show that $f^*(g'_{ab}) = g_{ab}|_O$. Let $p \in O$. We know that there is some $f' \in C$ with $p \in \text{dom}(f')$. So we compute that:

$$g'_{ab}|_{f'(p)} = (f \circ f'^{-1})^*(g'_{ab})|_{f'(p)} = f'^{-1*} \circ f^*(g'_{ab})|_{f'(p)}$$

The first equality follows since $f \circ f'^{-1} \in G$, and the second from properties of the pullback. This implies that $f'^*(g'_{ab})|_p = f^*(g'_{ab})|_p$. Since $f' \in C$ and therefore $f'^*(g'_{ab}) = g_{ab}|_{O'}$, this means that $f^*(g'_{ab})|_p = g_{ab}|_p$. Since $p \in O$ was arbitrary it must be that $f^*(g'_{ab}) = g_{ab}|_O$, and so $f \in C$. This gives us the right-to-left direction of the compatibility condition, and we have therefore shown that (S, C) is a locally G-structured space. $\square$

Proposition 3.2.1. *Let (M, g_{ab}) be a relativistic spacetime. If (S, C) is a locally G-structured space determined by (M, g_{ab}), then*

1. *The identity map 1_M is a diffeomorphism between (S, C^+) and M.*
2. *The coordinate transformation pseudogroup Γ associated with (S, C) is the isometry pseudogroup of (M, g_{ab}).*

Proof. Let $(\mathbb{R}^n, g'_{ab})$ be the representation of (M, g_{ab}) used in the construction of (S, C). We begin with 1. Consider a collection $\{c_i : U_i \to \mathbb{R}^n\}$ of elements of C such that the sets U_i cover S. Note that such a subset of C exists by the cover condition and that $\{U_i\}$ is an open cover both of the manifold M and of the manifold (S, C^+). Now for each i we consider the map $1_{U_i} = c_i^{-1} \circ c_i$, the composition of the map $c_i : U_i \to \mathbb{R}^n$ from an open subset of M to $\mathbb{R}^n$ and $c_i^{-1} : c_i[U_i] \to S$. We know that the former is a diffeomorphism by the definition of C, and the latter is a diffeomorphism by Proposition 2.2.2. So

1_{U_i} is a diffeomorphism between U_i (conceived of as a submanifold of M) and U_i (conceived of as a submanifold of (S, C^+)). Hence the gluing lemma for smooth maps (Lee, 2012, Corollary 2.8) implies that $1_M : M \to (S, C^+)$ is a diffeomorphism.

We now turn to 2. Suppose that $f : U \to V$ is a diffeomorphism between open sets U and V of M such that $f^*(g_{ab}) = g_{ab}$. We show that $f \in \Gamma$. Let $c : \mathrm{dom}(c) \to \mathbb{R}^n$ be a map in C such that $\mathrm{dom}(c) \cap V$ is non-empty (we know such a c exists by the cover condition) and consider the map $c \circ f : f^{-1}[\mathrm{dom}(c) \cap V] \to \mathbb{R}^n$, which we know is a diffeomorphism between its domain and range since both c and f are. Since $(c \circ f)_*(g_{ab}) = c_*(g_{ab}) = g'_{ab}$, it must be that $c \circ f \in C$. And hence it must be that $c^{-1} \circ (c \circ f) = f|_{f^{-1}[\mathrm{dom}(c) \cap V]}$ is in Γ_0. Since $f : U \to V$ is a diffeomorphism, we know that $f^{-1}[\mathrm{dom}(c) \cap V]$ is open because both $\mathrm{dom}(c)$ and V are. By the cover condition $V = \cup_{c \in C} \mathrm{dom}(c) \cap V$, and hence $U = \cup_{c \in C} f^{-1}[\mathrm{dom}(c) \cap V]$. The argument above implies that $f : U \to V$ is such that each $f|_{f^{-1}[\mathrm{dom}(c) \cap V]} \in \Gamma_0$. And hence $f \in \Gamma$ by clause (ii) of the definition of Γ.

Now on the other hand suppose that $h : U \to V$ is in Γ. We know that U and V are open in the topology on (S, C^+) by PG1. Since we have already shown that $1_M : M \to (S, C^+)$ is a diffeomorphism, it must be that U and V are open in the topology on M. And moreover, we know that h is a diffeomorphism between these open sets of M, since it is a diffeomorphism between open subsets of (S, C^+) and $1_M : M \to (S, C^+)$ is a diffeomorphism. It only remains to show that $h^*(g_{ab}) = g_{ab}$. Let $p \in U$. We know that $U = \cup_i U_i$ for open sets U_i and $h|_{U_i} \in \Gamma_0$ for each i. Since $p \in U_i$ for some i, we have that $h|^*_{U_i}(g_{ab})|_p = h^*(g_{ab})|_p$. Because it is in Γ_0, $h|_{U_i} = c^{-1} \circ d$ for some pair $c, d \in C$. This immediately implies that

$$h^*(g_{ab})|_p = h|^*_{U_i}(g_{ab})|_p = (c^{-1} \circ d)^*(g_{ab})|_p = d^* \circ c_*(g_{ab})|_p = d^*(g'_{ab})|_p = g_{ab}|_p$$

The first and second equalities follow from the immediately preceding discussion. The third follows from properties of the pullback and pushforward. The fourth and fifth follow by the definition of C. So $h^*(g_{ab}) = g_{ab}$ and therefore h is an isometry between open subsets of (M, g_{ab}), and hence an element of the isometry pseudogroup. □

Proposition 3.2.2. *Suppose that $f : S \to S'$ is an isomorphism between locally G- and G'-structured spaces (S, C) and (S', C') and let α be a tensor field (of arbitrary index structure) on S. Then α is implicitly defined by Γ if and only if $f_*(\alpha)$ is implicitly defined by Γ'.*

Proof. We show the right-to-left direction. The other direction follows analogously. Let $h \in \Gamma$ and $p \in \mathrm{dom}(h)$. Then since f is an isomorphism we know that $f \circ h \circ f^{-1} \in \Gamma'$ and hence $(f \circ h \circ f^{-1})^*(f_*(\alpha))|_{f(p)} = f_*(\alpha)|_{f(p)}$ since Γ' implicitly defines $f_*(\alpha)$. We now compute the following:

$$f_*(\alpha)|_{f(p)} = (f \circ h \circ f^{-1})^*(f_*(\alpha))|_{f(p)} = f_* \circ h^* \circ f^*(f_*(\alpha))|_{f(p)} = f_* \circ h^*(\alpha)|_{f(p)}$$

The second and third equalities follow from properties of the pullback. Applying f^* to both sides of the equation then implies that $h^*(\alpha)|_p = \alpha|_p$, so Γ implicitly defines α. $\square$

Proposition 3.2.3. *Let (M, g_{ab}) be a relativistic spacetime. Suppose that (S, C) and (S', C') are locally G-structured spaces determined by (M, g_{ab}). Then (S, C) and (S', C') are isomorphic.*

Proof. This follows from Proposition 3.2.1 in the same way that Proposition 3.1.3 followed from Proposition 3.1.1. $\square$

Proposition 3.2.4. *There are non-isometric relativistic spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ with trivial isometry pseudogroups.*

Proof. Consider the spacetimes $(\phi^{-1}[O], g_{ab}|_{\phi^{-1}[O]})$ and $(\phi^{-1}[O'], g_{ab}|_{\phi^{-1}[O']})$ defined in the proof of Proposition 3.1.4. We have already shown that they are not isometric and that both $\phi^{-1}[O]$ and $\phi^{-1}[O']$ are diffeomorphic to $\mathbb{R}^n$. It immediately follows from the fact that (M, g_{ab}) is Heraclitus that $(\phi^{-1}[O], g_{ab}|_{\phi^{-1}[O]})$ and $(\phi^{-1}[O'], g_{ab}|_{\phi^{-1}[O']})$ are too; since they are Heraclitus, they have trivial isometry pseudogroups. $\square$

Proposition 3.2.5. *Let (S, C) and (S', C') be locally G- and G'-structured spaces with trivial coordinate transformation pseudogroups Γ and Γ'. If the manifolds (S, C^+) and (S', C'^+) are diffeomorphic, then (S, C) and (S', C') are isomorphic.*

Proof. We know that there is a diffeomorphism $f : S \to S'$ and hence f satisfies condition 1 of the definition of an isomorphism. We show that f also satisfies condition 2. We need to show that the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection from Γ to Γ'. Let $s, s' \in \Gamma$ and suppose that $f \circ s \circ f^{-1} = f \circ s' \circ f^{-1}$. Since $f : S \to S'$ is a bijection, it must be that $s = s'$. Hence our map $s \mapsto f \circ s \circ f^{-1}$ is injective. Now let $s' \in \Gamma'$, so s' is the identity map 1_O on some open set $O \subset S'$. We see that $f^{-1} \circ 1_O \circ f = 1_{f^{-1}[O]}$. Since f is a diffeomorphism, $f^{-1}[O]$ is an open subset of S, and hence $1_{f^{-1}[O]}$ is in Γ. Since $f \circ 1_{f^{-1}[O]} \circ f^{-1} = 1_O$, our map is bijective, f satisfies condition 2, and hence f is an isomorphism between (S, C) and (S', C'). $\square$

Proofs in Section 4.1

Proposition 4.1.1. *Minkowski spacetime is determined by isometry.*

Proof. Let $(\mathbb{R}^4, g_{ab})$ be a relativistic spacetime with the same isometry group as Minkowski spacetime. It follows from (O'Neill, 1983, Proposition 23) that all flat and geodesically complete spacetimes with underlying manifold $\mathbb{R}^4$ are isometric. We will show that $(\mathbb{R}^4, g_{ab})$ has the same Levi-Civita derivative operator as $(\mathbb{R}^4, \eta_{ab})$. Since $(\mathbb{R}^4, \eta_{ab})$ is flat and geodesically complete, this will imply that $(\mathbb{R}^4, g_{ab})$ is too, which will in turn imply that the two spacetimes are isometric.

Let x^1, x^2, x^3, x^4 be the standard coordinates on $\mathbb{R}^4$. We write

$$g_{ab} = \sum_{i,j=1}^{4} \overset{ij}{g} d_a x^i d_b x^j$$

in these coordinates and consider the vector fields $(\frac{\partial}{\partial x^i})^a$. We know that these are Killing fields for $(\mathbb{R}^4, \eta_{ab})$, and moreover, that the flow maps Γ_t that they determine are diffeomorphisms $\mathbb{R}^4 \to \mathbb{R}^4$. Since $(\mathbb{R}^4, g_{ab})$ has the same isometry group as Minkowski spacetime, the Γ_t maps are isometries of $(\mathbb{R}^4, g_{ab})$, and hence Proposition 1.6.6 of Malament (2012) implies that the $(\frac{\partial}{\partial x^i})^a$ are Killing fields on $(\mathbb{R}^4, g_{ab})$ too. We compute that for each $k = 1, \ldots, 4$,

$$\mathbf{0} = \mathcal{L}_{(\frac{\partial}{\partial x^k})} g_{ab} = \mathcal{L}_{(\frac{\partial}{\partial x^k})} \sum_{i,j=1}^{4} \overset{ij}{g} d_a x^i d_b x^j = \sum_{i,j=1}^{4} (\mathcal{L}_{(\frac{\partial}{\partial x^k})} \overset{ij}{g}) d_a x^i d_b x^j$$

The first equality follows since $(\frac{\partial}{\partial x^k})^a$ is a Killing field on $(\mathbb{R}^4, g_{ab})$, the second from how we are writing g_{ab} in coordinates, and the third since $\mathcal{L}_{(\frac{\partial}{\partial x^k})} d_a x^i = \mathbf{0}$. This implies that $\mathcal{L}_{(\frac{\partial}{\partial x^k})} \overset{ij}{g} = 0$, which in turn implies via Proposition 1.6.2 of Malament (2012) that $(\frac{\partial}{\partial x^k})^a (\overset{ij}{g}) = 0$, and hence each scalar field $\overset{ij}{g}$ is constant.

We can now show that the Levi-Civita derivative operator for g_{ab} is just the coordinate derivative operator ∇ (see (Malament, 2012, Proposition 1.7.11)) for the standard coordinates x^1, x^2, x^3, x^4 on $\mathbb{R}^4$. We compute that

$$\nabla_n g_{ab} = \nabla_n (\sum_{i,j=1}^{4} \overset{ij}{g} d_a x^i d_b x^j) = \sum_{i,j=1}^{4} \nabla_n (\overset{ij}{g}) d_a x^i d_b x^j = \mathbf{0}$$

The first equality follows from writing g_{ab} out in $x_1, \ldots, x_4$ coordinates, the second since ∇ is the coordinate derivative operator (and therefore $\nabla_n d_a x^i = \mathbf{0}$ for all i), and the third since $\overset{ij}{g}$ is constant and hence $\nabla_n \overset{ij}{g} = \mathbf{0}$. One can easily show that ∇ is the Levi-Civita derivative operator for η_{ab}, so η_{ab} and g_{ab} have the same derivative operator. □