Quantization and the Preservation of Structure across Theory Change

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Abstract

This paper presents a novel sense in which theoretical structure has been preserved across the transition from classical to quantum physics. I import mathematical tools from category theory that have been used for structural comparisons in the context of theoretical equivalence and apply these tools to new situations involving theory change. The structural preservation takes the form of a categorical equivalence between categories of models of classical and quantum physics. I situate the significance of this structural preservation in terms of prospects for theory construction in quantum physics.

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1 Introduction

This paper presents a novel sense in which theoretical structure is preserved from classical to quantum physics, which makes three central contributions. First, while the traditional motivation for analyzing structural continuity across scientific change comes from scientific realism (e.g., Worrall, 1989; Ladyman, 1998), I argue for a distinct significance of the structural continuity established here in terms of its aid to theory construction. Second, while advocates of structural realism use model-theoretic tools of partial structures (da Costa and French, 1990) to analyze structural continuity across theory change, I propose a novel collection of mathematical tools as apt for comparing mathematized theories like classical and quantum mechanics. I show that category-theoretic tools can be imported from philosophical discussions of theoretical equivalence (e.g., Weatherall, 2019a,b) and employed in a way that sidesteps existing controversies concerning the relationship between category theory and structural realism (e.g., Lam and Wthrich, 2015). Third, the central results presented here provide a precise sense in which structure is preserved through functorial relations between categories of models of classical and quantum physics. This goes beyond existing statements in the philosophical literature by Thébault (2016) and Yaghmaie (2020) concerning structural preservation through quantization.

In §2, I place structural continuity across theory change in the context of the literature on scientific realism. In §3, I introduce category-theoretic tools for comparing structure. In §4, I describe the mathematical framework for quantization and the classical limit. In §5, I present the central results: categorical equivalences between categories of models of classical and quantum physics. In §6, I conclude with an interpretation of structural continuity.

2 Realism and Theory Change

Philosophical discussion of structural continuity across theory change arises in the context of *scientific realism*, the view that our best scientific theories are (probably, approximately) true. I briefly review structural continuity in the literature, although I ultimately argue for a distinct significance of structural continuity in §6.

Structural continuity has been proposed as grounds to save scientific realism from attacks based on discontinuities in the historical evolution of science. Major conceptual shifts during scientific theory change have been understood as leading to problems for scientific realism. The fact that once accepted scientific claims are now rejected as false casts some doubt that our current best science will stand the test of time (See, e.g., Laudan, 1981; Stanford, 2006).

Worrall (1989) argues that while one should not believe in aspects of scientific theories that change radically across revolutions, there are nevertheless stable aspects of theories that remain continuous through conceptual shifts. Worrall identifies *structure* as what is maintained across examples of theory change that involved shifts in even the kinds of objects one takes to instantiate those structures. He proposes *structural realism*, advocating that one should believe our best scientific theories accurately capture the structure of the world. This suggestion has led to a burgeoning literature (Ladyman, 1998; Frigg and Votsis, 2011; French, 2014; Ladyman, 2020; Wallace, 2021).

If one wants to reserve belief for structural aspects of scientific theories, one needs a precise and principled way of picking out the structure of a theory and determining whether it is preserved across theory change (Psillos, 2001; Stanford, 2006). Otherwise, the structure that should supposedly form the basis of our beliefs is vague or underdetermined. Since there is more hope of making structural continuity precise in specific examples, I turn attention to the particular case study in quantum physics that is my focus.

While the original motivating example Worrall (1989) used for his structural realism was the transition from Fresnel's theory of light to Maxwell's, others have endeavored to apply structural realism to quantum mechanics[1] and quantum field theory.[2] Indeed, quantum physics is a natural focus for studying structural preservation because quantum theories are often constructed through *quantization procedures* that relate a model of a physical system within the framework of classical physics to a model of the same system within the framework of quantum physics. In analyzing quantization procedures, Thébault (2016) identifies the Lie algebra structure defined by a Poisson bracket in classical physics as what is preserved in the transition to quantum physics. However, Yaghmaie (2020) rightly notes that several technical results (Gotay, 1999) imply that the Lie algebra structure cannot be preserved for all observables when mapping from classical to quantum physics. Moreover, while Thébault works in the context of what is known as geometric quantization (Woodhouse, 1997), Yaghmaie argues that the methods employed in geometric quantization give rise to a problem of underdetermination that undermines the structural realist's goals. Instead, Yaghmaie favors the methods of *deformation quantization*, which Feintzeig (2020, 2023) also argues are appropriate for making sense of the relation between classical and quantum physics. Deformation quantization comes in two mathematical forms: Yaghmaie's discussion employs the tools of *formal* deformation quantization (Kontsevich, 2003), while Feintzeig (2020, 2023) provides some reasons that *strict* deformation quantization (Rieffel, 1993; Landsman, 1998, 2007) is better suited to capturing the approximative correspondence between classical and quantum physics. I will proceed to use the framework of strict deformation quantization to compare classical and quantum physics in this paper.

Before introducing the details of deformation quantization, I first argue that the mathematics of quantization procedures does not align with the methods for structural comparison that have been proposed in the literature by da Costa and French (1990, 2003). This motivates using new tools for comparing structure, which I introduce in §3 and apply to deformation quantization in §4-5.

da Costa and French (1990, 2003) outline a framework of *partial structures* for the analysis of scientific theories along structural lines, and Bueno (2008) argues that this framework provides a way to analyze structural continuity across theory change. A partial structure is a set—the domain—with a collection of partial relations, where each partial relation is characterized by three extensions: (i) the objects in the domain that (we know) *do* satisfy the relation, (ii) the objects in the domain that (we know) *do not* satisfy the relation, and (iii) the objects that fall into neither of the previous two sets. A *partial isomorphism* is then a bijection between domains of partial structures that preserves which objects (we know) satisfy relations and which objects (we know) do not satisfy relations, but with no requirements on the objects that do not fall into those two sets. According to Bueno (2008, p.

[1]See, e.g., French and Ladyman (2003); Roberts (2011).

[2]See, e.g., Cao (2019, 2003); Saunders (2003a,b); French (2012).

229), one should model structural continuity with partial isomorphisms between particular models of the older theory and the newer theory. Bueno argues that the existence of a partial isomorphism shows that structure is preserved across theory change.

There are two issues with the tools of partial structures and partial isomorphisms for analyzing structural continuity through quantization procedures. Neither provides definitive reasons not to use the tools of partial structures for structural comparisons, but they point towards possible improvements.

First, neither models of classical physics nor models of quantum physics are typically given as partial structures. I will treat these models, as in the mathematical physics literature, as C*-algebras where the algebraic operations are everywhere defined and do not leave room for relations that are uncertain or indeterminate on some part of their domain. This consideration is, of course, defeasible because one could specify after the fact some subsets of the domains on which the relations are uncertain or indeterminate, thus reconstructing a partial structure from a C*-algebra. But this would be completely foreign to how the models are used in physics. I worry that turning such mathematical structures into partial structures is liable to be ad hoc and unfaithful to the use of the structures in physics.[3]

Second, while partial isomorphism is a criterion intended to compare individual models of an old theory to individual models of a new theory, contemporary discussions of structural comparison give reason to compare theories as a whole to one another (Barrett, 2020a, p. 395). In other words, contemporary strategies for comparing structure achieve greater generality by comparing the structure of all models of one theory to the structure of all models of another theory. This consideration is defeasible if one does not agree that such generality is needed, or if one wants to pursue this generality within the framework of partial structures.[4]

Regardless of the status of these problems with using partial structures for structural comparisons, I hope to demonstrate that one can do better on the two fronts specified. I wish to emphasize that it is not my intention here to definitively argue against other methods for analyzing structural continuity within philosophy. Rather, I hope to take the methods native to mathematical physics and use them to present one form of structural continuity that I take to be philosophically significant. Characterizing structure within the models actually used by mathematical physics is at least in line with a recent understanding of structural realism due to Wallace (2021). I will leave it to future discussions to determine which accounts of structural continuity are most perspicuous. The task of the current paper is to lay out a novel account of structural continuity across theory change that at least should be considered.

[3]In this case, one might want a guarantee that the partial structures that have been reconstructed actually capture physically relevant information from the original complete mathematical models. One might even be pushed to use category theoretic tools to establish that the reconstructed partial structures are somehow equivalent to the original models, and so one might be led to the same conclusion as in this paper. I thank an anonymous reviewer for this point.

[4]Indeed, one might imagine generalizing the search for partial isomorphisms between individual models to a search for other global relationships between the collections of partial structures. Category theoretic tools might also be useful here, and may lead to the same results. So it is not even clear that approaches to analyzing structural continuity using partial structures would yield significantly different results than the results of the current paper using C*-algebras and category theory. I thank an anonymous reviewer for this point.

While I have situated structural continuity within the existing literature on structural realism, my aims in this paper ultimately diverge from most contemporary discussions. The current literature splits along a divide between epistemic structural realism (roughly: all we can know is structure) and ontic structural realism (roughly: all that exists is structure) (Ladyman, 2020). I remain agnostic about the inference from structural continuity to either position. Instead, I return in §6 to an older thread, going back at least to remarks of Stein (1989) and Saunders (1993), who tie structural continuity to heuristics for theory construction. I will argue that structural continuity bears on theory construction in physics, while leaving implications for further aspects of realism open.

3 How to Compare Structure

The recent literature on theoretical equivalence discusses structural comparison between theories (Weatherall, 2019a,b; Dewar, 2022). Barrett (2020a,b) compares structure by comparing structure-preserving maps between models. The basic intuition is that if a theory has "the same" structure-preserving maps between models, then those models have "the same" structure to preserve. Indeed, Halvorson (2012, 2016) argues one should understand a scientific theory not as merely a collection of models (the semantic view of theories), but rather as at least coming equipped with structure-preserving maps between models encoding their interpretation and structure. On this approach, a scientific theory can be represented by what is known as a *category*.

I pause to remark on the controversial association between structural realism and category theory. While some discussions of category theory focus on using groups of structure-preserving morphisms to make sense of the structure of scientific theories (e.g., Landry, 2007), others (Bain, 2013; Lam and Wthrich, 2015; Eva, 2016; Lal and Teh, 2017) focus on a specific objection to ontic structural realism. These discussions focus on the claim by ontic structural realists that (structural) relations are more "fundamental" than objects, and a corresponding objection that one cannot make sense of relations without relata. Category theory enters these discussions as a mathematical framework that some claim (and others dispute) provides an understanding of relations as "fundamental". I mention this literature only to distinguish it from my goals in this paper. I will *not* touch upon any of these metaphysical issues, and my use of category theory is not related to that of these authors. Instead, I use category theory as a tool to make precise claims about structural continuity. I ultimately use the results to inform heuristics for theory construction rather than metaphysics.

A *category* $\mathbf{C}$ is a collection of *objects* and a collection of *arrows* or *morphisms*, each with a source and target. I denote a morphism by $f : A \to B$ for source A and target B. I denote the collection of morphisms with source A and target B by $\mathrm{Hom}_{\mathbf{C}}(A, B)$. A category comes equipped with an operation of composition for morphisms, denoted $\circ$, which is total in the sense that for each $f : A \to B$ and $g : B \to C$, there is an $h : A \to C$ such that $h = g \circ f$. Composition is associative, and moreover, each object A has a unique identity morphism 1_A whose composition with other morphisms leaves them unchanged. I use categories to represent scientific theories by taking models of a theory to form the class of objects of a category, with structure-preserving maps serving as the morphisms between models.

Example 1. *The category* **LCH** *has locally compact Hausdorff topological spaces as objects and continuous proper*[5] *maps between them as arrows.*

Example 2. *The category* $\mathbf{B^*Alg}$ *has Banach *-algebras with approximate identity as objects and non-degenerate*[6] **-homomorphisms between them as arrows.*

Example 3. *The category* $\mathbf{C^*Alg}$ *has C*-algebras as objects and non-degenerate *-homomorphisms between them as arrows.*[7]

Example 4. *The category* $\mathbf{AbC^*Alg}$ *has abelian C*-algebras as objects and non-degenerate *-homomorphisms between them as arrows.*

These examples provide tools to represent models of both classical and quantum physics. Models of classical physics can be understood as commutative C*-algebras in the category $\mathbf{AbC^*Alg}$, while models of quantum physics can be understood as noncommutative C*-algebras in $\mathbf{C^*Alg}$.

A model of classical physics in Hamiltonian form has a phase space, a smooth even-dimensional manifold M. The colllection $C_0(M)$ of continuous functions vanishing at infinity is a commutative C*-algebra with the supremum norm.[8] Moreover, $C_0(M)$ contains the dense subalgebra $C_c^\infty(M)$ of smooth, compactly supported functions carrying a Poisson bracket. Structure-preserving maps between models of classical physics in the categories I consider will be *-homomorphisms between such commutative C*-algebras preserving the smooth and Poisson structures.

A model of quantum physics has a Hilbert space $\mathcal{H}$, on which any norm closed subalgebra of the collection $\mathcal{B}(\mathcal{H})$ of bounded operators is a C*-algebra with the operator norm.[9] Corresponding to the classical Poisson bracket is the quantum commutator $[A, B] = AB - BA$ for $A, B \in \mathcal{B}(\mathcal{H})$, which is defined from the operator multiplication. Structure-preserving maps between models of quantum physics in the categories I consider will thus be *-homomorphisms between noncommutative C*-algebras that are obtained through quantization. In §4, I make precise what it means to obtain a C*-algebra through quantization.

Now I establish methods to compare structure between theories by comparing their categories of models. A *functor* $F : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ consists in two maps: one map between the objects of $\mathbf{C}$ and the objects of $\mathbf{D}$, and another map between the morphisms of $\mathbf{C}$ and the morphisms of $\mathbf{D}$. I will use the symbol F for both maps. A functor preserves sources and targets in the sense that if $f : A \to B$ is a morphism in $\mathbf{C}$, then $F(f) : F(A) \to F(B)$ in $\mathbf{D}$, i.e., if $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, then $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$.

[5] Proper maps are one appropriate choice for encoding the structure of locally compact spaces. Proper maps preserve the "point at infinity" in a non-compact, but locally compact Hausdorff space.

[6] A *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ is *non-degenerate* just in case $\alpha[\mathfrak{A}]\mathfrak{B}$ is dense in $\mathfrak{B}$. Non-degenerate *-homomorphisms are one appropriate choice for encoding the structure of normed *-algebras. Non-degeneracy guarantees that *-homomorphisms preserve approximate identities in non-unital Banach *-algebras.

[7] For background, see Kadison and Ringrose (1997) and Landsman (2017, Appendix C).

[8] The Gelfand representation theorem (see Ex. 6) shows this situation is generic. Every commutative C*-algebra is *-isomorphic to $C_0(M)$ for some locally compact Hausdorff topological space M. See Landsman (2017, §C.2-3).

[9] The Gelfand-Naimark theorem shows this situation is generic. Every C*-algebra is *-isomorphic to some closed subalgebra of $\mathcal{B}(\mathcal{H})$. See Kadison and Ringrose (1997, §4.5).

Moreover, a functor preserves arrow composition in the sense that if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathbf{C}$, then $F(g \circ f) = F(g) \circ F(f)$.

Example 5 (Enveloping C*-algebra)**.** *For any object $\mathfrak{A}_0$ in $\mathbf{B^*Alg}$ with norm $\|\cdot\|_0$, one can define a corresponding C*-algebra. First, let $\mathcal{P}(\mathfrak{A}_0)$ denote the set of pure states on $\mathfrak{A}_0$, the continuous positive linear functionals of unit norm. One can define a new norm $\|\cdot\|$ on $\mathfrak{A}_0$ that satisfies the C* identity by*

$$\|A\| = \sup_{\omega \in \mathcal{P}(\mathfrak{A}_0)} \omega(A^*A)^{1/2} \tag{1}$$

for any $A \in \mathfrak{A}_0$. Then $\|\cdot\|$ satisfies the C identity on $\mathfrak{A}_0$ and the completion with respect to $\|\cdot\|$ is a C*-algebra $\mathfrak{A} = \overline{\mathfrak{A}_0}$ called the* enveloping C*-algebra *of $\mathfrak{A}_0$.*

*Define a map $E : \mathbf{B^*Alg} \to \mathbf{C^*Alg}$ that takes each Banach *-algebra to its enveloping C*-algebra and each *-homomorphism between Banach *-algebras to its unique continuous extension. Then E is a functor (See Dixmier, 1977, §2.7).*

Define also a map $R : \mathbf{C^*Alg} \to \mathbf{B^*Alg}$ that leaves every object and arrow the same. Since each C-algebra is already a Banach *-algebra, R is also a functor.*

Example 6 (Gelfand duality)**.** *For any object $\mathfrak{A}$ in $\mathbf{AbC^*Alg}$, the set of pure states $\mathcal{P}(\mathfrak{A})$ with the weak* topology is a locally compact Hausdorff space. Moreover, for any non-degenerate *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$, the function $\hat{\alpha} : \mathcal{P}(\mathfrak{B}) \to \mathcal{P}(\mathfrak{A})$ defined by $\omega \mapsto \omega \circ \alpha$ is a continuous proper map.*

Define the opposite category $\mathbf{LCH}^{op}$ as the category of locally compact Hausdorff topological spaces with the direction of each arrow reversed. Define a map $H : \mathbf{AbC^*Alg} \to \mathbf{LCH}^{op}$ that takes each C-algebra $\mathfrak{A}$ to its pure state space $\mathcal{P}(\mathfrak{A})$ and each non-degenerate *-homomorphism α to the (opposite of the) continuous proper map $\hat{\alpha}$ that is its dual. Then H is a functor. (See Kadison and Ringrose (1997, §4.4) or Landsman (2017, §C.2-3).)*

By mapping structure-preserving morphisms in one category to structure-preserving morphisms in another category, a functor provides a standard according to which one can compare the structure that is preserved by those morphisms. To do so, I will use the heuristic that having "the same" morphisms implies having "the same" structure.

A functor must have the following special properties in order to identify morphisms in one category as encoding "the same" structure as morphisms in another category. Consider a functor $F : \mathbf{C} \to \mathbf{D}$.

- F is called *faithful* if for any $f, g : A \to B$ in $\mathbf{C}$, whenever $F(f) = F(g)$, it is also the case that $f = g$, i.e., F is injective from $\mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$.

- F is called *full* if for every morphism $g : F(A) \to F(B)$ in $\mathbf{D}$, there is some morphism $f : A \to B$ in $\mathbf{C}$ such that $F(f) = g$, i.e., F is surjective from $\mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$.

- F is called *essentially surjective* if for every object B in $\mathbf{D}$, there is some object A in $\mathbf{C}$ such that $F(A)$ is isomorphic to B in $\mathbf{D}$.

A functor is a *categorical equivalence* if it is full, faithful, and essentially surjective—in this case, there is a functor $G : \mathbf{D} \to \mathbf{C}$ that is "almost inverse" to F (See, e.g., Awodey, 2010, p. 172-3). The existence of a categorical equivalence indicates that $\mathbf{C}$ and $\mathbf{D}$ have "the same" morphisms relative to the chosen functor: essential surjectivity allows one to match all objects from $\mathbf{C}$ to all objects in $\mathbf{D}$ (up to isomorphism), whereas fullness and faithfullness guarantee that the morphisms between those objects stand in one-to-one correspondence.

Example 7 (Enveloping C*-algebra, continued)**.** *The functor* $E : \mathbf{B^*Alg} \to \mathbf{C^*Alg}$ *is faithful and essentially surjective, but not full. It is faithful because whenever the extensions of two continuous maps between Banach *-algebras agree, their restrictions to the original Banach *-algebra subspace also agree. It is essentially surjective because every C*-algebra is a Banach *-algebra whose enveloping C*-algebra is itself. It is not full because there may be *-homomorphisms between enveloping C*-algebras that do not preserve the initial Banach *-algebra subspace and so cannot be obtained by extending a continuous map between those original Banach *-algebras.*

On the other hand, the functor $R : \mathbf{C^*Alg} \to \mathbf{B^*Alg}$ *is full and faithful, but not essentially surjective. It is full and faithful since it acts as the identity on arrows. It is not essentially surjective because there are Banach *-algebras that are not C*-algebras.*

The functor E "defines" the structure of a C*-algebra from a Banach *-algebra. It fails to be a categorical equivalence because it "forgets" (Baez et al., 2004) the structure of the norm of the original Banach *-algebra in favor of the new C*-norm. Similarly, the functor R "defines" the structure of a Banach *-algebra from a C*-algebra. In this direction, the Banach *-algebra has all the same structure as the original C*-algebra. But R "forgets" the property captured in the C* identity by including C*-algebras in a category whose collection of objects is bigger and thus more general.

Example 8 (Gelfand duality, continued)**.** *The functor* H *is full, faithful, and essentially surjective. It is essentially surjective because for each each object* X *in* $\mathbf{LCH}^{op}$*, the abelian C*-algebra* $C_0(X)$ *of continuous functions vanishing at infinity has* $\mathcal{P}(C_0(X))$ *isomorphic to* X*. It is faithful because the duals of *-homomorphisms agree only when the original *-homomorphisms agree. It is full because every continuous proper map* $\hat{\alpha} : X \to Y$ *defines a non-degenerate *-homomorphism* $\alpha : C_0(Y) \to C_0(X)$ *by* $f \mapsto f \circ \hat{\alpha}$*, and* $\hat{\alpha}$ *is the dual map corresonding to this* α*.*

It follows that H *is a categorical equivalence, and hence there is an "almost inverse" functor* $K : \mathbf{LCH}^{op} \to \mathbf{AbC^*Alg}$*, which takes each locally compact Hausdorff space* X *to the C*-algebra* $C_0(X)$*. This functor* K *is also full, faithful, and essentially surjective.*

The functors H and K provide a precise sense in which abelian C*-algebras and locally compact Hausdorff spaces have the same structure. They establish that the structure of each locally compact Hausdorff space, as encoded in the structure-preserving maps between them, is "definable" from the structure of its C*-algebra of continuous functions vanishing at infinity. Conversely, the structure of each abelian C*-algebra, as encoded in the structure-preserving maps between them, is "definable" from the structure of its pure state space, understood as a locally compact Hausdorff space with the weak* topology.

Note that these correspondences hold only relative to a choice of categories and a choice of functor used for the comparison. I will not attempt to make any structural comparisons once and for all between physical theories; rather, I only make structural comparisons relative to given categories and functors. The central results discussed here consist in the existence of categorical equivalences between categories representing models of classical and quantum physics. The interpretation of these results depends strongly on the choice of categories and functors used to compare them, so in the next sections I include discussion of the philosophical significance of the chosen categories.

4 Mathematical Framework

In this section, I introduce the processes of quantization and the classical limit, which I will represent by functors.[10] I present general definitions for each process here, and then in the next section present the results that for certain classes of physical systems—corresponding to particular categories of models of classical and quantum physics—these functors form a categorical equivalence.

4.1 Quantization

To quantize a model of classical physics, one begins with a commutative C*-algebra of functions on a phase space and continuously deforms the product operation to arrive at a noncommutative C*-algebra of bounded operators on a Hilbert space. The resulting family of C*-algebras indexed by the parameter $\hbar$ forms a structure called a continuous bundle (See Dixmier (1977, Ch. 10) or Landsman (1998, §II.1.2)).

Definition 1 (continuous bundle of C*-algebras)**.** A *(uniformly)*[11] *continuous bundle of C*-algebras* over a *base space* $I \subseteq \mathbb{R}$, where I contains 0 as an accumulation point, is:

- a family of C*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar \in I}$ called *fibers*;
- a C*-algebra $\mathfrak{A}$ of *continuous sections*; and
- a family of surjective *-homomorphisms $\{\phi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar\}_{\hbar \in I}$ called *evaluation maps.*

Together, these are required to satisfy for each $a \in \mathfrak{A}$,

(i) $\|a\| = \sup_{\hbar \in I} \|\phi_\hbar(a)\|_\hbar$, where $\|\cdot\|_\hbar$ denotes the norm on the fiber algebra $\mathfrak{A}_\hbar$;

(ii) for each uniformly continuous and bounded function $f : I \to \mathbb{C}$, there is a section $fa \in \mathfrak{A}$ such that $\phi_\hbar(fa) = f(\hbar)\phi_\hbar(a)$;

(iii) the map $\hbar \mapsto \|\phi_\hbar(a)\|_\hbar$ is uniformly continuous and bounded.

Continuous bundles can be constructed by deforming the product in the direction of the Poisson bracket of a classical phase space with the following notion of a quantization map.

[10] See Landsman (2003) for an alternative quantization functor.

[11] See Steeger and Feintzeig (2021a, Appendix B) for details on different continuity conditions.

Definition 2 (strict deformation quantization)**.** A *strict deformation quantization* (over $I \subseteq \mathbb{R}$ containing 0 as an accumulation point) of a manifold M with a *-algebra $\mathcal{P} \subseteq C_b(M)$ of continuous, bounded functions carrying a Poisson bracket is:

- a family of C*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar \in I}$ and
- a family of linear *quantization maps* $\{\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar\}_{\hbar \in I}$, where $\mathcal{Q}_0$ is the identity.

Together, these objects must satisfy for each $f, g \in \mathcal{P}$:

(i) (von Neumann's condition) $\lim_{\hbar \to 0} \|\mathcal{Q}_\hbar(f)\mathcal{Q}_\hbar(g) - \mathcal{Q}_\hbar(fg)\|_\hbar = 0$;

(ii) (Dirac's condition) $\lim_{\hbar \to 0} \|\frac{i}{\hbar}\big[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)\big] - \mathcal{Q}_\hbar\big(\{f, g\}\big)\|_\hbar = 0$;

(iii) (Rieffel's condition) the map $\hbar \mapsto \|\mathcal{Q}_\hbar(f)\|_\hbar$ is continuous;

(iv) (Deformation condition) for each $\hbar \in I$, the map $\mathcal{Q}_\hbar$ is injective, its image $\mathcal{Q}_\hbar[\mathcal{P}]$ is closed under the product in $\mathfrak{A}_\hbar$, and $\mathcal{Q}_\hbar[\mathcal{P}]$ is dense in $\mathfrak{A}_\hbar$.

Every strict deformation quantization satisfying mild technical conditions defines a continuous bundle of C*-algebras.[12] The algebra of sections is generated by the maps $[\hbar \mapsto \mathcal{Q}_\hbar(f)]$ for each $f \in \mathcal{P}$ and the maps $\phi_\hbar$ are given concretely as evaluation of the sections at a particular value $\hbar \in I$.

Strict deformation quantization differs from formal deformation quantization, the framework Yaghmaie (2020) employs to discuss structural realism. Formal deformation quantization consists in the definition of an associative product on an algebra of formal power series, treating $\hbar$ as a formal parameter rather than a physical parameter with a numerical value. Formal deformation quantization has its own virtues, including existence and uniqueness theorems established by Kontsevich (2003), which Yaghmaie argues have significance for structural realism. However, in general, the formal power series employed by formal deformation quantization may not converge to genuine operators or functions, and so their interpretation as physical quantities is lacking. This also implies that we lack operator norm estimates or bounds governing the rescaling of quantities as $\hbar \to 0$, which means that formal deformation quantization lacks the crucial tools used by Feintzeig (2020) for making sense of the approximation of the structure of classical physics by quantum physics via uniform error bounds. Even Waldmann (2019), an expert working at the forefront of research on formal deformation quantization, argues that one must face the issue of whether the formal power series employed within that framework converge to genuine operators or functions in order to understand their significance for physics. The strict deformation quantization approach circumvents the convergence problem by directly defining those operators or functions and proving their existence, thus fulfilling the necessary mathematical preconditions for their physical interpretation.

In what follows, I consider only models of quantum physics obtained through deformation quantization. In doing so, I restrict attention to quantum systems that correspond to models of classical physics in the classical $\hbar \to 0$ limit.[13] For present purposes, I restrict attention to

[12]See Landsman (1998, §II.1.2) or Steeger and Feintzeig (2021a, Appendix A).

[13]See also Feintzeig (2022, 2023) for discussion of the prospects of intertheoretic reduction and its role in constructing quantum theories that correspond with classical physics.

models of quantum physics that correspond to models of classical physics because I focus on the question of whether the *structure* of those models is preserved in the quantum–classical transition. I set aside the question of whether the models or objects of classical physics are in one-to-one correspondence with the models or objects of quantum physics (essential surjectivity) by restricting attention to what will be the range of our quantization functor. I focus on the question of whether there is a one-to-one correspondence between the structure-preserving maps in classical and quantum physics (faithfulness and fullness), which is most relevant to structural comparisons.[14]

In taking models of quantum physics to consist in deformation quantizations, I include what might be considered "more structure" than that of quantum physics alone. If one thinks that the structure of quantum physics can be captured already by a single C*-algebra encoding the canonical commutation relations of quantum physics, then one might see the gluing together of many C*-algebras in a continuous bundle as "excess structure". A quantization and its associated continuous bundle adds to the algebraic structure of the canonical commutation relations further information about how physical quantities scale with changing values of Planck's constant $\hbar$, or in different systems of units (Feintzeig, 2020, 2023). I allow this scaling information as part of a model of quantum physics because in my examples the scaling information is determined by a substantive physical interpretation of the elements of a C*-algebra as quantities measured in associated physical units. For example, Feintzeig (2020, 2023) shows how one can extract this non-trivial scaling information from the physical interpretation of elements of the Weyl algebra as physical quantities. Hence, the framework of strict deformation quantization encodes physically relevant structure for quantum models, even if that structure goes beyond the canonical commutation relations.

One might also worry that deformation quantization comes with the "extra structure" of a particular quantization map because quantization maps are in general not unique for a given classical theory. One might object to choosing any particular quantization map—as I will—to represent models of quantum physics. In response, note that distinct quantization maps can still bear a relation of *equivalence* (See Landsman, 1998, p. 109), according to which equivalent quantizations generate the same continuous bundle of C*-algebras. This signifies that equivalent quantizations have the same asymptotic behavior in the classical $\hbar \to 0$ limit. Many of the results I present in what follows are invariant under changes between distinct, but equivalent quantization maps. In practice, a number of known quantization maps are equivalent in this sense, so non-uniqueness of quantization maps does not obviously stymie the below results concerning structural equivalence.

One might further worry that deformation quantization cannot succeed in providing a structural correspondence between classical and quantum physics, given the known obstructions to preservation of Lie algebra structure in quantization (Gotay, 1999; Yaghmaie, 2020). However, deformation quantization avoids the obstruction results by requiring that the Poisson bracket of the classical theory only match the commutator of the quantum theory *asymptotically* and *not exactly*. One cannot in general find a Lie algebra isomorphism

[14] I do not know of any examples of models of quantum systems for which it is known that they cannot be obtained through deformation quantization. If there were a model of quantum physics that could not be obtained through deformation quantization, this would imply that its classical limit did not exist. If there are any such cases, I suspect there would be significant controversy over whether they could count as genuine physical models of quantum systems.

between collections of observables in classical physics and quantum physics. But it would be too quick to infer that classical and quantum physics do not share structure; such an inference relies on what Barrett (2020b) calls the model isomorphism criterion for structural comparison, which he rightly criticizes as not being flexible enough to account for the theoretical equivalence of many agreed upon reformulations in physics. The obstruction results for quantization of Lie algebra structure motivate both the use of deformation quantization—for its more flexible *approximative agreement* of algebraic structures—as well as the use of category theory for structural comparisons—which avoids known problems with the model isomorphism criterion for structural comparison.

In this vein, it is worth noting that the categorical equivalence between classical and quantum physics that I discuss in this paper has a different philosophical significance than some other categorical equivalences considered by philosophers. Specifically, others (Weatherall, 2019a,b) have been interested in categorical equivalence as a standard to assess theoretical equivalence of different formulations of the same theory. Although I will display a categorical equivalence, it manifestly does *not* preserve empirical structure between classical and quantum physics, which is necessary to establish a full theoretical equivalence. In particular, classical and quantum physics make different empirical predictions, e.g., for observed energy spectra. Instead, the categorical equivalence merely provides an asymptotic or approximative correspondence between these empirical structures, as described by Feintzeig (2020).

4.2 The Classical Limit

In the opposite direction of quantization, the classical limit can be understood as the process of restricting a continuous bundle of C*-algebras obtained from quantization back to the commutative C*-algebra at $\hbar = 0$. Steeger and Feintzeig (2021a) show that the fiber algebra $\mathfrak{A}_0$ at $\hbar = 0$ can be reconstructed from a given bundle of C*-algebras $((\mathfrak{A}_\hbar, \phi_\hbar)_{\hbar \in I}, \mathfrak{A})$ over $I = (0, 1]$ containing only information about the quantum theory for $\hbar > 0$. To do so, consider the closed two-sided ideal $K_0 = \{a \in \mathfrak{A} \mid \lim_{\hbar \to 0} \|\phi_\hbar(a)\|_\hbar = 0\}$ of sections vanishing at $\hbar = 0$. Steeger and Feintzeig (2021a, §4) show that the quotient

$$\mathfrak{A}_0 = \mathfrak{A}/K_0 \tag{2}$$

is the unique limit point C*-algebra at $\hbar \to 0$ of the bundle up to *-isomorphism and that the quotient map $\phi_0 : \mathfrak{A} \to \mathfrak{A}/K_0$ defines the unique evaluation map at the fiber over $\hbar = 0$. This procedure allows one to reconstruct the fiber algebra $\mathfrak{A}_0$ of the classical theory at $\hbar = 0$ from the bundle of quantum algebras for $\hbar > 0$.

Deformation quantizations and their associated bundles of C*-algebras provide a framework for analysis of the classical limit. I do not, however, claim that this is the only mathematical framework for analyzing the classical limit. The reason I employ these mathematical tools is that they are perspicuous for philosophical engagement with the classical limit. For example, Feintzeig (2020) argues that deformation quantization helps provide an interpretation of scaling behavior in the classical limit as explanatory, and in doing so resolves philosophical puzzles around varying values of Planck's constant. Further, Steeger and Feintzeig (2021b) show that employing continuous bundles for analyzing the classical limit aids philosophical discussions of structural determination by defining a classical limit functor, which

I employ in what follows. I take this as sufficient justification for using this mathematical framework in the remainder of the paper.

I now have enough tools to define the action of quantization and the classical limit on objects. Quantization associates to a Poisson algebra $\mathcal{P}$ of functions on M a non-commutative C*-algebra $\mathfrak{A}_\hbar$ obtained by strict deformation quantization. On the other hand, the classical limit associates to a non-commutative C*-algebra $\mathfrak{A}_\hbar$ the unique commutative algebra $\mathfrak{A}_0$ functions on the phase space obtained as the $\hbar \to 0$ limit of a bundle. In order to draw structural comparisons, I further need a way to associate morphisms of classical and quantum models with one another.

One can take the classical limit of a morphism as follows. Consider two continuous bundles of C*-algebras $((\mathfrak{A}_\hbar, \phi_\hbar)_{\hbar\in I}, \mathfrak{A})$ and $((\mathfrak{B}_\hbar, \psi_\hbar)_{\hbar\in I}, \mathfrak{B})$ over $I = (0, 1]$ representing quantum systems for $\hbar > 0$. Suppose one has a family of morphisms $\alpha_\hbar : \mathfrak{A}_\hbar \to \mathfrak{B}_\hbar$ of the fibers for $\hbar > 0$ that lift to a *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ of the algebras of sections commuting with the evaluation maps in the sense that

$$\alpha_\hbar \circ \phi_\hbar = \psi_\hbar \circ \alpha \tag{3}$$

for each $\hbar > 0$. Steeger and Feintzeig (2021a, §5) show that in this situation the morphism is appropriately continuous in $\hbar$ so that there is a unique limit morphism $\alpha_0 : \mathfrak{A}_0 \to \mathfrak{B}_0$ obtained by factoring through the quotient $\mathfrak{A}/K_0$ and thus satisfying

$$\alpha_0 \circ \phi_0 = \psi_0 \circ \alpha. \tag{4}$$

This provides a direct way to associate morphisms of a model of quantum physics with morphisms of a model of classical physics through the classical limit.

Now I present the conditions under which one can take the classical limit of a morphism in the following definition. I associate with each family of quantization maps $\{\mathcal{Q}_\hbar\}_{\hbar\in I}$ a collection of *rescaling maps* $\{R^{\mathcal{Q}}_{\hbar\to\hbar'} : \mathcal{Q}_\hbar[\mathcal{P}] \to \mathcal{Q}_{\hbar'}[\mathcal{P}]\}_{\hbar,\hbar'\in I}$ defined by

$$R^{\mathcal{Q}}_{\hbar\to\hbar'} = \mathcal{Q}_{\hbar'} \circ (\mathcal{Q}_\hbar)^{-1} \tag{5}$$

for any $\hbar, \hbar' \in I$.

Definition 3 (morphisms)**.** Suppose one has two strict deformation quantizations $(\mathfrak{A}_\hbar, \mathcal{Q}_\hbar)_{\hbar\in I}$ and $(\mathfrak{B}_\hbar, \mathcal{Q}'_\hbar)_{\hbar\in I}$ over $I = (0, 1]$ of $\mathcal{P}$ and $\mathcal{P}'$, respectively. A *-homomorphism $\alpha_\hbar : \mathfrak{A}_\hbar \to \mathfrak{B}_\hbar$ between the fiber algebras at a fixed value $\hbar \in I$ is called

(i) *smooth* if $\alpha_\hbar\big[\mathcal{Q}_\hbar[\mathcal{P}]\big] \subseteq \mathcal{Q}'_\hbar[\mathcal{P}']$;

(ii) *scaling* if for every $\hbar' > 0$, the map

$$\alpha_{\hbar'} = R^{\mathcal{Q}'}_{\hbar\to\hbar'} \circ \alpha_\hbar \circ R^{\mathcal{Q}}_{\hbar'\to\hbar} \tag{6}$$

extends continuously to a *-homomorphism $\mathfrak{A}_{\hbar'} \to \mathfrak{B}_{\hbar'}$.

The smoothness condition says that a morphism preserves the additional structure of the collection of quantized smooth functions on which the Poisson bracket is defined. Insofar as the information that certain quantities are smooth, in addition to being merely continuous, is part of the physical theory, structure-preserving maps between quantum models should encode this structure. The scaling condition says that a morphism preserves the algebraic structure regardless of the numerical value of $\hbar$, where the rescaling maps are used to shift the morphism to different values of $\hbar' > 0$ in order to make the comparison. Insofar as the numerical value of Planck's constant $\hbar$ in a strict deformation quantization depends on a system of units (See Feintzeig, 2020, 2023), it is merely a conventional choice, which the status of a map as preserving the structure of the model should not depend on. Indeed, since it is typical in deformation quantization—and it holds true in all cases considered in this paper—that all of the algebras $\mathfrak{A}_\hbar$ for $\hbar > 0$ are *-isomorphic, one can understand the algebras for the quantum theory in different systems of units (different values of $\hbar > 0$) as having the same structure. By understanding morphisms of a quantum theory as *-homomorphisms satisfying the scaling condition, I am only requiring that these structure-preserving maps respect this structural sameness in different systems of units.[15]

If a *-homomorphism $\alpha_\hbar : \mathfrak{A}_\hbar \to \mathfrak{B}_\hbar$ is scaling, then the construction surrounding Eq. (3) provides a lift of the family $(\alpha_\hbar)_{\hbar \in I}$ to a morphism of the algebra of sections of the bundle and produces a unique limit morphism α_0 satisfying Eq. (4). If $\alpha_\hbar$ is smooth, then it follows that α_0 preserves the privileged Poisson subalgebra *and* the Poisson bracket (Steeger and Feintzeig, 2021a, Prop. 5.5).

I will restrict attention to smooth, scaling morphisms of deformation quantizations. I emphasize that such morphisms preserve more than the algebraic structure of a model of quantum physics. The scaling condition ensures that a morphism preserves the algebraic structure for any value of Planck's constant $\hbar > 0$. So with the interpretation given by Feintzeig (2020) of different values of $\hbar$ corresponding to different systems of units, this means that one can understand such a morphism to preserve the algebraic structure of a model of quantum physics in any system of units. In other words, scaling morphisms preserve the extra scaling structure described above that is encoded in deformation quantizations and their attendant bundles. Since this scaling structure represents *physically relevant* information governed by the interpretations of elements of a C*-algebra as physical quantities, this justifies a restriction to morphisms that preserve this structure. Moreover, since scaling morphisms are just those that do not depend on the system of units employed for the formulation of quantum physics they should appear as morphisms even in an "intrinsic" or unitless formulation of quantum theory (Dewar, 2021). I take these reasons to justify the restriction to scaling morphisms in the results of this paper.

With these tools for understanding quantization and the classical limit, I will proceed to characterize two categories of models of classical physics that can be quantized functorially, and whose quantization provides a categorical equivalence.

5 Categorical Equivalences

Here, I present categorical equivalences for two classes of quantum models constructed through strict deformation quantization (Feintzeig, 2024). In §5.1, I analyze quantization

[15] I leave it as an open question whether there even exist morphisms between fiber C*-algebras of a strict deformation quantization that do not satisfy the scaling condition in cases of interest. I have not been able to find morphisms between the fiber C*-algebras used in §5.1-5.2 that fail the scaling condition.

via the C*-Weyl algebra, and in §5.2, I analyze Rieffel's quantization for actions of $\mathbb{R}^d$. This vindicates the central claim of this paper by witnessing the structural continuity of these particular quantization procedures on these models.

5.1 The C*-Weyl Algebra for Linear Phase Spaces

One standard method for quantizing classical theories using the C*-Weyl algebra applies to systems whose phase space is the dual space V' (i.e., the collection of continuous linear functionals) of a topological vector space V with a symplectic form σ (i.e., a non-degenerate, bilinear, antisymmetric map $V \times V \to \mathbb{R}$). In this case, the Poisson *-algebra $\Delta(V, 0) \subseteq C_b(V')$ is generated by the functions $W_0(f) : V' \to \mathbb{C}$ for each fixed $f \in V$ defined by

$$W_0(f)(F) = e^{iF(f)} \tag{7}$$

for all $F \in V'$. The Poisson bracket on $\Delta(V, 0)$ is defined by the linear extension of

$$\{W_0(f), W_0(g)\} = \sigma(f, g)W_0(f + g) \tag{8}$$

for all $f, g \in V$. This algebra $\Delta(V, 0)$ is norm dense in the C*-algebra $AP(V')$ of continuous almost periodic functions on the phase space V'. This structure specifies the classical model.

The corresponding quantum model is obtained through the exponentiated Weyl form of the canonical commutation relations, which define for each $\hbar > 0$ a C*-algebra $\mathcal{W}(V, \hbar\sigma)$. A dense subalgebra $\Delta(V, \hbar)$ is generated freely by linearly independent elements of the form $W_\hbar(f)$ for $f \in V$ with multiplication and involution operations specified by

$$W_\hbar(f)W_\hbar(g) = e^{-\frac{i}{2}\hbar\sigma(f,g)}W_\hbar(f + g) \tag{9}$$
$$W_\hbar(f)^* = W_\hbar(-f) \tag{10}$$

for all $f, g \in V$. There is a unique maximal C*-norm on $\Delta(V, \hbar)$ and the C*-Weyl algebra $\mathcal{W}(V, \hbar\sigma) = \overline{\Delta(V, \hbar)}$ is defined as the completion of this dense subalgebra with respect to the C*-norm (See Petz, 1990).

In the special case where $V = \mathbb{R}^{2n}$, one can understand $\mathcal{W}(V, \hbar\sigma)$ through the standard Schrödinger representation π_S on $\mathcal{H}_S = L^2(\mathbb{R}^n)$. In this case, let $Q_j^\hbar$ and $P_j^\hbar$ denote the position and momentum operators

$$(Q_j^\hbar\psi)(x) = x_j \cdot \psi(x) \tag{11}$$
$$(P_j^\hbar\psi)(x) = i\hbar\frac{\partial}{\partial x_j}\psi(x) \tag{12}$$

for all $\psi \in L^2(\mathbb{R})$. Then π_S is the continuous linear extension of the representation

$$\pi_S(W_\hbar(a, b)) = e^{i\sum_{j=1}^n a_j \cdot P_j^\hbar + b_j \cdot Q_j^\hbar} \tag{13}$$

so that $\mathcal{W}(V, \hbar\sigma)$ can be understood as the C*-algebra generated by exponentials of configuration and momentum quantities.

The quantization maps $\mathcal{Q}_\hbar : \Delta(V, 0) \to \mathcal{W}(V, \hbar\sigma)$ are given for $\hbar \in [0, 1]$ by the linear extension of

$$\mathcal{Q}_\hbar(W_0(f)) = W_\hbar(f) \tag{14}$$

for all $f \in V$. These quantization maps define a strict deformation quantization on $M = V'$ for the Poisson algebra $\mathcal{P} = \Delta(V, 0) \subseteq \mathfrak{A}_0 = AP(V')$ and fiber algebras $\mathfrak{A}_\hbar = \mathcal{W}(V, \hbar\sigma)$ for $\hbar > 0$ (Binz et al., 2004).

One can define a category of classical models with linear phase spaces, as follows. This category will form the domain of a quantization functor.

Definition 4. Denote the following category by **LinClass**:

- *Objects* are pairs $\big(AP(V'), \Delta(V, 0)\big)$, where $AP(V')$ is the C*-algebra of almost periodic functions on the dual to a topological vector space V, and $\Delta(V, 0)$ is the dense Poisson subalgebra with Poisson bracket defined by a symplectic form σ_V.

- *Arrows* are *-homomorphisms $\alpha_0 : AP(V') \to AP(U')$ for symplectic topological vector spaces (V, σ_V) and (U, σ_U) that are *smooth* in the sense that

$$\alpha_0\big[\Delta(V, 0)\big] \subseteq \Delta(U, 0) \tag{15}$$

and *Poisson* in the sense that

$$\alpha_0\big(\{A, B\}_V\big) = \big\{\alpha_0(A), \alpha_0(B)\big\}_U \tag{16}$$

for all $A, B \in \Delta(V, 0)$.

Note that this category is general enough to include infinite-dimensional phase spaces representing linear classical field theories. The morphisms in this category preserve the structure of classical models at $\hbar = 0$ as symplectic phase spaces.

Similarly, one can define a category of quantum models corresponding to these linear phase spaces.

Definition 5. Denote the following category by **LinQuant**:

- *Objects* are deformation quantizations $\big(\mathcal{W}(V, \hbar\sigma), \Delta(V, \hbar), \mathcal{Q}_\hbar\big)_{\hbar \in (0,1]}$ of $\mathcal{P} = \Delta(V, 0)$.

- *Arrows* are smooth, scaling *-homomorphisms $\alpha_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, where $\mathfrak{A}_1 = \mathcal{W}(V, \sigma_V)$ and $\mathfrak{B}_1 = \mathcal{W}(U, \sigma_U)$ are the C*-Weyl algebras at $\hbar = 1$ for symplectic topological vector spaces V and U, respectively.

The morphisms in this category thus preserve the structure of the fully quantized models as non-commutative C*-algebras of operators at $\hbar = 1$.

The following result holds:

Theorem 1 (Feintzeig (2024))**.** *There are functors*

$$Q_W : \mathbf{LinClass} \leftrightarrows \mathbf{LinQuant} : L_W \tag{17}$$

providing a categorical equivalence.

The functors Q_W and L_W provide a one-to-one correspondence between the structure-preserving maps of each model in **LinClass** and **LinQuant**. Hence, this shows a sense in which, relative to the structure encoded in these choices of categories, classical and quantum models have shared structure, when compared with these choices of functors.

5.2 Rieffel's Quantization for Actions of $\mathbb{R}^d$

While quantization via the Weyl algebra is prominent among philosophers, it has limited applications and technical issues. Next, I consider a quantization prescription for a different algebra. To do so, I restrict to finite-dimensional phase spaces, thus losing the generality of the Weyl algebra for representing field theories. But this allows us to consider phase spaces that are manifolds and not necessarily linear spaces.

The method of quantization due to Rieffel (Rieffel, 1993) applies to classical systems whose phase space is a manifold M with a diffeomorphic action β of the Lie group $\mathbb{R}^d$. In what follows, I will assume the group $\mathbb{R}^d$ acts freely on M. In this case, I also assume the Lie group carries a symplectic form σ on $\mathbb{R}^d$, which corresponds to an antisymmetric matrix J_{jk} on the vector space $\mathbb{R}^d$, understood as the Lie algebra of the Lie group $\mathbb{R}^d$. In this case the Poisson *-algebra $C_c^\infty(M) \subseteq C_b(M)$ is the collection of smooth, compactly supported functions on the phase space. This algebra $C_c^\infty(M)$ is norm dense in the C*-algebra $C_0(M)$ of continuous functions vanishing at infinity on the phase space. The action of $\mathbb{R}^d$ on M defines an automorphic action τ of $\mathbb{R}^d$ on $C_0(M)$ by

$$\tau_x(f) = f \circ \beta_x. \tag{18}$$

The subalgebra $C_c^\infty(M)$ carries a corresponding infinitesimal action of the Lie algebra $\mathbb{R}^d$ by smooth vector fields ξ_X for $X \in \mathbb{R}^d$ given by

$$\xi_X(f) = \frac{\partial}{\partial t}_{|t=0} \tau_{tX}(f) \tag{19}$$

for all $f \in C_c^\infty(M)$. The Poisson bracket on $C_c^\infty(M)$ is then defined from the infinitesimal action of the Lie algebra and the symplectic form σ for all $f, g \in C_c^\infty(M)$ by

$$\{f, g\} = \sum_{j,k} J_{jk} \xi_{X_j}(f) \xi_{X_k}(g), \tag{20}$$

where the vectors $\{X_k\}_{k=1}^d$ form a basis for the Lie algebra $\mathbb{R}^d$. This structure specifies the classical model.[16]

The corresponding quantum model is obtained by deforming the product on $C_c^\infty(M)$. Define $\mathcal{P}_\hbar(M)$ to be the vector space with involution $C_c^\infty(M)$ with the new multiplication operation $\star_\hbar$, sometimes called the Moyal product, defined by

$$f_\hbar \star_\hbar g_\hbar = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(f) \tau_y(g) e^{i\hbar\sigma(x,y)} \tag{21}$$

where I use the notation $f_\hbar, g_\hbar \in \mathcal{P}_\hbar(M)$ to distinguish these from the identical elements $f, g \in C_c^\infty(M)$. Rieffel (1993) shows that this expression can be made well-defined in terms of oscillatory integrals, and that one can define a C*-norm on $\mathcal{P}_\hbar(M)$ so that the completion

$$\mathfrak{A}_\hbar(M) = \overline{\mathcal{P}_\hbar(M)}$$

[16] The methods developed by Rieffel (1993) for quantization apply more broadly, even to deforming products on non-commutative C*-algebras carrying actions of $\mathbb{R}^d$. The methods have been further generalized by Landsman (1998, 1999) to cases where the construction is employed locally, including Riemannian manifolds, principal bundles, and Lie groupoids. Bieliavsky and Gayral (2015) have provided a generalization of the quantization prescription for a much wider class of group actions.

with respect to this norm is a C*-algebra. Note that each C*-algebra $\mathfrak{A}_\hbar(M)$ also carries a strongly continuous group action of $\mathbb{R}^d$, which I denote by $\tau^\hbar$, picked out as the unique continuous extension of the group action τ on $\mathcal{P}_\hbar$ (Rieffel, 1993, Thm. 5.11, p. 44). Likewise, there is an infinitesimal action of the Lie algebra, which I denote by $\xi^\hbar$ on the subalgebra $\mathcal{P}_\hbar(M)$.

For example, in the case where $M = \mathbb{R}^{2n}$ with the group action τ for $d = 2n$ by translations, one has $\mathfrak{A}_\hbar(\mathbb{R}^{2n}) \cong \mathcal{K}(L^2(\mathbb{R}^n))$. One can also understand this algebra through the standard Schrödinger representation of $\mathfrak{A}_\hbar(\mathbb{R}^{2n})$ on $L^2(\mathbb{R}^n)$, which I now denote $\tilde{\pi}_S$, given by the continuous extension of

$$\tilde{\pi}_S(f_\hbar) = \int_{\mathbb{R}^{2n}} \frac{d^n a d^n b}{(2\pi)^n} (\mathcal{F}f)(a,b)\pi_S(W_\hbar(a,b)) \tag{22}$$

for $f_\hbar \in \mathcal{P}_\hbar(\mathbb{R}^{2n})$. Here, $\mathcal{F}f$ denotes the Fourier transform of the function $f \in C_c^\infty(\mathbb{R}^{2n})$ and $\pi_S(W_\hbar(a,b))$ is the Schrödinger representation of the element $W_\hbar(a,b)$ in the Weyl algebra $\mathcal{W}(\mathbb{R}^{2n}, \sigma)$ as given by Eq. (13).

The quantization maps $\mathcal{Q}_\hbar : C_c^\infty(M) \to \mathfrak{A}_\hbar(M)$ are given for $\hbar \in [0,1]$ by

$$\mathcal{Q}_\hbar(f) = f_\hbar \tag{23}$$

for all $f \in C_c^\infty(M)$. These quantization maps define a strict deformation quantization on M with $\mathcal{P} = C_c^\infty(M)$ and fiber algebras $\mathfrak{A}_\hbar(M)$ for $\hbar > 0$.

To define the categories of classical and quantum models suitable for Rieffel quantization, I will need to specify when a morphism of a C*-algebra (either $C_0(M)$ or $\mathfrak{A}_\hbar(M)$) is compatible with a group action. Suppose one has a *-homomorphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ between two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ carrying group actions by $\mathbb{R}^d$ and $\mathbb{R}^{d'}$, respectively. I now denote the infinitesimal action of the Lie algebra by ξ (corresponding to the action ξ or $\xi^\hbar$, as above.) I will call α *compatible with the group actions* if for each $X \in \mathbb{R}^{d'}$, there is a $Y \in \mathbb{R}^d$ such that $\xi_X \circ \alpha = \alpha \circ \xi_Y$ on the domain of ξ_X and ξ_Y (i.e., on $C_c^\infty(M)$ or $\mathcal{P}_\hbar$).

Now I define a category of classical models suitable for Rieffel quantization.

Definition 6. Denote the following category by **RClass**:

- *Objects* are triples $(C_0(M), C_c^\infty(M), \tau)$, where $C_0(M)$ is the C*-algebra of continuous functions vanishing at infinity on a manifold M carrying a strongly continuous, free action τ of $\mathbb{R}^d$ on $C_0(M)$ arising from a diffeomorphic action on M by Eq. (18). $C_c^\infty(M)$ is a dense Poisson subalgebra with Poisson bracket defined by the symplectic form σ on $\mathbb{R}^d$ by Eq. (20).

- *Arrows* are *-homomorphisms $\alpha_0 : C_0(M) \to C_0(N)$ for manifolds M and N that are (i) compatible with the group actions, (ii) *smooth* in the sense that

$$\alpha_0\big[C_c^\infty(M)\big] \subseteq C_c^\infty(N) \tag{24}$$

and (iii) *Poisson* in the sense that

$$\alpha_0\big(\{A,B\}_M\big) = \big\{\alpha_0(A), \alpha_0(B)\big\}_N \tag{25}$$

for all $A, B \in C_c^\infty(M)$.

This category is general enough to include non-linear phase spaces. The morphisms in this category preserve the structure of classical models at $\hbar = 0$ as phase spaces.

Likewise, I define a category of quantum models corresponding to Rieffel's quantization.

Definition 7. Denote the following category by **RQuant**:

- *Objects* are deformation quantizations $\left(\mathfrak{A}_\hbar(M), \mathcal{P}_\hbar(M), \mathcal{Q}_\hbar, \tau^\hbar\right)_{\hbar\in(0,1]}$ of $\mathcal{P} = C_c^\infty(M)$, as given by the discussion around Eq. (21) and (23).
- *Arrows* are *-homomorphisms $\alpha_1 : \mathfrak{A}_1(M) \to \mathfrak{A}_1(N)$ between the C*-algebras $\mathfrak{A}_1(M)$ and $\mathfrak{A}_1(N)$ at $\hbar = 1$ obtained as the Rieffel quantizations of $C_c^\infty(M)$ and $C_c^\infty(N)$, respectively, where the morphisms are (i) compatible with the group actions, (ii) smooth, and (iii) scaling.

The morphisms in this category preserve the structure of the fully quantized models as non-commutative C*-algebras of operators at $\hbar = 1$.

The following result holds:

Theorem 2 (Feintzeig (2024))**.** *There are functors*

$$Q_R : \mathbf{RClass} \leftrightarrows \mathbf{RQuant} : L_R \tag{26}$$

providing a categorical equivalence.

The functors Q_R and L_R provide a one-to-one correpsondence between the structure-preserving maps of each model in **RClass** and **RQuant**. Hence, this shows a sense in which, relative to the structure encoded in these choices of categories, classical and quantum models share structure, when compared with these choices of functors.

6 Significance for Theory Construction

Now that I have presented Thms. 1 and 2, I conclude by discussing their significance. By presenting a structural correspondence between classical and quantum theories (relative to the choices of categories and functors used to compare them), I have in some sense vindicated the claims of structural preservation across theory change that structural realists draw upon as evidence for their position. However, I believe that these structural comparisons have a philosophical significance beyond structural realism. I argue here that structural comparison is a worthy pursuit even for one who is skeptical about or uninterested in the realism debates.

Stein (1989) provides motivation for skepticism about the realism debates by arguing that one cannot coherently distinguish between sufficiently sophisticated forms of realism and antirealism. However, Stein ultimately endorses the idea that there is a significance to finding structure that is maintained across instances of theory change. Some of Stein's remarks fit in with those of structural realists, including the following excerpt, which I quote at length (Stein, 1989, p. 58):

> [I]n the development of physics in [...] our own century—an [...] astonishing discovery has emerged; more astonishing because it involves our understanding of *ordinary, familiar things*, and because the key to this understanding turns out to lie in mathematical structures, and *improved* understanding in *structural "deepening"*, of just such kind as characteristically occur in pure mathematics itself. And, I should add, more astonishing not least because of the circumstance I am concerned to emphasize, that in this structural deepening what tends to persist—to remain, as it were, quasi-invariant through the transformation of theories—is on the whole (and especially in what we think of as the "deepest"—or most "revolutionary"—transformations) not the features most conspicuous in referential semantics: the substances or "entities" and their own "basic" properties and relations, but the more abstract mathematical forms.

These claims of structural continuity echo those of Worrall (1989). But whereas Worrall and other structural realists takes structural continuity as evidence for realism, Stein takes a different tack. He finds significance for structural continuity in *theory construction* (Stein, 1989, p. 57):

> I agree wholeheartedly [...] that *we have learned*—that is to say, scientists have indeed learned, in their practice; and in our philosophical reflections upon science, we *should* by now have learned explicitly—that successful scientific theories are to be taken very seriously as clues to the deeper understanding of phenomena, i.e. in the search for better and more fundamental theories.

The claim is that one should use the structure of our current physical theories as a guide in the search for new physics. I take up this suggestion and argue that structural continuity from classical to quantum physics has significance for the construction of new models.

First, notice that the quotes presented from Stein are slightly ambiguous. The term "scientific theory" might refer to two different kinds of things, which I will distinguish by calling them a *theoretical framework* and a *model*. On my usage, classical Hamiltonian mechanics is a theoretical framework, while quantum mechanics is a different theoretical framework. On the other hand, different dynamical equations define different models in each framework. So for example, there is a model in the framework of quantum mechanics for a single charged particle under a central Coulomb force as in the hydrogen atom, while there is a distinct model for electromagnetically charged Fermionic matter fields governed by the Dirac equation, and a distinct model for bosonic matter governed by the Klein-Gordon equation, and so on. One of my claims is that it is worth thinking philosophically not only about transitions between different theoretical frameworks (as is more common in the realism literature), but also about the construction of new models.

In the construction of quantum theories, guidance is needed both at the level of frameworks and models. Theoretical physicists do not have agreement about how to construct new quantum theories, as evidenced by the proliferation of different approaches to quantum gravity.[17] In fact, some approaches seem to attempt to fit quantum gravity into the existing theoretical framework of quantum mechanics, while other approaches seem to seek a new

[17] See, e.g., Callender and Huggett (2001).

theoretical framework entirely. So guidance is needed for applying the theoretical framework of quantum physics to construct models of phenomena that are currently outside its purview. Further, guidance is needed in constructing mathematical models in the theoretical framework of quantum physics that represent phenomena that already fall under the purview of quantum field theory and the standard model of particle physics. There is not yet a mathematically rigorous model capturing quantum Yang-Mills theory, the model of quantum field theory underlying all inter-particle forces in the standard model. Although the predictions of the standard model are well enough understood to be highly confirmed by empirical evidence, the Clay Institute for Mathematics lists as one of the Millenium Prize Problems the task of providing a rigorous mathematical formulation of quantum Yang-Mills theory reproducing a basic property called the "mass gap" (Jaffe and Witten, 2000; Douglas, 2004). Similarly, the profileration of different methods for constructing models in the framework of quantum field theory (Summers, 2016) shows that scientists have not yet learned *precisely how* to take successful quantum theories as clues for the construction of new models. My suggestion, then, is that we should construct new models of quantum physics by extending quantization procedures that are known to preserve structure. In this way, bringing structural continuity to bear on the construction of models of quantum theory can have an impact on important and outstanding questions in the foundations of physics.

My vision for how philosophical work in the tradition of the realism debates might make a difference for scientific methodology in practice aligns closely with that of Stanford (2018). Stanford translates anti-realism into a liberal stance concerning ongoing science in which one should expect new theories to look very different from previous theories. In contrast, my conclusion in this paper is conservative in its recommendation to construct new physical theories that *do* look like those of the past and present. I do not see this as a disagreement because unlike in Stanford's discussion, the task at hand in theoretical physics I am drawing attention to is *not* to determine how one might project "forward" from methods for constructing quantum theories to methods for constructing future theoretical frameworks. Rather, the task I see is to determine how to project "sideways" by applying existing methods for constructing models within the already known theoretical framework of quantum physics to construct new models within the same theoretical framework. The quantization prescriptions analyzed in this paper provide examples of how certain models in the framework of quantum physics are already constructed from corresponding models in the framework of classical physics in a way that is guided by and preserves theoretical structure. My suggestion is that the results of this paper establishing structural preservation from classical to quantum physics give us reason to conservatively employ essentially these same procedures at least as a first attempt in the construction of new models. Stanford may be correct that drastic changes are needed to construct new theoretical frameworks, but this conclusion need not apply to the construction of new models within the known framework of quantum physics.

I am proposing two different positions for consideration, both of which I understand as analogues of structural realism that move away from questions about metaphysics and even what we ought to believe. Instead, these analogues of structural realism concern the more pragmatic question of what methodology we ought to take going forward in science, thus paralleling Stanford's methodological position. The two different analogues of structural realism that I propose advocate for, respectively, seeking out structural continuity in either (i) the construction of new models of an existing framework (projecting sideways), or (ii) the construction of an entirely new theoretical framework (projecting forwards). I myself believe both positions are plausible enough to be worth further discussion, but in this paper I have only provided arguments bearing on the first position concerning the construction of new models. So in this paper, I maintain agnosticism twice over—first, I am agnostic about the traditional realist's questions concerning what exists and what to believe (so that my conclusions are orthogonal to the majority of the structural realism literature) and second, I am agnostic about the appropriate methodology for constructing new theoretical frameworks, although I think the methodology of aiming for structural continuity warrants further investigation. Ultimately, the modest position I advocate for is that we should aim for structural continuity when projecting sideways to construct new models in the existing framework of quantum physics.

Note that aiming for structural continuity from classical to quantum models involves more than merely agreeing to formulate new physics in a broadly quantum mechanical framework. In contemporary formulations, constructing models of quantum field theory involves defining a non-commutative product on a relevant algebra of field quantities, which in turn corresponds to a choice of quantization map (Fredenhagen and Rejzner, 2015; Hawkins and Rejzner, 2020). The methodological force of aiming for structural continuity as I advocate in this paper involves using the structure of classical physics to guide the construction of such algebraic structures. Indeed, I believe such considerations from correspondence with classical physics already play a role in mathematical physics. The detailed ways in which mathematical physicists use information from classical physics to inform the construction of models of quantum field theory deserves systematic philosphical attention.

The construction of new models of quantum physics by these means will not necessarily ensure the desired outcome (e.g., an accurate representation of the phenomena), but it would provide at least a conservative understanding of the mathematical formulation of such models of quantum field theory—or more generally, new models of quantum physics—that aligns with our current understanding of the mathematical formulation of models of quantum theory. I believe that this is a good place to start, and I hold out hope that this will clarify issues faced in generating new physics.

One might be skeptical that quantization and structural continuity will play a role in the construction of models of quantum Yang-Mills theory. Physicists often emphasize instead the importance of the renormalization group in resolving divergence issues in the construction of quantum theories. One might object that renormalization group methods seem independent of quantization. However, contemporary mathematical approaches to quantum field theory reveal that the framework of deformation quantization applies to many field theory models including Yang-Mills gauge theory models (Fredenhagen and Rejzner, 2013, 2015). The process of renormalization can be understood in causal perturbation theory as resolving ambiguities in the products of operators at the same spacetime point. The definition of the non-commutative product that resolves these renormalization ambiguities also corresponds to a choice of quantization map (Hawkins and Rejzner, 2020), which is thus informed by the corresponding model of classical physics. Recent work even shows promise for resolving large order divergences in perturbation theory by defining the non-commutative product in a way that allows the coupling constant in an interacting field theory to take on a numerical, rather than formal value (Hawkins and Rejzner, 2020). These last mentioned results take place in the framework of formal, rather than strict deformation quantization, so they treat $\hbar$ as a formal parameter. Further work remains in the transition from formal to strict quantization, which would allow us to understand the quantum model in terms of genuine operators. Still, the recent literature in mathematical physics shows that the correspondence of classical and quantum structure has a role to play in quantum field theory.

What, specifically, does one learn from the central results of this paper—that quantization can be understood as a categorical equivalence—for the construction of quantum theories? First, by understanding what structures are preserved from classical physics through quantization in known models, one gains insight into the structures one needs to specify in classical theories in order to quantize them in the same manner. Second, by understanding quantization as a functor, there is a sense in which one employs *the same* prescription for theory construction across a variety of model systems, rather than ad hoc methods that vary from case to case. Third, by establishing a categorical equivalence one is assured that the known quantization prescriptions preserve the structure of classical models, which provides motivation for generalizing *these* methods for theory construction to a class of models wide enough to encompass cases of interest, e.g., classical field theories with infinite dimensional phase spaces and non-linear dynamics. I hope to have shown that structural preservation between classical and quantum physics, as established in this paper, is significant for the construction of quantum theories.

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