Why the global phase is not real

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Abstract

In this paper, I present a new analysis of the meaning of the phase in quantum mechanics. First, I give a simple but rigorous proof that the global phase is not real in ψ-ontic quantum theories. Next, I argue that a similar strategy cannot be used to prove the reality of the global phase due to the existence of the tails of the wave function. Finally, I argue that the relative phase is not a nonlocal property of two regions together, and adding a relative phase to one local branch of a superposition only changes the local properties at the boundary of the region of the branch.

1 Introduction

It is a standard view in quantum mechanics that the global phase is not real, i.e. two wave functions that differ only in the global phase represent the same physical state, and thus the space of physical states has the structure of a projective Hilbert space rather than that of a linear Hilbert space. The main reason motivating this view is that the change in the global phase cannot be measured by experiments, and the empirical predictions of quantum mechanics are not sensitive to the change in the global phase either. However, this reason is not sufficient. It has been widely thought that there may exist unobservable physical properties in a realist quantum theory. For example, in Bohmian mechanics, the trajectories of the Bohmian particles are unobservable in principle (Goldstein, 2021).[1] In this paper, I will give a

[1]Note that Bohmian mechanics is different from de Broglie's original double solution program. For a helpful introduction of the latter, see Colin et al (2017) and Croca et al (2023).

rigorous proof of the unreality of the global phase that is independent of its unobservability in ψ-ontic quantum theories.[2] Moreover, I will argue that the existence of the tails of the wave function will block a similar argument for the reality of the global phase and also help clarify the meaning of the relative phase.

2 A simple proof of the unreality of the global phase

In the following, I will first give a rigorous proof that the global phase is not real in ψ-ontic quantum theories. There are three reasons to restrict my analysis to ψ-ontic quantum theories. The first reason is that there are already strong ψ-ontic theorems (Pusey, Barrett and Rudolph, 2012; Hardy, 2013; Gao, 2024). According to the ψ-ontic view that these theorems prove, two wave functions which differ not only in the global phase represent different physical states, but it is still unclear whether two wave functions which differ only in the global phase represent different physical states or whether the global phase is real. The second reason is that if the wave function is not real (e.g. as required by the ψ-epistemic view), then it will be insignificant to further prove the unreality of the global phase, and in some cases it will be obvious that the global phase is not real either.[3] The last reason is that it is easier to find a rigorous proof of the unreality of the global phase in ψ-ontic quantum theories. A general proof for all quantum theories still needs to be found.

Suppose there are two types of non-interacting particles 1 and 2 being in a product state $|\psi_1\rangle \otimes |\psi_2\rangle$. Consider two situations. One is that a unitary interaction is introduced to add a global phase ϕ to $|\psi_1\rangle$, where $\phi \in (0, 2\pi)$, and the state of the two particles becomes $e^{i\phi} |\psi_1\rangle \otimes |\psi_2\rangle$. The other is that another unitary interaction is introduced to add a global phase ϕ to $|\psi_2\rangle$, and the state of the two particles becomes $|\psi_1\rangle \otimes e^{i\phi} |\psi_2\rangle$. Since the two particles are of different types, one can introduce a unitary interaction to add a global phase only to the wave function of one particle (and not to the wave function of the other particle). Then we have the relation $e^{i\phi} |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle \otimes e^{i\phi} |\psi_2\rangle$.

Now if the global phase is real, then the unitary interaction that adds a global phase ϕ to $|\psi_1\rangle$ will change the physical state of particle 1, and the unitary interaction that adds a global phase ϕ to $|\psi_2\rangle$ will change the physical state of particle 2. This means that the changed physical states

[2]For a recent discussion of this topic, see Wallace (2022) and Gao (2022).

[3]For example, if the ψ-epistemic view requires that two wave functions that differ only in the global phase are compatible with the same ontic state or physical state, then it will be obvious that the global phase is not real, since the two wave functions that differ in the global phase do not represent different physical states.

of the two particles in the above two situations will be different. But they are represented by the same wave function $e^{i\phi}|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle \otimes e^{i\phi}|\psi_2\rangle$. This contradicts the ψ-ontic view.[4] Therefore, the global phase is not real in ψ-ontic quantum theories.

It can be seen that the unreality of the global phase results from the fact that the global phase of a product state of two particles does not uniquely determine the global phase of each particle. For example, there are infinitely many identical wave functions for which the sum of the global phases of particles 1 and 2 is ϕ, and three of them are $e^{i\phi}|\psi_1\rangle \otimes |\psi_2\rangle$, $|\psi_1\rangle \otimes e^{i\phi}|\psi_2\rangle$ and $e^{i\phi/2}|\psi_1\rangle \otimes e^{i\phi/2}|\psi_2\rangle$. If the global phase is real (which means that two wave functions of a particle which differ in the global phase will represent different physical states of the particle), then these identical wave functions will represent different physical states. But this contradicts the ψ-ontic view.

Finally, it is worth pointing out that the above proof of the unreality of the global phase also applies to the product state of two properties (e.g. position and spin) of a single quantum system. Thus, we can also prove that the global phase of the universal wave function is not real.

3 Can we similarly prove the reality of the global phase?

It seems that one can also use a similar strategy to prove the reality of the global phase in ψ-ontic quantum theories. Suppose there is a superposition of two spatially separated wavepackets of a particle $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Consider two situations. One is that a unitary transformation is applied in the region of $|\psi_1\rangle$, which adds a phase ϕ to this branch, where $\phi \in (0, 2\pi)$, and the superposition becomes $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$. The other is that a unitary transformation is applied in the region of $|\psi_2\rangle$, which adds a phase $-\phi$ to this branch, and the superposition becomes $\frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$. We have the relation $e^{i\phi}|\psi_1\rangle + |\psi_2\rangle = e^{i\phi}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$. Now if the two superpositions in these two situations, which differ by a global phase factor, correspond to two different physical states, then we can prove that the global phase is real.

On the ψ-ontic view, the two superpositions $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ correspond to different physical states, so do $\frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. In other words, the unitary transformation that changes the phase of each branch of the initial superposition also changes the underlying physical state of the particle. Moreover, when as-

[4] If the wave function is complete, then the wave function uniquely determines the physical state, and thus different physical states cannot be represented by the same wave function. If the wave function is not complete and there are hidden variables, different physical states that include the same hidden variables cannot be represented by the same wave function either.

suming locality for product states,[5] the unitary transformation applied in one region does not change the physical state of the particle in other regions. Then, the unitary transformation that changes the phase of $|\psi_1\rangle$ only changes the physical state of the particle in the region of $|\psi_1\rangle$, and the unitary transformation that changes the phase of $|\psi_2\rangle$ only changes the physical state of the particle in the region of $|\psi_2\rangle$. Since the regions of $|\psi_1\rangle$ and $|\psi_2\rangle$ are separated, the changed physical states in the above two situations are different. This proves the reality of the global phase in ψ-ontic quantum theories.

There is a potential objection to the above proof based on the second-quantized description of quantum states. In this description, the above superposition of a particle, $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, can be rewritten in an entangled state $\frac{1}{\sqrt{2}}(|1\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2)$, where $|1\rangle_1$ and $|1\rangle_2$ are the one-particle states which describe the regions 1 and 2 with one particle, and $|0\rangle_1$ and $|0\rangle_2$ are the vacuum states which describe the regions 1 and 2 without the particle. Then, each region is described not by a pure state, but by a mixed state $\frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|)$. As a result, a phase transformation in each region does not change the state of the region represented by this mixed state.

However, it can be argued that this objection is not valid. The key is to notice that the density matrix formulation contains no information about the global phase of the wave function. Not only a mixed state but also the density matrix of a pure state such as $|1\rangle\langle 1|$ is not changed by a phase transformation. Then, if we assume that the density matrix of a pure state is a complete representation of the physical state, we will already exclude the possibility that the global phase is real. But this assumption has not been justified. Although the density matrix formulation is enough for empirical predictions, it may not contain the whole truth about the ontology of quantum mechanics. In this sense, that a phase transformation does not change the density matrix of one region does not imply that it does not change the physical state of the region.

The real issue with the above proof is that it ignores the tails of the wave function. Since the physical interactions are always finite, the wave function of a particle will have infinitely long tails in the universe. Although the tails of the wave function can be omitted for all practical purposes if they are sufficiently small, they cannot be ignored in the above proof, no matter how small they are. The reason is that only when the tails of each wavepacket in the superposition do not exist in the region of the other wavepacket, can a unitary transformation be applied to add a phase *only* to one branch of the

[5] Locality for product states says that for two systems being in a product state, the ontic state of one system (e.g. a particle) in one region is not affected by the other system in the other region (e.g. a system which implements a unitary transformation there) via action at a distance, and it holds true in existing ψ-ontic quantum theories such as Bohmian mechanics, the many-worlds interpretation and collapse theories of quantum mechanics (Gao, 2024).

superposition, but not to the other branch of the superposition. It is this possibility that leads to the relation $e^{i\phi}|\psi_1\rangle + |\psi_2\rangle = e^{i\phi}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$ and further makes the proof go through. If we take the tails of each wavepacket into consideration, then we cannot derive the above relation, and thus the above proof of the reality of the global phase cannot go through.

In fact, the above proof of the unreality of the global phase cannot go through either due to the same reason when the two particles are of the same type (e.g. they are both electrons) and the unitary transformations are local. In this case, we have two independent particles 1 and 2 being in a product state $|\psi_1\rangle \otimes |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are two spatially separated (nomalized) wave functions. Since the two particles are of the same type, a local unitary transformation applied in the region of $|\psi_1\rangle$, which adds a phase ϕ to (the bulk of) this branch, will also add a phase ϕ to the tails of the other branch $|\psi_2\rangle$ when taking the tails of the wave function into consideration. Then we will not have the exact relation $e^{i\phi}|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle \otimes e^{i\phi}|\psi_2\rangle$, and thus the proof cannot go through.

4 Is the relative phase nonlocal?

In this section, I will argue that taking the tails of the wave function into consideration will also help understand the nature of the relative phase.

The phase ϕ in the superposition $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ is often called the relative phase. This denomination seems to suggest that the relative phase is a relative property or a nonlocal property of the two branches of the superposition together. Moreover, it seems that the unreality of the global phase requires that changing of relative phase must change the nonlocal properties of two regions together. In other words, the relative phase must be nonlocal. An argument for this view can be formulated as follows, which is similar to the above (wrong) proof of the reality of the global phase. On the ψ-ontic view, the two superpositions $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ correspond to different physical states, so do $\frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Moreover, if changing of relative phase only changes local properties, then the added relative phases in the two states $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$ will change the local properties in the two regions of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively, and thus these two states will correspond to different physical states. But this contradicts the unreality of the global phase, since these two states differ only by a global phase factor.

As noted above, the issue with this argument is that it ignores the tails of the wave function. If taking the tails of each branch in a superposition into consideration, then we cannot add a relative phase only to one branch of the superposition. A local unitary transformation which adds a relative phase to (the bulk of) one branch will also add the same phase to the tails of the other branch. This means that we cannot generate the two states $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$, which differ only by a global phase factor, from the original state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Thus the above argument cannot go through.

When realizing that a local unitary transformation can only add a position-dependent phase to a wave function, we can more readily see that the (constant) relative phase, which exists only in an approximate sense, is not a nonlocal property, or changing of relative phase does not change the nonlocal properties of two regions together. First, the relative phase ϕ of the superposition $\frac{1}{\sqrt{2}}(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle)$ is not a local property inside the region of $|\psi_1\rangle$, since it is also the global phase of the wave function $|\psi_1\rangle$, and the global phase is not real. Next, the derivative of the phase with respect to position, $\nabla S(x,t)$, is a local physical property, since we have the relation $\nabla S(x,t) = mj(x,t)/\rho(x,t)$, and the density $\rho(x,t)$ and the flux density $j(x,t)$ are local physical properties (on the ψ-ontic view). Third, when a local unitary transformation adds a relative phase ϕ to $|\psi_1\rangle$, the phase of the state changes inside the region of the bulk of $|\psi_1\rangle$, but it keeps unchanged outside the region (i.e. for the tails of $|\psi_1\rangle$), and thus the derivative of the phase with respect to position at the boundary of this region changes. Since the density is not changed by the adding of the relative phase, the flux density must change at the boundary of the region. Thus, adding a relative phase ϕ to $|\psi_1\rangle$ will change the flux density, a local physical property, at the boundary of the region of $|\psi_1\rangle$.[6] Since the two regions of $|\psi_1\rangle$ and $|\psi_2\rangle$ are separated, a local physical property at the boundary of one region is not a nonlocal property of the two regions together.

5 Conclusion

In this paper, I show that the unreality of the global phase in ψ-ontic quantum theories can be proved based on the fact that the global phase of a product state does not uniquely determine the global phase of each component state. However, a similar strategy cannot be used to prove the reality of the global phase due to the existence of the tails of the wave function. Moreover, I argue that the relative phase is not a nonlocal property, and adding a relative phase to one local branch of a superposition only changes the local properties at the boundary of the region of the branch.

[6] Since the density and the flux density can be measured locally for an ensemble of identically prepared systems, the relative phase can also be measured locally (cf. Aharonov and Vaidman, 2000).

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References

[1] Aharonov, Y. and Vaidman, L. (2000). Nonlocal aspects of a quantum wave. Phys. Rev. A 61, 052108.

[2] Colin, S., Durer, T. and Willcox, R. (2017). L. de Broglie's double solution program: 90 years later. Annales de la Foundation Louis de Broglie, 42 (1), 19-71.

[3] Croca, J.R., Castro, P., Gatta, M. et al. (2023). Proposed Experiments to Clarify the Real Nature of the Quantum Waves. Foundations of Physics 53, 14.

[4] Gao, S. (2022). A Conjecture on the Origin of Superselection Rules - with a Comment on "The Global Phase Is Real". http://philsci-archive.pitt.edu/20456/.

[5] Gao, S. (2024). Locality Implies Reality of the Wave Function: Hardy's Theorem Revisited. https://philsci-archive.pitt.edu/23005/.

[6] Goldstein, S. (2021). Bohmian Mechanics, The Stanford Encyclopedia of Philosophy (Fall 2021 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/fall2021/entries/qm-bohm/.

[7] Hardy, L. (2013). Are quantum states real? International Journal of Modern Physics B 27, 1345012.

[8] Pusey, M., Barrett, J. and Rudolph, T. (2012). On the reality of the quantum state. Nature Physics 8, 475-478.

[9] Wallace, D. (2022). On the reality of the global phase. http://philsci-archive.pitt.edu/20455/.