How (Not) to Define Inertial Frames

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Caspar Jacobs*

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Abstract

It is nearly impossible to open a textbook on Newtonian mechanics without encountering the concept of inertial frames: the frames that are privileged by the theory's dynamics. In this paper, I argue that extant definitions of inertial frames are unsatisfactory. I criticise two common definitions of inertial frames: *law-based definitions*, according to which inertial frames are simply those in which the laws are true, and *structure-based* definitions, according to which inertial frames are those that are 'adapted' to spatiotemporal structure. I then offer a new, *symmetry-based* definition of inertial frames. This definition offers a non-conventional way of specifying the dynamically privileged frames. The result clarifies the foundations of Newtonian mechanics and accounts for the empirical success of coordinate-dependent formulations of it.

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*Institute for Philosophy, Leiden University, Nonnensteeg 1–3, 2311 BE Leiden, Netherlands, c.a.jacobs@phil.leidenuniv.nl

> But no person whose mode of thought is logical can rest satisfied with [Newton's law]... How does it come about that certain [frames of reference]... are given priority over other [frames of reference]...? *What is the reason for this preference?*
>
> Einstein (1954, pp. 71–2)

1 Introduction

Newton's laws are not just true (insofar as they are true) *tout court*. Rather, they hold true within certain frames of reference: 'inertial' ones. Consider a coordinate-dependent formulation of Newton's second law: $\mathbf{F} = m\mathbf{a}$. Without further information, this is incomplete in the same way that the statement "the house is on the left" is without further context: a point of view is required to evaluate this expression. In the case of Newton's laws, this 'point of view' is a frame of reference (hence the expression, sometimes used, of 'referring' the laws to a certain frame). The full statement of the theory is: *within (and only within) the inertial frames*, Newton's laws are satisfied.

I claim that the usual definitions of inertial frames are insufficient to 'complete' coordinate-dependent formulations of Newton's laws. I will distinguish two types of definitions. On the first, inertial frames are grounded in the laws: they are those frames in which Newton's laws are satisfied (§3). This definition is too liberal: for almost any world there exists *some* frame of reference in which Newton's laws are satisfied. On the second type of definition, inertial frames are grounded in spatiotemporal structure: they are frames that are 'adapted' to said structure (§4). Again, however, almost any world turns out to satisfy the laws in *some* adapted frame. This would trivialise Newton's theory.

I will offer a different, *symmetry-based* definition of inertial frames: inertial frames are those frames that 'mesh' with the dynamical symmetries of the theory (§5). On this view, inertial frames are jointly grounded in dynamical *and* spatiotemporal structure.

Foundational discussions of classical mechanics typically involve coordinate-*free* formulations in the language of differential geometry (Friedman, 1983; Malament, 2012). The correct definition of inertial coordinates may seem irrelevant. But coordinate-based versions of classical mechanics are both historically and philosophically significant: discussions of mechanics have proceeded, both in the past and often at present, in terms of coordinates.[1] Despite their coordinate-dependence, these formulations seem to correctly identify the content of Newtonian mechanics. Of course, such formulations presume certain geometric concepts, such as that of a vector quantity. But their equations relate those quantities as expressed in a system of coordinates. If I am correct that common definitions of inertial frames fail, then it is a puzzle how coordinate-dependent versions of Newtonian mechanics could work. This paper offers a solution to that puzzle.

Before I move on, I will clarify 'Newtonian mechanics': it is a theory that describes the motion of point-like massive particles under forces. The kinematics of the theory are that of 'Galilean' spacetime.[2] We can thus represent spacetime as a differentiable manifold, M, on which are defined a spatial 'metric' h^{ab}, a temporal 'metric' t_{ab} and an affine connection ∇. For any pair of points one can meaningfully speak of the duration between them, and for any pair of points at the *same* time one can meaningfully speak of the distance between them. But there is no meaningful notion of distance between points at *different* times, and so one can say neither how far nor how fast particles move across time. The connection does, however, provide an objective standard of acceleration. It is of course also possible to define Galilean spacetime directly in terms of coordinates (Wallace, 2019).

The dynamics of Newtonian mechanics consist of Newton's laws (Morin, 2008, 51):

1. NI: A body moves with constant velocity (which may be zero) [insofar as] acted on by a [net external] force ($\frac{d\mathbf{v}}{dt} = 0$).[3]
2. NII: The time rate of change of the momentum of a body equals the [net external] force acting on the body ($\mathbf{F} = m\mathbf{a}$).
3. NIII: For every force on one body, there is an equal and opposite force on another body ($\mathbf{F}_{12} = -\mathbf{F}_{21}$).

The coordinate expressions of these laws hold only when referred to a certain class of privileged frames. In a rotating frame, for instance, $\mathbf{F} = m\mathbf{a}$

[1] For a defence of coordinate-based approaches, see Wallace (2019).

[2] Or, perhaps, 'Maxwellian' spacetime; cf. Saunders (2013).

[3] Hoek (2022) convincingly argues that 'insofar as' is a more faithful translation of the original Latin than the standard 'unless', but the difference does not matter for the purposes of this paper.

fails to hold due to the presence of so-called 'fictitious forces'. It is true that one can alter the form of the laws to account for such forces: the altered laws hold for a particular class of non-inertial frames. These expressions are syntactically more complex than the standard ones due to the presence of additional terms. Strictly, then, Newton's laws hold true *in their simplest form* only when referred to inertial frames. The inertial frames are of particular interest because within them Newton's laws are afforded a particularly simple formulation.[4]

Without any force law Newtonian mechanics is just a framework, not a theory. The *law of universal gravitation* determines the gravitational force $\mathbf{F}_{12}$ that one particle exerts on another, given their masses m_1, m_2 and positions $\mathbf{x}_1, \mathbf{x}_2$:

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{|\mathbf{x}_2 - \mathbf{x}_1|^3}(\mathbf{x}_2 - \mathbf{x}_1) \tag{1}$$

where G is the gravitational constant. When one conjoins the law of universal gravitation to Newtonian mechanics, the result is Newtonian Gravitation. Of course, there are other forces than gravity, but I will not consider those here.

2 Frames and Coordinates

I first draw a distinction between (inertial) *frames* and (inertial) *coordinates*. I mostly follow Earman and Friedman's (1973) account.[5] The only point of departure lies in the definition of inertial coordinates.

Firstly, define:

> *Frame of reference*: an identification of points of space over time, i.e. a time-like vector field X which 'threads' the manifold M.

In effect, a choice of frame amounts to a choice of which bodies to regard as being at rest.

Secondly, define:

> *Coordinate system*: a smooth and injective function x^μ from the manifold M into $\mathbb{R}^4$, such that x^0 is constant across surfaces of simultaneity.

[4] Weatherall (2021) criticises this notion of a 'simplest form' of an equation, partly for the same reason that I object to law-based definitions below.

[5] For different ways of drawing this distinction, see Torretti (1983, §1.4) or Brown (2005, §2.3).

This definition assumes the existence of a foliation of spacetime into hyperplanes of simultaneity, which follows from the requirement that forces cannot act backwards in time (Brown, 2005, §2.2.3).

Coordinate systems are connected to reference frames as follows:

> *Adapted coordinates:* a coordinate system x^μ is *adapted* to a frame of reference F iff x^i = constant ($i = 1, 2, 3$) along the integral curves of X.

Put differently, coordinates are adapted to a frame whenever the spatial coordinates of the bodies that are considered at rest within that frame are constant over time.

With this connection between frames and coordinates, it is now possible to define *inertial* frames. Following Earman and Friedman:

> A frame of reference F is *inertial* iff there exists an inertial coordinate system adapted to F.

This definition appeals to inertial coordinates, which I have not yet defined—their definition is the topic of this paper. Earman and Friedman define them in terms of an affine connection stipulated to vanish in inertial coordinates. This is a structure-based definition of inertial coordinates, which I will discuss in §4.

The notion of inertial coordinates is prior to that of inertial frames: the latter are defined in terms of the former. So, although physicists often speak of the laws holding within inertial *frames*, it seems more appropriate to speak of the laws holding within a system of inertial *coordinates*. It is a consequence of the invariance of Newtonian mechanics under Galilean transformations that if the laws hold in one inertial coordinate system adapted to F, then so they do in any other. The difference therefore does not matter much in practice. Indeed, Brown (2005, 2.3) simply defines inertial frames as equivalence classes of inertial coordinates systems. For this reason, I will use the expressions 'inertial frame' and 'inertial coordinates' interchangeably in what follows.

In what follows, I will use the notational convention that the spatial coordinates of a particle i at time t in a coordinate system x^μ are represented by a position vector $\mathbf{x}_i(t) = (x_i^1(t), x_i^2(t), x_i^3(t))$. The velocity of i is then defined as $\mathbf{v}_i(t) \coloneqq \frac{d\mathbf{x}_i}{dt} \equiv (\frac{dx_i^1}{dt}, \frac{dx_i^2}{dt}, \frac{dx_i^3}{dt})$, so $\mathbf{v}_i$ is the coordinate derivative of $\mathbf{x}_i$ with respect to t. Acceleration is likewise defined as the coordinate derivative of $\mathbf{v}_i$ with respect to t: $\mathbf{a}_i(t) \coloneqq \frac{d\mathbf{v}_i(t)}{dt}$

These vector quantities are by definition coordinate-dependent. For $\mathbf{x}$ and $\mathbf{v}$ this is no surprise, since absolute position and velocity are not Galilean-invariant. But acceleration *is* invariant, so it may seem odd to define it as the derivative of a coordinate-dependent quantity. After all, the kinematical structure of Galilean spacetime enables one to define acceleration 'intrinsically' as an invariant tensorial quantity. Once so defined, an equation such as $\mathbf{F} = m\mathbf{a}$ is independent of coordinates. It simply equates two tensor fields. Yet acceleration is not treated this way in the standard formulation of classical mechanics under discussion. For if acceleration is defined intrinsically, the second law will hold no matter what coordinates are used. This would contradict the fanukuar claim that $\mathbf{F} = m\mathbf{a}$ only holds in inertial coordinates, because otherwise one has to account for fictitious forces. Indeed, inertial frames are often *defined* as those in which the laws, in their simple form, hold true. But if those laws are coordinate-independent, they will hold true in any arbitrary frame. Therefore, a coordinate-based formulation of classical mechanics must define acceleration in a coordinate-dependent way. Whenever $\mathbf{a}$ occurs in this paper, then, it is the second coordinate derivative of a position vector.

3 Law-Based Definitions

On *law-based definitions*, inertial frames are defined in terms of the satisfaction of the laws. In particular, it is common to see inertial frames defined as those in which Newton's *first* law holds true. This is the standard view found in many physics textbooks (Blagojevic, 2001; Morin, 2008; Pfister and King, 2015), as well as foundational philosophical works (Nagel, 1961; Brown, 2005).

3.1 Laws and Inertial Frames

In more detail, the standard view holds that Newton's first law defines (or allows one to construct) a class of inertial frames, namely those in which force-free bodies move (or would move) uniformly. It is not the first law by itself that defines inertial frames, since a force-law is also required to identify the force-free bodies. Another way to see this is that NI is invariant under projective transformations, whereas inertial frames are related by the subset of Galilean transformations. How to identify these force-free bodies independently remains a subtle question. For the sake of argument, I will assume that one can independently characterise force-free bodies, for example as those far away from each other; see Eisenbud (1958); Pfister (2004);

Brown (2005) for further discussion.

Having found the inertial frames, one can evaluate the second and third law with respect to them. This view expressed well by Morin's (2008, 52) limerick:

> For things moving free or at rest,
> Observe what the first law does best.
> It defines a key frame,
> 'Inertial' by name,
> Where the second law then is expressed.

In fact, NI does more than just offer a definition: it also asserts that inertial frames exist (this entails the actual or counterfactual possibility of free particles). This provides a sense in which the first law is more than a definition.

In summary, a law-based approach defines inertial coordinates as follows:

> *Inertial coordinate system (law-based)*: a coordinate system in which force-free bodies move with constant velocity (i.e. $\frac{d\mathbf{v}}{dt} = 0$ for them).

The inertial frames are those that admit of inertial coordinates. One can refer the second and third law to them. For example, the second law will read:

> NII-LAW: Within those frames in which force-free bodies move with constant velocity (i.e. $\frac{d\mathbf{v}}{dt} = 0$ for them), $\mathbf{F} = m\mathbf{a}$.

On this view the first law is not a consequence of the second law. The first law asserts that there exist certain frames with respect to which the second law is supposed to hold. The second law thus does not even make sense without the first law to define those frames.

3.2 Too Many Inertial Frames

Unfortunately, this popular definition of inertial frames fails. It is too liberal: there are inertial frames in which NII-LAW holds true even for patently non-Newtonian worlds.

Consider the following pair of worlds:

- W: a world in which n free particles each move with uniform velocity (with respect to the affine background structure);
- W^*: a world exactly like W, except that one of the particles—call it 'Curvy'—moves haphardly about in a non-linear fashion (with respect to the affine background structure).

The first world, W, is Newtonian by stipulation. The second world, W^*, is clearly *not* Newtonian. As Morin (2008, 52) puts it: "we can't have a bunch of free particles moving with constant velocity while another one is doing a fancy jig". But that is exactly what Curvy is doing in W^*. If the law-based account correctly identifies the inertial frames, however, then W and W^* will both satisfy Newton's laws even in their simplest form—as I will now show. Therefore, law-based definitions do not correctly identify the inertial frames.

It is helpful here to distinguish between a particle's *coordinate acceleration* and its *physical acceleration*. The former refers to the value of $\mathbf{a}$ for some particle within a coordinate system x^μ, while the latter refers to the particle's acceleration with respect to the affine structure independently from any coordinate system. The same physical acceleration has different coordinate representations in different coordinate systems.

Consider first W. Since the physical acceleration of all free particles in W is zero, W is Newtonian. Suppose that it is possible to construct an inertial coordinate system x^μ such that the coordinate accelerations of all particles are zero: $\mathbf{a}_i = 0$ for all i. By stipulation, $\mathbf{F}_i = 0$ too, and hence $\mathbf{F}_i = m\mathbf{a}_i$ for all i. NII-LAW is satisfied; W is Newtonian.

Consider W^* next. It may seem that NII-LAW must fail to hold in W^* when referred to the same coordinates. But since x^μ is a function defined on the points of W and not those of W^*, it is impossible to compare (coordinate) accelerations in W and W^* directly. Instead, one must independently construct a coordinate system for W^*. The problem is that it turns out to be possible to construct an inertial coordinate system for W^* within which NII-LAW is satisfied. To see this, assume that there exists a diffeomorphism (i.e. a smooth bijection between spacetime points), ϕ, that maps the linear trajectory of Curvy in W onto the haphazard trajectory of the same particle in W^*.[6] If $\gamma(\tau)$ represents the trajectory of Curvy in W, then Curvy's trajectory in W^* is represented by $\gamma^*(\tau) := \phi \circ \gamma(\tau)$, where τ is a dimensionless parameter. Next, define the coordinate system $x'^\mu := x^\mu \circ \phi^{-1}$. By

[6] This assumption is without (much) loss of generality: it requires only that W and W^* concur on whether Curvy's trajectory intersects the trajectories of any other particle.

construction, the coordinates x^μ assigns to $\gamma(\tau)$ are the same as those that x'^μ assigns to $\gamma^*(\tau)$. It follows that the coordinate accelerations of the free particles in W^* with respect to x'^μ are the same as those of the particles in W with respect to x^μ, namely zero. Thus x'^μ is an inertial coordinate system for W^*. Because $\mathbf{F} = 0 = m\mathbf{a}$ in this coordinates system, NII-LAW is satisfied in W^*.[7]

But this verdict is clearly incorrect: W^* is a world in which the physical acceleration of a force-free particle, Curvy, is not zero, contrary to the first law! This is a *reductio ad absurdum* of the law-based definition of inertial frames.

The above story may seem to rest on a confusion: acceleration should be defined with respect to the affine structure, which is unaffected by coordinate transformations.[8] But as I pointed out in the previous section, this implies a coordinate-free expression of the second law which is true independently from one's chosen coordinates. Such a coordinate-free expression equally cannot privilege x^μ over x'^μ. The law-based approach therefore fails either way.

The objection to law-based definitions generalises: one can apply an *arbitrary* diffeomorphism to the particle trajectories of a Newtonian world—even ones subject to forces, unlike in the above toy example. There is always a coordinate transformation that 'undoes' this diffeomorphism, such that Newton's laws hold in the same form with respect to the 'primed' coordinates. Pooley (2013, fn. 88) notes this possibility:

> Suppose, for example, that the only spatiotemporal information one retains is that which is common to all coordinatizations of the particle trajectories obtainable from an initial inertial coordinate system by smooth but otherwise arbitrary coordinate transformations that preserve the timelike directedness of the trajectories. [...] Many Newtonian worlds involving complex histories of relative distances and interactions will be topologically equivalent to histories where all particles maintain constant distance from one another.

However, Pooley's objection is slightly different from mine. Pooley argues that if arbitrary coordinates are allowed, Newton's laws are not the *simplest*

[7] If you are inclined to think that x'^μ is obviously faulty because it is not 'adapted' to spacetime's affine structure—such a claim is characteristic of structure-based definitions, which I discuss in the next section.

[8] I thank an anonymous reviewer for this point.

ones. For example, one can almost always find coordinates such that all trajectories 'seem uniform', in which case the simplest law is that $\mathbf{v} = 0$ for all particles. The present objection, on the other hand, applies *even if* Newton's laws are the simplest ones in some arbitrary coordinate systems. The problem is rather that it is too easy to find coordinates in which the laws are at least as simple as those of Newton.

The advocate of a law-based definition might adopt a form of functionalism in response, such as Knox's (2013) 'inertial frame functionalism'. This type of functionalist claims that force-free bodies *define* inertial trajectories, so that there is no real sense in which Curvy's trajectory in W^* is not uniform; force-free bodies provide a 'coordinative definition' of the world's inertial structure (DiSalle, 1990). This entails that there just are no worlds that differ only over whether some force-free particle moves inertially or not. This position is more radical than simple relationism, since W and W^* differ over the distance between Curvy and the other particles. I do not find it plausible that such worlds could not exist, and so will not further discuss this approach here.

4 Structure-Based Definitions

I noted that a diffeomorphism between W and W^* need not preserve spatiotemporal structure. In particular, it need not preserve metrical structure. But the distances between particles in W according to x^μ are the same as the distances between particles in W^* according to x'^μ, so if the former correctly represents distances then the latter must *mis*represent them. And if the x'^μ coordinates misrepresent distances, surely one should not evaluate the laws with respect to them.

The requirement that appropriate coordinates do not only make the laws true but also match the world's metrical structure is expressed by Brown (2005, 18):

> The coordinates x^μ are special not just because the equation of motion expressed in terms of them takes [a] special simple form [...]; the coordinates x^i (i = 1, 2, 3) should also be special in relation to the metrical properties of space. When Newton talks of uniform speeds, he means equal distances being traversed in equal times, and these distances are meant in the sense of Euclid.

On *structure-based definitions*, inertial frames are partially defined in terms of some spatiotemporal structure, such as the Euclidean metric.

It seems that historical definitions of inertial frames due to Neumann, Lange and Mach are in part structure-based, as they require inertial coordinates to respect the metrical structure of space. Of Neumann's construction, Barbour (1989, 669) writes that it is "explicitly constructed from the observable relative distances and relative velocities", and of Mach that he "accepted distance measurements as given" (685). Since the aim of this paper is not historical, however, I will not further comment on these matters.

In more detail, procedure of a structure-based definition is to (i) stipulate some spatiotemporal structure, (ii) claim that certain coordinates best represent this structure, and (iii) restrict the inertial coordinates to just those ones.

I am sceptical of step (ii): I see no reason to believe that certain numerical representations of, say, metrical structure are intrinsically—that is, independent of dynamical considerations—better than others.[9] This means that the satisfaction of the laws becomes dependent on one's choice of representational convention. Just as law-based definitions, structure-based definitions of inertial frames fail to distinguish worlds in which Newtonian mechanics is true from worlds in which it is false.

4.1 Structure and Inertial Frames

The claim that certain coordinates are 'adapted' to spatiotemporal structure is widespread:

> Every spacetime will have a preferred set of frames that reflects the structure inherent in the spacetime. (Earman, 1989, 29)

> The intrinsic geometrical structure of space and time according to Newton entails that special sets of coordinates exist. [...] the existence of such convenient coordinates [...] follow[s] from the spacetime structure itself. (Maudlin, 2012, 31-2)

> Both substantivalists and relationalists will view certain coordinate systems as kinematically privileged in the sense of being optimally adapted to the particular spatiotemporal quantities that they each recognize. (Pooley, 2013, 528)

As the final quote illustrates, such claims are neutral between substantivalism and relationism. Of course, these positions disagree over *which* spa-

[9]I should note that Brown may well concur with this point, since on his dynamical approach spacetime structure depends on dynamical structure.

tiotemporal structure inertial coordinates are adapted to. The substantivalist posits an affine connection; a coordinate system is adapted to the connection whenever trajectories that are straight with respect to this connection are parametrised by linear equations. The relationist, meanwhile, typically only posits a weaker Leibnizian spatiotemporal structure, which consists just of a temporal and spatial metric.[10]

It would seem that substantivalism and relationism must differ over the definition of adapted coordinates. But the issue is more subtle. Earman and Friedman (1973, 339) show that these procedures pick out the *same* class of frames: either (i) one stipulates that x^μ is adapted to affine structure (i.e. the connection vanishes); or (ii) one stipulates that x^μ is adapted to Leibnizian structure *and* that the first law holds within these coordinates.[11] Therefore, regardless of whether or not a connection is posited one can define inertial frames as those that are adapted to metrical structure and in which force-free bodies move uniformly. Since adaptation to metrical structure is common between substantivalism and relationism, I will focus on it in what follows.

The remainder of this paper concerns the definition of 'Leibnizian coordinates': coordinates adapted to metrical structure. I also focus on the spatial metric, for simplicity; adaptation to the temporal metric is to be treated analogously. Once one has defined a class of 'Leibnizian' coordinates, one can define the class of inertial coordinates by appeal to the first law. But I claim that structure-based definitions cannot even correctly characterise the Leibnizian coordinates, which dooms their effort to define inertial frames.

In summary, a structure-based approach defines inertial coordinates as follows:

> *Inertial coordinate system (structure-based)*: a coordinate system that is adapted to metrical structure, and in which force-free bodies move with constant (coordinate) velocity.

The inertial frames are again those that admit of inertial coordinates. The second law then reads:

> NII-STR: Within those frames adapted to the metric and in which force-free bodies move with constant (coordinate) velocity, $\mathbf{F} = m\mathbf{a}$.

[10] The fact that relationism takes seriously spatiotemporal structure does not mean that it believes in the existence of spacetime; cf. North (2018).

[11] In their paper, these correspond to Def. 4 and Def. 6 of inertial frames respectively.

Because NII-STR is stronger than NII-LAW, it promises to rule out the problematic coordinate systems discussed in the previous section.

4.2 Which Metric?

It is still unclear what it means for a coordinate system to be 'adapted' to the metric. I believe that there is no unequivocal notion of adaptation. Whether coordinates are adapted to metrical structure depends on the way this structure is represented. This is a matter of convention. I resist the claim that certain coordinate systems are intrinsically better adapted to some structure than others. On different conventions, different coordinates are adapted to the same metric. Problematically, for some of these conventions there exist adapted coordinate systems within which Newton's laws are satisfied even in patently non-Newtonian worlds.

Unfortunately, little has been written on this crucial notion of adaptation. Sometimes, it is suggested that coordinates adapted to the Euclidean metric are such that the physical distance between points should equal their Pythagorean distance:

> In Euclidean space, a frame is 'adapted' to some reference body if it is at rest at the origin of the frame, the axes are orthogonal and distances along the axes equal to the distances from the body. (Huggett, 2006, 46)

> The ways in which a coordinate system can be *adapted* to these quantities is straightforward [...] spatial coordinates are chosen so that, for all particles i, j and for all times, $|\mathbf{x}_i - \mathbf{x}_j| = r_{ij}$, where r_{ij} is the instantaneous inter-particle distance between i and j. (Pooley, 2013, 529)

Both authors claim that within an adapted coordinate system, the Pythagorean distance $|\mathbf{x}_i - \mathbf{x}_j| := \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$ should equal the physical distance between particles (in some chosen unit), where x_i and x_j are the position vectors in coordinates x^μ of particles i and j respectively. Call a coordinate system x^μ *adapted to the Pythagorean metric* iff, for any pair of particles i, j, the distance between i and j in some chosen unit is equal to $|\mathbf{x}_i - \mathbf{x}_j|$. (I will shortly explain why I call this metric 'Pythagorean' and not 'Euclidean'.) The coordinates adapted to the Pythagorean metric are the familiar Cartesian ones. The requirement that coordinates are adapted to this metric thus rules out the problematic coordinate systems from the previous section. If it could be shown that Cartesian coordinates are uniquely well-adapted to the world's metrical structure, the structure-based approach might succeed.

But the issue is more complicated. Distinguish between *metric functions* and their *coordinate representations*. A metric function on a space X is a function $d : X \times X \to \mathbb{R}$ from pairs of points into real numbers such that:[12]

$$d(i,j) = 0 \iff i = j \tag{2}$$
$$d(i,j) = d(j,i) \tag{3}$$
$$d(i,k) \leq d(i,j) + d(j,lk) \tag{4}$$

The value of $d(i,j)$ then represents the distance between i and j as expressed in some particular unit.

In addition to these axioms, the *Euclidean metric* also satisfies Ptolemy's inequality:

$$d(i,j) \cdot d(k,l) + d(j,k) \cdot d(i,l) \geq d(i,k) \cdot d(j,l) \tag{5}$$

The Euclidean metric represents distances in a Newtonian world, since the geometry of three-dimensional hyperplanes of simultaneity of Galilean space-time is Euclidean.

This definition is independent of coordinates: d is a function from points themselves to real numbers, not from their coordinates. In particular, the Euclidean metric defined here is *not* the Pythagorean metric discussed above, although they are often identified. The former is a function of pairs of points, the latter of pairs of position vectors. It is the former metric that codifies the theory's physical content, namely the physical distances between points or particles. The latter metric only defines their coordinate distance.

However, it is often convenient to *represent* a metric as a function on coordinates. We will say that a function $r : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ represents a metric d in a coordinate system x^μ iff $r(\mathbf{x}_i, \mathbf{x}_j) = d(i,j)$. If d satisfies the axioms for a metric then so does r, so r itself is a metric on $\mathbb{R}^3$—but not on X.

It is easy to see that r represents the Euclidean metric in a coordinate system x^μ iff x^μ is adapted to r. Therefore a representation r of a metric d defines a class of adapted coordinates, namely those in which r does represent d. For the Pythagorean metric, this is the class of Cartesian coordinates: if d assigns to each pair of points their Euclidean distance in some particular unit, then the function $r(\mathbf{x}_i, \mathbf{x}_j) := |\mathbf{x}_i - \mathbf{x}_j|$ represents d iff x^μ is

[12]The space X here is a hyperplane of simultaneity of the manifold M.

a Cartesian coordinate system. This explains why the authors quoted above focus on the Pythagorean metric.

However, the Pythagorean metric is not the only representation of Euclidean distance. Given an arbitrary diffeomorphism ϕ of X, one can define another representation as follows. First, notice that ϕ induces a coordinate transformation $x^\mu \to x'^\mu$ such that $x'^\mu(p) = x^\mu(\phi(p))$. In brief, x'^μ assigns the same coordinates to p as x^μ does to $\phi(p)$. Second, define a function r_ϕ such that $r_\phi(\mathbf{x}'_i, \mathbf{x}'_j) \equiv r(\mathbf{x}_i, \mathbf{x}_j)$. By construction, r_ϕ represents the Euclidean metric in x'^μ iff r represents the same metric in x^μ. Conversely, this means that r_ϕ defines a *different* class of coordinates from r. Whenever ϕ is not an isometry of the metric space $\langle X, d\rangle$, the coordinates adapted to r_ϕ are distinct from those adapted to r. The upshot is that which coordinates are adapted to the Euclidean metric depends on the way one numerically represents that metric. Although this point is mathematically trivial, it is not often noted by philosophers; van Fraassen (1970, §1.3) is an exception.

The central problem for structure-based definitions is that whether NII-STR is satisfied depends on the way in which Euclidean distance is represented. If one chooses to represent physical distances by the Pythagorean metric, then NII-STR is satisfied in certain worlds. But if one chooses to represent physical distances by some other metric, NII-STR may fail to hold in those very same worlds. Whether those worlds count as Newtonian, by the light of the structure-based definition of inertial frames, thus depends on which numerical representation one chooses. (If you believe that there is no problem here because one of those representations is clearly superior: I address that response below.)

For an illustration, consider again the pair of worlds presented in §3:

- W: a world in which n free particles each move with uniform velocity;
- W^*: a world exactly like W, except that one of the particles—call it 'Curvy'—moves haphardly about in a non-linear fashion.

W is a Newtonian world; W^* is not. We have seen that $\mathbf{F} = m\mathbf{a}$ is true in W with respect to the coordinates x^μ, but that it is *also* true in W^* with respect to the coordinates x'^μ. The structure-based definition must therefore rule that x'^μ is inadmissible because it does not reflect the Euclidean distances between particles. This is indeed the case if one were to impose the condition that $d(i,j) = |\mathbf{x}_i - \mathbf{x}_j|$. But recall that $x'^\mu = x^\mu \circ \phi^{-1}$ for some diffeomorphism ϕ from W to W^*. If one instead were to impose the condition that $d(i,j) = r_\phi(\mathbf{x}_i, \mathbf{x}_j)$, then it is x^μ that is inadmissible. Under that condition, NII-STR is satisfied not in W but in W^*. There is no physical reason to use r rather than r_ϕ: both functions represent the same Euclidean metric, so the choice between them is only a matter of representational convention. Just like the law-based definition discussed in the previous section, then, structure-based definitions run the risk of erroneously classifying certain patently non-Newtonian worlds as Newtonian.

The core of this objection to structure-based definitions is that no representation is better than any other. Before I move on to my symmetry-based proposal for the definition of inertial frames, let me discuss two responses that would privilege certain representations. First, the *pragmatist* response claims that Cartesian coordinates are simpler or more convenient. Maudlin (2012, 31-2), for instance, writes that “[i]n the most convenient coordinatizations of Newtonian space and time, the acceleration of a trajectory through time is proportional to the second derivative of the spatial coordinates with respect to the time coordinate.” But how does one characterise simplicity here? One cannot define the simplest coordinates as those in which the laws have their simplest form, as that would reduce to a law-based definition. The most straightforward definition of simple coordinates is that they are the most convenient: “[b]y a convenient frame, I mean one in which the calculations will be easy to do” (Maudlin, 2012, 171). But whether calculations are easy seems to provide a merely subjective account of simplicity, which should not play a role in our formulation of the theory. It does not seem unlikely, for example, that some alien community of scientists finds it much easier to carry out calculations in non-Cartesian coordinates. Although convenience may explain why *we* prefer Cartesian coordinates, it does not explain why the laws are true in their simple form in just those coordinates.

The second, *naturalist* response is that certain coordinates ‘naturally’ represent Euclidean distance. North (2021), for instance, believes that Cartesian coordinates are more natural because they “have straight, mutually orthogonal coordinate axes”, and that their “numerical values reflect the relative locations of the points in a particularly clear manner.” Sometimes, ‘naturalness’ seems to reduce to simplicity. But other times, North states that certain coordinates ‘respect’ spatiotemporal structure better than others. For example, she writes that it is “better to use coordinate systems whose continuity matches the continuity structure—the topology—of the space” (cf. Maudlin (2012, 27)). It is not just easier to use continuous coordinates, North believes, but such coordinates more perspicuously reflects the continuity of space itself. Likewise, Cartesian coordinates are said to more perspicuously reflect the Euclidean metric.

I find this response unsatisfactory for several reasons.[13] Firstly, the notion of 'naturalness' is far from clear. What reason is there to believe that some alien community of scientists would not find non-Cartesian coordinates more natural? Secondly, North's claim that certain coordinates better 'reflect' some structure seems to *presuppose* a representational convention of that very structure. Consider a map of the Earth. The map seems to misrepresent Earth's curvature: the Earth is spherical, the map is flat. It is well-known that as a consequence, maps must distort features such as relative land mass. The Mercator projection, for instance, distorts the relative size of the continents. North would presumably say that the 'map-coordinates' cannot reflect the geometry of the Earth perspicuously. But there is a sense in which *any* map offers an entirely accurate representation of the Earth—once one has adopted an appropriate representational convention. Nguyen (2020, 1027) makes this point for the Mercator projection: "Features like 'being of equal area' on the map, don't have to be interpreted as representing 'being of equal area' on the Earth's surface. In fact, if one had a sufficiently good understanding of the projection used to create the map, then one could provide an interpretation function that delivered truths about area properties of the Earth, despite the dissimilarities between these and the area properties of the map." On the convention that the area of a continent on the map is proportional to the area of a continent on Earth, the map's coordinatisation of the Earth's surface is mal-adapted. But on the alternative convention that the proportionality depends on the continent's latitude, the map's coordinates are perfectly well-adapted. The map-coordinates only seem unnatural when one tries to judge the relative area of the continents by a convention not appropriate to the map. The same is the case for the topological features of space. On the convention that a discontinuity in coordinates represents a discontinuity in spacetime, the map's coordinates are mal-adapted to spacetime's topological structure. But on the alternative convention that the $-180°$ and $180°$ coordinates represent adjacent locations, the map does represent the Earth as round. (Compare this to a clock face: the fact that the number 1 does not come after the number 12 does not mean that one o'clock does not follow noon!) Therefore, an appeal to natural representation cannot save the structure-based approach.

[13] For another critical response to North, see Barrett (2022).

5 Symmetry-Based Definitions

In response to the failure of standard definitions of inertial frames, I want to propose a different definition: a *symmetry-based* one. As far as I am aware, this type of definition has not been suggested before. The account that comes closest Landau and Lifshitz's (1976, 5) definition of an inertial frame as one "in which space is homogeneous and isotropic and time is homogeneous". But it is left unclear what it means for space to be homogeneous or isotropic 'in' an inertial frame. The symmetry-based account I propose elucidates what it means for a frame to possess these features.

Moreover, it is unclear what justifies this demand that coordinates are homogeneous and isotropic. I base this demand on the dynamical symmetries of Newtonian mechanics, namely the invariance of the laws under translations and rotations. Put more precisely, the account I propose justifies the choice of the Pythagorean metric as a privileged representation of Euclidean distance within the context of Newtonian mechanics because it meshes with the theory's dynamical laws. Unlike structure-based definitions, it does not claim that certain spatiotemporal structures are intrinsically better represented by some coordinates. Rather, certain coordinates mesh better with the dynamics. If the dynamics were different, different coordinates would be privileged—*even if structure of spacetime is kept fixed.* If the laws were spherically symmetric around a dynamically special point, for example, then spherical coordinates would mesh better with the theory's dynamics even if space were still Euclidean. Similarly, the appropriate metric for Lorentz's aether theory is one that is invariant under the theory's relativistic symmetries—the Lorentz transformations—despite the fact that this theory was set on a classical spacetime (cf. Bradley (2021)). For these reasons I consider the symmetry-based account a novel approach that succeeds where the above definitions fail.

5.1 Symmetry Constraints

Recall that the laws of Newtonian mechanics (in their simple form) are invariant under spatial and temporal translations, as well as under spatial rotations. It does not matter for the satisfaction of the laws whether one uses some set of coordinates x^μ or a different set of coordinates x'^μ related to the first by a transformation of the 'Newton group' (Pooley, 2013):

$$\mathbf{x} \to \mathrm{R}\mathbf{x} + \mathbf{c}; \quad t \to t + d \tag{6}$$

where $\mathbf{c}$ and d are constant and R is an orthogonal matrix with determinant ± 1. If the laws are true when referred to a coordinate system x^μ, then so they are when referred to a coordinate system related to x^μ by these transformations.[14]

Crucially, this is true even when one uses non-standard coordinates, such as the ones adapted to r_ϕ from the previous section. This is because the translation- and rotation-invariance of Newtonian mechanics is a consequence of the form of the laws themselves. So, if there is some world in which the laws of Newtonian mechanics are satisfied in certain non-Cartesian coordinates, then the laws of Newtonian mechanics remain satisfied when *those* coordinates are translated or rotated.

The fact that the laws are invariant under these transformations means that it should not matter which coordinates are chosen from an equivalence class closed under the action of the Newton group. This is true for the standard Cartesian coordinates. In particular, the Pythagorean metric is itself invariant under translations and rotations in that $|\mathbf{x}_i - \mathbf{x}_j| = |\mathbf{x}'_i - \mathbf{x}'_j|$ whenever x^μ and x'^μ are related by a Newtonian transformation. For example, it is invariant under a translation $\mathbf{x} \to \mathbf{x} + \mathbf{c}$ since $|(\mathbf{x}_i + \mathbf{c}) - (\mathbf{x}_j + \mathbf{c})| = |\mathbf{x}_i - \mathbf{x}_j|$. When one uses the Pythagorean metric to represent distances, then, it does not matter whether one uses one system of coordinates or another one related to the first by a translation or rotation.

But the same is not the case for alternative representations of the metric. Consider an arbitrary representation r_ϕ as defined above. Generally—when ϕ is not an isometry—the effect of a translation or rotation on a coordinate system is to *change* the distances between particles: the distance between i and j as determined by r_ϕ in a non-Cartesian coordinate system x^μ is different from the distance between i and j as determined by the same r_ϕ in the transformed coordinate system x'^μ. In arbitrary coordinates the difference between transformed coordinates *does* matter, contrary to the fact that these transformations are symmetries of Newtonian mechanics. In other words, non-Cartesian representations make it seem as if certain coordinate systems are better adapted to the distances between particles than others even when those coordinates are symmetry-related, contrary to the symmetry-invariance of the dynamics. Yet another way to make the point is that it is desirable for the theory's active symmetries (symmetries of the laws) to match the theories passive symmetries (coordinate transformations): this is

[14] Of course, the laws are also invariant under boosts, which leads to the *Galilei group*. But since I have restricted the discussion to Leibnizian structure only, I will set these aside for now.

the case whenever the representation of the metric is invariant under the action of the Newton group.[15] It is on this basis that non-standard coordinates are ruled out on the symmetry-based approach.

5.2 Derivation of the Pythagorean metric

Based on the dynamical symmetries Newtonian mechanics, it is reasonable to constrain the coordinate representation of Euclidean distance as follows:

Translation Invariance: $r(\mathbf{x}_i, \mathbf{x}_j) = r(\mathbf{x}_i + \mathbf{c}, \mathbf{x}_j + \mathbf{c})$.

Rotation Invariance: $r(\mathbf{x}_i, \mathbf{x}_j) = r(\mathrm{R}\mathbf{x}_i, \mathrm{R}\mathbf{x}_j)$.

From Translation Invariance, it follows that $r(\mathbf{x}_i, \mathbf{x}_j) \equiv f(\mathbf{x}_i - \mathbf{x}_j)$. From Rotation Invariance it follows that the distance does not depend on the direction but only on the magnitude of the difference $\mathbf{x}_i - \mathbf{x}_j$, so $r(\mathbf{x}_i - \mathbf{x}_j) \equiv g(|\mathbf{x}_i - \mathbf{x}_j|)$. [16]

These invariance principles do not yet yield the Pythagorean metric. For example, the discrete metric $r(\mathbf{x}_i, \mathbf{x}_j) = 1$ for $i \neq j$ and 0 otherwise also satisfies them. But with one further assumption one can derive the Pythagorean metric up to a proportionality factor:

Absolute Homogeneity: $r(\alpha\mathbf{x}_i, \alpha\mathbf{x}_j) = |\alpha| r(\mathbf{x}_i, \mathbf{x}_j)$.

This principle states that the metric scales with coordinates. This may seem controversial: scaling transformations are not dynamical symmetries of Newtonian mechanics.[17] The effect of a scaling is to increase the distance between all particles by a constant factor. But if all particles were, say, twice as far away from each other, then the gravitational attraction between

[15]See Gomes (2022) for a similar idea applied to the diffeomorphism invariance of GR.

[16]There is a more technical way of putting this point. Instead of a function $r(\mathbf{x}_i, \mathbf{x}_j)$, we can think of a metric as represented by a tensor that assigns at any point p a scalar to every pair of vectors X_p, Y_p from the tangent space at p. The Euclidean metric tensor as represented in Cartesian coordinates is invariant under translations, rotations and reflections. But not all metric tensors are so invariant. The spherical metric, for instance, varies under translations because it has a distinguished origin. The requirement that the representation of the metric is invariant under rotations and translations is then equivalent to the requirement that the metric tensor in adapted coordinates is proportional to $\mathrm{diag}(1, 1, 1)$. This is just the requirement put in by hand by Earman and Friedman (1973), but they have not justified it on the basis of symmetries or in another way.

[17]But see Gryb and Sloan (2021) for a different perspective, calling such transformations 'dynamical similarities'.

them would be weaker. The trajectories of the particles would differ as a result.

However, scalings *are* symmetries when considered as passive transformations. A passive transformation is a mere change of units, say from metres to inches. It does not affect the actual trajectories. Importantly, the value of the gravitational constant, G, changes under a passive scaling because it has dimensions proportional to $[L]^3$. The increase in distances is therefore balanced by a higher value for G. Because we are now concerned with passive transformations, the numerical representation of the metric itself must also transform. This just amounts to a change of units. If the scale factor is equal to 100, for example, the transformation is a change from metres to centimetres. When we conceive of scaling transformations as passive, Absolute Homogeneity is uncontroversial.

It is easy to see that $r(|\mathbf{x}_i - \mathbf{x}_j|)$ satisfies Absolute Homogeneity iff $r(|\mathbf{x}_i - \mathbf{x}_j|) \equiv k|\mathbf{x}_i - \mathbf{x}_j|$, which is just the Pythagorean metric up to a multiplicative constant. The constant k reflects our freedom to choose a unit of length.

We have thus derived that in symmetry-adapted coordinates the distance r_{ij} between particles i and j as measured in some unit is proportional to the Pythagorean distance $|\mathbf{x}_i - \mathbf{x}_j|$. This is just the requirement formulated by Huggett and Pooley, but here it is justified rather than asserted. The notion of adaptation is defined in terms of invariance under dynamical symmetries. The coordinates for which this is the case are the familiar Cartesian ones. Therefore, the Cartesian coordinates are uniquely adapted to the Leibnizian structure of spacetime, given the dynamics of Newtonian mechanics. The Cartesian coordinates are preferable for purely physical reasons.

Let me briefly compare this account to that of Wallace (2019), who uses the passive symmetries of dynamical equations in a somewhat similar manner. Where Wallace uses dynamical symmetries to determine a theory's spacetime structure—metric incluis—my approach assumes the existence of a metric function and uses dynamical symmetries to constrain the coordinate representation of this function. Although our approaches have a similar spirit, they answer slightly different questions.

5.3 Defining Inertial Frames

The above procedure gives us only a class of 'Leibnizian' coordinates. In order to define inertial frames, it is also required that coordinates are adapted to spacetime's inertial structure. But recall that Earman and Friedman offered a definition of inertial frames as those that are adapted to Leibnizian spatiotemporal structure *and* in which Newton's first law is satisfied. This definition was problematic because their particular notion of adaption—essentially the demand that coordinates are Pythagorean—was left unmotivated. But now that this demand is justified it is possible to follow suit and define the inertial coordinates as follows:

> *Inertial coordinate system (symmetry-based)*: a coordinate system that is adapted to a symmetry-invariant metric, and in which force-free bodies move with constant (coordinate) velocity.

The inertial frames are those frames that admit of inertial coordinates adapted to them. The second law then reads:

> NII-SYM: Within those frames that are adapted to a symmetry-invariant metric and in which force-free bodies move with constant (coordinate) velocity, $\mathbf{F} = m\mathbf{a}$.

I have thereby shown that any inertial coordinate system is adapted to the metrical structure of spacetime, in the sense that physical distances between particles as measured in some unit are proportional to the Pythagorean distance between their coordinates. From the dynamical symmetries of a theory one can construct a coordinate system that is unique up to time-*dependent* translations and rotations. The additional stipulation that the first law must hold constrains this to an equivalence class of frames that is closed under time-*independent* translations and rotations as well as boosts: the Galilean transformations. These frames are the inertial ones, and within them the laws hold true in their simplest form.

What if one were to consider the laws in a more complex form? It is possible that those expressions have different symmetries than the Galilean ones. In that case the coordinate representation of the Euclidean metric must also remain invariant under different transformations, so different coordinate systems are adapted. This does not pose a problem: of course an expression of the laws in different coordinates requires a different coordinate representation of the metric! The form of the laws and the inertial frames are determined jointly. I leave it open whether there is any reason other than convenience to prefer one expression of the laws over another. Given an expression of the laws, however, there is always a uniquely privileged class of inertial coordinates relative to it, determined by its symmetries.

6 Conclusion

I have discussed three definitions of inertial frames. The first two definitions—law-based and structure-based ones—are typically found in foundational treatments of classical mechanics, but both are deficient: they fail to pick out the correct space of physically possible worlds. I then presented a novel, *symmetry-based* definition which does pick out the correct space of possibilities. In particular, symmetry considerations uniquely determine a numerical representation of the Euclidean metric, from which one can define the class of Cartesian coordinates.

In close, recall that problems with the inertial frame concept have led some philosophers to move away from a coordinate-dependent formulation of Newtonian mechanics towards a coordinate-independent formulation. But this disregards the fact that physics has used the inertial frame concept successfully for centuries.[18] To quote Brown (2005, 23):

> In their influential 1973 article on Newton's first law of motion, John Earman and Michael Friedman claimed that no rigorous formulation of the law is possible except in the language of 4-dimensional geometric objects. But the appearance of systematic studies of the 4-dimensional geometry of Newtonian spacetime is relatively recent [...]. It is curious that so much success had been achieved by the astronomers in applying Newton's theory of universal gravity to the solar system [...] well before this date. How could this be if the astronomers were unable to fully articulate the first law of motion, and hence the meaning of inertial frames? [...] How tempting it is in physics to think that precise abstract definitions are if not the whole story, then at least the royal road to enlightenment.

I concur with Brown that the history of physics has shown that it is far too easy to dismiss the inertial frame concept. However, I am more positive about the possibility of 'precise abstract definitions'. I hope to have shown that one *can* offer a precise and correct definition of inertial frames, based on fairly abstract symmetry principles. These results put coordinate-dependent formulations of Newtonian mechanics on a surer footing and further emphasise the central role of symmetries in physics.

References

Barbour, J. 1989. *Absolute or Relative Motion? A Study From the Machian Point of View of the Discovery and the Structure of Dynamical Theories.* Cambridge

[18] Of course, Newton's own formulation of his theory made no appeal to inertial frames (Maudlin, 2012, 24ff).

University Press.

Barrett, T. W. 2022. Coordinates, Structure, and Classical Mechanics: A review of Jill North's Physics, Structure, and Reality. http://philsci-archive.pitt.edu/20366/.

Blagojevic, M. 2001. *Gravitation and Gauge Symmetries.* CRC Press.

Bradley, C. 2021. The Non-Equivalence of Einstein and Lorentz. *British Journal for the Philosophy of Science 72*(4): 1039–1059.

Brown, H. R. 2005. *Physical Relativity: Space-Time Structure From a Dynamical Perspective*. Oxford University Press UK.

DiSalle, R. 1990. Conventionalism and the Origins of the Inertial Frame Concept. *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association* 1990: 139–147.

Earman, J. 1989. *World Enough and Spacetime.* MIT press.

Earman, J. and M. Friedman. 1973. The Meaning and Status of Newton's Law of Inertia and the Nature of Gravitational Forces. *Philosophy of Science 40*(3): 329–359.

Einstein, A. 1954. *Relativity.* London: Methuen.

Eisenbud, L. 1958. On the Classical Laws of Motion. *American Journal of Physics 26*(3): 144–159.

Friedman, M. 1983. *Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science*. Princeton University Press.

Gomes, H. 2022. Same-diff? Conceptual similarities between gauge transformations and diffeomorphisms. Part I: Symmetries and isomorphisms.

Gryb, S. and D. Sloan. 2021. When scale is surplus. *Synthese 199*(5): 14769–14820.

Hoek, D. 2022. Forced Changes Only: A New Take on the Law of Inertia. *Philosophy of Science*: 1–17.

Huggett, N. 2006. The Regularity Account of Relational Spacetime. *Mind 115*(457): 41–73.

Knox, E. 2013. Effective spacetime geometry. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 44*(3): 346–356.

Landau, L. D. and E. M. Lifshitz. 1976. *Mechanics: Volume 1* (3rd edition ed.). Amsterdam Heidelberg: Butterworth-Heinemann.

Malament, D. B. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago: Chicago University Press.

Maudlin, T. 2012. *Philosophy of Physics: Space and Time* (Reprint edition ed.). Princeton Oxford: Princeton University Press.

Morin, D. 2008. *Introduction to Classical Mechanics: With Problems and Solutions* (1st edition ed.). Cambridge, UK ; New York: Cambridge University Press.

Nagel, E. 1961. *The Structure of Science: Problems in the Logic of Scientific Explanation*. Harcourt, Brace & World.

Nguyen, J. 2020. It's Not a Game: Accurate Representation with Toy Models. *British Journal for the Philosophy of Science 71*(3): 1013–1041.

North, J. 2018. A New Approach to the Relational–Substantival Debate. In *Oxford Studies in Metaphysics: Volume 11*, eds. K. Bennett and D. Zimmerman, Chapter Oxford Studies in Metaphysics Volume 11. Oxford University Press.

North, J. 2021. *Physics, Structure, and Reality*. Oxford, New York: Oxford University Press.

Pfister, H. 2004. Newton's First Law Revisited. *Foundations of Physics Letters* 17: 49–64.

Pfister, H. and M. King. 2015. *Inertia and Gravitation: The Fundamental Nature and Structure of Space-Time*. Lecture Notes in Physics. Springer International Publishing.

Pooley, O. 2013. Substantivalist and Relationalist Approaches to Spacetime. In *The Oxford Handbook of Philosophy of Physics*, ed. R. Batterman. Oxford University Press.

Saunders, S. 2013. Rethinking Newton's Principia. *Philosophy of Science 80*(1): 22–48.

Torretti, R. 1983. *Relativity and Geometry*. Elsevier Science & Technology Books.

van Fraassen, B. C. 1970. *An Introduction to the Philosophy of Time and Space*. Random House.

Wallace, D. 2019. Who's Afraid of Coordinate Systems? An Essay on Representation of Spacetime Structure. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 67: 125–136.

Weatherall, J. O. 2021. Two dogmas of dynamicism. *Synthese 199*(2): 253–275.