A Hyperintensional Two-Dimensionalist Solution to the Access Problem

David Elohim*

Written: May 7, 2022; Revised: April 7, 2024

Abstract

I argue that the two-dimensional hyperintensions of epistemic topic-sensitive two-dimensional truthmaker semantics provide a compelling solution to the access problem.

I countenance an abstraction principle for epistemic hyperintensions based on Voevodsky's Univalence Axiom and function type equivalence in Homotopy Type Theory. I apply, further, modal rationalism in modal epistemology to solve the access problem. Epistemic possibility and hyperintensionality, i.e. conceivability, can be a guide to metaphysical possibility and hyperintensionality, when (i) epistemic worlds or epistemic hyperintensional states are interpreted as being centered metaphysical worlds or hyperintensional states, i.e. indexed to an agent, when (ii) the epistemic (hyper-)intensions and metaphysical (hyper-)intensions for a sentence coincide, i.e. the hyperintension has the same value irrespective of whether the worlds in the argument of the functions are considered as epistemic or metaphysical, and when (iii) sentences are said to consist in super-rigid expressions, i.e. rigid expressions in all epistemic worlds or states and in all metaphysical worlds or states. I argue that (i) and (ii) obtain in the case of the access problem.

In his (1973), Benacerraf inquires into how the semantics of mathematics might interact with the theory of knowledge for mathematics. He raises the inquiry concerning how knowledge of acausal abstract objects such as those of mathematics (e.g. numbers, functions, and sets) is possible, assuming that the best theory of knowledge is that deployed in the empirical sciences and thus presupposes a condition of causal interaction. This is known in the literature in philosophy of mathematics as the access problem. Field (1989) generalizes Benacerraf's problem by no longer presupposing the condition of causal interaction, and inquiring into what might explain the reliability of mathematical beliefs. Clarke-Doane (2016) has argued that the Benacerraf-Field problem might no longer be thought to be pressing in light of mathematical beliefs satisfying conditions of safety and sensitivity. A belief is safe if it could not easily have been

*I changed my name from Hasen Joseph Khudairi to David Elohim, in April, 2024. Please cite this paper and my published book and articles under 'Elohim, David'.

different. A belief is sensitive if, had the contents of the belief been false, we would not believe them. Mathematical beliefs are thus sensitive, because mathematical truths are metaphysically necessary, true at all worlds. Clarke-Doane quotes David Lewis, who writes: [I]f it is a necessary truth that so-and-so, then believing that so-and-so is an infallible method of being right. If what I believe is a necessary truth, then there is no possibility of being wrong. That is so whatever the subject matter [...] and no matter how it came to be believed' (1986: 114-115). Mathematical beliefs are safe, because mathematical truths hold at all nearby worlds, indeed at all of them, and 'there are reasons to think that we could not have easily had different mathematical beliefs. Our "core" mathematical beliefs might be thought to be evolutionarily inevitable. Given that our mathematical theories best systematize those beliefs, there is a "bootstrapping" argument for the safety of our belief in those theories' (24).

Clarke-Doane provides a compelling case against there being an issue of the reliability of mathematical belief given that such belief is safe and sensitive. If the access problem might yet be thought to be a live issue, however, I argue in this paper that it can be solved by availing of the epistemic interpretation of a particular semantics, namely epistemic two-dimensional semantics. The semantics accounts for the truth-conditions of mathematical formulas, while also having an epistemic interpretation of the intensional, and, as I will argue, hyperintensional parameters relative to which those formulas receive their semantic values. Thus epistemic two-dimensional semantics can account for the convergence between the semantics and theory of knowledge for mathematics. Furthermore, however, epistemic two-dimensional semantics countenances two-dimensional intensions, which are functions from epistemically possible worlds to metaphysically possible worlds to extensions. In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: 'Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If $[A]_1$ and $[A]_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if $[A]_1$ epistemically necessitates that $[A]_2$ subjunctively necessitates S. A good heuristic here is to ask "If $[A]_1$ is the case, then if $[A]_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1([A]_1 \rightarrow \Box_2([A]_2 \rightarrow S))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively'.

Two-dimensional intensions thus provide a conduit from conceivability to metaphysical possibility, and can thus explain the connection between the conceivability of mathematical formulas and their metaphysical possibility. In previous work, I have availed of two-dimensional intensions to account for the interaction between the epistemic and objective or metaphysical profiles of abstraction principles, set-theoretic axioms (including large cardinal axioms), Orey i.e. undecidable propositions, indefinite extensibility, and rational intuition. However, by bridging the epistemic and metaphysical universes, the two-dimensional intensions of epistemic two-dimensional semantics can explain how our epistemic states about mathematical formulas can be a guide to their metaphysical profiles. In this way, epistemic two-dimensional semantics provides a solution to the access problem. Because mathematical truths are metaphysically necessary, this approach to the access problem treats the issue as one in the epistemology of modality. Later in the paper, when hyperintensional resources are availed of, the topics of truthmakers for mathematical truths will be relevant to capturing their distinctively mathematical subject matter. Topic-sensitive two-dimensional hyperintensions are similarly such that epistemic states can be a guide to metaphysical states for mathematical truths, given the satisfaction of a number of other conditions specified below.

I will define epistemic possibility via the notion of apriority, such that ϕ is epistemically possible iff ϕ is primary conceivable, where primary conceivability ($\diamond$) is the dual of apriority ($\neg\blacksquare\neg$, i.e. not apriori ruled out). So epistemic possibility is the dual of apriority i.e. epistemic necessity, i.e. not apriori ruled out. Chalmers (2002) distinguishes between primary and secondary conceivability. Secondary conceivability is counterfactual, so rejecting the metaphysical necessity of the identity between Hesperus and Phosphorus is not secondary conceivable. Primary conceivability targets epistemically possible worlds considered as actual rather than counterfactual worlds. Chalmers also distinguishes between positive and negative conceivability and prima facie and ideal conceivability. A scenario is positively conceivable when it can be imagined with perceptual detail. A scenario is negatively conceivable when nothing rules it out apriori, as above. A scenario is prima facie conceivable when it is conceivable 'on first appearances'. E.g. a formula might be prima facie conceivable if it does not lead to contradiction after a finite amount of reasoning. A scenario is ideally conceivable if it is prima facie conceivable with a justification that cannot be defeated by subsequent reasoning (op. cit.).

Chalmers distinguishes between deep and strict epistemic possibilities. He writes: '[W]e might say that the notion of *strict epistemic possibility* – ways things might be, for all we know – is undergirded by a notion of *deep epistemic possibility* – ways things might be, prior to what anyone knows. Unlike strict epistemic possibility, deep epistemic possibility does not depend on a particular state of knowledge, and is not obviously relative to a subject' (62). About deep epistemic necessity, he writes: 'For example, a sentence s is deeply epistemically possible when the thought that s expresses cannot be ruled out a priori / This idealized notion of apriority abstracts away from contingent limitations' (66). All references to epistemic possibility in this paper will be to Chalmers' notion of deep epistemic possibility.

Chalmers defines epistemic possibility as (i) not being apriori ruled out (2011:

63, 66),[1] i.e. as the dual of epistemic necessity i.e. apriority (65),[2], and as (ii) being true at an epistemic scenario i.e. epistemically possible world (62, 64). He accepts a Plenitude principle according to which: 'A thought T is epistemically possible iff there exists a scenario S such that S verifies T' (64). Chalmers advances both epistemic and metaphysical constructions of epistemic scenarios. In the metaphysical construction of epistemic scenarios, epistemic scenarios are centered metaphysically possible worlds (69). Canonical descriptions of epistemically possible worlds on the metaphysical construction are required to be specified using only 'semantically neutral' vocabulary, which is 'non-twin-earthable' by having the same extensions when worlds are considered as actual or counterfactual (Chalmers, 2006: §3.5). In the epistemic construction of epistemic scenarios, they are sentence types comprising an infinitary ideal language, M, with vocabulary restricted to epistemically invariant expressions (Chalmers, 2011: 75). He defines epistemically invariant expressions thus: '[W]hen s is epistemically invariant, then if some possible competent utterance of s is epistemically necessary, all possible competent utterances of s are epistemically necessary' (op. cit.). The sentence types in the infinitary language must also be epistemically complete. A sentence s is epistemically complete if s is epistemically possible and there is no distinct sentence t such that both s $\wedge$ t and s $\wedge$ $\neg$t are epistemically possible (76). The epistemic construction of epistemic scenarios transforms the Plenitude principle into an Epistemic Plenitude principle according to which: 'For all sentence tokens s, if s is epistemically possible, then some epistemically complete sentence of [M] implies s' (op. cit.).

The thesis of 'weak modal rationalism' states that conceivability can be a guide to 1-possibility, i.e. conceivability entails 1-possibility or truth at a centered metaphysically possible world (2002). Thus conceivability can be a guide to metaphysical possibility on the metaphysical construction of epistemically possible worlds. In the hyperintensional setting, epistemic states might be analyzed as centered metaphysical states.

However, in his (2002) and (2010), Chalmers argues that 1-, i.e. epistemic, possibility entails 2-, i.e. metaphysical, possibility, in the case when the primary and secondary intensions for physics and consciousness coincide. Thus, there is no gap between the epistemic and metaphysical profiles for expressions involving physics or consciousness, and the conceivability about scenarios concerning them will entail the 1-possibility and the 2-possibility of those scenarios. In the hyperintensional setting, one works with hyperintensions, i.e. functions from topic-sensitive truthmakers to extensions, rather than intensions.

Finally, in his (2012), Chalmers defines a notion which he refers to as super-rigidity: 'When an expression is epistemically rigid and also metaphysically rigid (metaphysically rigid *de jure* rather than *de facto*, in the terminology of Kripke

[1] 'One might also adopt a conception on which every proposition that is not logically contradictory is deeply epistemically possible, or on which every proposition that is not ruled out a priori is deeply epistemically possible. In this paper, I will mainly work with the latter understanding' (63).

[2] 'We can say that s is deeply epistemically necessary when s is a priori: that is when s expresses actual or potential a priori knowledge' (65).

1980), it is *super-rigid*' (Chalmers, 2012: 239). He writes: 'I accept Apriority/Necessity and Super-Rigid Scrutability. (Relatives of these theses play crucial roles in "The Two-Dimensional Argument against Materialism"' (241). The Apriority/Necessity Thesis is defined as the 'thesis that if a sentence S contains only super-rigid expressions, s is a priori iff S is necessary' (468), and Super-Rigid Scrutability is defined as the 'thesis that all truths are scrutable from super-rigid truths and indexical truths' (474). This is thus a third way for conceivability to be a guide to metaphysical possibility. The epistemic necessity i.e. apriority of a sentence involving only super-rigid expressions is such that it converges with the metaphysical necessity of that sentence. In the hyperintensional setting, super-rigidity is replaced by a hyper-rigidity condition specified below.

What is the status, with regard to the primary and secondary intensions and their interaction, of the natural numbers? It is usually argued that numbers have metaphysically necessary being (see e.g. Hale and Wright, 1992, 1994; Hale, 2013: ch. 7; for a counterargument, however, see Field, 1993), and Elohim (ms) argues that the metaphysical, i.e. maximally objective, necessity of the existence of numbers is yet consistent with there being non-maximal, thus not metaphysical, yet still objective modalities relevant to the potential expansion of the domain of numbers. If numbers are metaphysically necessary, then this is consistent with their having a necessary 2-intension, i.e. function from metaphysically possible worlds to extensions. However, there are reasons to think that numbers are not epistemically rigid, and thus not super-rigid. Chalmers (2012: 367) suggests that expressions and concepts for the numbers are epistemically rigid, i.e. refer to the same objects throughout epistemic modal space. In response, Benacerraf's (1965) dilemma can be raised, according to which there are reductions of the natural numbers to both von Neumann ordinals (e.g., $2 = \{\emptyset,\{\emptyset\}\}$), as well as Zermelo ordinals (e.g., $2 = \{\{\emptyset\}\}$), and there is no reason to prefer one reduction to the other. If so, then the epistemic profile of the numbers diverges in each account, and the numbers are thus not epistemically rigid and subsequently not super-rigid.

Thus, the link between epistemic states concerning mathematics and objective or metaphysical states concerning mathematics might be effected either by (i) the metaphysical construction of epistemically possible worlds, such that conceivability is a guide to 1-, i.e. epistemic, possibility which is defined as a centered metaphysically possible world; or (ii) the primary and secondary intensions of mathematical expressions coinciding.

One of the unfortunate things about the limits of mathematical knowledge is that it is perfectly conceivable for mathematical objects to have substantially diverging metaphysical profiles despite being partially described by the axioms and abstraction principles of our current mathematical theories. However, the epistemic profile of such axioms and abstraction principles are – given conditions of consistency e.g. – at least a partial guide to the metaphysical profiles of numbers. So, 1-possibility and 2-possibility might coincide in the case of mathematical objects. Barring that hypothesis, however, epistemic states concerning mathematical entities and truths can be a guide to the corresponding metaphysical states in light of (i) above, i.e. the metaphysical construction of epistemic scenarios, i.e. countenancing epistemically possible worlds as centered metaphysically possible worlds.

Hyperintensionality in Chalmers' epistemic two-dimensional intensional semantics can be countenanced via what he refers to as 'structured' intensions, i.e. intensions for each component expression of a sentence, rather than there being an intension for the sentence taken as a whole. However, there are other degrees of hyperintensionality which it would be ideal to capture. One dimension of hyperintensionality can concern sentences being true at parts of worlds rather than at whole worlds themselves. Thus, e.g., 'snow is white or it is not the case that snow is white' and 'grass is green or it is not the case that grass is green' are necessarily equivalent, but have different contents. In truthmaker semantics, this is owing to the two sentences being made true by different parts of worlds. These parts of worlds which verify and falsify sentences can thus be considered hyperintensional truthmakers and falsemakers (Fine, 2013, 2017,a-c).

Another dimension of hyperintensionality which it would be ideal to capture concerns subject matters. Subject matters are called topics in the literature, and capture the aboutness of atomic formulas. Thus, contents can be defined as pairs of intensions, i.e. functions from worlds to extensions, as well as topics which compose via mereological fusion (Berto, 2018, 2019; Canavotto et al, 2020; and Berto and Hawke, 2021).

In this paper, I will advance a version of epistemic two-dimensional semantics which is a truthmaker semantics and which is topic-sensitive.

According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle$S, P, $\leq$, v$\rangle$, where S is a non-empty set of states, P is the subspace of possible states where states s and t comprise a fusion when s $\sqcup$ t$\in$P, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^-\rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Fine 2017a,b; Hawke and Özgün, 2023). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, 2023):

s $\vdash$ p if s$\in[\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash \neg$p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv \neg$p if s $\vdash$ p

(s falsifies not p, if s verifies p);

s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u

(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);

s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q

(s falsifies p and q, if s falsifies p or s falsifies q);

s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q

(s verifies p or q, if s verifies p or s verifies q);

s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u (s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);

s $\vdash$ $\forall$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$

[s verifies $\forall$xϕ(x) "if it is the fusion of verifiers of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (Fine, 2017c)];

s $\dashv$ $\forall$xϕ(x) if s $\dashv$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall$xϕ(x) "if it falsifies one of its instances" (op. cit.)];

s $\vdash$ $\exists$xϕ(x) if s $\vdash$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists$xϕ(x) "if it verifies one of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (op. cit.)];

s $\dashv$ $\exists$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\dashv$ ϕ(a$_1$), ..., s$_n$ $\dashv$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$ (op. cit.)

[s falsifies $\exists$xϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if s $\vdash$ p if s$\in$$[\![p]\!]$;

s inexactly verifies p if and only if s $\triangleright$ p if $\exists$s'$\leq$S, s' $\vdash$ p; and

s loosely verifies p if and only if, $\forall$v, s.t. s $\sqcup$ v $\vdash$ p, where $\sqcup$ is the relation of compatibility (35-36);

s $\vdash$ Aϕ if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u$\in$P and u' $\vdash$ ϕ, where Aϕ denotes the apriority of ϕ; and

s $\dashv$ Aϕ if and only if there is a v$\in$P such that for all u$\in$P either v $\sqcup$ u$\notin$P or u $\dashv$ ϕ;

s $\vdash$ A(Aϕ) if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u $\in$P and u' $\vdash$ ϕ and there is a u"$\in$P such that u' $\sqcup$ u"$\in$P and u" $\vdash$ ϕ;

s $\vdash$ A($\forall$xϕ(x)) if and only if for all u$\in$P there is a u'$\in$P such that u$\vdash$ [$u'\vdash$ $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and u' = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$];

s $\vdash$ A($\exists$xϕ(x)) if and only if or all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash$ ϕ(a)] for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, $\leq$*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![$P$]\!]^+$ = 1 and $[\![$p$]\!]^-$ = 0. Thus, M = $\langle$S, S*, D, $\leq$, $\leq$*, V, V*$\rangle$. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s$\in$S,s*$\in$S* $\vdash$ p iff:
(i) $\exists$c$_s$$[\![p]\!]^{c,c}$ = 1 if s$\in$$[\![p]\!]^+$; and
(ii) $\exists$i$_{s*}$$[\![p]\!]^{c,i}$ = 1 if s*$\in$$[\![p]\!]^+$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(**) M,s∈S,s*∈S* $\vdash$ p iff:
(i) $\forall$c'$_s$$[\![\text{p}]\!]^{c,c'}$ = 1 if s∈$[\![\text{p}]\!]^+$; and
(ii) $\forall$i$_{s*}$$[\![\text{p}]\!]^{c,i}$ = 1 if s*∈$[\![\text{p}]\!]^+$

The foregoing provides a two-dimensional hyperintensional semantic framework within which to interpret the values of a proposition:

s is a two-dimensional exact truthmaker of p if and only if (*);

s is a two-dimensional inexact truthmaker of p if and only if ∃s'≤S, s→ s', s' $\vdash$ p and such that

∃c$_{s'}$$[\![\text{p}]\!]^{c,c}$ = 1 if s'∈$[\![\text{p}]\!]^+$, and

∃i$_{s*}$$[\![\text{p}]\!]^{c,i}$ = 1 if s*∈$[\![\text{p}]\!]^+$;

s is a two-dimensional loose truthmaker of p if and only if, ∃t, s.t. s ⊔ t, s ⊔ t $\vdash$ p:

∃c$_{s\sqcup t}$$[\![\text{p}]\!]^{c,c}$ = 1 if s'∈$[\![\text{p}]\!]^+$, and

∃i$_{s*}$$[\![\text{p}]\!]^{c,i}$ = 1 if s*∈$[\![\text{p}]\!]^+$.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in an epistemic state space;

- Subjunctive Hyperintension:

  $\texttt{sec}_{v_@}(x) = \lambda i.[\![x]\!]^{v_@,i}$, with i a state in metaphysical state space I;

- 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda i [\![\text{x}]\!]^{s,i} = 1$.

Following the presentation of topic models in Berto (op. cit.), atomic topics comprising a set of topics, T, record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all x, y, z∈T, the following properties are satisfied: idempotence (x ⊕ x = x), commutativity (x ⊕ y = y ⊕ x), and associativity [(x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)] (Berto, 2018: 5). Topic parthood is a partial order, ≤, defined as ∀x,y∈T(x ≤ y $\iff$ x ⊕ y = y) (op. cit.: 5-6). Atomic topics are defined as follows: Atom(x) $\iff$ ¬∃y < x, with < a strict order. Topic parthood is thus a partial ordering such that, for all x, y, z∈T, the following properties are satisfied: reflexivity (x ≤ x), antisymmetry (x ≤ y ∧ y ≤ x → x = y), and transitivity (x ≤ y ∧ y ≤ z → x ≤ z) (6). A topic frame can then be defined as {W, R, T, ⊕, t}, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ, atomic formulas, p, q, r (p_1, p_2, ...), and a set of atomic topics, Utϕ = {p_1, ... p_n}, the topic of ϕ, t(ϕ) =

$\oplus$Utϕ = t(p$_1$) $\oplus \ldots \oplus$ t(p$_n$) (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus t(ϕ) = t($\neg\neg\phi$), tϕ = t($\neg\phi$), t($\phi \wedge \psi$) = t(ϕ) $\oplus$ t(ψ) = t($\phi \vee \psi$) (op. cit.).

The diamond and box operators can then be defined relative to topics:
$\langle$M,w$\rangle \Vdash \diamond^t\phi$ iff $\langle$R$_{w,t}\rangle(\phi)$
$\langle$M,w$\rangle \Vdash \Box^t\phi$ iff [R$_{w,t}$](ϕ), with
$\langle$R$_{w,t}\rangle(\phi)$:= {w'$\in$Wt'$\in$T | R$_{w,t}$[w', t'] $\cap\, \phi \neq \emptyset$ and t'(ϕ) $\leq$ t(ϕ)
[R$_{w,t}$](ϕ) := {w'$\in$Wt'$\in$T | R$_{w,t}$[w', t'] $\subseteq \phi$ and t'(ϕ) $\leq$ t(ϕ).

We can then combine topics with truthmakers rather than worlds, thus countenancing a multi-hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- Topic-Sensitive Epistemic Hyperintension:

  $\texttt{pri}_t(x) = \lambda s\lambda t.[\![x]\!]^{s\cap t,s\cap t}$, with s a truthmaker from an epistemic state space.

- Topic-Sensitive Subjunctive Hyperintension:

  $\texttt{sec}_{v_@\cap t}(\mathrm{x}) = \lambda w\lambda t.[\![x]\!]^{v_@\cap t,w\cap t}$, with w a truthmaker from a metaphysical state space.

- Topic-Sensitive 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s\lambda w\lambda t[\![\mathrm{x}]\!]^{s\cap t,w\cap t} = 1$.

I will now specify a homotopic abstraction principle for (hyper-)intensions. The philosophical significance of this implicit definition will be to provide an epistemic conduit for one's grasp of (hyper-)intensions as abstract objects. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of both analyticity (§2) and synonymy (§15). Carnap writes that: '[A]n expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension' (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).[3] The current approach differs from Carnap's by modeling the equivalence relation necessary for an abstraction principle for epistemic intensions on Voevodsky's (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.[4] In the following section, I define, then, a class of models for Epistemic Modal Algebra.

[3]For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

[4]Note further that, by contrast to Carnap's approach, epistemic intensions are here distinguished from linguistic intensions (cf. Author (ms$_1$,), for further discussion), and the current work examines the philosophical significance of the convergence between epistemic intensions and formal, rather than natural, languages. For topological Boolean-valued models

Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, a domain of propositions, and an accessibility relation:
F = $\langle$X,R$\rangle$;
X = $(X_x)_{x \in X}$; and
R = $(Rxy)_{x,y \in X}$ iff $R_x \subseteq X_x$ x X_x, s.t. if Rxy, then $\exists$o$\subseteq$X, with x$\in$o s.t. $\forall$y$\in$o(Rxy),
where the set of points accessible from a privileged node in the space is said to be open.[5] A model defined over the frame is a tuple, M = $\langle$F,V$\rangle$, with V a valuation function from subsets of points in F to propositonal variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:
M,x $\Vdash \Box\phi$ iff $\exists$o$\subseteq$X, with x$\in$o, such that $\forall$y$\in$o M,y $\Vdash \phi$.

Homotopy Theory

Homotopy Theory countenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.[6]

Reflexivity

$\forall$x,y:A$\forall$p(p : x $=_A$ y) : τ(x,y,p), with A and τ designating types, 'x:A' interpreted as 'x is a token of type A', p $\bullet$ q is the concatenation of p and q, $\texttt{refl}_x$: x $=_A$ x for any x:A is a reflexivity element, and e:$\prod_{x:A}\tau(a,a,\texttt{refl}_\alpha)$ is a dependent function[7]:
$\forall\alpha$:A$\exists$e(α) : $\tau(\alpha, \alpha, \texttt{refl}_\alpha)$;
p,q : (x $=_A$ y)
$\exists$r$\in$e : p $=_{(x=_A y)}$ q
$\exists\mu$: r = $_{(p=_{(x=_A y)} q)}$ s.

of epistemic set theory – i.e., a variant of ZF with the axioms augmented by epistemic modal operators interpreted as informal provability and having a background logic satisfying S4 – see Scedrov (1985), Flagg (1985), and Goodman (1990).

[5]In order to ensure that the Kripke semantics matches the topological semantics, X must further be Alexandrov; i.e., closed under arbitrary unions and intersections. Thanks here to Peter Milne.

[6]The definitions and proofs at issue can be found in the Univalent Foundations Program (op. cit.: ch. 2.0-2.1). A homotopy is a continuous mapping or path between a pair of functions.

[7]A dependent function is a function type 'whose codomain type can vary depending on the element of the domain to which the function is applied' (Univalent Foundations Program (op. cit.: §1.4).

Symmetry

$\forall A \forall x,y{:}A \exists H_{\Sigma}(x{=}y \to y{=}x)$
$H_{\Sigma} := p \mapsto p^{-1}$, such that
$\forall x{:}A(\mathtt{refl}_x \equiv \mathtt{refl}_x{}^{-1})$.

Transitivity

$\forall A \forall x,y{:}A \exists H_{T}(x{=}y \to y{=}z \to x{=}z)$
$H_{T} := p \mapsto q \mapsto p \bullet q$, such that
$\forall x{:}A[\mathtt{refl}_x \bullet \mathtt{refl}_x \equiv \mathtt{refl}_x]$.

Homotopic Abstraction

$\prod_{x:A}B(x)$ is a dependent function type. For all type families A,B, there is a homotopy:

$H := [(f \sim g) :\equiv \prod_{x:A}(f(x) = g(x)]$, where
$\prod_{f:A \to B}[(f \sim f) \wedge (f \sim g \to g \sim f) \wedge (f \sim g \to g \sim h \to f \sim h)]$,
such that, via Voevodsky's (op. cit.) Univalence Axiom, for all type families A,B:U, there is a function:
$\mathtt{idtoeqv} : (A =_U B) \to (A \simeq B)$,
which is itself an equivalence relation:
$(A =_U B) \simeq (A \simeq B)$.

Abstraction principles for epistemic hyperintensions take, then, the form of function type equivalence:

- $\exists f,g[f(x) = g(x)] \simeq [f(x) \simeq g(x)]$.[8]

[8] Observational type theory countenances 'structure identity principles' which are type equivalences between identification types, and the theory is said to be observational because the type formation rules satisfy structure preserving definitional equality. Higher observational type theory holds for propositional equality. 'The idea of higher observational type theory is to make these and analogous structural characterizations of identification types be part of their definitional inference rules, thus building the structure identity principle right into the rewrite rules of the type theory' (2023: https://ncatlab.org/nlab/show/higher+observational+type+theory). Shulman (2022) argues that higher observational type theory is one way to make the Univalence Axiom computable. Wright (2012c: 120) defines Hume's Principle as a pair of inference rules, and higher observational type theory might be one way to make first-order abstraction principles defined via inference rules, although not higher-order abstraction principles, computable. The Burali-Forti paradox could be circumvented, because the target abstraction principles wouldn't be based on isomorphism like the Univalence Axiom. See Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entrain the paradox. I avoid the Burali-Forti paradox in my abstraction principle for two-dimensional hyperintensions because the definition is not augmented to second-order logic like in the abstractionist foundations of mathematics, is instead taken in isolation, and the definition defines functions from sets of epistemic states taken as actual to sets of metaphysical states to extensions.

It is easy to see that mathematical sentences - whether arbitrary formulas, axioms, or Orey sentences - can be evaluated two-dimensionally, such that their epistemic profile can be a guide to their metaphysical profile. The two-dimensional hyperintensions of mathematical sentences capture the interaction between the epistemic and objective profiles of the foregoing sentences.

The foregoing proposal also differs from full-blooded platonism in the following respects. According to full-blooded platonism, if a mathematical formula is consistent and thus logically possible, as well as for whatever objects are logically possible, then those formulas are true and those objects exist (Balaguer, 1998). Formulas such as the Continuum Hypothesis (CH) which states that all infinite sets of reals have the cardinality of either the natural numbers or the real numbers, as well as the negation of CH are both logically possible and thus are actually satisfied in different universes (Balaguer, 2001: 97).

Epistemic two-dimensional semantics differs from full-blooded platonism in concerning epistemic possibilities rather than logical possibilities, as well as metaphysical possibility rather than existence. Thus, primary intensions are functions from epistemically possible worlds considered as actual to extensions. So only epistemically possible worlds considered as actual can be a guide to metaphysical possibility, by contrast to the case of full-blooded platonism according to which any logically possible object or formula actually exists or is true.

A second point of departure from full-blooded platonism is that epistemic two-dimensionalism is consistent with monism about the universe of sets, i.e. there being a cumulative hierarchy of sets comprising a single universe. This contrasts to the set-theoretic pluralism entrained by the unsettled yet logically possible status of both CH and $\neg$CH as in full-blooded platonism.

Finally, two-dimensional intensions can be availed of as a bridge between what Cantor (1883/1996: §8) refers to as 'immanent' mathematical reality and 'transient' mathematical reality. Immanent reality concerns what exists relative to the 'understanding', whereas transient reality concerns what exists relative to the 'external world' (op. cit.). Immanent reality is constrained by conditions of coherence and consistency. Cantor famously argues that mathematics is free to stipulate the existence of any objects or concepts which satisfies those conditions and that they are thus possessed of immanent reality. He leaves it as an open question what conditions need to be satisfied in order for immanent reality to be connected to metaphysics or transient reality, although he appeals to the 'unity of the all to which we ourselves belong' (op. cit.) in order to account for their convergence. Two-dimensional intensions are natural candidates for bridging the divide between conceivability and metaphysics, and thus provide a more satisfying explanation of how immanent and transient reality might converge than Cantor's own.

References

Balaguer. M. 1998. *Platonism and Anti-Platonism in Mathematics.* Oxford University Press.

Balaguer. M. 2001. A Theory of Mathematical Correctness and Mathematical Truth. *Pacific Philosophical Quarterly*, 82.

Benacerraf, P. 1965. What Numbers could not be. *Philosophical Review*, No. 74, Vol. 1.

Benacerraf, P. 1973. Mathematical Truth. *Journal of Philosophy*, 70.

Berto, F. 2018. Aboutness in Imagination. *Philosophical Studies*, 175.

Berto, F. 2019. The Theory of Topic-Sensitive Intentional Modals. In I. Sedlar and M. Blicha (eds.), The Logica Yearbook, 2018. College Publications.

Berto, F., and P. Hawke. 2021. Knowledge relative to Information. Mind, 130: 517.

Burali-Forti, C. 1897/1967. A Question on Transfinite Numbers (tr. J. van Heijenoort). In van Heijenoort (ed.), *From Frege to Gödel.* Harvard University Press.

Canavotto, I., F. Berto, and A. Giordani. 2020. Voluntary Imagination: a Fine-grained Analysis. *Review of Symbolic Logic.*

Cantor, G. 1883/1996. *Foundations of a General Theory of Manifolds*, tr. W.B. Ewald. In Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Vol. II.* Oxford University Press.

Chalmers, D. 2002. Does Conceivability Entail Possibility? In T. Gendler and J. Hawthorne (eds.), *Conceivability and Possibility.* Oxford University Press.

Chalmers, D. 2006. The Foundations of Two-dimensional Semantics. In M. Garcia-Carpintero and J. Macia (eds.), *Two-dimensional Semantics.* Oxford University Press.

Chalmers, D. 2010. *The Character of Consciousness.* Oxford University Press.

Chalmers, D. 2011. The Nature of Epistemic Space. In A. Egan and B. Weatherson (eds.), *Epistemic Modality.* Oxford University Press.

Chalmers, D. 2012. *Constructing the World.* Oxford University Press.

Clarke-Doane, J. 2016. What is Benacerraf's Problem? In F. Pataut (ed.), *Truth, Objects, Infinity.* Springer.

Field, H. 1989. *Realism, Mathematics, and Modality.* Basil Blackwell.

Field, H. 1993. The Conceptual Contingency of Mathematical Objects. *Mind*, 102.

Fine, K. 2013. Truth-maker Semantics for Intuitionistic Logic. *Journal of Philosophical Logic*, 43.

Fine, K. 2017a. A Theory of Truthmaker Content I: Conjunction, Disjunction, and Negation. *Journal of Philosophical Logic*, 46:6.

Fine, K. 2017b. A Theory of Truthmaker Content II: Subject-matter, Common Content, Remainder, and Ground. *Journal of Philosophical Logic*, 46:6.

Fine, K. 2017c. Truthmaker Semantics. In B. Hale, C. Wright, and A. Miller (eds.), *A Companion to Philosophy of Language*. Blackwell.

Hale, B. 2013. *Necessary Beings*. Oxford University Press.

Hale, B., and C. Wright. 1992. Nominalism and the Contingency of Abstract Objects. *Journal of Philosophy*, 89.

Hale, B., and C. Wright. 1994. A Reductio ad Surdum? Field on the Contingency of Mathematical Objects. *Mind*, 103:410.

Hawke, P., and A. Özgün. 2023. Truthmaker Semantics for Epistemic Logic. In F. Faroldi and F. van de Putte (eds.), *Kit Fine on Truthmakers, Relevance, and Non-Classical Logic*. Sringer.

Hazen, A. 1985. Review of Crispin Wright's *Frege's Conception of Numbers as Objects*. *Australasian Journal of Philosophy*, 63:2.

Hodes, H. 1984. Logicism and the Ontological Commitments of Arithmetic. *Journal of Philosophy*, 81:3.

Shulman, M. 2022. Towards third generation HOTT. slides.

Wright, C. 2012. Frege and Benacerraf's Problem. In R. DiSalle, M. Frappier, and D.Brown (eds.), *Analysis and Interpretation in the Exact Sciences: Essays in Honour of William Demopoulos*. Springer.