A Puzzle About General Covariance and Gauge

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Abstract

We consider two simple criteria for when a physical theory should be said to be "generally covariant", and we argue that these criteria are not met by Yang-Mills theory, even on geometric formulations of that theory. The reason, we show, is that the bundles encountered in Yang-Mills theory are not natural bundles; instead, they are gauge-natural. We then show how these observations relate to previous arguments about the significance of solder forms in assessing disanalogies between general relativity and Yang-Mills theory. We conclude by suggesting that general covariance is really about functoriality.

1. Introduction

Few observations would find more universal assent among relativists of the past century than that one should prefer – or insist on – generally covariant theories. Of course, over the same period there has been little agreement about what "general covariance" means, much less whether it imposes a substantive constraint on physical theorizing (Norton, 1993; Earman, 2006; Pooley, 2009). On one side are followers of Kretschmann (1917), who famously argued (contra Einstein) that general covariance is trivial because *any* remotely plausible physical theory can be reformulated in generally covariant form; and on the other side have been dozens of attempts to identify some non-trivial, physically well-motivated principle, satisfied by general relativity and affiliated theories but not others, that captures working physicists'

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sense that general covariance imposes a difficult-to-meet constraint on theorizing, with real consequences for things like conservation principles (Freidel and Teh, 2022).

We will not attempt to adjudicate the many decades of dispute on this topic. (By our reckoning, it is now 11 and counting!) Instead, we wish to identify one sense in which general covariance, or something very much like it, *does* amount to a non-trivial constraint on theories—a constraint, we suggest, that is so non-trivial that it is not even satisfied by widely accepted current physics. Specifically, we will argue that (classical) Yang-Mills theory is not generally covariant.[1] This claim, in turn, has consequences for how we interpret physical geometry, the meaning of diffeomorphism invariance, the role of principal bundles and solder forms in Yang-Mills theory, and the meaning of "gauge" in contemporary physics.

Of course, in order to defend this claim, we will need to say what we mean by "general covariance". We will do that in section 2. In the following section, we will present our argument that Yang-Mills theory is not generally covariant (even though it admits a "geometrical" formulation and has other features quite similar to general covariance). Then, in section 4, we will introduce the formalism of *natural bundles* (Salvioli, 1972; Terng, 1978; Kolář et al., 1993), which extends and generalizes the geometric object program (Schouten and Haantjes, 1936; Nijenhuis, 1952). As we explain there, the arguments of sections 2 and 3 may be neatly summarized as follows: whereas the tangent structures to a manifold used in general relativity form natural bundles, the vector and principal bundles encountered in Yang-Mills theory are not natural.

The remainder of the paper will address responses and consequences. In section 5, we will argue that the need to generalize and thereby restore naturality for Yang-Mills theories provides a new perspective on the role that principal bundles play in those theories; it also clarifies the role of the frame bundle in general relativity. Then, in section 6, we will argue

[1]For background on this theory, its interpretation, and the formalism we adopt here for discussing it, see Weatherall (2016), which builds on earlier work by Palais (1981) and Bleecker (1981).

that the foregoing arguments also provide a new perspective on the role of solder forms in some analyses of tangent structures (c.f. Anandan, 1993; Healey, 2007; Weatherall, 2016). Finally, we will conclude with some remarks about how naturality bears on older debates about the meaning and status general covariance and preview some further ways in which the perspective offered here can help clarify issues in the foundations of general relativity and Yang-Mills theory.

2. What is General Covariance?

As alluded to above, general covariance is a famously vexed concept. Many versions of general covariance, as a criterion that a physical theory may or may not satisfy, have been proposed, and many arguments have been made that those criteria are incoherent, trivial, or inadequate. For present purposes, though, the key issues in the debate – namely, whether some particular account does or does not provide a non-trivial and plausibly reasonable constraint – may be set aside. Our discussion, at least in the next several sections, concerns only certain core ideas connected to general covariance that are apparently shared among virtually all participants in the debate. In other words, we will consider just minimal necessary conditions for general covariance; many authors have argued that (substantive) general covariance requires more than just these weak necessary conditions (Norton, 1993; Pooley, 2009). Our goal is to show that in fact widely accepted theories fail to satisfy even these weak necessary conditions.

For our purposes, the conceptual core of general covariance is the requirement that the principal relationships posited by a theory must be preserved under coordinate transformations. In other words, whether some physically meaningful assertion holds true cannot depend on a choice of coordinate system; were some relationship to hold when expressed in one coordinate system, then the same relationship, expressed in different coordinates using suitably transformed representations of the objects involved in the assertion, would still hold (and in fact, express the same fact). From a modern perspective (Misner et al., 1973; Wald, 1984, c.f.), the requirement that some object has the appropriate transformation properties under coordinate transformations to enter into such relationships is often replaced by the requirement that the object admit a *coordinate-independent* characterization, and that the relationships involving such objects can be expressed using just these coordinate-independent objects.[2] Coordinate transformations, then, may be re-interpreted as implementing diffeomorphisms acting on these coordindate-independent relationships, and invariance under coordinate changes can be interpreted as the requirement that physically meaningful relationships between coordinate-independent objects preserve their truth value under the action of diffeomorphisms.

Thus, we will assume that a theory is rightly described as generally covariant *only if*:

1. the objects involved in the principal claims and relationships of that theory are (or can be) expressed in a way that does not depend on particular choices of coordinate system; and
2. those objects have well-defined actions under diffeomorphisms, or changes or coordinate system.

As we said above, we take these to be weak necessary conditions expected to hold of any generally covariant theory. Any full account of general covariance would require more. At very least, it would require that the principal claims and relationships asserted by a theory are preserved under the actions of diffeomorphisms on the objects concerned.[3] But even with

[2] Our approach is especially close to Misner et al. (1973, p. 48), who identify general covariance with the requirement that "every physical quantity must be describable by a geometric object" in the sense of Nijenhuis (1952); and they attribute the first clear articulation of this view Veblen and Whitehead (1932). (See also Misner et al., 1973, p.302-3.) These remarks are very closely connected to our arguments here and, especially, in section 4.

[3] Much more can be said about what makes a "theory", understood as a system of differential equations, generally covariant in the sense suggested here, but we leave that discussion of "natural theories" to future work because it is not necessary for the present points. (Fatibene and Francaviglia (2003) offer one approach for theories derived from a variational principle.)

this further requirement, we do not not claim to have captured all of the desiderata that have been required by various authors assessing whether general covariance is a plausible and substantive constraint on theories.

Consider an example. General relativity is surely a canonical example of a generally covariant theory. What does this mean? Let M be a smooth, four-dimensional manifold representing events in space and time, and let g_{ab} be a smooth, Lorentz-signature metric on M representing spatio-temporal relations between those events. (The pair (M, g_{ab}) will be called a *relativistic spacetime* in what follows.) Finally, suppose there is some distribution of matter throughout space and time, whose energy and momentum properties can be represented by a smooth tensor field T^{ab}. Then according to general relativity, Einstein's equation must hold, relating the metric and its associated curvature to the energy-momentum tensor T^{ab}:

$$R_{ab} - \frac{1}{2}Rg_{ab} = kT_{ab},$$

where R_{ab} is the Ricci curvature tensor, R is scalar curvature, and k is a constant related to Newton's gravitational constant and the speed of light.

As we have just described it, general relativity clearly meets the two criteria we have set out. The objects implicated in Einstein's equation – the metric, curvature, and mass-energy tensor – are all tensor fields, which we have presented in a coordinate independent way. Moreover, diffeomorphisms act on all of these fields in a well-defined way, via the push-forward construction. Though it is not immediately relevant, we also note that the relationship expressed by Einstein's equation is preserved under that action by diffeomorphisms. That is, if $\varphi : M \to N$ is a diffemorphism between M and some other manifold N, then we have

$$R_{ab} - \frac{1}{2}Rg_{ab} = kT_{ab} \qquad \Leftrightarrow \qquad \varphi_*(R_{ab} - \frac{1}{2}Rg_{ab}) = \varphi_*(kT_{ab}),$$

where φ_* is the pushforward along φ. Thus, Einstein's equation is "coordinate-independent" in the required way, and the theory is generally covariant.

The example helps show how weak our conditions are, at least for theories formulated using tensor fields. General covariance is automatic for such theories. To show a theory is generally covariant, one need only rewrite it in the language of tensor calculus. It was on these grounds that Kretschmann (1917) argued that general covariance is trivial. At very least, one might think he established that *these* conditions are trivial, and so the whole of the dispute about general covariance is whether there are other, stronger criteria that should be imposed on top of these.[4] Many arguments are available that aim to show more is required. We set those aside because we will presently argue that even these conditions are not satisfied for realistic (and widely accepted) theories.

3. Yang-Mills Theory is not Generally Covariant

Models of Yang-Mills theory consist of a relativistic spacetime (M, g_{ab}), and a principal G-bundle $G \to P \to M$ over M. The central objects of Yang-Mills theory then consist in a principal connection ω on $G \to P \to M$ (along with its associated curvature two-form), and sections of associated (vector) bundles $P \times_G V \to M$, which represent matter fields. Sections $\sigma : M \to P \times_G V$ of an associated bundle can, in turn, be associated with charge-current densities J on P. These enter into the Yang-Mills equation

$$\star D \star \Omega = J$$

where $\star$ is the Hodge star operator relative to g_{ab} (pulled back to P along π), D the exterior covariant derivative relative to ω, and Ω the curvature two-form of ω. For example, the

[4]For just one example: (Wald, 1984, p. 57-8) takes for granted the two conditions required here, and adds further that "the metric of space is the only quantity pertaining to space that can appear in the laws of physics". Later he allows that quantities pertaining to space that are determined by the metric, such as a derivative operator, are also compatible with general covariance.

structure group of electromagnetism is $U(1)$, so models of electromagnetism consist in a principal $U(1)$ bundle $U(1) \to EM \to M$ over M and a principal connection ω on EM. In the simplest case, where matter is represented by a complex scalar field, the vector space V is a copy of $\mathbb{C}$ equipped with a faithful representation of $U(1)$, and the associated bundle $EM \times_{U(1)} \mathbb{C} \to M$ has fibers isomorphic to $\mathbb{C}$.

On this way of presenting Yang-Mills theory, it does satisfy our condition (1). In particular, the objects involved in the principle claims and relationships of Yang-Mills theory, including connections on a principle bundle and sections of an associated bundle, can be characterized in a coordinate-independent way as per the above.

However, notice that the coordinates at issue here are importantly different from the coordinates at issue in the claim that general relativity admits a coordinate-independent characterization. Sections of an associated bundle can be characterized in a way that is independent of the choice of coordinatization of the base and total spaces, and have well-defined actions under vector bundle morphisms. But the coordinates at issue are not (just) the coordinates of the base space. This raises a problem for how to understand the action of smooth maps on the base space on these objects. In particular, given a diffeomorphism $\varphi : M \to M$ which acts on the base space of some vector bundle $B \to M$, and given two sections $\sigma, \sigma' : M \to B$ of that bundle, there is in general no way to say whether or not those sections are "related by the diffeomorphism φ". As a result, Yang-Mills theory does not satisfy our condition (2), at least on one plausible way of understanding what the relevant class of diffeomorphisms for condition (2) is. Note also that this failure to satisfy (2) does not depend on the details of the dynamics of Yang-Mills theory. Rather, it is a simple consequence of the fact that the theory is formulated using structures defined on a (non-natural) principal bundle.

To make this concrete, consider the following example. Suppose we have two diffeomorphic manifolds M and N and (smooth) complex scalar fields on each, i.e. smooth sections

σ_M, σ_N of the bundles $EM_M \times_{U(1)} \mathbb{C} \to M$ and $EM_N \times_{U(1)} \mathbb{C} \to N$ on M, N respectively. Let $\varphi : M \to N$ be a diffeomorphism. One might then ask: is σ_N the image of σ_M under the action of φ? There is no way to answer this question. The reason is that φ does not act on points in the bundle space $EM_M \times_{U(1)} \mathbb{C}$. Indeed, there is no well-defined, unambiguous way of saying what it would mean for φ to act on $EM \times_{U(1)} \mathbb{C}$. What is needed is some canonical way of associating to each diffeomorphism φ a unique bundle morphism (ψ, φ), but in general, to do so would require further structure, such as a flat connection or a preferred global trivialization; and different choices of that additional structure would yield different associations of diffeomorphisms to bundle morphisms. This illustrates our claim that Yang-Mills theory, even in its "geometric formulation", does not satisfy our condition (2).

It is worth emphasising the difference between real and complex scalar fields in this respect. Naïvely, one might think that if this argument that Yang-Mills theory is not generally covariant works, it works too well, in that a version of this problem would also apply to smooth real scalar fields, of the sort found in general relativity. Recall that smooth real scalar fields, i.e. maps $M \to \mathbb{R}$, can always be thought of as smooth (global) sections of the trivial smooth rank-1 vector bundle $\mathbb{R} \to B \stackrel{\pi_M}{\to} M$. So by parity of reasoning (so the thought goes): diffeomorphisms $\varphi : M \to N$ act only on the base space, but not on B, smooth real scalar fields are smooth (global) sections $\sigma : M \to B$, and therefore diffeomorphisms do not act on those fields and so one cannot say whether two such fields are "related by a diffeomorphism".

What has gone wrong here? The crucial point is that unlike the associated bundle $EM \times_{U(1)} \mathbb{C} \to M$ of Yang-Mills theory, there is always a canonical way of lifting diffeomorphisms $\varphi : M \to N$ on M to bundle morphisms on $\mathbb{R} \to B \stackrel{\pi_M}{\to} M$. (As we will see in section 4, this is deeply related to the fact that the bundle $\mathbb{R} \to B \stackrel{\pi_M}{\to} M$ is, in a certain intuitive sense, "constructed" out of the base space M whereas the bundle $EM \times_{U(1)} \mathbb{C} \to M$ is not.)

To see this, first recall that any two complete ordered fields $\mathfrak{R}_1 = (R_1, +_1, \times_1, \leq_1, 0_1, 1_1)$, $\mathfrak{R}_2 = (R_2, +_2, \times_2, \leq_2, 0_2, 1_2)$ are uniquely isomorphic (by standard results in analysis). Now suppose that $\mathbb{R} \to B \overset{\pi_M}{\to} M$ is a trivial smooth rank-1 vector bundle. The foregoing implies that not just is B isomorphic to $M \times \mathbb{R}$ (since it is a trivial bundle with $\mathbb{R}$ fibres), but there is in fact a *unique* isomorphism $\chi : B \to M \times \mathbb{R}$ which preserves the complete ordered field structure on the fibres of B. In turn, this means that B comes canonically equipped with a projection map $\pi^M_{\mathbb{R}}$ onto the second factor (since $M \times \mathbb{R}$ does, so we just pull it back by χ). So we can define a canonical bundle morphism (ψ, φ) by requiring that it preserve this projection onto the second factor (in the sense that we now have $\pi^N_{\mathbb{R}} \circ \psi = \pi^M_{\mathbb{R}}$ as well as $\pi_N \circ \psi = \phi \circ \pi_M$), which gives us $\psi = \varphi \times \mathrm{id}_{\mathbb{R}}$. In other words: whilst the diffeomorphism φ is defined to act only on the base space, the way that we have constructed the bundle $\mathbb{R} \to B \overset{\pi_M}{\to} M$ means that diffeomorphisms φ on the base space are always canonically associated with a unique action on the total space, and hence induce bundle morphisms.

This makes it clear why no analogous construction works for the bundle $EM \times_{U(1)} V \to M$. For one, the space $EM \times_{U(1)} \mathbb{C}$ need not be isomorphic to $M \times \mathbb{C}$; even if it is, we cannot use this fact to pin down a unique vector bundle morphism unless we first specify a choice of isomorphism $\chi : EM \times_{U(1)} \mathbb{C} \to M \times \mathbb{C}$, since the field $\mathbb{C}$ has a non-trivial automorphism group.

4. Naturality, Geometricity, and Covariance

One way to understand what is going on here is to introduce the concept of a *natural bundle*. This is a way of capturing the idea of a "geometric object", developed by Schouten and Haantjes (1936), Nijenhuis (1952) and others in the middle part of the century.[5] The basic observation is that the bundles encountered in general relativity typically are natural, whereas the ones encountered in Yang-Mills theory are not natural.

[5] Recall fn. 2.

Roughly speaking, natural bundles are "species" of bundles that depend (only) on the structure of their base space, in the sense that (a) given any (suitable) smooth manifold, one can uniquely define a bundle of the relevant species over that manifold; and (b) (suitable) smooth maps acting on base spaces "lift" to bundle morphisms between the natural bundles defined over them. (The term "suitable" in each of these clauses will be clarified presently.) What is intended here is clearest when one considers examples. Take, for instance, tangent bundles. Every manifold M determines a smooth bundle $TM \to M$, whose fibers at each point are the tangent space at that point, and whose sections are smooth vector fields. Moreover, well-behaved smooth maps $\varphi : M \to N$ between manifolds determine smooth maps between the tangent bundles of the manifolds, via the pushforward construction. Thus, tangent bundles realize the two properties we have isolated. Other examples abound. Cotangent bundles, (tangent) frame bundles, (tangent) tensor bundles, bundles of metrics, bundles of $k-$forms for fixed k, and bundles of connections are all natural. (So are the bundles in which real scalar fields are valued.)

The ideas just sketched can be made precise using the language of category theory. Let $\mathcal{M}_n$ denote the category of smooth, $n-$dimensional manifolds, with smooth embeddings as morphisms.[6] Let $\mathcal{F}$ denote the category whose objects are smooth fiber bundles and whose morphisms are smooth bundle morphisms. Then a *natural bundle* (over $n-$manifolds) is a functor $F : \mathcal{M}_n \to \mathcal{F}$ such that (1) for every object M of $\mathcal{M}_n$, FM is a bundle whose base space is M; and (2) for every morphism $\varphi : M \to N$ of $\mathcal{M}_n$, $F\varphi$ is of the form (φ_*, φ), where the maps φ_* induces from fibers of FM to fibers of FN are diffeomorphisms.[7]

The two informal conditions sketched above get realized in the requirements for func-

[6]Our definition of $\mathcal{M}_n$ here most closely follows Palais and Terng (1977). Kolář et al. (1993) define the category of n-manifolds to have local diffeomorphisms as its arrows. These are immersions but not necessarily injective.

[7]Another way of doing this would be to include, as morphisms of $\mathcal{F}$, only smooth bundle morphisms that act as diffeomorphisms on fibers. This is the approach taken by Palais and Terng (1977). Some treatments also impose a "regularity" condition, but a classic result due to Epstein and Thurston (1979) establishes that that condition is automatic in the presence of the others.

toriality: a natural bundle associates bundles with manifolds and bundle morphisms with embeddings. (Moreover, it does so in a way that preserves composition and identity.) The "suitable" provisos in the informal discussion, meanwhile, are made precise with our definitions of the categories, specifically with the choices of objects and arrows of $\mathcal{M}_n$. Note that for some natural bundles one might wish to study in physics (especially, general relativity), we must modify the category $\mathcal{M}_n$ to require manifolds satisfy further conditions. For instance, while Lorentzian metrics have a well-defined behavior under pushforwards along embeddings, not every $n-$manifold admits any Lorentz-signature metric (Geroch and Horowitz, 1979; O'Neill, 1983).

Note that the terminology is a bit odd. A natural bundle, officially, is not a bundle at all; rather, it is a functor that associates bundles with each manifold. This is why we wrote above of "species" of bundles. Although it is an abuse of language, we will use "natural bundle" to refer to both natural bundle functors and to the objects in the image of those functors. This is similar to using the phrase "the tangent bundle" to refer both to a general construction procedure and to specific bundles over specific manifolds that arise from that construction procedure.

We have already discussed several examples of natural bundles familiar from general relativity. Indeed, one can easily check that all of the standard examples of fields that one encounters in relativistic field theories – spacetime metrics, derivative operators, curvature tensors, stress-energy tensors, electromagnetic field strengths, real scalar fields, and so on – can be seen as sections of natural bundles over spacetime. We claim that this fact is deeply connected to the general covariance of theories involving these structures. In fact, we take the natural bundle framework to provide a more precise specification of the two necessary conditions we identified in section 2. The objects under consideration in a generally covariant theory have to exhibit "diffeomorphism" covariance in the sense made precise by the fact that a natural bundle is functorial over smooth manifolds.

This framework also allows us to restate the claims of section 3. Neither the principal bundles nor the associated vector bundles encountered in Yang-Mills theory are natural bundles. In fact, there are several barriers to naturality. One is that when we define the bundles used in Yang-Mills theory, it is common to specify only the fiber type, and not the global topology of the bundle. We say, for instance, that we are considering an $SU(2)$ theory, which implies the fibers are $SU(2)$ torsors, but implies nothing about whether the bundle is trivial. But for general manifolds, there is not necessarily a unique principal bundle (viz., vector bundle) with given fibers available, and which bundle one chooses will determine the space of global field configurations. This failure of uniqueness, meanwhile, creates problems for defining the bundles using a functor, since a functor must assign at most one object in its codomain to each object in its domain. Now, admittedly, the bare technical problem can be overcome, for instance by specifying that one is considering only trivial bundles. This can always be done, but it comes at an interpretational cost, because it effectively rules out certain classes of global field configuration associated with non-trivial bundles (or, more generally, ones with different global topology than those in the codomain of the functor). More generally, if one wishes to allow that field configurations associated with different bundles over a single base space are all in some sense possible configurations of a field with Yang-Mills charge, then one cannot take those fields to be sections of a (single) natural bundle over that base space.[8]

The other barriers to naturality are arguably deeper. The bundles encountered in Yang-Mills theory are not generally constructed from (just) the structure of the base space. This, in turn, means that there is no generally applicable and uniform – i.e., no natural – way to lift diffeomorphisms to act on the fibers of these bundles. Preserving the structure of the base space is not enough to preserve the structure of the bundle. Some further choice is needed

[8]There are deep and under-explored issues, here, about global topology of principal bundles and the physical possibility of certain global configurations of matter that we are setting aside for present purposes.

to identify fibers that were otherwise associated with different points, and in general, that choice can be made in many different ways. Of course, we have already made this argument in concrete detail for the case of complex scalar fields. Now, though, we see that the problem manifests as a failure of functoriality.

5. Gauge Naturality

Thus far, we have argued that there is an important sense in which Yang-Mills theories, by virtue of being formulated on principal bundles and associated vector bundles on which base space diffeomorphisms do not act, are not generally covariant. This situation presents a puzzle for anyone who defends general covariance. Surely we cannot give up on Yang-Mills theory simply because it fails to meet some abstract principle of physical theorizing, given its enormous empirical and theoretical success. On the other hand, giving up on general covariance is also a hard bullet to bite, especially given how fundamental the two necessary conditions we identified in section 2 appear to be. In particular, as discussed in section 3, the failure of naturality has far-reaching consequences for assessing the physical equivalence of field configurations on diffeomorphic manifolds. This, in turn, raises other questions, such as how to assess the well-posedness of partial differential equations set on such bundles.[9]

For these reasons, we do not propose dropping general covariance, so much as reconsidering precisely what it demands. As we have seen, the two necessary conditions for general covariance that we presented in section 2 can be restated as the requirement that certain structures should be natural, in the sense of being functorial. And as we have shown, generically vector bundles are *not* natural in this sense, at least over their base spaces. But it turns out that these vector bundles can be reconstrued as natural, by adopting a different

[9]Here is what we have in mind. As we know from Einstein's equation, subtle issues regarding physical equivalence of solutions can arise when trying to determine whether a system of equations has unique solutions for some initial data. Without clear criteria for equivalence of solutions, it is hard to see how to get started in analyzing uniqueness properties of those solutions.

perspective on what sorts of maps should be required to lift to act on them. This idea can be made precise using the formalism of *gauge natural bundles*. The key move is to change the category that acts as the domain of the natural bundle functor, so that the objects in that category are not the base space of the bundle under consideration but rather principal bundles, for some fixed structure group G, over that base space. Doing so leads to the notion of a gauge natural bundle (Kolář et al., 1993, Ch. 12).[10]

We proceed similarly to as before. We define a category $\mathcal{PB}_n(G)$ whose objects are principal $G-$bundles over n dimensional manifolds and whose arrows are principal bundle morphisms whose action on the base space is a smooth embedding. (Thus we have a full functor $B : \mathcal{PB}_n(G) \to \mathcal{M}_n$, taking each object in $\mathcal{PB}_n(G)$ to its base space, and taking arrows to their underlying smooth embedding.) Then a *gauge natural bundle* is a functor $F : \mathcal{PB}_n(G) \to \mathcal{F}$ satisfying the following conditions: (1) the action of F on objects preserves their base space, i.e., it takes principal bundles over a manifold M to fiber bundles over M; (2) the action of F on arrows preserves their action on the base space; and (3) for every object $\pi : P \to M$ of $\mathcal{PB}_n(G)$ and open set $U \subseteq M$, the inclusion arrow $(i, 1_M)$, which takes the subbundle $\pi^{-1}[U] \to U$ into $P \to M$, is mapped to the inclusion arrow taking $q^{-1}[U] \to U$ into $F(\pi : P \to M)$, where q is the projection map associated with $F(\pi : P \to M)$.

This definition is abstract. The key examples of gauge natural bundles for present purposes – that is, for the purposes of interpreting matter theories in Yang-Mills theory – are the vector bundles associated to a principal bundle.[11] One can construct these bundles systematically by fixing a vector space V and a representation $\rho : G \to GL(V)$ of G on V. (More generally, one can consider any fixed manifold S with a left G action.) One then considers, for each principal bundle $G \to P \to M$ in $\mathcal{PB}_n(G)$, the product manifold $P \times V \to P$. This structure can be viewed as a trivial bundle with typical fiber V over P,

[10] See also Fatibene and Francaviglia (2003) for a more accessible discussion of these ideas.

[11] In fact, the construction we presently discuss is generic at "0th order"; we discuss higher order associated bundles in the next section.

though for present purposes we will keep the full product structure, so that in fact we are considering a trivial bundle with fixed global trivialization.

The representation of G on V, along with the right action of G on P, determines a right action of G on $P \times V$, by $(x, v) \cdot g = (x \cdot g, \rho(g^{-1}) \cdot v)$ for each $g \in G$. Let $E = (P \times V)/G$ be the smooth manifold that results by quotienting by this action, so that points of E are equivalence classes $[x, v]$ of points related by the action. Since the action of G on $P \times G$ is fiber preserving over G, the projection $\pi : P \to G$ determines a projection $\pi_f : E \to M$. Finally, for any smooth G principal bundle morphism $f : (P \to M) \to (P' \to M')$, the map $(f, 1_V) : P \times V \to P' \times V$ determines a smooth bundle morphism between the quotients $(P \times V)/G \to M$ and $(P' \times V)/G \to M'$.[12] These two constructions together can be shown to define a functor from $\mathcal{PB}(G)$ to $\mathcal{F}$ that satisfies the conditions set forth above.

Gauge natural bundles are like natural bundles in the sense that they associate bundles with manifolds uniformly across different manifolds, in a way that is compatible with the manifold structure (as reflected by the functoriality of the construction); and because they give rise to a notion of "pushforward" along maps in $\mathcal{PB}_n(G)$. Now, though, both the assignment of bundles and the pushforward depends on more than just the base space and maps acting on base spaces; they also depend on a principal bundle over the base space and arrows between principal bundles. Why should the principal bundle structure help here? As emphasized by various authors (e.g. Kolář et al., 1993; Weatherall, 2016; Gomes, 2024), a principal bundle associated with a vector bundle can be thought of as a *bundle of frames*, or basis fields, for that vector bundle. (We make this idea precise and elaborate on it below.) Specifying information about those frames and how they transform is the missing piece in resolving the issues raised in the previous section.

Consider, for instance, how the issue of uniqueness raised above, concerning the global

[12]These quotients are the same bundles we encountered above, for which we previously used the notation $P \times_G V \to M$. Here we are emphasizing the construction procedure, and so the fact that these are quotients by a G action is especially salient.

topology of a non-natural bundle, is addressed here. It remains the case that one can define many vector bundles, with different global topologies, with a given typical fiber over a generic manifold. But that freedom corresponds exactly to the freedom to define different principal G bundles with different group representations acting on those fibers over the same manifold. Thus, by the construction above, one gets a *different* vector bundle over M for each principal $U(1)$ bundle over M. Similarly, the problem of how to define the action of diffeomorphisms on non-natural bundles is resolved by also specifying how elements of arbitrary bases at each point transform. That extra information uniquely determines how fiber elements in an associated bundle transform.

To see what this means in more concrete detail, consider again the example of the complex scalar field discussed above. Since the complex numbers come equipped with a preferred Hermitian product, they can be thought of as carrying a representation of $U(1)$ that preserves that product.[13] (This is the "fundamental" representation of $U(1)$.) To think of complex scalar fields as a natural bundle over manifolds of dimension n, then, one can begin with the category of $U(1)$ bundles over n dimensional manifolds, $\mathcal{PB}_n(U(1))$, and then define a functor via the construction above for associated bundles, yielding, for each object $U(1) \to P \to M$ of $\mathcal{PB}_n(U(1))$, a one dimensional complex vector bundle over M. Arrows in $\mathcal{PB}_n(U(1))$ specify not just an action on the base spaces of each bundle, but also specify how to identify complex phases between fibers at domain and codomain points, by specifying how bases transform.

Before proceeding, we note a connection between the remarks here and an argument from Weatherall (2016), to the effect that the principal bundles in Yang-Mills theory are "auxiliary" or "secondary" structure. The idea is that it is the (associated) vector bundles that represent the possible states of matter, and it is the connections on those bundles that determine the physically relevant standards of constancy for those matter fields. The

[13] One could also proceed by considering principal $GL(1, \mathbb{C})$ bundles.

principal bundles, meanwhile, serve only to coordinate the Yang-Mills connections across different types of matter that participate in the same Yang-Mills force. They do not represent matter or its possible states directly. Weatherall analogies this situation to the sense in which a coach plays an auxiliary coordinating role vis a vis players on the field. Gomes (2024) goes even further, arguing that principal bundles are "epiphenomenal" in Yang-Mills theory because in fact, all matter properties can be valued in a single vector bundle, with fibers $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}$, and so principal bundles are not needed even to coordinate between distinct vector bundles.

These analyses bears revisiting in light of the role of principal bundles in defining gauge natural bundles. While it is true that principal bundles do not represent possible states of matter directly, the role they do play in Yang-Mills theory is nonetheless robust and important. What we have seen in this section is that the vector bundles in which matter takes its properties are, in the sense described above, determined by the principal bundles to which they are associated—in a way analogous to how natural bundles are determined by their base space. We also now see that what it means for matter valued in two different vector bundles to participate in the same Yang-Mills force on a given spacetime is for them to be images of the same principal bundle under two different gauge natural bundle functors. Likewise, the bundles associated with a given type of matter across different base spaces depends not just on the base space, but also on a choice of principal bundle over that space. And it is principal bundle morphisms that play the role of smooth maps on the base space in considerations of general covariance for associated vector bundles. We conclude that while principal bundles may be viewed as auxiliary or secondary in one important sense, there are other senses in which they are of central importance.

6. Solder Forms

We now turn to an application of the ideas presented thus far. Several authors have discussed an apparent disanalogy between Yang-Mills theory and general relativity (Anandan, 1993; Healey, 2007; Weatherall, 2016). As various authors have emphasized, like Yang-Mills theory, general relativity can be understood as a theory of connections on principal bundles (Trautman, 1980; Weatherall, 2016). The relevant principal bundles in that case are the (tangent) frame bundles over $GL(n, \mathbb{R}) \to LM \to M$, where $GL(n, \mathbb{R})$ is the (real) general linear group in n dimensions. As in Yang-Mills theory, matter fields are represented by sections of an associated bundle; for example, if V is an n-dimensional vector space equipped with a faithful representation of $GL(n, \mathbb{R})$, then the associated bundle $V \to LM \times_{GL} V \to M$ is (canonically) isomorphic to the tangent bundle $TM \to M$, and sections of $V \to LM \times_{GL} V \to M$ correspond to vector fields on M. Similarly, the cotangent bundle and bundles of rank-(r, s) tensor fields on M can all be understood as associated to the frame bundle.

This takes us to the apparent point of disanalogy between general relativity and Yang-Mills theory. We just said that the associated bundle $V \to LM \times_{GL} V \xrightarrow{\pi} M$ is isomorphic to the tangent bundle $TM \to M$. In fact, there are many isomorphisms between these bundles, but the constructions of the tangent bundle, frame bundle, and associated bundle together determine a canonical one, relative to certain choices made in those constructions. This canonical isomorphism equips the frame bundle with a solder form θ, which is a linear isomorphism, at each $u \in LM$, between $T_{\pi(u)}M$ and V, and is equivariant with respect to the $GL(n, \mathbb{R})$ action on LM.

As Weatherall (2016) notes, this construction is very general, in the sense that *any* frame bundle, including ones constructed from vector bundles that are not tangent to a manifold, comes equipped with a solder form that fixes an isomorphism between the vector bundle and associated vector bundles (of the same dimension) to its frame bundle. In in more detail, let $V \to B \to M$ be any vector bundle over M. We can construct the frame bundle

$GL(V) \to LB \stackrel{\wp}{\to} M$ for B. Since the associated bundle $LM \times_{GL} V \to M$ is isomorphic to B, we can equip LB with an equivariant one-form which defines, at each $u \in LB$, a linear isomorphism between the fiber of B at $\wp(u)$ and the fiber of $LM \times_{GL} V$ at $\wp(u)$. The construction procedures fix a preferred isomorphism, and thus a preferred solder form, in just the same way as for the tangent bundle.

But principal bundles do not generally carry a solder form. They do so only insofar as they are viewed as (subbundles of) the frame bundle for some particular vector bundle. And the principal bundles in Yang-Mills theory are not always thought of as frame bundles. For some authors, this fact reveals a sense in which general relativity is disanalogous to Yang-Mills theory. For example, Anandan (1993) and Healey (2007), the lack of a solder form on principal bundles in general shows that Yang-Mills theory does not admit the same kind of geometrical interpretation as general relativity. Anandan even argues that the existence of the solder form "breaks" gauge invariance of the gravitational field. Healey, meanwhile, takes the solder form in general relativity to partly motivate his endorsement of a holonomy interpretation of Yang-Mills theory but not generaly relativity. Weatherall (2016), meanwhile, drawing on the above construction, emphasizes that every principal bundle in Yang-Mills theory *can* be thought of as a subbundle of a frame bundle, and in that sense does carry a solder form. For Weatherall, the important difference is that LM is soldered specifically to the tangent bundle, rather than that there is a solder form at all; and he argues that this difference does not support the conclusions drawn by Healey and Anandan.

To make the connection between this debate and our discussion of naturality, we need to introduce one more bit of machinery. Let M be a smooth (real) manifold of dimension n. An r-frame at $p \in M$ is an invertible r-jet $j_0^r f$, $f : \mathbb{R}^n \to M$, $f(0) = p$, where 0 denotes the zero element of $\mathbb{R}^n$.[14] The set of all r-frames on M $L^r M$ is a principal fiber bundle (the r-frame

[14]Recall that if M and N are smooth manifolds, and $f : U \to N$, $g : V \to N$ are smooth maps defined on open neighbourhoods U, V of some $p \in M$, then f and g are said to be r-equivalent at p iff they agree

bundle) over M with projection map π^r_0, $\pi^r_0(j^r_0 f) = f(0)$ and structure group $GL^r(n,\mathbb{R})$, where $GL^r(n,\mathbb{R})$ is the group of invertible r-jets $j^r_0 f$, $f:\mathbb{R}^n \to \mathbb{R}^n$ with group multiplication as composition of r-jets.[15] It is straightforward to check that $GL^1(n,\mathbb{R}) = GL(n,\mathbb{R})$ and L^1M is canonically isomorphic to LM.

How does this help? Let M be a smooth manifold of dimension n and let S be a smooth manifold which carries a (left) $GL^r(n,\mathbb{R})$ action. We can construct the associated bundle $L^rM \times_{GL^r} S \to M$ as above. We now invoke a classic result in the theory of natural bundles (known as the finite order theorem, due originally to Palais and Terng (1977), see, e.g., Kolář et al. (1993)), which states that *any* natural bundle over M can be constructed in this way.

It not be immediately clear that theorem illuminates the connection between solder forms and naturality, since the frame bundles to which natural bundles are associated are generally higher-order, and solder forms are only defined on first-order frame bundles. However, there is a well-known generalisation of the solder form to higher-order frame bundles (introduced by Kobayashi (1961)). Again, let $GL^r(n,\mathbb{R}) \to L^rM \to M$ be the frame bundle of order r over M. There is then a family of projection maps $\pi^r_s : L^rM \to L^sM$, $s \leq r$. We can define a canonical equivariant one-form θ^r on L^rM which assigns, to each $p \in L^rM$, a linear isomorphism between $T_{\pi^r_{r-1}(p)}L^{r-1}M$ and $\mathbb{R}^n \oplus \mathfrak{gl}^{r-1}(n,\mathbb{R})$, where $\mathfrak{gl}^r(n,\mathbb{R})$ is the Lie algebra of $GL^r(n,\mathbb{R})$. To see this, let $p = j^r_0 f \in L^rM$, for some $f:\mathbb{R}^n \to M$. Then f induces a homeomorphism $h : U \subset L^{r-1}\mathbb{R}^n \to L^{r-1}M$ on an open neighbourhood U of the

on all their partial derivatives up to order r at p (in any local coordinate charts containing p, $f(p)$, $g(p)$). An r-jet at p is an equivalence class $[f]_p$ of smooth maps which are r-equivalent at p, and the r-jet at p containing f is denoted $j^r_p f$. For any smooth maps $f : U \subset M \to N$ and $g : V \subset N \to P$, if $f(p) = q$, then $j^r_q g \circ j^r_p f := j^r_p(g \circ f)$. An r-jet $j^r_p f$, $f : U \subset M \to N$ is invertible iff there exists an r-jet $j^r_q g$, for some $g : V \subset N \to M$ satisfying $q = f(p)$, such that $j^r_q g \circ j^r_p f = j^r_p \mathrm{id}_M$ and $j^r_p f \circ j^r_q g = j^r_q \mathrm{id}_N$. To simplify notation in what follows, we will assume that if $j^r_p f$ is an r-jet, then f is always a local smooth map, i.e., $f : M \to N$ denotes $f : U \subset M \to N$ for some (unspecified) open neighbourhood U of p. We note that our definitions of r jets and r frames, here, makes use of $\mathbb{R}^n$, as opposed to a generic vector space V, which might be viewed as more general and/or geometric (and more consistent with the style elsewhere in the paper). But for present purposes, nothing turns on this choice, and we adopt it for simplicity and consistency with the literature.

[15]Our terminology follows e.g. Kolář et al. (1993), though note that L^rM is also sometimes called the bundle of holonomic r-frames (to distinguish it from, e.g. the bundle of (non-holonomic) r-frames obtained by $r-1$ times recursively taking the bundle of first jets of sections of LM).

identity element e^r of $GL^r(n,\mathbb{R})$ via $h(j^{r-1}_q g) = j^{r-1}_q(f \circ g)$ for all $g : \mathbb{R}^n \to \mathbb{R}^n$. In turn, h induces a linear isomorphism $\tilde{p} : T_{e^{r-1}} L^{r-1}\mathbb{R}^n = \mathbb{R}^n \oplus \mathfrak{gl}^{r-1}(n,\mathbb{R}) \to T_{\pi^r_{r-1}(p)} L^{r-1}M$, which is independent of the choice of representative f of the equivalence class, i.e., depends only on p. Then at each $p \in L^r M$, θ^r is defined by the condition

$$\tilde{p}(\theta^r(\xi)) = T\pi^r_{r-1}\xi$$

for all $\xi \in T_p L^r M$.

We now combine these two facts: any natural bundle over M is associated to some (possibly higher-order) frame bundle over M; and any frame bundle (of any order) carries a canonical one-form which reduces to the solder form in the case of first-order frame bundles. So the fact that e.g. the bundles of rank-(r,s) tensor fields considered in general relativity are associated to a bundle that carries a solder form is just a consequence of the fact that they are all natural bundles, in the sense that any bundle that cannot be understood as associated to a bundle which carries a canonical one-form could not possibly be natural. Conversely, the fact that principal bundles of Yang-Mills theory are not equipped with a solder form (and are not associated to any bundle which does) is a reflection of the fact that they are not natural. In this way, the solder form vs. no solder form distinction can be seen as an indirect way of probing the natural vs. non-natural bundles distinction, which, we have argued, is an important point of disanalogy between general relativity and Yang-Mills theory.

Already, this illuminates quite substantially the sense in which the solder form on the frame bundle of general relativity is a real point of disanalogy with Yang-Mills theory. With this in mind, now consider Weatherall's point that the frame bundle over *any* vector bundle comes equipped with a solder form. We have seen that whilst arbitrary principal bundles are not natural, vector bundles associated to any such principal bundle are gauge natural. One also has a collection of gauge natural (higher-order) 'frame bundles' (usually called principal prolongations of the bundle) defined as follows (Kolář et al., 1993, Ch. XII). Let M be a smooth manifold of dimension n, and let $G \to P\pi \to M$ be a principal bundle. An r-frame at $p \in P$ is an invertible r-jet $j^r_{(0,e)}f$, $f : \mathbb{R}^n \times G \to P$, $f(0, e) = p$, where 0 again denotes the zero of $\mathbb{R}^n$ and e is the identity element of G. The set of all r-frames on P, $W^r P$, is a principal bundle (the rth-order principal prolongation of P) over M with projection map π^r_0, $\pi^r_0 f = f(0)$ and structure group $W^r(n, \mathbb{R}, G)$, where $W^r(n, \mathbb{R}, G)$ is the group of invertible r-jets $j^r_{(0,e)}f$, $f : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ with group multiplication as composition of r-jets.[16]

Now let $G \to P\pi \to M$ be a principal bundle and let S be a smooth manifold with a (left) $W^r(n, \mathbb{R}, G)$ action. Again, we can construct the associated bundle $W^r P \times_{W^r} S \to M$. Moreover, one has an exact analogue of the finite order theorem for gauge natural bundles (see Eck (1981) for a first version of the theorem, which was strengthened by Kolář et al. (1993)): any gauge natural bundle over P can be constructed in this way. Not only this, but (higher-order) frame bundles over arbitrary principal bundles also carry a canonical one-form (introduced by Kolář (1971)). Let $W^r(n, \mathbb{R}, G) \to W^r P \to M$ be the principal prolongation of order r of $G \to P \to M$. Again, there is a family of projection maps $\pi^r_s : W^r P \to W^s P$, $s \leq r$. We can define a canonical equivariant one-form θ^r on $W^r P$ which assigns, to each $p \in W^r P$, a linear isomorphism between $T_{\pi^r_{r-1}(p)} W^{r-1} P$ and $\mathbb{R}^n \oplus \mathfrak{w}^{r-1}(n, \mathbb{R}, G)$, where $\mathfrak{w}^r(n, \mathbb{R}, G)$ is the Lie algebra of $W^r(n, \mathbb{R}, G)$. The isomorphism (and corresponding one-form) is constructed exactly as before, replacing L^r with W^r and $\mathbb{R}^n$ with $\mathbb{R}^n \times G$ throughout, and noting that $T_{e^{r-1}} W^{r-1}(\mathbb{R}^n \times G) = \mathbb{R}^n \oplus \mathfrak{w}^{r-1}(n, \mathbb{R}, G)$.

Again, summarising: any gauge natural bundle over $G \to P \to M$ is associated to some (possibly higher-order) principal prolongation of P, and any principal prolongation of P (of any order) carries a canonical one-form. In this way, one can understand Weatherall's point

[16]Note that $W^r(n, \mathbb{R}, G)$ can also be defined as the semidirect product $W^r(n, \mathbb{R}, G) = GL^r(n, \mathbb{R}) \rtimes J^r_n G$, where $J^r_n G$ is the (Lie) group of r-jets $j^r_0 f$, $f : \mathbb{R}^n \to G$ with group multiplication defined via $j^r_0 f \circ j^r_0 g := j^r_0(f \cdot g)$, where $\cdot$ here denotes group composition in G (see Kolář et al., 1993, Ch. XII).

that the solder form is nothing special about the bundle of frames over M (as opposed to the bundle of frames on the total space of some arbitrary principal bundle P) precisely as a way of saying that something like general covariance (as understood here) can be restored for Yang-Mills theory by observing that one can always construct gauge natural bundles over some arbitrary principal bundle. It also provides a deeper explication of Weatherall's point that the real disanalogy between Yang-Mills theory and general relativity is not that there exists a solder form, but that the frame bundle of general relativity is soldered to the tangent bundle, rather than any other bundle—or in other words, that it is natural, rather than gauge natural.

7. Conclusion: General Covariance Revisited

Our primary goal in this paper was to extract, from the long and vexed literature on general covariance, certain precise necessary conditions; and then to argue that those conditions are non-trivial because in fact they are violated by Yang-Mills theory. We then argued that something like general covariance, properly generalized, could be restored for Yang-Mills theory, but only by moving to a more general mathematical setting; and we showed how this perspective could provide fruitful insight into other different debates, concerning the status and significance of principal bundle and solder forms in understanding the relationship between general relativity and Yang-Mills theory.

We have not attempted to give a general or complete account of general covariance. As we have noted, to do so, we would need to say much more about what it means for a *theory* to have the right sort of behavior under diffeomorphisms, whereas our focus has been on properties of the objects that our theories posit. We postpone that more complete discussion for future work. Even so, we will conclude by suggesting that the (somewhat preliminary) arguments and observations offered here offer a valuable perspective on what general covariance and related issues, such as coordinate-independence, are really about.

General covariance is often discussed in terms of coordinate transformations and coordinate independence. But we take the lesson of the forgoing discussion to be that more important than coordinate independence is how the objects we use in constructing physical theories depend on one another—and, in particular, how those objects depend on a spacetime manifold and other, related structures. In fact, coordinate independence turns out not to be fully probative, because as we have seen, objects can have coordinate independent characterizations without having well-defined behavior under coordinate transformation (or, better yet, smooth maps). This is because objects can be coordinate-independent simply because they do not have the right relationship with a manifold to be candidates to depend on (manifold) coordinates. The complex vector bundles we discussed above demonstrate this.

Talk of "structures" and "dependence" is evocative, but not very precise. But the key idea in the present case is explicated using the formalism of natural and gauge-natural bundles. We suggest that it is naturality, or more precisely, still, functoriality, that captures the core of what general covariance is concerned with. The difference between the sorts of objects and relationships we encounter in general relativity – the generally covariant ones – and those in Yang-Mills theory is that the former, but not the latter, involve objects that depend only on a manifolds that represents space and time; whereas the latter depends on further structure on top of that. Functoriality reflects this idea by enforcing the requirement that our constructions can be applied uniformly across manifolds, in a way that is compatible with those maps that we take to preserve manifold structure. Coordinate- or frame-dependent constructions fail to be functorial (over manifolds) because they require further information that is not generally preserved by diffeomorphisms. This is why those constructions fail to be generally covariant. But that is not the only way to fail to be generally covariant, and the natural bundle formalisms shows very clearly why.

Acknowledgments

JOW is grateful to Henrique Gomes and David Malament for conversations on material related to this paper. EM acknowledges the support of Balliol College, Oxford, and the Faculty of Philosophy, University of Oxford. We are grateful to Henrique Gomes for comments on a previous draft.

References

Anandan, J., 1993. Remarks concerning the geometries of gravity and gauge fields, in: Hu, B.L., Ryan, M.P., Vishveshwara, C.V. (Eds.), Directions in General Relativity. Cambridge University Press, New York, pp. 10–20.

Bleecker, D., 1981. Gauge Theory and Variational Principles. Addison-Wesley, Reading, MA. Reprinted by Dover Publications in 2005.

Earman, J., 2006. Two challenges to the requirement of substantive general covariance. Synthese 148, 443–468.

Eck, D.J., 1981. Gauge-natural bundles and generalised gauge theories. Memoirs of the American Mathematical Society 33.

Epstein, D.B.A., Thurston, W.P., 1979. Transformation groups and natural bundles. Proceedings of the London Mathematical Society 38, 219–236.

Fatibene, L., Francaviglia, M., 2003. Natural and Gauge Natural Formalism for Classical Field Theories. Kluwer Academic Publishers, Boston, MA.

Freidel, L., Teh, N.J., 2022. Substantive general covariance and the einstein–klein dispute: A noetherian approach, in: Read, J., Teh, N.J. (Eds.), The Philosophy and Physics of Noether's Theorems. Cambridge University Press, Cambridge, UK, pp. 274–295.

Geroch, R.P., Horowitz, G.T., 1979. Global structure of spacetimes, in: Hawking, S.W., Israel, W. (Eds.), General Relativity : An Einstein Centenary Survey. Cambridge University Press, Cambridge, pp. 212–293.

Gomes, H., 2024. Gauge theory is about the geometry of internal spaces. ArXiv:2404.10461 [hep-th].

Healey, R., 2007. Gauging What's Real. Oxford University Press, Oxford, UK.

Kobayashi, S., 1961. Canonical forms on frame bundles of higher order contact, in: Allendoerfer, C.B. (Ed.), Differential Geometry, Providence, Rhode Island. pp. 186–193.

Kolář, I., 1971. Canonical forms on the prologations of principal fibre bundles. Rev. Roumaine Math. Pures Appl. 16, 1091–1106.

Kolář, I., Michor, P.W., Slovák, J., 1993. Natural Operations in Differential Geometry. Springer-Verlag, New York.

Kretschmann, E., 1917. über den physikalischen sinn der relativitätspostulate. a. einstein'ss neue und seine ursprüngliche relativitätstheorie. Annalen der Physik 53, 575–614.

Misner, C.W., Thorne, K.S., Wheeler, J.A., 1973. Gravitation. W. H. Freeman & Co., New York, NY.

Nijenhuis, A., 1952. Theory of geometric objects. Ph.D. thesis. University of Amsterdam.

Norton, J.D., 1993. General covariance and the foundations of general relativity: Eight decades of dispute. Reports of Progress in Physics 56, 791–861.

O'Neill, B., 1983. Semi-Riemannian geometry, With applications to relativity. Academic Press, New York.

Palais, R.S., 1981. The Geometrization of Physics. Institute of Mathematics, National Tsing Hua University, Hsinchu, Taiwan. Available at http://vmm.math.uci.edu/.

Palais, R.S., Terng, C.L., 1977. Natural bundles have finite order. Topology 16, 271–277.

Pooley, O., 2009. Substantive general covariance: Another decade of dispute, in: Suárez, M., Dorato, M., Rédei, M. (Eds.), EPSA Philosophical Issues in the Sciences: Launch of the European Philosophy of Science Association. Springer, pp. 197–209.

Salvioli, S.E., 1972. On the theory of geometric objects. Journal of Differential Geometry 7, 257–278.

Schouten, J.A., Haantjes, J., 1936. On the theory of geometric objects. Proceedings of the London Mathematical Society 42, 356–376.

Terng, C.L., 1978. Natural vector bundles and natural differential operators. American Journal of Mathematics 100, 775–828.

Trautman, A., 1980. Fiber bundles, gauge fields, and gravitation, in: Held, A. (Ed.), General Relativity and Gravitation. Plenum Press, New York, pp. 287–308.

Veblen, O., Whitehead, J.H.C., 1932. The foundations of differential geometry. Cambridge University Press, Cambridge.

Wald, R.M., 1984. General Relativity. University of Chicago Press, Chicago, IL.

Weatherall, J.O., 2016. Fiber bundles, yang-mills theory, and general relativity. Synthese DOI:10.1007/s11229-015-0849-3.