Phenomenology and independence

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May 13, 2024

In this brief note I will try to develop the following thesis: Gödel's program includes a rich and exciting task for the philosopher that has been overlooked by the majority of the philosophers of set theory (let alone set theorists). *Gödel's program*[1] intends, in a nutshell, to *solve* Cantor's Continuum Hypothesis (hereafter, CH) as legitimate *problem* by means of the addition of new axioms to ZFC that satisfy some criteria of *naturalness* and that, moreover, allow to derive either CH or its negation. Hence, the view encapsulated by such program clashes violently with other attitudes towards the status of CH, like those defending that CH is a problem but is solved by the independence phenomenon itself[2], those that argue that CH is a vague statement and therefore is ill-posed as a problem[3] and, finally, those that regard the axiom-adding proposals as incapable of settling the question[4].

Gödel's idea of *naturalness* as a criterion for adding new axioms already appears in [10][5]. Nevertheless, the *locus classicus* of Gödel's program is found in [13]. For our exegetical purposes it is interesting to note that, between the publication of [13] and [15], it is estimated that Gödel wrote [14], where his thoughts on phenomenology were more or less systematically presented and where, as we claim, *Gödel's phenomenological program* is to be found. Following the interpretative tools provided in [21], we may distinguish two well-differentiated periods of Gödel's thought, the *pre-Husserlian* (alternatively, 'Leibnizian') and the *Husserlian* one, separated by the so-called 'phenomenological turn'[6]. A stronger claim would consist in establishing an analogy between this division and the brands of realism that Gödel can be read as endorsing, the *ontological* and the *epistemological* one, respectively[7].

Let us begin with the form of Gödel's program presented in [13]. Here, the rather subtle notion of naturalness is refined into two halves, namely, *extrinsic* and *intrinsic* criteria, that have guided the contemporary debates in the philosophy of set theory[8]. Roughly, the first ones have to do with the purely practical features that candidate axioms may bear while the others are intended to provide something close to a set of *semantic* grounds for the acceptance of such candidates. Now, in turn, the possibility of solution of CH is defended as follows:

> [...] on the basis of the point of view here adopted, a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of π) would by no means solve the problem. For if the meanings of the primitive terms of set theory are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. [13]

The aforementioned 'basic point of view' can be summarized along the following lines: (i) undecidability can only constitute the end of the story when no meaning has been given to the primitive ideas of the corresponding formal system, (ii)

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[1] As [31] calls it.

[2] Like Cohen himself.

[3] Like [5].

[4] Like [17]. I have omitted others that simply regard CH as an uninteresting question or, rather, as a question which is less interesting than others: see [30].

[5] There, one reads: 'The proposition A [the constructibility axiom $V = L$] added as a new axiom seems to give a *natural* [emphasis is mine] completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way'.

[6] I am leaving aside many relevant details on Gödel's thought and its contact with the phenomenological trend. In addition to [21] I redirect the reader to [1] and [32] for an exhaustive study of these topics.

[7] This terminology is due to [21].

[8] See, e.g., [7].

sets may be (somehow naively but without any danger) understood *qua* pluralities and (iii) an access to sets is secured through mathematical intuition, which works in a similar way to that of perception in the case of physics[9]. I wish to defend that this position can be clearly labeled as ontological realism. Maddy and others may have found in this conception some early form of naturalism[10] but, at least for me, Gödel is making a quite Leibnizian argument in spirit: (1) CH refers to a state of affairs in a determined reality, (2) such realm requires *consistency*, hence (C) CH is either true or false. Of course, enabling CH as a legitimate problem would remain as an impotent unless its very same possibility of solution is also enabled. As Gödel puts it, '[s]uch a belief is by no means chimerical, since it is possible to point out ways in which the decision of a question, which is undecidable from the usual axioms, might nevertheless be obtained'[11]. It is here where Gödel's program, in its usual formulation, is made explicit[12]

In [15], the axiom-adding program is once again vindicated. Even if the Husserlian terminology is ubiquitous, one should not be misled by how some notions as 'intuition' and 'the given' appear there: Gödel defends once again his specific form of ontological realism and *only later* provides the weaker restatement that may be labeled as epistemological:

> However, the question of the objective existence of the objects of mathematical intuition [...] is not decisive for the problem under discussion here. The mere *psychological fact* [emphasis is mine] of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. What, however, perhaps more than anything else, justifies the acceptance of this criterion of truth in set theory is the fact that continued appeals to mathematical intuition are necessary [...] for obtaining unambiguous answers to the questions of transfinite set theory [...]. This follows from the fact that for every axiomatic system there are infinitely many undecidable propositions of this type. [15]

Here, the possibility of solution of CH and, correspondingly, of its status as legitimate question, is secured by appealing to a 'psychological fact'[13]. What could these cryptic remarks mean? This question leads us to the precise formulation of Gödel's phenomenological program in [14] that we have mentioned before.

Gödel's treatment of phenomenology in [14] is preceded by a (somewhat Hegelian) discussion on the independence phenomenon and some usual attitudes towards it[14]. Against Hilbert's program, which places the emphasis on syntactical and mechanical procedures, he claims that 'the certainty of mathematics is to be secured [...] by cultivating (deepening) knowledge of the abstract concepts themselves which lead to the setting up of these mechanical systems, and further by

[9] The details of this position, developed during the Leibnizian phase, can be found in [11].

[10] Maddy [26] focuses on the perception of sets and intends to give a naturalized account of this exotic faculty of perception. It is true that this reading is partially justified by, e.g., the following passage from in [13]: 'When theorems about all sets (or the existence of sets in general) are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of sets of integers, etc. (respectively, that there either exist sets of integers, or sets of sets of integers, or ... etc., which have the asserted property). This concept of set, however [...] has never led to any antinomy whatsoever: that is, the perfectly *"naive"* and *uncritical* [emphasis is mine] working with this concept of set has so far proved completely self-consistent'. Also, Gödel's attack against intuitionism arguing that it 'is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be completely given in mathematical intuition' can be read as the trademark slogan of Maddy's naturalism: mathematics go first.

[11] See [13], §3.

[12] On the one hand, the intrinsic criteria: '[...] the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". [...] there may exist, [...] other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts'. Gödel then puts the example of an axiom that satisfies the maximality criterion presented in [27]. On the other, the extrinsic ones: '[...] even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its [...] fruitfulness in consequences, in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. [...] There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems [...] that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory'. Much has been said regarding this notion of pragmatic 'success'. See, for example, [3].

[13] Apart from the treatment in [26], much has been discussed around Gödel's notion of intuition. See for example [9], where some comparisons with Poincaré appear. As it can be seen below, Poincaré's idea of the working unconscious in mathematics somehow resonates in Gödel's point of view.

[14] One could include Cohen, Feferman and Hamkins along those that wish to surrender to the 'nihilistic consequences [...] in accord with the spirit of time' [14].

seeking, according to the same procedures, to gain insights into the solvability, and the actual methods for the solution, of all meaningful mathematical problems'[15] It is worth noting how this coincides with the previous quote and the nature of intrinsic criteria, as formulated in [13][16].

At first, Gödel provides a negative characterization of the method required for deepening such knowledge, namely, that it should not be pursued 'by trying to give explicit definitions for concepts and proofs for axioms, since for that one obviously needs other undefinable abstract concepts and axioms holding for them'. Rather, the desired method should imply in 'a clarification of meaning that does not consist in giving definitions'[17]. Therefore, Gödel's idea of this systematic method cannot be purely formal (i.e. merely dealing with explicit axioms, definitions and theorems), for the formal elements such as the candidates for natural new axioms of ZFC are in any case the *outcome* of such method, not one of its means[18]. It is here where Gödel ties the fulfilment of the purely mathematical program to the phenomenological one:

> Now in fact, there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc. [...] it is or in any case should be a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. [14]

This is the core of what we have come to call 'Gödel's phenomenological program': new axioms will be implied by the intentional analyses of the primitive ideas of ZFC[19], and these will be reasonable candidates for finding further expansions of such formal system[20]. On the other hand, the mathematical side of Gödel's program remains as consisting in trying to decide CH by means of these axioms[21]. But one thing must be clear: if we wish to preserve the notion of intrinsic criteria, in the sense intended by Gödel, we must necessarily deal with this phenomenological program and its implicit difficulties.

Up until this point we have been following the usual interpretative sources, even when we have awkwardly tried to divide Gödel's program in two halves. The point that we want to make now is that, as long as we want to elude any sign of mysticism, we must commit to Gödel's phenomenological program in an *open* way. Since here, as is recognized by Gödel, we are surrounded by philosophical problems without the aid of any formal system whatsoever, it is tempting to just take an appealing candidate for a new axiom and merely establish its naturality, without making explicit the way in which such result has been obtained[22]. Here we stand in need of a radical and serious analysis, as Husserl originally demanded, that is to be made clear enough if we deem rigour as an important value in our argumentation[23].

As far as I know, intrinsic criteria have only received a serious amount of attention by Hauser. As a whole, [21] is devoted

[15]See [14].

[16]This also coincides with the later [16]: 'there *do* exist unexplored series of axioms which are analytic in the sense that they only explicate the content of the concepts occurring in them, e.g., the axioms of infinity in set theory, which assert the existence of sets of greater and greater cardinality or of higher and higher transfinite types and which only explicate the content of the general concept of set. These principles show that ever more (and ever more complicated) axioms appear during the development of mathematics. For, in order only to understand the axioms of infinity, one must first have developed set theory to a considerable extent'. The notion of 'analytic' in Gödel has also received renovated attention. See [11] for Gödel's treatment of this notion.

[17]Again, see [14].

[18]Of course, the relationship between both may go in a zigzagging fashion: see the already mentioned quote of [16]. The development in the formal framework may affect the phenomenological enterprise and reciprocally.

[19]The relevance of the notion of 'analysis' for Gödel is notorious, as we have noted before. Even during his Leibnizian period, in [13] §3, one can read, regarding the mathematical difficulties that appeared in the search for a proof of CH, that 'there are also deeper reasons behind it and that a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as "set", "one-to-one correspondence", etc.) and of the axioms underlying their use'. Feferman [5] actually claims that the vagueness of CH is found in the vagueness of these terms (occurring in its formulation).

[20]See [14]: '[...] it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident. [...] every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate'.

[21]This is the characterization also given in [21].

[22]In [2], for instance, one can find a set of concrete criteria for studying the naturalness of some axioms (e.g. that they take the form of reflection principles) but no justification (apart from the reference to Wao's and Gödel's conversations) is given.

[23]Gödel did not give, in fact, any examples on how he intended his method to work, as it is pointed out in [21]. This is why the treatment mentioned in the previous footnote is, philosophically, insufficient.

to build an exhaustive and coherent picture that subsumes both the Gödelian traces of Platonism with the Husserlian insights in which Gödel saw a promising tool of analysis. I myself stand pessimistic regarding this project, not as a matter of principle[24] but as for its philosophical implications is concerned. While in [20] one can find a detailed exposition on how some Husserlian ideas may be employed in the analysis of the degree of evidence of the axiom of choice, the corresponding (expected, I would like to say) investigation in the case of CH is lacking in [19] on the pretext that Gödel's brand of Platonism is apparently enough to settle the question without further research. Additionally, in [18], one may detect a restriction of the phenomenological program that runs against the adjective 'open' that we used before, by incorporating additional requirements for the candidates for axioms[25].

On the other hand, it is unclear exactly where Gödel's phenomenological program stands with respect to Maddy's philosophy of set theory, but I suspect that it does somewhere in the middle of her First and Second Philosophers. Through the lens of her naturalism, Maddy finds more adequate to investigate extrinsic criteria instead of the intrinsic ones, for '[e]xtrinsic reasons are easy to recognize; they involve the consequences of a given axiom candidate, its fruits, if you will', so 'we need to assess the prospects of finding a new axiom that is well-suited to the [purely mathematical, methodological] goals of set theory and also settles CH'[26]. Now, of course, Maddy can also be read as defending a 'foundational role' of set theory along its methodological goals[27]. This provides a certain tension which arises from the fact that set theory is, under Maddy's conception, simultaneously *foundation* and *branch*. This, *prima facie*, seems an innocent statement, but further explanations should be provided by Maddy's naturalist in order to explain this duality for, provided the soundness of this objection, one should carefully distinguish between the mathematical and the philosophical features of set theory, and this certainly does not look to be an easy task[28]. Therefore, it may be the case that extrinsic criteria depend on a elucidation of some key aspects of mathematical practice and that, too, they deserve the same amount attention, although one of a different nature, as the intrinsic ones[29].

I find myself more comfortable when interpreting some of Husserl's key points, as Rota does, through the comparative reading of other philosophers and, more prominently, Wittgenstein[30]. In this way, we can clarify what Gödel's phenomenological program demands as some kind of 'conceptual analysis', as it is normally put nowadays, of the primitive

[24] For instance, as it is argued in [22], it is quite possible that '[w]hat Gödel called intuition is much more like Husserl's *Wesensschau* than Husserl's notion of intuition [...] How such intuition operates as to give us any real insights remains a mystery. [...] Husserl's notion of intuition allows manipulation and experimentation in thought. Gödel's does not, wherefore it is only a pale shadow of Husserl's ideas'. Similar difficulties regarding the metaphysical access to the abstract realm (in contraposition to a merely intentional relation with mathematical items) may also arise.

[25] Here I am of course referring to Freiling's famous 'philosophical proof' of CH. Regarding this, Hauser first argues that '[t]he ZFC axioms are an adequate formal framework of mathematics in the sense that (i) every mathematical statement is expressible in the language of set theory with *set* and *membership* as the only primitives and (ii) every theorem of classical mathematics is formally derivable from ZFC'. What status do these assumptions bear? Are they empirical? It could be *a priori* possible that, through an intentional analysis of the primitive notions of set theory, others could be made necessary as well as the laws governing their relations with the already known ones. Even if Hauser later recognizes that alternative foundations may be successful, any technical difficulty that they may have at the moment is decisive, in virtue of suppositions (i) and (ii), to discard them as equally satisfactory as ZFC. But, once again, any intentional analysis is lacking and the argumentation is complemented with hypothesis whose justification is unclear, once the standards of rigour of phenomenology are conceded, as Hauser does in [21] or [20].

[26] See [7]. I will not enter here into the debate on how relevant Gödel considered the extrinsic criteria in comparison to intrinsic ones. In [15] one can see a reluctant defense of those criteria (refining the exposition of [13]), but it could be perfectly defended that this is only due to the results known in Gödel's time. See [2] for an illustration on how intrinsic and extrinsic criteria may work in a complementary way.

[27] See [27]. It is interesting to note that an early form of the criterion of MAXIMIZE is found in [13], footnote 23: 'only a maximum property would seem to harmonize with the concept of set explained in footnote 14'. In footnote 14, one reads: 'The operation "set of x's" (where the variable "x" ranges over some given kind of objects) [...] can only be paraphrased by other expressions involving again the concept of set, such as: "multitude of x's", "combination of any number of x's", "part of the totality of x's", where a "multitude" ("combination", "part") is conceived of as something which exists in itself no matter whether we can define it in a finite number of words (so that random sets are not excluded)'. This, going back to [11], seems a continuation of the conception of sets *qua* pluralities. But how can one claim that this conception of set is purely methodological? One could argue that it is a *philosophical pretext* for a methodologically fruitful conception, which is unclear that goes directly against Maddy's defence of *mathematics go first*.

[28] Woodin, for example, concedes that philosophical *motivations* cannot be fully eluded in set theory. Tiles has argued that only dealing with extrinsic criteria is circular, since the very same idea of 'foundation' implies some degree of *evidence* (see [3] for an assessment of these kind of arguments). I believe that Gödel would stand very close to this position. Hence, if the working mathematician is an agnostic with regarding foundations (as is usually regarded), the set theorist without philosophical motivations would be closer to a bad theist.

[29] Again, see [3] for an exhaustive discussion of extrinsic criteria.

[30] See [29]. Rota motivates the Husserl's *Fundierung* as a logical notion through the *function* and *facticity* duality that one may find in Wittgensteinian language games (i.e. an arbitrary item and its role in the game).

ideas of set theory[31]. More precisely, I believe that the following remarks of Wittgenstein shed some light on the nature of the analysis that Gödel pursues[32]:

> The investigation of the rules of the use of our language, the recognition of these rules, and their clearly surveyable representation amounts to, i.e. accomplishes the same thing as, what one often wants to achieve in constructing a phenomenological language.
>
> Each time we recognize that such and such a mode of representation can be replaced by another one, we take a step toward that goal.
>
> [...] What we're missing isn't a more precise scrutiny [...] nor the discovery of a process *behind* the one that is observed superficially (that would be the investigation of a physical or psychological phenomenon), but clarity in the grammar of the description of the old phenomenon. Because if we looked more closely we would simply see something *else*, and would have made no advance on our problem. *This* experience, and not another, is what needs to be described[33]. [33]

Therefore, what we seek here through the process of analysis is a grammatical elucidation; every phenomenological consideration here *is* or, at least, should be grammatical in essence. Note that, through the ideas of [29], one can distinguish the purely phenomenological requirements present in Gödel's phenomenological program from the elements that are closer to his preferred from of Platonism, independently from the Husserlian terms he may use to describe and defend it[34].

But not only this, one can find fragments in Husserl's works that may provide additional insights to the characterization of Gödel's program, particularly in [23], although this is an interpretative task that will not be carried out here[35].

As a fresh start, one could propose to analyze, for example, how the requirement that the candidates for new axioms *must* take the form of reflection principles has to do with the Cantorian conception of the Absolute in set theory[36]. Following Gödel, a careful exposition should make clear how, in our usual way of dealing with sets, such notion would present itself in consciousness. But I deem it more relevant to study two intertwined key notions for the whole question on CH, namely, those of *problem* and *solution*. Let me provide a first grammatical analysis of such notions, that should be followed, in any case, by further and better efforts.

My main claim is that, regarding the status of CH, 'problem' and, respectively, 'solution' receive radically different roles and that our task is to carefully distinguish them[37], even before trying to unveil the analytic unfolding of the primitive notions of set theory. Let me concede, following Rota's spirit, that CH has somewhat of a *flavour* for set theorists[38]. Normally, a 'good' proof for a theorem lets its flavour to be perceived with clarity, while a technical lemma may difficult such acquaintance and will likely be regarded as unnatural or ugly. Also, normally, we seek to corroborate our *evidence* of

[31] I am aware that Koellner has also talked about conceptual analysis in [25], though in a different way.

[32] I concede that the identification between Wittgenstein's, Husserl's and Gödel's conception of phenomenology is not exempt of difficulties. For me, Rota's analysis of *Fundierung* allows to make a clear bridge between the first two. Regarding the last one, see [32].

[33] It is instructive to compare some remarks by Gödel, specially in [15], with Wittgenstein's conception of phenomenology as grammar. Compare, e.g., the famous Gödelian preconception of sets 'forcing upon us' with §96 in [33]: 'Does something *force* me into the interpretation that the tree that I see through my window is bigger than the window? That depends on how I use the words "bigger" and "smaller" [italics are mine]'.

[34] For example, compare Gödel's Platonism, together with his notion of perception and sets *qua* pluralities with [33]§94: »If a true circle within one's visual field is in some sense inconceivable, then the sentence "There never is a true circle within a visual field" must be the same kind of sentence as "There is never a high C within a visual field"«.

[35] In [23] Husserl investigates the completeness of axiom systems and the *transition through the ideal*, where the case of adding axioms to a formal system is considered. For example, in <K I 26/43> one can read: 'the determinate equivocality can once again be eliminated by the joint force of the axioms, so that we are enabled univocally to determine new and ever new elements from given elements (and here that can only mean elements assumed as given and, as it were, named by means of proper names) on the basis of the axioms, and consequently to regard them likewise as given. An axiom system which, in this manner, delimits a general sphere of univocally determinate existence, and thus contains forms - whether simple or more complicated - of univocal determination of objects, out of which forms, upon arbitrary substitution of given values for the indeterminate signs for givens in general, new and ever new givens of a derived kind result: Of such an axiom system we say that it has a domain'. Determining the exact meaning of these lines is the aforementioned (difficult) task. Husserl's *Doppelvortrag* has been interpreted as dealing with the modern notions of completeness and categoricity. I claim that it may also help in particular contexts of the axiom-adding issues.

[36] See [25] for a detailed account of reflection principles. See [24] for details regarding Cantor's conception of the Absolute.

[37] For, as in [29], '[c]onfusing function with facticity in a *Fundierung* relation is a case of *reduction*. Reduction is the most common and devastating error of reasoning in our time. Facticity is the essential support, but it cannot upstage the function it *founds*'.

[38] I mean, even when it independence from ZFC is a hard fact. This follows from Rota's conception that *there is much more* to mathematics than what is conveyed through axiomatic formal systems. But Rota does not fall into realism or naturalism of any kind.

an statement (which is closely connected to its flavour) through a chain of definitions and proofs *in* a determined formal system. Here, a problem is simply any statement of which we have some degree of flavour but no formal proof, and a solution will consist in a proof of such statement. Another key feature is that the solution of the problem will open a range of possibilities, it will make the concepts involved (and, ultimately, the primitive ideas of the corresponding formal system) richer and deeper through renovated interactions[39].

But the situation with CH changes slightly, for it seems that, in this case, the solution is confused with the *possibility* of solution[40]. A solution of CH will not open possibilities *upwards* (i.e. in the direction of new theorems and definitions) but *downwards* (i.e. in the direction of new axioms). The problem here is not a statement that stands in need of proof *in* a formal system but a statement whose flavour exceeds the formal system itself and asks for new rules in the game[41]. In particular, there is nothing left but faith in asserting that, for establishing new laws governing the primitive ideas, it is impossible for new ones to come along as natural[42] or, alternatively, a rigorous exam that would provide a full justification for such belief.

A further difficulty here is a predominant specific refinement of the term 'solution' that can be found, e.g. in [6], i.e. a *solution* for $(H, \in)$ is ZFC $+ \{A_\alpha \mid \alpha \in \Lambda\}$, where $\{A_\alpha \mid \alpha \in \Lambda\}$ is a family of axioms compatible with the large cardinal axioms and which make the properties of $(H, \in)$ forcing-invariant. This shows how deep the usual shift in the meaning of 'solution' is. On the other hand, it remains clear that the negation of the problem itself is not the same thing as a bad candidate for its solution. That is, adding CH or its negation as new axioms *must* have a different degree of plausibility than adding $V = L$ or PD, even when these do not count today as *legitimate* solutions for CH[43].

Finally, let me consider a more specific case. As it is widely known, one of the most serious attempts of settling CH has been provided by Woodin's work[44]. In few words, Woodin's work may divided in two parts: (i) Providing grounds for $\neg$CH through the Ω-conjecture and (ii) Providing grounds for CH through structural properties of the so-called Ultimate-L. Of course, this approach seems refreshing in comparison to Gödel's program since Woodin is able to connect purely mathematical, methodological or technical aspects of other theories to the decision of CH. This is the spirit of *mathematical traction*[45]. But one should recognize here a form of reduction and an indication that the deep and exciting project of finding new axioms through a phenomenological analysis is left aside once again. Even if the truth of the Ω-conjecture would settle the truth value of CH, the Gödelian quest for the evidence of CH would simply translate into another for the

[39] Here I am following Rota's phenomenology of mathematical truth, beauty and proof; see [29].

[40] Consider, e.g. [34]: 'So, is the Continuum Hypothesis solvable? Perhaps I am not completely confident the "solution" I have sketched is the solution, but it is for me convincing evidence that there *is* a solution'.

[41] Perhaps this phenomenon is only of relevance for the set theorist because they are *too* interested in axiomatics, as Rota argues. Of course, the other component would be the self-imposed 'foundational' requirements, which renders the quest of axioms as a crucial investigation. More crucial seems, rather, to acknowledge how in the problem of deciding CH the usual language game of theorems and proofs shifts in a profound manner.

[42] After all, if we follow the Wittgensteinian insight that axioms force upon ourselves *jointly*, it is perfectly imaginable that a set of new axioms together with a new primitive idea (in addition to the already known ones) force upon ourselves in the very same fashion (moreover, compare with Husserl's fragment quoted above). What is strange is that Freiling's mental experiment, which is in fact an eidetic variation on the concept of set, has been discarded by Hauser as a matter of principle and not in virtue of a parallel, yet incompatible, eidetic variation. What is also relevant here is how the decidability of CH in the second-order logic formulation of ZFC has been philosophically rejected because second-order logic relies on the very concept of set. Here the issue is: does not this also apply to first-order logic? It seems that here, as with the attitude towards Freiling's argument, we are just facing tradition, as R. Asensi has pointed out in conversation (see [8] for an account of the emergence of the first-order logic tradition). Then, we can conclude, the reluctance of Hauser is a matter of mere inclination or philosophical inertia.

[43] Another interesting question regarding the status of CH, that I will not treat, is if here we are dealing with a similar situation as, for instance, with Church-Turing Thesis. Gödel considered relevant the fact that 'with this concept [of Turing machine] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion [the one of 'algorithm'], i.e., one not depending on the formalism chosen' [12]. As [21] acknowledges, Gödel seems to believe that if we had nothing 'sharp' to begin with (i.e. an informal notion of computability), how we arrived at such absolute characterization would remain a complete mystery. Despite this, it has been argued that, since Church-Turing Thesis connects two concepts of different nature, its status is mixed or, at least, contains some empirical content (see, e.g., Kalmar's infamous article). The consequences of a similar analysis for CH are a matter of future research but it is possible that the status of CH may be more philosophically convoluted that it may seem superficially, and that we are seeking for an empirical justification instead of a *proof*.

[44] See [28] and [4] for more details.

[45] In [28] it is even proposed that '[t]he philosophical pluralism/non-pluralism debate [...] could be settled on the basis of answers to precise mathematical questions. These are questions about structures that we already know and which we can define, questions about the $\mathrm{HOD}^{L(A,\mathbb{R})}$. These questions about $\mathrm{HOD}^{L(A,\mathbb{R})}$ are not independent from ZFC; independence would cease to be an issue'.

one of the setting in which such conjecture is formulated[46]. Remember that, after all, Gödel's intended analysis should not consist in merely formal or mathematical aspects. Even when granting that Woodin is doing a *deeply motivated* piece of mathematics, such work should not be considered as a phenomenological analysis of the notions of set theory. A further reason for this is the interesting fact that Woodin justifies the solvability of CH in the following way:

> Of course, for the dedicated skeptic there is always the "widget possibility". This is the future where it is discovered that instead of sets we should be studying widgets. Further, it is realized that the axioms for widgets are obvious and, moreover, that these axioms resolve the Continuum Hypothesis (and everything else). For the eternal skeptic, these widgets are the integers (and the Continuum Hypothesis is resolved as being meaningless). [34]

That is, he does so by negating any possible *eidetic variation* of the primitive notions and, therefore, against the *openness* that we argue that Gödel's program, in essence, has[47].

The conclusion is that, when we take Gödel's phenomenological program in a liberal sense, it seem possible for new axioms to take exotic forms, even introducing new primitive ideas for set theory. In other words, there is *a priori* no decisive way of granting that intrinsic criteria will sanction, say, Woodin's proof and reject Freiling's: new assumptions need to be made, and the nature of these is the very same problem at stake[48]. This strikes us as something strange, yet one also realizes that CH, for example, deals with the reals numbers, which may be argued to be not completely set-theoretic in essence. This 'set-theoretic essence' is the main philosophical issue behind the notion of naturalness; the branch-duality problem makes it certainly unbearable. The confident naturalist will regard this assumption-making as a simple fact of mathematical activity, its (a)philosophical and true justification: set theorists do not take Freiling's proof as *the* solution for CH. But if we only take empirical facts belonging to set-theoretical practice as relevant then it seems very likely that, through new (purely mathematical) practice, CH will take a wide variety of new flavours, and that the very notion of 'natural' (in the strong sense favored by Gödel) will be undermined, i.e. no longer descriptive, for different areas of the same theory will consider incompatible extensions to be methodologically natural[49]. Then, Gödel's program, as a whole, will be compromised and the conception of CH as a legitimate problem in the traditional sense will shift. The only way to escape this possibility is through a rigorous and continuous analysis, be it negative or positive, of the use that we make of the basic concepts of set theory[50].

[46] What goes on when we interpret a mathematical result in such-and-such way? One tends to regard such-and-such (mathematical) phenomenon as reasonable from such-and-such (philosophical) perspective. But it remains possible that another phenomenon may be given and, despite this, it will be interpreted alternatively. In other words: we are relying in a philosophical interpretation of some (particular) mathematical phenomena, and this interpretation guides later our attitude towards purely philosophical questions. The aforementioned branch/foundation duality thus derives in the mathematical/philosophical one with even more force. If we follow [29], we may say that philosophical and mathematical truth are different in essence and, moreover, that identifying both is a clear-cut example of reduction by elimination of *Fundierung*.

[47] Compare with Hauser's quote from [18] that we mentioned before.

[48] Another issue that we have not fully treated here is the existence of a third kind of criteria, namely, the *empirical* ones: set theory may have deep implications in the mathematical frameworks employed by, say, physicists. Even if these criteria may be regarded as a subset of the extrinsic ones (since, after all, we are dealing with mathematical consequences of set theory), they also bear some special content (provided their role in empirical theories). Hence, one could defend that CH is an open problem by recurring to this content, since it could be *a priori* possible that the work of physicists could provide evidence for it (or, alternatively, its negation). This topic is deeply interesting but exceeds our present scope, for it involves an analysis of the methodological behaviour of both pure and applied mathematics (in particular, of set theory and its empirical consequences in the sense above). This approach is due to C. Straffelini (personal communication).

[49] This is a restatement of what is argued in [17]. The argument can be read as stating how the *purely* methodological features of set theory cannot lead to a resolution of CH in the traditional sense. We claim, additionally, that this traditional conception belongs to a different realm, see below.

[50] The problematic can be rephrased as follows: in CH there is a severe change in the standards of the language game that is being played. That is, moves regarding this game will no longer be *interior* but *exterior*. If these two notions correspond to 'mathematical' and 'philosophical', respectively, is yet to be explained. What is confusing is how the terms 'problem', 'solution', etc. are used without distinguishing between the two (we claim: well-distinguished) contexts of use. This leads to the mixed situation regarding Freiling's proof and traditional set theory.

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