What is Fundamental in Fundamental Physics?

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Abstract

Metaphysicians as well as philosophers of science often turn to particle physics for a description of the most fundamental entities in our universe. The common assumption is that physics readily provides a clear account of both what those fundamental building blocks are and how they come together to form more complicated objects, and, conversely, how compound objects can be seen as being composed of those fundamental entities. I argue that this picture contains a major difficulty because quantum theories allow for more than one metaphysically meaningful procedure to decompose a system into parts, fundamental or otherwise. I will identify and interpret two such procedures, mathematically given by the direct sum and the tensor product, and show that they lead to different results for what the parts of a quantum system are. This shows that there are conventional choices involved in finding the fundamental parts of an object which have not yet been widely recognised by either metaphysicians or philosophers of science. I take my findings to provide a sense in which, as a result, particle physics on its own is not enough to determine the fundamental ontology of the world.

1 Introduction

Metaphysicians, as well as philosophers of science, often turn to modern particle physics for an account of the most fundamental entities in our universe. Tahko (2018, p. 1) observes that many think "that particle physics aims to describe the fundamental level of reality, which contains the basic building blocks of nature." Oppenheim and Putnam (1958, p. 9) put elementary particles at the very bottom of their mereological hierarchy of material entities. Inman (2017, p. 75) claims that "[t]hough the strong reductive letter of Oppenheim and Putnam's account of the mereological ordering of reality has been largely abandoned [...], many contemporary philosophers are apt to endorse something similar in spirit" and points to Kim (1998, p. 15) who asserts that "[t]he bottom level is usually thought to consist of elementary particles, or whatever our best physics is going to tell us are the basic bits of matter out of which all material things are composed" and that "[t]he ordering relation that generates the hierarchical structure is the mereological (part-whole) relation."

The common assumption is that particle physics readily provides an account of what those fundamental building blocks are, how they come together to form more complicated objects, and, conversely, how compound objects can be seen as being composed of those fundamental entities. Even those who examine the mereology of quantum theories in more detail, such as Calosi and Tarozzi (2014), tacitly assume that matters are settled in physics regarding how to decompose a given system into its fundamental constituents in the quantum realm.

I argue that the picture is more complicated: quantum theories allow for more than one metaphysically meaningful procedure to decompose a system into subsystems, fundamental or otherwise. In particular, I will explore two decomposition procedures for quantum systems and show that they lead to different results for a system's constituents, thus giving rise to conflicting conceptions of fundamentality. From this, I will conclude that fundamental physics on its own cannot provide an account of the fundamental level of reality and that more interpretational and metaphysical work needs to be done in order to arrive at such a description.

More concretely, I will argue that there are situations where the formal description of a given system in particle physics can be decomposed according to two very different mathematical procedures: the tensor product decomposition identifies subsystems of the system with clusters of properties that are independent of each other in the sense that a measurement on one of the clusters does not disturb measurements on other ones.[1] On the other hand, the direct sum decomposition describes the compound system as a mixture of subsystems, each of which differs in one of the fundamental properties particle physics predicts quantum systems to have, like electric charge, or colour charge. For example, the simple model used to describe the spin structure of a hydrogen atom can be viewed as either two independent spin-$\frac{1}{2}$ degrees of freedom (the electron and the proton) or a mixture of a spin-0 system and a spin-1 system, where the theory does not predict which of the two possibilities will actually obtain. That is, the behaviour of a hydrogen atom is equally determined by either two independent spin-$\frac{1}{2}$ systems or a mixture of a spin-0 and a spin-1 system. These represent two very different ways of decomposing the system into its fundamental constituents with disagreeing results. In this way, the metaphysics of the systems is underdetermined by the physics. Hence, metaphysicians and philosophers of physics need to specify in more detail which of the conceptions of fundamentality they refer to when claiming that particle physics describes the fundamental level of reality, or we need an account of how two conflicting notions of fundamentality can coexist. I view this as another way in which one cannot "read off" one's metaphysics from physics alone, as French (1998, p. 93) argues.

I will proceed as follows: in Section 2, I will first discuss the metaphysics of fundamentality and extract what appears to lie at the core of various accounts, namely a relational concept of metaphysical priority. Then, I outline the formal description of the basic objects of inquiry of particle physics, quantum systems. The two sections after that then rehearse the two ways such a quantum system could be

[1] There is a rich hierarchy of such independence notions that the tensor product satisfies, of which I will only briefly mention a few below.

decomposed from a general perspective and examine how each can be justified as corresponding to a relation characterizing ontological hierarchy. Section 5 will then look at the framework of group representations, which is heavily employed in particle physics and enables us to see exactly where the two notions of fundamentality will clash. Using this, I will show in Section 6 that the two notions disagree on the fundamental constituents that they ascribe to some systems and conclude that this is in conflict with the expectation that particle physics can settle the question of fundamentality for a naturalistic metaphysics.

2 Fundamentality and Quantum Systems

The term "fundamental" is used in a wide variety of senses in the metaphysics literature, commonly[2] denoting that something is "basic or primitive" (Tahko 2018, p. 1). Most approaches to fundamentality are relational at their core: as Schaffer (2010, p. 36) observes, "[a]nyone who is interested in what is fundamental [...] must understand some notion of priority." That is, fundamentality is connected to a priority relation that holds between the more and less fundamental entities.

This relation is often taken to be that of *grounding* (see e. g. Cameron (2016), Mehta (2017), and Schaffer (2009), or Bianchi and Giannotti (2021) for an account of relational fundamentality based on ontic structural realism) although this is not accepted by everyone (e. g. Wilson 2014). Some think that there can be multiple such relations: Bennett (2017), for example, argues for a plurality of "building relations" that apply in different circumstances. Others (e. g. Fine (2001) and Wilson (2014)) argue that fundamentality is a primitive notion not further analysable, though it is still characterizable in terms of other relations. Whatever the details of the account of fundamentality might be, common to most approaches—and the only necessary assumption for my argument—is that they are based on some relation. For simplicity, I refer to this relation as the *priority relation*—the reader is welcome to substitute their favourite fundamentality relation if they wish.

This relation, in turn, can result from of a notion of *(de-)composition* of systems. If a system is decomposable into subsystems, these subsystems are prior to the compound system and can, thus, be regarded as more fundamental. In the following, I will look at possible decomposition procedures in quantum theories and will argue that they give rise to priority relations that can characterise fundamentality. Hence,

[2]For a survey of notions of fundamentality in metaphysics and philosophy of physics, see Morganti (2020a,b).

from the fact that there are different and often incompatible ways of decomposing a given system into its constituents (fundamental or otherwise), I will conclude that particle physics does not fix one account of fundamentality.

Given a priority relation, there are, then, two different ways considered in the literature to define what is fundamental: On the one hand, one can say that x is fundamental if and only if there is no (other) y that is prior to x—this is sometimes referred to as the *independence* conception of fundamentality (e. g. in Tahko (2018) and Bennett (2017, ch. 5)), but I will instead refer to this as non-decomposability to avoid confusions with the notion of independence used in physics. On the other hand, one can take x to be fundamental just in case it is a member of a set (often called a *minimal basis*) $\mathcal{B}$, which is such that for every other entity $y \notin \mathcal{B}$, there are some $b \in \mathcal{B}$ which are prior to y (and which are the only objects prior to y). How the two definitions are related is a topic of debate,[3] but, again, note that both of them employ a priority relation.

Applied to particle physics and using the terminology of (de-)compositions, the "independence" conception of fundamentality corresponds to the claim that elementary particles are not decomposable into even more fundamental particles in that they don't have structure which could be used to divide them up even further. The notion that elementary particles form a minimal basis of the material world corresponds to the claim that all of matter is composed of these particles, in other words, that they feature as the "building blocks of reality."

Both of these claims are frequently made in the literature; however, in the following, I will argue that one is misled in thinking that particle physics readily provides a conceptual account of them. I will examine two procedures that relate descriptions of quantum systems to compositions of systems that can be thought of as being prior to the compound system—one based on the direct sum, '$\oplus$', and the other one applying the tensor product, '$\otimes$'—and will show that each can be given a relevant metaphysical interpretation. In the context of the mathematical framework of group representations, it will be shown that these notions of fundamentality disagree on the fundamental constituents of some systems.

Before we can continue, however, we need to clarify some technical concepts. The basic theoretical framework which underpins particle physics is quantum theory,[4] which describes the kinematics and dynamics of (quantum) systems. For our

[3]See e. g. Leuenberger (2020), who argues that whether the two definitions agree on the entities they designate as fundamental depends on other metaphysical commitments.

[4]I shall refer to non-relativistic quantum mechanics and quantum field theory both as quantum

purposes, a *quantum system* is described by a state space $\mathcal{H}$, which is a complex, separable Hilbert space, together with an algebra of observables $\mathcal{A}$, standardly[5] chosen to consist of a suitable algebraic completion[6] of a set of self-adjoint operators on $\mathcal{H}$—sometimes this will be the full algebra of bounded operators of $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$. Whereas for the following two sections, the choice of algebra and Hilbert space are somewhat more independent, we will restrict this in Section 5, and consider $\mathcal{H}$ and $\mathcal{A}$ to be given by viewing the system as a representation of the symmetry group of its degrees of freedom. Additionally, one could introduce dynamics to describe the evolution of the system in question over time. I shall not include this in my considerations since if fundamentality cannot be determined even in the static, non-evolving case, the situation will only become more complicated if the dynamical behaviour of the system is considered as well.[7]

Together, these mathematical objects allow one to calculate expectation values of observables in a given state, which are interpreted to represent the mean outcome of the experiments associated with those observables, as well as transition probabilities, which specify the likelihood of the system to transition from one state into another. As a simple example, the Hilbert space $\mathcal{H} = \mathbb{C}^2$, together with an algebra $\mathcal{A}$ generated by the Pauli-spin-matrices, describe a spin-$\frac{1}{2}$ system with spin as its only degree of freedom.[8] This models, for example, the spin of an electron or proton, or an abstract qubit.

Many metaphysical conceptions of what the fundamental entities of the world might be are compatible with this characterization of a "system": object ontologies, for example, can take them to be descriptions of material entities. Alternatively, there are constructions of Hilbert spaces that allow for interpretations that take facts or propositions as the fundamental constituents of reality,[9] and similarly one

theories and will be more specific if I need to pick either one of them.

[5]For generalizations, see Roberts (2018).

[6]Different choices for this completion might be justified—however, nothing in my argument hinges on this choice.

[7]One might argue that only through including interactions in the picture one can find that some systems are non-fundamental: for example, that the hydrogen atom is composed of a nucleus and an electron can be shown by trying to "kick out" the electron from the bound state. While this might be true in some cases, it does not pertain to my argument directly, which concerns the ambiguity in decomposing a system of which one already has a theoretical description that acknowledges the non-fundamentality of the system. That is, while the question of whether a system is fundamental might depend on such dynamic considerations, once one recognises the composite structure of a system, one still runs into the problem of underdetermination of how to decompose the systems.

[8]Note that in this case, the algebra of observables is the full algebra of (bounded) operators on $\mathbb{C}^2$, that is, $\mathcal{A} = \mathcal{B}(\mathcal{H}) \cong \mathbb{C}^{2\times 2}$.

[9]See e. g. Jauch (1968) for a construction of the formalism starting with a system of propositions.

can adopt other metaphysical views. Motivated partly by considerations about symmetry groups that we will focus on in Section 5, many take structural realism as their chosen ontology of quantum physics, like Esfeld and Lam (2011), French and Ladyman (2003), Lyre (2004), and McKenzie (2020) or propose metaphysical holism: Esfeld, Lazarovici, et al. (2017). Here, I shall use the term *system* to refer to a broad variety of what fundamental entities could be, thereby remaining neutral on the debate on the correct ontology of quantum theories. For example, regardless of whether one considers quantum field theories to be about particles or fields, my considerations apply in both cases, *mutatis mutandis*, although I will use the term particles throughout. Again, the reader is welcome to substitute their favourite ontology of quantum theories.

We are now ready to examine the decomposition procedures available in quantum physics in general, starting with the tensor product in the next section and continuing with the direct sum in Section 4.

3 Tensor Product Decomposition

The tensor product is introduced in first textbooks on quantum mechanics as the standard way to model compound systems, it is one of the central notions in the literature on entanglement,[10] and it is widely used in constructions in quantum field theory as well. I start by reviewing the formal construction and will then look at one way to justify the tensor product as a metaphysically meaningful priority relation.

3.1 The Tensor Product

The tensor product for a quantum system can be formed for both Hilbert spaces and algebras of observables. One way to obtain the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is to consider the Hilbert space which, as a vector space, is spanned by vectors of the form $e_i \otimes f_j$, where e_i denote the basis vectors of $\mathcal{H}_1$ and f_j denote the basis vectors of $\mathcal{H}_2$. The inner product on the tensor product space is given by the product of the individual inner products: $\langle u \otimes x, v \otimes y \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} := \langle u, v \rangle_{\mathcal{H}_1} \langle x, y \rangle_{\mathcal{H}_2}$, where $u, v \in \mathcal{H}_1$ and $x, y \in \mathcal{H}_2$. For finite-dimensional complex vector spaces $\mathbb{C}^m$ and $\mathbb{C}^n$, their tensor product is isomorphic to the vector space $\mathbb{C}^{mn}$, so for our example of $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$, we would find the tensor product of two spin-$\frac{1}{2}$ systems to be $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$.

[10]See for a conceptual overview Earman (2015).

The algebra of observables of this joint system can be constructed as the algebraic closure[11] of the set of operators $\{A_1 \otimes I_2 : A_1 \in \mathcal{A}_1\} \cup \{I_1 \otimes A_2 : A_2 \in \mathcal{A}_2\}$, where I_1 and I_2 are the identity maps on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Conversely, if one constructs the tensor product algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ there are natural embeddings of the factor algebras into the tensor product given by $\iota_1 \colon \mathcal{A}_1 \to \mathcal{A}, a \mapsto a \otimes 1_{\mathcal{A}_2}$ and similarly for $\mathcal{A}_2$, so that the choice above for algebras of observables agrees with the standard tensor product of the factor algebras. Note also that there is an isomorphism relating $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Thus, if the algebra of observables consists of all bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the tensor product of the algebras of observables will again be all the bounded operators of the tensor product space.

Consider again our example of the Hilbert space $\mathcal{H} = \mathbb{C}^4$ with the algebra of observables consisting of all bounded operators, i. e. $\mathcal{A} \cong \mathbb{C}^{4\times 4}$. Then this system could be viewed as modelling two spin-$\frac{1}{2}$ systems since $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, and the corresponding algebras would again be the full algebras of bounded operators on each $\mathbb{C}^2$ subspace.

3.2 Independence

How can one interpret the tensor product decomposition of a given quantum system into components? One way to arrive at this construction is the requirement that the subsystems be what I shall simply call *independent* of each other. There are many ways to spell out precisely what one means by the independence of subsystems, from statistical to logical or operational independence. For an overview of such conditions in the context of quantum theories see for example Hamhalter (1998) and Rédei and Summers (2010).

The common idea is that subsystems of a compound system should be suitably decoupled from each other to be identified as proper constituents of the compound system. Hence, one tries to find clusters of properties (or, in the language of quantum mechanics: subalgebras of the algebra of observables) such that a measurement on one cluster does not disturb the results of measurements on another one. The accounts of independence in the literature differ in spelling out what this requirement should exactly amount to mathematically and conceptually. I will only mention two such accounts: the commutativity of observables and the so-called split property.

[11] In the finite-dimensional case, this just means taking the algebraic closure of the set of operators. In other cases, however, one has to choose a norm in which the set is supposed to be closed—a complication that can be addressed, but I shall not be concerned with this here.

One of the simplest conditions expressing independence used in the framework of quantum theories is that the observables for each of the subsystems commute: that is, for all $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$, it holds that $[A_1, A_2] = 0$, where $\mathcal{A}_1, \mathcal{A}_2$ are the algebras of observables associated with the respective subsystems, embedded in the algebra of the compound system. One can arrive at this requirement in multiple ways. Malament (1996, p. 5, footnote 5) for example shows that two observables commuting is equivalent to the conditional probabilities, conditioned on the measurement of the respective other observable, being equal to the non-conditional probabilities. To illustrate, consider two observables A, B and the probabilities of the values of these observables being in certain sets $a, b \subseteq \mathbb{R}$, denoted by $P(A \in a)$ and $P(B \in b)$. Consider now the probability $P(B \in b|A \in a)$, i. e. the conditional probability of the value of the observable B being in set b, given one already measured the system with respect to observable A, and that value was in a. This can be calculated using the so-called "Lüders rule", and Malament shows that $P(B \in b|A \in a) = P(B \in b)$ is equivalent to the associated operators A and B commuting. That is, if (and only if) the observables are commuting, the predictions of the theory for outcomes of measurements of these observables are independent of each other in the sense that even if one measures A, the resulting probability distribution for B will remain as if one did not measure A and vice versa.

Unfortunately, the commutativity of the algebras of observables of the component system in the compound system is not enough to guarantee that the compound system is the tensor product of the component systems. One needs stronger conditions on the algebras, and various such requirements are discussed in the literature. I shall only mention the case of one of the strongest of these conditions[12], the *split-property*: two von Neumann algebras[13] $\mathcal{A}_1, \mathcal{A}_2$, are said to satisfy the split property just in case there exists a Type I factor $\mathcal{F}$ such that $\mathcal{A}_1 \subset \mathcal{F} \subset \mathcal{A}_2'$, where $\mathcal{A}_2'$ denotes the *commutator* of $\mathcal{A}_2$, that is, all observables in $\mathcal{A}$ that commute with all elements of $\mathcal{A}_2$. In the case where both $\mathcal{A}_1$ and $\mathcal{A}_2$ are Type I factor algebras, which is the case in non-relativistic quantum mechanics or when we are dealing with finite-dimensional Hilbert spaces, the split property implies that the smallest

[12] Discussed in both Summers (2009) and Earman (2015).

[13] A von Neumann algebra is an algebra of bounded operators on a Hilbert space that is closed in the so-called weak operator topology (for finite-dimensional Hilbert spaces, any algebra of operators is a von Neumann algebra). A central result for von Neumann algebras is that each such algebra is isomorphic to a direct integral (a generalization of the direct sum) of so-called factors, which are classified into three Types (I-III). Every full $\mathcal{B}(\mathcal{H})$ and any subalgebra of $\mathcal{B}(\mathcal{H})$ for a finite-dimensional $\mathcal{H}$ is a Type I factor, and in the following, only such algebras are considered. See for more details on this topic e. g. Haag (1996, Section III.2).

algebra containing both $\mathcal{A}_1$ and $\mathcal{A}_2$, commonly denoted as $\mathcal{A}_1 \vee \mathcal{A}_2$ is isomorphic to the tensor product of the two (in the sense of von Neumann algebras).[14] That is, in this case, the compound system containing only the two component systems is given by the tensor product.

Hence, in the cases of interest here, if two von Neumann algebras satisfy independence in the form of the split property, then the compound algebra will be the tensor product of the two. This motivates the tensor product as a construction from the requirement of independence of components in a compound system. Conversely, if one is given a system and one wants to find its components, then one can look for those subalgebras, that are independent of each other in that they satisfy the split property.

Note, that not every decomposition of a Hilbert space into a tensor product can be given a physical interpretation. Additionally, such a decomposition is not unique for a system: there could be several different ways how to view a given Hilbert space as the tensor product of factor spaces. This notion is explored in the literature on "virtual subsystems" in quantum information theory, see for example Zanardi (2001).[15] However, it should be noted that this is a different claim from my main argument. Whereas virtual subsystems concern the non-uniqueness of decompositions once one has chosen a fixed decomposition procedure (namely the tensor product), I argue here that the tensor product is not the only way to look at the decomposition of a quantum system in principle.

In sum, considering the fact that the tensor product is often used by physicists to model the relation of a compound system to its subsystems, and since this can be understood conceptually as arising from the independence of the component systems, I take the tensor product to give rise to a priority relation in the sense of Section 2. This gives rise to a meaningful notion of fundamentality in quantum physics: a compound system can be broken into its fundamental components by considering tensor product factors as associated with independent systems, and, if it cannot be broken down any further it should be considered fundamental. Conversely, fundamental systems can be composed into compound systems by using the tensor product construction.

[14] See for example theorem 4.1 in Summers (2009, p. 8), together with the fact that for Type I factors the tensor product of von Neumann algebras agrees with the algebraic tensor product used here.

[15] That these systems might not be so "virtual" after all is suggested in contexts of quantum optics, see for example Reck et al. (1994).

4 Direct Sum Decomposition

The other construction, which I claim is suitable to give rise to a metaphysical priority relation characterizing a notion of fundamentality, emerges in systems featuring so-called *superselection rules*. For our purposes,[16] these are systems where the algebra of observables is, in a specific way, a proper subset of the full algebra of all bounded operators on the Hilbert space, i. e. cases in which $\mathcal{A} \subsetneq \mathcal{B}(\mathcal{H})$. In these circumstances, the Hilbert space will decompose into a direct sum of *superselection sectors*, which one can then interpret as components prior to the compound system.

In the following, I will outline the technical construction first and then present a way to interpret and motivate it. Then, in Section 5, I will give another justification using the mathematical framework of group representation theory, which is motivated forcefully by its use in the Standard Model of particle physics. This will allow us to get to the most powerful formulation of my argument in Section 6: we will see that the two decomposition procedures can actually disagree on the very same systems.

4.1 The Direct Sum

The direct sum of two Hilbert spaces $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$ is given by the Hilbert space whose basis is the disjoint union of the bases of the summands. That is, if e_i are the basis elements of $\mathcal{H}_1$ and f_j are the basis elements of $\mathcal{H}_2$, then $\mathcal{H}$ is spanned by the elements $\{e'_i, f'_j\}$, where the dash denotes that even if some $e_q = f_l$, they are taken to be distinct elements in the direct sum space. The vectors in the sum vector space can then be written as $x = \sum_i c_i e'_i + \sum_j k_j f'_j$, although one usually uses the notation $e_i \oplus f_j$ instead of $e'_i + f'_j$. The inner product of this space is extended accordingly as the sum of the inner products of the component spaces. For finite-dimensional Hilbert spaces $\mathbb{C}^m, \mathbb{C}^n$, this means that $\mathbb{C}^m \oplus \mathbb{C}^n \cong \mathbb{C}^{m+n}$, that is, the dimension of the direct sum is the sum of the dimensions of the summands. The algebras of observables of the sum vector space arise naturally as the direct sums of the algebras of the summand spaces; in the finite-dimensional case, the operators are realised as block matrices.

It is important to remember that the description of a quantum system is always given by a Hilbert space *and* an algebra of observables, together. Consider again the toy example of $\mathcal{H} = \mathbb{C}^4$. This space can be viewed as the direct sum $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$

[16] See Earman (2008) for an outline of different ways to define superselection and philosophical considerations thereof.

of $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. However, even if one considers the two summand spaces $\mathcal{H}_i$ to be equipped with algebras of observables that contain all bounded operators of $\mathbb{C}^2$ (which, of course, are just all two-by-two matrices), then the direct sum of those algebras of observables would still not be the full algebra of four-by-four matrices. Put differently, if the quantum system is modelled by the Hilbert space $\mathcal{H} = \mathbb{C}^4$ equipped with an algebra of observables that contains all bounded operators of $\mathbb{C}^4$, then the direct sum construction is not available as a decomposition of quantum systems. Hence, this decomposition, similar to the case of the tensor product, is not available in all situations. However, we will see that this decomposition is always available if one restricts the class of systems to the ones being given by group representations, which, as Section 5 argues, is a very natural choice in particle physics.

Note also that the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ formally is the same vector space as the direct sum of the two spaces, $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$. However, the basis vectors of the direct sum are given by the disjoint union of the summand bases, i. e. $\mathbb{C}^4 \cong \langle e_1, e_2, f_1, f_2 \rangle$ whereas, in the tensor product space, the basis is $\mathbb{C}^4 \cong \langle e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2 \rangle$. Hence, the relationship between the compound space and the component spaces is entirely different in the two cases.

4.2 Superselection and Mixed States

How can one interpret this construction physically? As mentioned in the introduction to this section, the situations[17] in which this decomposition is available are those in which superselection occurs. In these cases, the algebra of observables is a proper subset of all bounded operators of the Hilbert space, missing, in particular, those operators that would superpose states from different superselection sectors or transform a state contained in one of the sectors into a state from a different sector. The sectors are subspaces, and together they exhaust the whole Hilbert space.

If the Hilbert space and algebra of observables arise from the direct sum of two Hilbert spaces and associated algebras respectively, then it is easy to see that in general superselection occurs: all observables in this case are of the form $A_1 \oplus A_2$ for $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ where $\mathcal{A}_1, \mathcal{A}_2$ are the algebras of observables of the two component

[17]For the following mathematical characterizations of these notions, we shall restrict ourselves to basic quantum mechanics, explicitly excluding quantum field theory—the mathematics for the latter case is considerably more difficult, but the conceptual conclusions for our purposes remain largely the same. See Earman (2015, Section 2) for an outline of the differences for superselection rules, and Ruetsche (2004, Section 3) in the case of mixed states. I am also only considering here what Earman (2015) calls “weak superselection”, ignoring other senses of superselection.

systems. Since they act separately on the direct sum components of vectors, that is $(A_1 \oplus A_2)(\psi_1 \oplus \psi_2) = (A_1\psi_1) \oplus (A_2\psi_2)$, components from different subspaces cannot be mixed by such observables. States that are, nevertheless, given by superpositions from different sectors are examples of what one calls *mixed states*.[18]

Systems in mixed states built from states in different sectors suggest an interpretation in terms of being non-fundamental: in a sense, they are "forbidden" combinations of different component states of a system. Although the interpretation of mixed states is a topic of philosophical debate, I will follow Ruetsche (2004, Sect. 3) and outline some that are often brought forward in the contexts that are relevant to our discussion here.[19] Systems that allow for superselection are usually interpreted as *mixtures* of component systems and model situations in which either the exact state of the system is not specified[20] or there are multiple systems in an ensemble, each in a unique state. That is, on the former interpretations, the mixture models the state of a system before a measurement of a property that is not known to allow for superpositions (such as the charge quantum numbers), but the exact value of that property is unknown. A mixed state on this account represents a system before a measurement that could distinguish in which sector that system is. The latter interpretations take mixed states to model an ensemble of systems, each in a definite state, with the mixed state describing the whole ensemble. Baker and Halvorson (2010, p. 103) interpret the direct sum $X \oplus Y$ of two systems, each considered to have a determined quantum charge, as modelling "a mixture of possible charges, so that [the system] may have either charge X or charge Y; the theory doesn't tell us which". That is, the compound system $X \oplus Y$ is interpreted as being decomposable into the two more fundamental systems X and Y, and the theory does not explicitly tell which of the two possibilities is realised. Whatever interpretation one favours, on all of them the system itself is compound and the components of the mixture are structures metaphysically prior to and required for the very definition of the compound system. Thus, one can interpret the compound system as being decomposable into the constituents given by this procedure.

[18] Formally, mixed states are defined on the algebra of observables as those states, which can be written as a non-trivial convex combination of other states—but I shall not get into too deep technical details here. For our purposes, it suffices to think of mixed states as those arising from superpositions of states in different superselection sectors. See, again, both Earman (2015) and Ruetsche (2004) for details.

[19] For a more detailed review see Ruetsche (2004). We are here only dealing with cases of what she calls "ordinary quantum mechanics".

[20] That is, not specified for whatever reason—mixed states are sometimes used to model situations in which the specific state is not known, or in which one wants to be ignorant about it.

One response to this interpretation of the direct sum as a decomposition procedure of physical systems might be to point out that states in a direct sum *can* be mixed states, but they do not need to be: the system could also be in a pure state contained in one of the superselection sectors, and then there is no interpretation in terms of ensembles or mixtures. I think this is not relevant to the case brought forward here on the level of the state space and algebra of observables the system *is* still structurally conceived as a mixture. Thus, this description is either inaccurate (and thus should be improved before assessing the metaphysical implications of it) or the system at hand has the structure of a compound system.

5 The Standard Model and Group Representations

In the previous sections, I have introduced two different possible decomposition procedures in quantum physics: a system might turn out to be the tensor product of independent subsystems, or it might be a direct sum and represent a mixture of component systems. In some cases, though, the notions are somehow separate and apply in different circumstances. For example, if the algebra of observables is the full algebra of bounded operators, then there cannot be a direct sum decomposition. One might respond, that in practice it will be clear which one a system is: a compound of independent subsystems or a mixture of possibilities.

First, note that this is not an objection to my argument: it still holds that quantum physics does not give rise to a unique notion of fundamentality because it does not feature a unique decomposition procedure. The two accounts of decomposition that I present here are conceptually vastly different, so it is still interesting to look at how naturalistic metaphysical conceptions of fundamentality should incorporate this fact. However, I will show in the following that within the framework of group representation theory, which is one of the foundations of modern particle physics, the two notions disagree in a very strong sense: I will discuss how a system can be decomposed via both the direct sum as well as the tensor product constructions in the same circumstances, with different constituent systems arising. This, then, is a clear problem for the hope for a simple and unique priority relation for the metaphysics of fundamentality arising directly from particle physics.

In this section, we will look at the Standard Model of Particle Physics and see how the direct sum decomposition arises naturally from the ontology of the theory: the first two subsections will introduce the mathematical framework of group representation theory, and subsection 5.3 then covers how so-called irreducible representations arise naturally as the fundamental entities of particle physics on the direct sum decomposition. In Section 6 I will finally show how the two decompositions disagree and discuss some philosophical consequences of this incompatibility.

5.1 The Standard Model

The *Standard Model of particle physics* comprises a set of theories that together predict several different types of elementary particles,[21] neatly arranged according to a few properties such as mass, spin, electric charge and other "generalised charges." Each elementary particle is characterised by the values of these properties: the electron, for example, is a spin-$\frac{1}{2}$ particle with an electric charge of -1, weak isospin of $-\frac{1}{2}$, weak hypercharge of -1, a mass of about 9.11×10^{-31}kg and a colour charge of $\mathbf{1}$. Mathematically, these quantum numbers correspond to labels of so-called irreducible representations of symmetry groups, so the colour charge of the electron $\mathbf{1}$ is not the natural number 1, but the label of the one-dimensional representation of the global colour gauge group, and similarly for the other charges.[22] It is in this sense that one can say that the ontology of the Standard Model is determined by these symmetry groups and their representations.

This account is the basis for a widespread "definition" of elementary particles, summarised here by Ne'eman and Sternberg:[23]

> Ever since the fundamental paper of Wigner on the irreducible representations of the Poincaré group, it has been a (perhaps implicit) definition in physics that an elementary particle 'is' an irreducible representation of the group, G, of 'symmetries of nature'.

[21]The counting depends a bit on the author: Griffiths (2008, p. 50) counts 61 including the then-to-be-discovered Higgs particle; Thomson (2013, Ch. 1) describes the more usual 12 fermions and five bosons. The differences are due to whether one considers certain particles as states of one unified particle or as separate particles proper—an issue that certainly deserves more attention, but is not in the scope of this paper.

Furthermore, despite using the term "particle" a few times in this section—because it is how those systems in particle physics are standardly denoted—I do not want to commit to any further consequences one might attach to this notion. For a recent overview of the discussions around particles in quantum theories see for example Fraser (2021) and for an overview of arguments against a particle interpretation in quantum field theory see Kuhlmann (2010, Ch. 8).

[22]For an alternative account of why the quantum numbers or charges have to be labels of representations see Baker and Halvorson (2010). However, no part of my argument depends on the differences.

[23]Ne'eman and Sternberg (1991, p. 327), quoted in Roberts (2011, p. 51).

This account of particles has been widely discussed in the philosophy literature, see for example Kantorovich (2003) and McKenzie (2014). The paper that Ne'eman and Sternberg refer to, Wigner (1939), however, does not deal with elementary particles as such but is focused on a related mathematical problem: the classification of all unitary representations of the Poincaré[24] group. The motivation that Wigner gives for this endeavour is that unitary representations of the spacetime symmetry group can, to a certain extent, replace the equations of motion for quantum systems that are placed in such a relativistic spacetime.[25] Hence, he argues, by enumerating all possible unitary representations of the spacetime symmetry group, one gets a classification of all equations of motion, i. e. all possible dynamics in relativistic spacetime. Wigner proves in the paper that in order to find all possible unitary representations of the Poincaré group, it suffices to find the irreducible unitary representations (or short: *irreps*), as they serve as the building blocks for all other representations. Then, he shows that the irreducible representations of the Poincaré group can be classified by only two parameters: $m \in \mathbb{R}, \sigma \in \frac{1}{2}\mathbb{Z}$, which thus can be used as labels for the representations and are physically interpreted as the mass and spin of the particle described by the representation.[26]

This is a short characterization of what is often referred to as "Wigner's conception of particles".[27] The other parameters of elementary particles in the Standard Model, like charge and colour-charge, arise similarly from other symmetry groups, called *internal* symmetries. In the following, I shall describe in more detail how one extends Wigner's account to the other properties of elementary particles and will give a brief account of why representations in general, and irreps in particular can characterise those physical systems. We will see that quantum systems in particle physics can be modelled by group representations, which provide both the Hilbert space as well as the algebra of observables—instead of specifying $\mathcal{H}$ and $\mathcal{A}$ separately.

[24]In the paper, Wigner refers to what one nowadays calls the "Poincaré group" as the "inhomogeneous Lorentz group," whereas the "homogeneous" Lorentz group is what is known today simply as the Lorentz group.

[25]The short argument is that the Poincaré group includes the time-translations of a system which, of course, need to agree with the dynamics of the system and vice versa. I shall not deal with the details of this here but refer to Roberts (2022, Ch. 4).

[26]Note, that not all possible irreps are physically meaningful, cf. Sternberg (1995, p. 147f).

[27]For a more detailed overview see e. g. Kuhlmann (2010, pp. 87ff.).

5.2 Symmetries and Representations

Symmetry groups are used in mathematics to describe the structure of objects by collecting transformations that leave that structure invariant. The above-mentioned Poincaré group $\mathcal{P}$ is the symmetry group of Minkowski spacetime, which is the mathematical representation of spacetime according to special relativity, that is, $\mathbb{R}^4$ equipped with the Lorentz metric, often written as $\mathbb{R}^{1,3}$. $\mathcal{P}$ contains the transformations of $\mathbb{R}^{1,3}$ that leave the Lorentz distance between two spacetime points invariant: it comprises translations in space and time, rotations in space, parity and time reversal, and so-called Lorentz boosts, which describe the transition of a reference frame at rest to one moving at a constant speed. Similarly, the group of unitary operators on a Hilbert space $\mathcal{U}(\mathcal{H})$ contains the operators that preserve the Hilbert space structure (that is, the linear vector space structure and the inner product).[28] Hence, if $\mathcal{H}$ is used to model a quantum system, $\mathcal{U}(\mathcal{H})$ can be viewed as the symmetry group of the quantum system itself.

Consider now a quantum system with spatio-temporal degrees of freedom, that is, a position in spacetime. Those degrees of freedom cannot be arbitrarily implemented in the state space because they must respect the structure of spacetime itself. That is, one expects the symmetries of spacetime to be reflected in how the spatio-temporal degrees of freedom are implemented in the Hilbert space representation of the quantum system. This leads to the demand that the symmetries of spacetime shall not alter the basic structure of the quantum system under consideration. In other words, one expects the symmetries of spacetime to correspond to symmetries of the quantum system.

Mathematically, this connection between spacetime and quantum symmetries is expressed by way of *unitary group representations*:[29] A group representation is a homomorphism from a group G to the operators on a vector space, and if this vector space is a Hilbert space and all operators in the image of the map are unitary, it is called a unitary representation: $\pi\colon G \to \mathcal{U}(\mathcal{H})$. That is, π realises the abstract symmetry transformations in G as concrete unitary operators on $\mathcal{H}$.

[28] In fact, also anti-unitary operators preserve the full Hilbert space structure. However, only the unitary operators form a group, and if one uses projective representations (see below), the distinction becomes void. For a more thorough explanation of why one disregards anti-unitaries see e. g. Roberts (2022, Section 3.4).

[29] Actually, the relevant representations are not the unitary ones but the *projective* representations. This is a technical subtlety that does not bear any conceptual significance but would complicate this treatment significantly. As is customary in philosophical treatments of this matter, I thus stick to unitary representations.

Requiring $\mathcal{H}$ to carry a representation of the Poincaré group thus gives one the means to say formally that a system has spatio-temporal degrees of freedom, that is, a position in spacetime. Conversely, if one has a quantum system whose Hilbert space carries a representation of the Poincaré group, then some of the symmetries of this quantum system can be readily interpreted as being implementations of the spacetime symmetries, connected to how the spatio-temporal degrees of freedom of the system are implemented in $\mathcal{H}$.[30] Thus, we arrive at what Roberts (2022, Chapter 2) calls the *representation view*: a quantum system has spatio-temporal degrees of freedom if and only if its Hilbert space carries unitary representations of the Poincaré group.[31]

One can extend this to internal degrees of freedom (or quantum charges):[32] the structure of the spaces in which they take their values should be preserved in the description of a quantum system. For example, colour charges take values in a space whose structure group is $SU(3)$, so one expects a representation of $SU(3)$ on the Hilbert space of any quantum system that is said to have colour charges. Just as in the case of spatio-temporal degrees of freedom, one thus says that a quantum system has a given degree of freedom if and only if its Hilbert space carries a representation of the symmetry group of the space in which this degree of freedom takes its values. One might call this the *general representation view*.

Carrying a unitary representation has two main consequences for the description of a quantum system. On the one hand, it implements the assumption that a system has certain degrees of freedom, as just discussed. On the other hand, it fixes a property that remains invariant under the symmetry transformations, namely the possible representations the Hilbert space carries—a property, which, obviously, cannot be changed by application of a symmetry transformation on that space.[33] Hence, if one can label all the unitary representations of a group, then one can label the various quantum systems having those degrees of freedom and use these labels as the *quantum numbers* describing the structure of a system—as we have already seen in the case of the Poincaré group, where the labels are identified as mass and

[30]Note, that this argument can also be run in the converse direction: by systematizing the invariance behaviour of physical systems we can infer the symmetries of spacetime, see also Roberts (2022, Ch. 5).

[31]Or the Galilean group, for a non-relativistic spacetime setting; see Castellani (1998) and Lévy-Leblond (1963).

[32]Cf. Roberts (2022, p. 179).

[33]Technically, these invariants can be identified with the eigenvalues of so-called Casimir operators that commute with all other symmetries in a given representation—and in the case of irreps below, will be multiples of the identity.

spin; in the case of internal symmetry groups, one gets the other quantum charges, like colour charge. This shows how the quantum numbers are not ordinary numbers but labels of group representations.[34]

Hence, by specifying the quantum numbers of a system, one determines a Hilbert space $\mathcal{H}$, together with a set of unitaries $\pi(G) \subseteq \mathcal{U}(\mathcal{H})$ on it. However, a group representation can provide more structure: one can also construct a corresponding algebra of observables from the group representation structure. To be more precise, one can look at the group's Lie algebra,[35] whose generators will give rise to self-adjoint operators via the lie algebra representation,[36] which in turn can be used to generate an algebra of observables. This way, the available observables derive directly from the global structure group defining the system's available degrees of freedom. Hence, I take the general representation view to imply that the labels of a symmetry group fully specify the structure of a quantum system: the available states (the Hilbert space) as well as the algebra of observables, which thus consists of all and only of combinations of observables that are determined by the assumed degrees of freedom. That is, the choice of the algebra of observables is restricted if one considers the description of the quantum system to arise from the group representation of its degrees of freedoms' symmetry groups. In the following, we will thus view a quantum system as a group representation, and not just as the more arbitrary choice of a Hilbert space $\mathcal{H}$ and algebra of observables $\mathcal{A}$.

In sum, the ontology of the Standard Model is captured by the relevant symmetry groups. The Poincaré group describes spatio-temporal degrees of freedom, giving rise to the quantum charges of mass and spin; the internal symmetry groups $U(1)$ and $SU(2)$ together describe the electroweak degrees of freedom and give rise to weak isospin and weak hypercharge; and the internal symmetry group $SU(3)$ describes colour-related degrees of freedom characterizing the colour quantum number. It is in this way that the "symmetry group of nature" provides a "definition" of the

[34] Note the connection between the charge and the degrees of freedom: saying that the electron has colour charge **1** expresses that the electron is in the **1**-representation of $SU(3)$, from which it follows that it does not have any colour-related degrees of freedom; whereas a quark, which is in the **3**-representation of $SU(3)$, does indeed have colour degrees of freedom.

[35] The Lie algebra of a Lie group is an algebra associated with the group, representing infinitesimal group transformations near the identity. See Fuchs and Schweigert (1997, Ch. 4) for essential definitions.

[36] A Lie algebra representation is, similarly to a group representation, a structure-preserving map into the operators of a Hilbert space. There is a one-to-one correspondence between unitary group representations and Lie algebra representations for so-called simply connected groups. $SU(n)$ are all simply connected, for the Poincaré group and $U(1)$ there are separate arguments as to why one gets operators representing the measurable properties associated with the degrees of freedom of a quantum system from the symmetry transformations.

different types of elementary particles.

5.3 Irreducible Representations as Fundamental Systems

Note that one can define the direct sum of group representations analogously to the case of Hilbert spaces and algebras of observables: given $\pi_1 \colon G \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2 \colon G \to \mathcal{U}(\mathcal{H}_2)$, their direct sum is a new representation, $\Pi \colon G \to \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ where $g \mapsto \pi_1(g) \oplus \pi_2(g)$. Note that in general $\mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2) \subsetneq \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. We will define the tensor product of group representations similarly in Section 6.

Then, a unitary[37] representation $\pi \colon G \to \mathcal{U}(\mathcal{H})$ is called *irreducible* (or, as already mentioned, an *irrep*) just in case there are no proper non-trivial subspaces of $\mathcal{H}$ that are invariant under the operators in $\pi(G)$, that is, if there are no subspaces that themselves would furnish a representation of G. Irreps have a significant role within the class of all unitary representations of a given group: as mentioned above, Wigner showed that the irreps of $\mathcal{P}$ function as basic building blocks for all other unitary representations of $\mathcal{P}$. To be more precise, he found that any representation of the Poincaré group is isomorphic to a direct sum of irreps. The Peter-Weyl theorem[38] proves the same for another important class of groups, the so-called compact Lie groups—all the internal symmetry groups that come up in particle physics are of this type. That is, all relevant symmetry groups in particle physics have the property that any of their unitary representations can be decomposed into a direct sum of irreps.

Hence, irreps can be considered fundamental amongst the representations of a given group according to one of the metaphysician's definitions of fundamentality, as discussed in Section 2: they are the basic building blocks of all other representations of that group, with respect to a priority relation given by the composition based on the direct sum of group representations. It is also easy to see that irreps can further be considered fundamental according to the second account of fundamentality, in that they cannot be the direct sums of other representations and hence are not decomposable into more fundamental particles.

If one combines this formal sense of fundamentality with the general representation view set out in the subsection before, one gets the following: every kind of quantum system in particle physics is given by a group representation (irreducible or reducible). By the above-mentioned theorems by Wigner and Peter-Weyl, any

[37] The definition holds for any group representations. However, we restrict our attention to unitary ones.

[38] See e. g. Sternberg (1995, p. 179).

such system is decomposable into a direct sum of irreps. Hence, every system in particle physics is either irreducible or decomposable into a direct sum of irreps. On the representation view, those components correspond to component systems carrying the specified degrees of freedom. Overall, one can say that the decomposition into a direct sum supplies one way of a decomposition of physical systems into their fundamental constituents. Following "Wigner's definition", those fundamental components are the elementary particles.

Note that the resulting notion of decomposition is relative to the choice of the symmetry group G. This might be viewed as an advantage in that it allows for a bespoke notion of fundamentality depending on what properties of systems one is interested in—for example, one can capture the idea that something is not decomposable with respect to specific properties, while still being decomposable with respect to a larger set of properties. On the other hand, one could see this as a disadvantage because it does not guarantee, strictly speaking, an ultimate notion of fundamentality—one can always introduce a larger symmetry group, which will lead to a new set of irreps.

However, I view this more as an expression of the fact that physicists might in the future discover that systems, which were previously considered fundamental, turn out to actually be compound systems. This was the case with atoms, which were once thought to be what their name suggests—indivisible—but are now considered to be compound systems composed of various "elementary" particles. With respect to a "full" symmetry group, describing—in the words of Ne'eman and Sternberg—all "symmetries of nature", the notion of fundamentality would be ultimate in the sense of including all possible properties that can be used to discern parts of systems. I do not want to take a position here on whether such a full group is discoverable. For our purposes, it is enough to conclude that the theorems by Peter-Weyl and Wigner provide a decomposition procedure given by the direct sum of irreps, that explains the particle ontology of the Standard Model and is distinct from the decomposition procedure arising from the tensor product construction.

Overall, it was shown in this section that, based on the representation view, a quantum system in particle physics can be described by a unitary representation of the structure groups of the degrees of freedom that the system is assumed to have, which in turn specifies the state space and algebra of observables of the system. This allows a decomposition of the system into a direct sum of irreps, which thus can be interpreted as the fundamental components of that system since the irreps cannot be decomposed any further on the direct sum decomposition account.

6 Incompatible Decompositions

In the previous section, I showed how the direct sum of group representations arises naturally in the context of the Standard Model of particle physics. Now, we shall also apply the tensor product construction to group representations and then see how these two decomposition procedures can disagree. I start by defining the tensor product for group representations and then look at the example of the addition of angular momenta from elementary quantum mechanics. I will, however, present it slightly differently from how it is usually done in textbooks to highlight the discrepancy between the two possible decompositions. I will conclude the section with some philosophical considerations which follow from what has been discussed so far.

Similar to the direct sum of group representations, the tensor product of group representations $\pi_{1,2} \colon G \to \mathcal{B}(\mathcal{H}_{1,2})$ is given by the representation $\Pi \colon G \to \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ such that $\Pi(g) = \pi_1(g) \otimes \pi_2(g)$. This construction agrees with the ordinary tensor product of Hilbert spaces. However, the algebra of observables obtained from the tensor product of the group representations is distinct from the tensor product of the individual algebras of observables. In general, it won't be the case that if $\pi_i(G) = \mathcal{B}(\mathcal{H}_i)$ that the algebra of the compound system will again be the all bounded operators, i. e. $\Pi(G) \neq \mathcal{B}(\mathcal{H})$. This is a crucial difference to the simple tensor product algebras of observables as it will lead to situations where both decompositions discussed above are available on the same system; this will happen if the tensor product is a reducible representation, which generally is the case.

I will again refer to the familiar example of the system that is described by the Hilbert space $\mathbb{C}^4$. Assume that it carries a reducible representation of $SU(2)$, the group characterising the structure of quantum systems with angular momentum.[39] It now turns out that there are two ways this representation could be broken down into irreps: either into the tensor product of two spin-$\frac{1}{2}$ representations (both being $\mathbb{C}^2$ with the full algebra of operators as observables) or into the direct sum of a spin-0 and a spin-1 system. That is, if one is handed a quantum system whose Hilbert space is $\mathbb{C}^4$ and whose algebra of observables has a specific form, together with the information that this quantum system has a $SU(2)$-structured degree of freedom, one has two possible ways to distinguish constituents of this system: on the one hand as two independent spin-$\frac{1}{2}$ subsystems, and on the other hand as the mixture of a spin-0 with a spin-1 system. Neither the formalism nor any observable

[39] This information suffices to fix both the Hilbert space as well as the algebra of observables since there is only one reducible representation of $SU(2)$ on $\mathbb{C}^4$.

in the algebra can tell which one is the “correct” decomposition.

The connection to the addition of angular momenta[40] becomes apparent when considering the total angular momentum of an electron in an atom, which is described as a degree of freedom with structure group $SU(2)$. It has two contributions: the spin of the electron itself and the orbital angular momentum arising from it being bound to the nucleus of the atom. These are two independent degrees of freedom, both described by $SU(2)$. Hence, modelling the system as the tensor product of two spin-$\frac{1}{2}$ system is appropriate. However, the total angular momentum of the system is not fully determined: it can either be that of a spin-0 or that of a spin-1 system, depending on whether the two angular momentum vectors are aligned or not. That is, from this point of view, one is dealing with a mixed system, the constituents of which are a spin-0 and a spin-1 system. Hence, this system, depending on which fundamentality relation one takes, is either composed of two spin-$\frac{1}{2}$ systems or a spin-0 and a spin-1 system. Physical considerations alone cannot straightforwardly give a unique answer to the question of what the fundamental constituents of this system are.

That the two decompositions will disagree in other cases too is easy to see, at least for systems described by finite-dimensional Hilbert spaces: the dimension of a tensor product space equals the product of the dimensions of the constituent spaces, whereas the dimension of the direct sum equals the sum of the dimensions of the summands. For some structure groups there even exist formulas to calculate the different decompositions, commonly known as the Clebsch-Gordan formulas:[41] consider again the case of $SU(2)$. We already saw that the irreps can be indexed by half-integer numbers, so formally one can write $\frac{1}{2}$ to refer to the spin-$\frac{1}{2}$ representation. Exploiting the fact that one thus can use half-integer numbers to refer to the irreps, one can formally write the Clebsch-Gordan Formula in the case of $SU(2)$ for $X, Y \in \frac{1}{2}\mathbb{N}$:[42]

$$X \otimes Y \cong \bigoplus_{Z=|X-Y|}^{|X+Y|} Z,$$

where Z increases in steps of 1. That is, the tensor product of irreps X and Y is isomorphic in the sense of group representations to the direct sum of irreps Z, where Z ranges from $|X-Y|$ to $|X+Y|$ and where X, Y, Z are now taken to be just

[40] See, e. g. Griffiths (1994, Section 4.4.3, pp. 165ff)

[41] As seen before, such a decomposition is possible for *any* compact Lie group. Explicit formulas and calculations exist for several special cases relevant to particle physics, most notably $SU(n)$.

[42] See for this case Baker and Halvorson (2010, p. 103) or more generally Larkoski (2019, Section 3.3).

ordinary numbers from the half-integers. For our example of two spin-$\frac{1}{2}$ systems, or, equivalently, a spin-0 and a spin-1 system, the formula reads $\frac{1}{2} \otimes \frac{1}{2} \cong 0 \oplus 1$. All representations involved here—X, Y and all of the Z—are irreducible.

It is important to note that the above analysis does not merely say that the mathematical formalism is ambiguous as to how the physical system "actually" breaks down into subsystems, and a proper physical inspection will show whether one is dealing with a system that is *either* a mixture of its fundamental constituents *or* consists of independent fundamental components. Note, that one does not have any other means to inspect the system other than via measurements modelled by the observables in the algebra of observables. So assuming that the mathematical framework is both correct and complete, we can follow that both decompositions are equally possible.

One response to this could be to conclude that these considerations merely show that the same mathematical framework can be used to describe different physical situations, like both sound waves and electromagnetic waves can be described by the same wave equations. In the same way, one might argue, the two decomposition procedures above describe two different physical situations that just happen to be described by the same Hilbert space and algebra of observables. Nevertheless, the situation in particle physics is different from other cases: in the latter situation, there are other physical and metaphysical ways to differentiate the systems—for example, by observing that the waves propagate in different materials. In the case of particle physics, however, the group representation (including the Hilbert space and the algebra of observables) is supposed to be a complete description of the physical system. Hence, if one takes the theory as it is, there is no additional information about the systems to be gained that could distinguish the cases.

I wish to close with some remarks on how one could react to this argument. The minimal conclusion to take away, indeed, is that fundamentality in fundamental physics is not as easy as one might think when just pointing to particle physics as an explication of how the world "is made up of fundamental building blocks." My argument has shown that conventional choices are involved in composing and decomposing systems in quantum theories. As mentioned above, this is a worry orthogonal to the already existing choices one faces if one has settled on the tensor product as the preferred decomposition procedure, as is discussed in the literature on virtual subsystems.[43]

In this sense, I agree with critics of the slogan that one has to simply "read off

[43] See, again, Zanardi (2001).

one's metaphysics from one's physics,"[44] since it is clear that a bare mathematical formalism does not uniquely determine the metaphysics. However, the nature of the argument also shows that such metaphysical projects need to be based on what physics tells us, agreeing with pushes for a "naturalised" ontology like Ladyman and Ross (2007).

7 Conclusion

I have argued that contrary to popular opinion, particle physics does not provide an account of "the fundamental" because one cannot uniquely determine the fundamental constituents of the systems described by particle physics. This was shown by discussing two possible ways of decomposing a quantum system: the direct sum and the tensor product. In the context of group representations, a widely used framework within particle physics, the contrast is especially stark: the same system is decomposable into different "fundamental" constituents, depending on the decomposition procedure one chooses. This shows that there is an element of convention and choice necessary to determine the basic building blocks of a concrete object in the material world. This goes against the idea that particle physics simply presents us with an account of the fundamental and shows another way of the underdetermination of metaphysics by physics.

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[44] Again, see French (1998, p. 93).

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