Length Abstraction in Euclidean Geometry

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Abstract

Given synthetic Euclidean geometry, I define *length* $\lambda(a, b)$ (of a segment ab), by taking equivalence classes with respect to the congruence relation, $\equiv$: i.e., $\lambda(a, b) = \lambda(c, d) \leftrightarrow ab \equiv cd$. By geometric constructions and explicit definitions, one may define the *Length structure*, $\mathbb{L} = (\mathbb{L}, \mathbf{0}, \oplus, \preceq, \bullet)$, "instantiated by Euclidean geometry", so to speak. One may show that this structure is isomorphic to the set of non-negative elements of the one-dimensional linearly ordered vector space over $\mathbb{R}$. One may define the notion of a *numerical scale* (for length) and a *unit* (for length). One may show how *numerical scales for length* are determined by Cartesian coordinate systems. One may also obtain a derivation of *Maxwell's quantity formula*, $Q = \{Q\}[Q]$, for lengths.

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1 Introduction

It's an over-arching methodological rule in science that quantities introduced be *measurable*:

> It is an important principle of physics that no quantity should be introduced into the theory which cannot, at least in principle, be measured. Newton's Laws involve not only the concepts of velocity and acceleration, which can be measured by measuring distances and times, but also the new concepts of mass and force. To give the laws a physical meaning, we have, therefore, to show that these are measurable quantities. (Kibble & Berkshire (1996): §1.3, p. 8)

The outcome of such measurements (experiments) are then called "*measurement reports*", like "the period of Halley's comet, on average, is 76 years", or "the temperature of the cosmic microwave background is 2.7 K" or "the rest mass of an electron is 9.1×10^{-31} kg" and so on. (A list of measurement reports corresponds to a *relational database*. For example, the database of planetary distances and planetary period measurements which Kepler took to corroborate his three laws.) Consider Nelson's Column:

Figure 1: Nelson's Column

As one can see from the photo, the column looks to be about 20 or 25 times as high as a person. Assuming people are around 2 m height, Nelson's Column must be something like 40 m to 50 m in height. In fact, it's about 52 m. More accurately, the measurement reports are:

$$\text{The length (height) of Nelson's Column} = 5159\,\text{cm}. \tag{1}$$
$$\text{The length (height) of Nelson's Column} = 2031\,\text{inch}. \tag{2}$$

What does an expression such as

"1 cm" "1 inch" "52 m" "5159 cm" ...

refer to? Not numbers, to be sure. But not physical objects, either. It also seems clear that the linguistic expression "5159 cm" must refer to the result of "multiplying" the real number 5159 by whatever the expression "1 cm" refers to:

$$5159 \text{ cm} = 5159 \cdot 1\,\text{cm} \tag{3}$$

$$1 \text{ cm} \quad \text{is a unit length} \tag{4}$$

But:

(Q1) What is this "multiplication", denoted $\cdot$, exactly?
(Q2) And what is a "unit length" (e.g., 1 cm), exactly?

I wish to explain what these mean.[1] I wish to explain what "*the length of* line segment *ab*" means, by working up from *geometry* as our background and introducing lengths as abstract entities, introduced by abstraction: a length is an equivalence class over the geometrical equivalence relation of *congruence* between line segments ("having the same length as"). The approach to analysing length that we develop here is sketched briefly and informally by Paolo Mancosu as follows:

> Consider the notion of equality of segments in Euclid. Starting from the congruence relation between segments, a contemporary mathematician might naturally introduce length using a definition by abstraction such as '$\lambda(a) = \lambda(b)$ iff a is congruent to b' (with the option of explicitly defining length by means of equivalence classes or other devices or simply accepting lengths as new entities, as Peano does). But Euclid does not do this and he simply says, in common notion 4, that two segments are equal if they 'coincide with one another' ('Things which coincide with one another are equal to one another'). Then in the midst of the proof of proposition I.4 we find the converse being implicitly used for segments ('if two segments are equal they coincide with one another'). Is the notion of equality of segments taken to be primitive or is it introduced by abstraction (for it is not defined explicitly)? If we exclude the former case then, if there is a definition by abstraction of equality of segments, it is at best implicit, for what we are originally given is not a definition introduced by an 'if and only if' (and a fortiori not a definition by abstraction). Moreover, there is no mention of the class of segments that have in common the property of being congruent (which in a more contemporary setting could be used to define $\lambda(a)$, namely the length of a). (Mancosu (2016): 23)

In this article, we work out the details.

[1] As it turns out, it is a form of scalar multiplication by a real, just as one has, e.g., $3\mathbf{e}_1 + 7\mathbf{e}_2$ in a vector space. This scalar multiplication is defined below, at Definition 5 as applied to the system $\mathbb{L}$ of lengths. As explained at Definition 8, a unit length is any non-zero positive element of $\mathbb{L}$. 1 cm is simply such a unit length. A change of unit is a change of basis for $\mathbb{L}$.

2 Euclidean geometry

We shall proceed as follows. We are going to focus on two-dimensional Euclidean geometry, as axiomatized synthetically by Alfred Tarski (Tarski (1948); Tarski (1959); Tarski & Givant (1999)). This synthetic geometry has two synthetic geometrical primitives, a 3-place predicate $\mathtt{B}$ and a 4-place predicate $\equiv$, where:

$\mathtt{B}(a, b, c)$ means:	"the point b lies on a straight line (inclusively) between the points a and c",
$ab \equiv cd$ means:	"the segment ab is as long as the segment cd is" (or, equivalently, "the segments ab and cd are congruent").

We shall call this theory $\mathsf{EG}(2)$. N.B., we mean the second-order theory, not the first-order theory, which I call $\mathsf{EG}_0(2)$. The precise axioms of this theory $\mathsf{EG}(2)$ are given as follows:

Definition 1. $L(\mathtt{B}, \equiv)$ is the first-order language over the signature $\{\mathtt{B}, \equiv\}$. $L_2(\mathtt{B}, \equiv)$ is the (monadic) second-order language over the signature $\{\mathtt{B}, \equiv\}$. The implicit logical axioms are Extensionality for point sets and (monadic) Comprehension for point sets. The non-logical axioms of $\mathsf{EG}(2)$ in $L_2(\mathtt{B}, \equiv)$ are the following eleven:

E1.	$\mathtt{B}$-Identity	$\mathtt{B}(p, q, p) \to p = q$.
E2.	$\equiv$-Identity	$pq \equiv rr \to p = q$.
E3.	$\equiv$-Transitivity	$pq \equiv rs \wedge pq \equiv tu \to rs \equiv tu$.
E4.	$\equiv$-Reflexivity	$pq \equiv qp$.
E5.	$\equiv$-Extension	$\exists r\, (\mathtt{B}(p, q, r) \wedge qr \equiv su)$.
E6.	Pasch	$\mathtt{B}(p, q, r) \wedge \mathtt{B}(s, u, r) \to \exists x\, (\mathtt{B}(q, x, s) \wedge \mathtt{B}(u, x, p))$.
E7.	Euclid	$\mathtt{B}(a, d, t) \wedge \mathtt{B}(b, d, c) \wedge a \neq d \to \exists x\, \exists y\, (\mathtt{B}(a, b, x) \wedge \mathtt{B}(a, c, y) \wedge \mathtt{B}(x, t, y))$
E8.	5-Segment axiom	$(p \neq q \wedge \mathtt{B}(p, q, r) \wedge \mathtt{B}(p', q', r') \wedge pq \equiv p'q' \wedge qr \equiv q'r' \wedge ps \equiv p's' \wedge qs \equiv q's') \to rs \equiv r's'$.
E9.	Lower dimension	There exist three points which are not collinear (i.e., not $\mathtt{co}_2$).
E10.	Upper dimension	Any four points are coplanar (i.e., $\mathtt{co}_3$).
E11.	Continuity Axiom	$\exists r\, (\forall p \in X)\, (\forall q \in Y)\, \mathtt{B}(r, p, q) \to \exists s\, (\forall p \in X)\, (\forall q \in Y)\, \mathtt{B}(p, s, q)$

Note that E11, the *Continuity Axiom*, is a single second-order axiom, not an axiom scheme. The *first-order theory* $\mathsf{EG}_0(2)$ is obtained by removing the Continuity Axiom, and replacing it with all instances of the *Continuity Axiom Scheme*, with $\varphi(p)$ and $\theta(p)$ formulas of $L(\mathtt{B}, \equiv)$:

Continuity Axiom Scheme $\quad \exists r\, \forall p \forall q\, (\varphi(p) \wedge \theta(q) \to \mathtt{B}(r, p, q)) \to \exists s\, \forall p \forall q\, (\varphi(p) \wedge \theta(q) \to \mathtt{B}(p, s, q))$

These theories have quite different meta-mathematical properties. $\mathsf{EG}_0(2)$ is complete and hence decidable (Tarski (1948)). However, $\mathsf{EG}(2)$ is incomplete and interprets very powerful mathematical theories (roughly, the second-order theory of real numbers).

But, fortunately, we can avoid working at the ground level based on the austere system of synthetic axioms stated in Definition 1.[2] Instead, for we shall work entirely from the

[2] A monograph that does do this in a lot of detail is Schwabhäuser et al. (1983).

Representation Theorem for $\mathsf{EG}(2)$ instead. Furthermore, we don't state this model-theoretically. Instead, we assume our ordinary ambient mathematical theory, along with the $\mathsf{EG}(2)$ axioms as primitive. In this context, we prove representation directly. We assume as primitive a unary predicate $\mathtt{point}(p)$, with the set $\mathbb{P}$ defined as $\{p \mid \mathtt{point}(p)\}$. The $\mathsf{EG}(2)$ axioms are assumed to hold on $\mathbb{P}$. We let B be defined as $\{(p, q, r) \in \mathbb{P}^3 \mid \mathtt{B}(p, q, r)\}$. We let $\equiv$ be defined as $\{(p, q, r, s) \in \mathbb{P}^4 \mid pq \equiv rs\}$. (This overloads $\equiv$, of course.)

So, we are assuming $\mathsf{EG}(2)$ is true. Equivalently, we assume:[3]

$$(\mathbb{P}, B, \equiv) \models \mathsf{EG}(2). \tag{5}$$

Definition 2. We define, for coordinate points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \mathbb{R}^2$ (where $\mathbf{x} = (x_1, x_2)$, and so on):

$$\begin{aligned} B_{\mathbb{R}^2}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= (\exists \alpha \in [0, 1])\, (\mathbf{y} - \mathbf{x}) = \alpha(\mathbf{z} - \mathbf{x})) && (6) \\ \Delta_2(\mathbf{x}, \mathbf{y}) &:= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} && (7) \\ \equiv_{\mathbb{R}^2} (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) &:= \Delta_2(\mathbf{x}, \mathbf{y}) = \Delta_2(\mathbf{z}, \mathbf{w}) && (8) \end{aligned}$$

The first relation $B_{\mathbb{R}^2}$ is the *Cartesian coordinate betweenness relation* on $\mathbb{R}^2$. The second (a function) Δ_2 is the *Cartesian coordinate Euclidean metric* on $\mathbb{R}^2$. The third relation $\equiv_{\mathbb{R}^2}$ is the *Cartesian coordinate congruence relation* on $\mathbb{R}^2$.

Theorem 1 (Line Representation Theorem)**.** Let O, I be points, with $O \neq I$. Let $\ell = \ell(O, I)$, the line through O and I. Let $+, \times, \leq$ be the *geometrical* addition, multiplication and order operations on ℓ.[4] Then there is a unique function $\varphi : \ell \to \mathbb{R}$ such that:

(1) $\varphi(O) = 0$.
(2) $\varphi(I) = 1$.
(3) φ is a bijection.
(4) for all $p, q \in \ell$: $\varphi(p) + \varphi(q) = \varphi(p + q)$.
(5) for all $p, q \in \ell$: $\varphi(p) \times \varphi(q) = \varphi(p \times q)$.
(6) for all $p, q \in \ell$: $\varphi(p) \leq \varphi(q) \leftrightarrow p \leq q$.

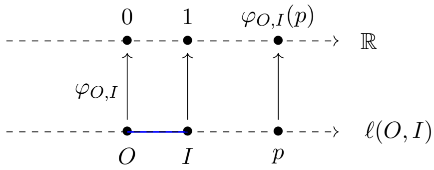
Consequently, φ is an isomorphism, and we denote this isomorphism $\varphi_{O,I}$, to register its dependence on O, I. It might be called "*the local coordinate system*" on the line $\ell(O, I)$. Following Burgess (Burgess & Rosen (1997): 107), we may call the two points O, I "benchmarks".

[3] $\mathsf{EG}(2)$, having 11 axioms, can be considered a single sentence. Then, the provable equivalence of the geometrical sentence $\mathsf{EG}(2)$ with the semantical sentence $(\mathbb{P}, B, \equiv) \models \mathsf{EG}(2)$ is an example of a Tarski T-sentence, analogous to the equivalence of "snow is white" and " 'snow is white' is true".

[4] See Bennett (1995) for the definitions of geometrical additional $+$ and multiplication $\times$ on a line given by two parameters. Here, we overload the symbols. Context will always disambiguate. These definitions go back to Hilbert (1899). See the proof of Theorem 1 in Tarski (1959) for the definition of the geometrical order $\leq$ on a line (in terms of betweenness). A similar definition is explained in Szmielew (1983).

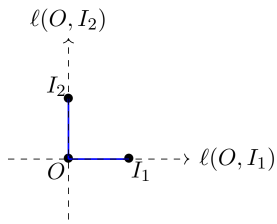
We may indicate this isomorphism $\varphi_{O,I}$ from the line $\ell(O, I)$ to the real numbers like this:

Figure 2: Line representation: $\varphi_{O,I}$



Definition 3. Let O, I_1, I_2 be three points in $\mathbb{P}$. Then we say that the triple (O, I_1, I_2) is a *Euclidean 2-frame* in $\mathbb{P}$ if $O \neq I_1$, $O \neq I_2$, $I_1 \neq I_2$, and $OI_1 \equiv OI_2$ and $OI_1 \perp OI_2$ (note: perpendicularity $\perp$ is definable using $\equiv$). Again, following Burgess, we can call these three points "benchmarks".

Figure 3: Euclidean 2-frame



Alfred Tarski's Theorem 1 in Tarski (1959) states the core Representation Theorem for $\mathsf{EG}_0(2)$—i.e., the first-order version of $\mathsf{EG}(2)$. Its proof is sketched there. And in much more detail in Schwabhäuser et al. (1983). Tarski's Theorem 1 in Tarski (1959) is for the *first-order* (or "elementary") synthetic theory, $\mathsf{EG}_0(2)$. But we are interested in second-order theory, $\mathsf{EG}(2)$. For the second-order theory, the Representation Theorem is the following:

Theorem 2 (Representation Theorem for D=2 Synthetic Euclidean Geometry)**.** Let (O, I_1, I_2) be a Euclidean 2-frame. Then there is a unique function $\Phi : \mathbb{P} \to \mathbb{R}^2$ such that

(1) $\Phi(O) = (0, 0)$.
(2) $\Phi(I_1) = (1, 0)$.
(3) $\Phi(I_2) = (0, 1)$.
(4) Φ is a bijection.
(5) for all $p, q, r \in \mathbb{P}$: $\mathsf{B}(p, q, r)$ iff $B_{\mathbb{R}^2}(\Phi(p), \Phi(q), \Phi(r))$.
(6) for all $p, q, r, s \in \mathbb{P}$: $pq \equiv rs$ iff $\Delta_2(\Phi(p), \Phi(q)) = \Delta_2(\Phi(r), \Phi(s))$.

This Φ is called *the Cartesian coordinate system* determined by the Euclidean 2-frame O, I_1, I_2. Any function Φ satisfying (4), (5), (6) is an *isomorphism*:

$$(\mathbb{P}, B, \equiv) \overset{\Phi}{\cong} (\mathbb{R}^2, B_{\mathbb{R}^2}, \equiv_{\mathbb{R}^2}). \tag{9}$$

Given a Cartesian coordinate system $\Phi : \mathbb{P} \to \mathbb{R}^2$, one may define the *coordinate length* $\Delta_\Phi(a, b)$ of the line segment ab.

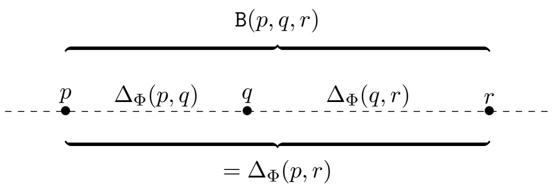
Definition 4. $\Delta_\Phi(a, b) := \Delta_2(\Phi(a), \Phi(b))$.

Below we shall need two small lemmas about coordinate distances:

Lemma 1. $\mathtt{B}(p, q, r) \leftrightarrow \Delta_\Phi(p, q) + \Delta_\Phi(q, r) = \Delta_\Phi(p, r)$.

Thus, q lies (inclusively) between p and r if and only if the sum of the coordinate distance from p to q with the coordinate system from q to r is equal to the coordinate distance from p to r:

Figure 4: Lemma 1



The formula $[\Delta_\Phi(p, q) + \Delta_\Phi(q, r) = \Delta_\Phi(p, r)]$ is rather interesting. This is because, for any Cartesian coordinate systems Φ, Ψ, we have:

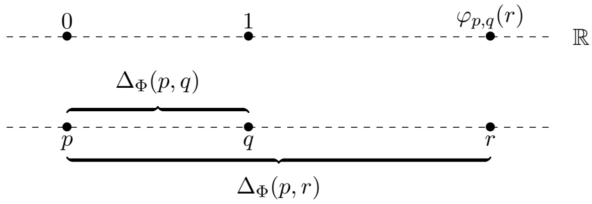
$$\overbrace{[\Delta_\Phi(p, q) + \Delta_\Phi(q, r) = \Delta_\Phi(p, r)]}^{\text{covariant formula}} \leftrightarrow \overbrace{[\Delta_\Psi(p, q) + \Delta_\Psi(q, r) = \Delta_\Psi(p, r)]}^{\text{covariant formula}} \tag{10}$$

This is an example of *covariance*.

Lemma 2. If $p \neq q$ and either $\mathtt{B}(p, r, q)$ or $\mathtt{B}(p, q, r)$, then: $\varphi_{p,q}(r) = \frac{\Delta_\Phi(p,r)}{\Delta_\Phi(p,q)}$.

Thus, under the stated side conditions, the quantity $\varphi_{p,q}(r)$ is equal to the *ratio* of the coordinate distance p to r and the coordinate distance p to q:

Figure 5: Lemma 2



The ratio $\frac{\Delta_\Phi(p,r)}{\Delta_\Phi(p,q)}$ is *coordinate invariant*. That is, if $p \neq q$ and p, q, r are collinear, then, for any coordinate systems Φ, Ψ, we have:

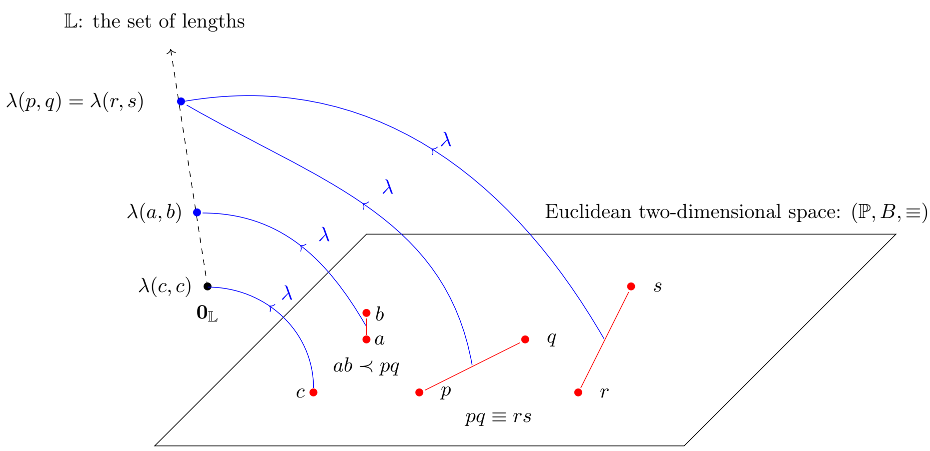
$$\frac{\Delta_\Phi(p,r)}{\Delta_\Phi(p,q)} = \frac{\Delta_\Psi(p,r)}{\Delta_\Psi(p,q)} \tag{11}$$

3 Definitions

We next wish to define length. Not *coordinate* length: that's easy (Definition 4). But genuine *coordinate-free*, *unit-independent* length. Any length will be the value of a binary function λ:

$$\lambda : \mathbb{P}^2 \to \mathbb{L} \tag{12}$$

Figure 6: Lengths of Line Segments



Each segment (which can be identified with the ordered pair of its endpoints) is mapped by λ to a length in $\mathbb{L}$. To be clear, we do not assume that a length is a *number*.

Rather, a length $\lambda(a, b)$ is an coordinate-independent abstract object associated with a segment ab between two points a and b.

We next define, using geometrical methods in Euclidean geometry, a *zero length* ($\mathbf{0}$), a *length addition operation* ($\oplus$), a *linear order* ($\preceq$), and a *scalar multiplication* by non-negative reals ($\bullet$). Assembled together, we obtain a certain structure, the *Length Quantity*:[5]

$$\mathbb{L} := (\mathbb{L}, \mathbf{0}, \oplus, \preceq, \bullet). \tag{13}$$

(Here, I follow the usual mathematical practice of conflating the name of a *structure* with the name of its underlying *carrier set.*)

From the definition of lengths, we shall immediately obtain the required abstraction principle for length:[6]

$$\lambda(a, b) = \lambda(c, d) \leftrightarrow ab \equiv cd \tag{14}$$

Above, we defined the *coordinate length* $\Delta_\Phi(a, b)$ of a segment ab, given a coordinate system Φ. But what is the relationship between the *coordinate length* $\Delta_\Phi(a, b)$ of a segment ab, and its *length* $\lambda(a, b)$?

Definition 5. Lengths are equivalence classes (of ordered pairs):

$$[(a, b)]_\equiv \quad := \quad \{(c, d) \in \mathbb{P}^2 \mid cd \equiv ab\} \tag{15}$$

$$\lambda(a, b) \quad := \quad [(a, b)]_\equiv \tag{16}$$

$$\mathbb{L} \quad := \quad \{\lambda(a, b) \mid a, b \in \mathbb{P}\} \tag{17}$$

Next, we define the *zero* length:

$$\mathbf{0}_\mathbb{L} \quad := \quad \lambda(a, a) \tag{18}$$

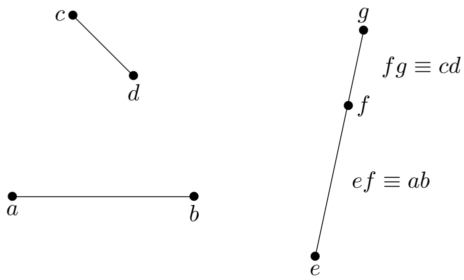
Next, we define *addition* of lengths:

[5] There is a minor terminological conflation, in measurement theory literature, between the notion of *Length*—as a Quantity or quantitative property—and the notion of specific or *individual lengths*, like 3 cm and 25 m. I shall simply acquiesce in this, but sometimes capitalize and italicize, using "*Length*" to mean the quantitative property as a whole, and "length" to mean a specific element of Length. Some authors, and this is standard in physics, instead say that *Length* is a "*quantitative property*" (e.g., Eddon (2013)) or say that *Length* is a "*physical attribute*". On our view, *Length* is a certain kind of abstract structure, related to a linearly ordered vector space, and specific lengths are "vectors" in that space, and units are basis vectors.

[6] Instead of defining length by equivalence classes, as done here, one could take the abstraction axiom (14) as a *primitive* (along with a surjectivity axiom: each length has a witness), then develop the same theory as we give—though it is far more work, as we don't have an ambient mathematical theory to exploit. The methods required for this are explained in Burgess (1984).

$$\lambda(a,b) \oplus_{\mathbb{L}} \lambda(c,d) = \lambda(e,g) \quad := \quad (\exists f \in \mathbb{P})\,(\mathtt{B}(e,f,g) \wedge ef \equiv ab \wedge fg \equiv cd) \tag{19}$$

Figure 7: Definition of length addition



Order $\preceq_{\mathbb{L}}$ is then defined straightforwardly in terms of addition:

$$\lambda(a,b) \preceq_{\mathbb{L}} \lambda(c,d) \quad := \quad (\exists e,f \in \mathbb{P})\,(\lambda(a,b) \oplus_{\mathbb{L}} \lambda(e,f) = \lambda(c,d)) \tag{20}$$

Finally, we define *scalar multiplication* of lengths by a non-negative real x. This requires two cases: $a = b$ and $a \neq b$. First, if $a = b$:

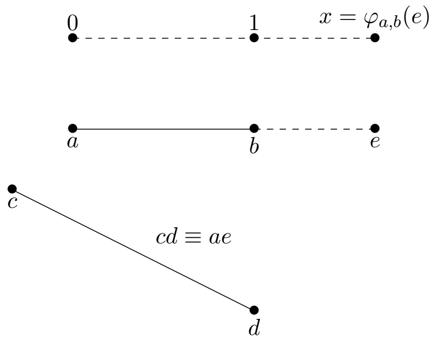
$$x \bullet_{\mathbb{L}} \lambda(a,b) \quad := \quad \mathbf{0}_{\mathbb{L}} \tag{21}$$

Second, if $a \neq b$:

$$\lambda(c,d) = x \bullet_{\mathbb{L}} \lambda(a,b) := \quad (\exists e \in \mathbb{P})\,((\mathtt{B}(a,b,e) \vee \mathtt{B}(a,e,b)) \wedge cd \equiv ae \wedge \varphi_{a,b}(e) = x) \tag{22}$$

We depict this as follows:

Figure 8: Definition of scalar multiplication by a real $x \in \mathbb{R}_0^+$



We now drop the subscripts, and assemble these operations into a package:

Definition 6. $\mathbb{L} := (\mathbb{L}, \mathbf{0}, \oplus, \preceq, \bullet)$. This the *Length structure.*

We shall prove below (Theorem 3) that the Length structure $\mathbb{L}$ is (isomorphic to) exactly the *pointed positive cone of the one-dimensional vector space over* $\mathbb{R}$.

Definition 7. A *length* l is an element of $\mathbb{L}$.

For us, lengths are not "properties", or "attributes". Rather, they are simply elements of the structure $\mathbb{L}$ (i.e. equivalence classes of segments). For example, 5 cm is a length: an equivalence class of segments.

Definition 8. A *unit length* is a positive (hence, non-zero) element of $\mathbb{L}$.

For example, 1.616255×10^{-35} m is a unit length (which physicists call the Planck length). Physicists usually denote it l_P.

4 Theorems

Equipped with the above definitions, we can prove a series of theorems about the Length structure $(\mathbb{L}, \mathbf{0}, \oplus, \preceq, \bullet)$. For example:

Lemma 3. The following hold for $(\mathbb{L}, \mathbf{0}, \oplus, \preceq, \bullet)$, for any $l, l_1, l_2, l_3 \in \mathbb{L}$, and $x, y \in \mathbb{R}^+_0$:

(1) $\mathbf{0} \preceq l$.
(2) $l_1 \preceq l_2 \wedge l_2 \preceq l_1 \to l_1 = l_2$.
(3) $l_1 \preceq l_2 \wedge l_2 \preceq l_3 \to l_1 \preceq l_3$.
(4) $l_1 \preceq l_2 \vee l_2 \preceq l_1$.
(5) $\exists l_2\,(l_1 \prec l_2)$.
(6) $l_1 \prec l_2 \to \exists l_3\,(l_1 \prec l_3 \prec l_2)$.
(7) If X is a non-empty bounded set of lengths, then $\sup X$ exists.
(8) $l \oplus \mathbf{0} = \mathbf{0} \oplus l = l$.
(9) $l_1 \oplus l_2 = l_2 \oplus l_1$.
(10) $l_1 \oplus (l_2 \oplus l_3) = (l_1 \oplus l_2) \oplus l_3$.
(11) $l_1 \preceq l_2 \to (l_1 \oplus l_3 \preceq l_2 \oplus l_3)$.
(12) $x \bullet (y \bullet l) = xy \bullet l$.
(13) $1 \bullet l = l$.
(14) $x \bullet (l_1 \oplus l_2) = x \bullet l_1 \oplus x \bullet l_2$.
(15) $(x + y) \bullet l = x \bullet l \oplus y \bullet l$.
(16) $0 \bullet l = \mathbf{0}$.

The results (1)–(6) imply that the reduct $(\mathbb{L}, \mathbf{0}, \preceq)$ is an unbounded linear order with a least element. Result (7) states that $\preceq$ is "Dedekind order complete". So, $(\mathbb{L}, \mathbf{0}, \preceq)$ is an *unbounded Dedekind order-complete, linear order with a least element.* Results (8)–(10) state that the reduct $(\mathbb{L}, \mathbf{0}, \oplus)$ is a *commutative monoid.* Result (11) states the compatibility of $\oplus$ and $\preceq$. Consequently, results (1)–(11) state that $(\mathbb{L}, \mathbf{0}, \oplus, \preceq)$ is an *ordered commutative monoid which is Dedekind order complete.* The results (11)–(15)

are "vector space" axioms, but we here do not have a vector space over $\mathbb{R}$, as there are no negative lengths.

In principle, we could use these conditions (1)–(16) to prove Theorem 3 below. Although these axioms are slightly different from Hölder's, this would reproduce Hölder's Theorem (Hölder (1901)). However, it is far easier to prove Theorem 3 directly, as we shall do. Then results (1)–(16) are mostly easy consequences of Theorem 3.

Lemma 4. Immediate from the definitions we have an abstraction principle, (A1), and a surjectivity principle, (A2):

$$\text{(A1)} \qquad \lambda(a,b) = \lambda(c,d) \leftrightarrow ab \equiv cd. \tag{23}$$

$$\text{(A2)} \quad (\forall l \in \mathbb{L})\,(\exists a, b \in \mathbb{P})\,(l = \lambda(a,b)). \tag{24}$$

Lemma 5. Let $\Phi : \mathbb{P} \to \mathbb{R}^2$ be a standard Cartesian coordinate system. Let $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+_0$ be the coordinate distance function relative to Φ. Then:

$$\lambda(a,b) = \lambda(c,d) \quad \leftrightarrow \quad \Delta_\Phi(a,b) = \Delta_\Phi(c,d) \tag{25}$$

Proof. The Representation Theorem implies that $ab \equiv cd \leftrightarrow \Delta_\Phi(a,b) = \Delta_\Phi(c,d)$, and, using (A1), we obtain the claim. □

Lemma 6. $\lambda(a,b) = \mathbf{0}$ if and only if $a = b$.

Proof. By Lemma 5, $\lambda(a,b) = \lambda(c,c) \leftrightarrow \Delta_\Phi(a,b) = \Delta_\Phi(c,c)$. But $\lambda(c,c) = \mathbf{0}$ and $\Delta_\Phi(c,c) = 0$. So, $\lambda(a,b) = \mathbf{0} \leftrightarrow \Delta_\Phi(a,b) = 0$. But $\Delta_\Phi(a,b) = 0$ if and only if $a = b$. So, $\lambda(a,b) = \mathbf{0} \leftrightarrow a = b$. □

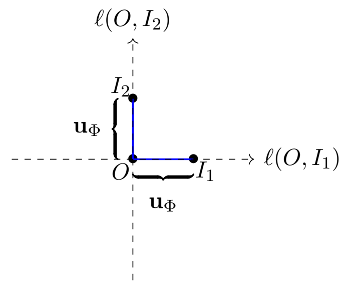
Definition 9. Let Φ be a Cartesian coordinate system based on Euclidean 2-frame (O, I_1, I_2). Let $\mathbf{u}_\Phi := \lambda(O, I_1)$. I.e., the length of OI_1.

Lemma 7. Let Φ be a Cartesian coordinate system based on Euclidean 2-frame (O, I_1, I_2). Then $\mathbf{u}_\Phi$ is a unit length.

Proof. Since $O \neq I_1$, we have $\lambda(O, I_1) \neq \mathbf{0}$, by Lemma 6. Hence, $\mathbf{u}_\Phi$ is a unit length. □

Thus, any Cartesian coordinate system determines a unique unit length. It is essentially the length of the segment from the origin O to the point labelled 1 on the x-axis:

Figure 9: Unit length $\mathbf{u}_\Phi$ of a Cartesian coordinate system



The next definition is the key to all the following theorems.[7]

Definition 10. Fix a standard Cartesian chart $\Phi : \mathbb{P} \to \mathbb{R}^2$ with coordinate distance function $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}_0^+$. We define a function,

$$h_\Phi : \mathbb{L} \to \mathbb{R}_0^+ \tag{26}$$

as follows. For any $l \in \mathbb{L}$, if $l = \lambda(a, b)$, we define:[8]

$$h_\Phi(l) \quad := \quad \Delta_\Phi(a, b) \tag{27}$$

In particular, for any points $a, b \in \mathbb{P}$:

$$h_\Phi(\lambda(a, b)) \quad = \quad \Delta_\Phi(a, b) \tag{28}$$

Lemma 8. $h_\Phi(\mathbf{u}_\Phi) = 1$.

Thus, the *coordinate length*, relative to the coordinate system Φ, of the unit length $\mathbf{u}_\Phi$ is 1, as expected.

Definition 11. Let $(\mathbb{R}, 0, +, \leq, \bullet)$ be the one-dimensional linearly ordered vector space over $\mathbb{R}$. Let $\mathcal{Q}$ be its restriction to the set $\{r \in \mathbb{R} \mid 0 \leq r\}$ of non-negative elements (this is a pointed convex cone).

The fact that base quantities in physics—Mass, Length, Time—are *one-dimensional vector spaces*, or something like the non-negative fragment of a vector space, has always seemed intuitively obvious to me. And then composite or derived quantities (Velocity,

[7] It took me an agonizingly long time to figure how to get this to work. I knew that the relationship expressed by (28) must be true, and so I attempted to prove it, fruitlessly. Eventually, I realised that (28) is really just a definition. When that became clear, the theorems below all fall out immediately.

[8] This is independent of the representative ab. For if $l = \lambda(a, b)$ and $l = \lambda(c, d)$, then $ab \equiv cd$; whence, $\Delta_\Phi(a, b) = \Delta_\Phi(c, d)$.

Frequency, and so on) are *tensor products* and *duals*. I have never seen this stated in any of the philosophy of science literature, or in the Representational Theory of Measurement literature, or the metaphysics of quantities literature. On my side, though, Terence Tao:[9]

> For instance, to continue the example of the $\{M, L, T\}$ system of dimensions from the previous section, we can postulate the existence of three one-dimensional real vector spaces $\{V^M, V^L, V^T\}$ (which are supposed to represent the vector space of possible masses, lengths, and times, where we permit for now the possibility of negative values for these units). As it is physically natural to distinguish between positive and negative masses, lengths, or times, we endow these one-dimensional spaces with a total ordering (obeying the obvious compatibility conditions with the vector space structure), so that these spaces are *ordered one-dimensional real vector spaces*. However, we do not designate a preferred unit in these spaces (which would identify each of them with $\mathbb{R}$). (Tao (2012): paragraph 69)

Tao says that "we can **postulate** the existence of three one-dimensional real vector spaces $\{V^M, V^L, V^T\}$". In general, this postulation is definitely right for the foundations of physics. But here we actually *prove* that the length structure, built into Euclidean geometry so to speak, is the positive cone of a linearly ordered one-dimensional real vector space. That is, we prove that the Length structure is isomorphic to $\mathcal{Q}$ (this is the main theorem of this paper):

Theorem 3. Let Φ be a Cartesian coordinate system. Let h_Φ be the function defined above. (We drop the $_\Phi$ subscript for clarity.) Then:

$$\mathbb{L} \overset{h}{\cong} \mathcal{Q}. \tag{29}$$

Proof. We must verify six claims, as follows:

Claim 1: h is injective.

Pick two lengths, say $l_1 = \lambda(a, b)$ and $l_2 = \lambda(c, d)$. Let $h(l_1) = h(l_2)$. Hence, $\Delta_\Phi(a, b) = \Delta_\Phi(c, d)$. Hence, $\lambda(a, b) = \lambda(c, d)$, and so $l_1 = l_2$.

Claim 2: h is surjective.

Let $x \in \mathbb{R}^+_0$. Now $\Delta_\Phi : \mathbb{P}^2 \to \mathbb{R}^+_0$ is surjective. Hence, there exists $a, b \in \mathbb{P}$ such that $x = \Delta_\Phi(a, b)$. Hence, $x = h(\lambda(a, b))$. Hence, we have $l \in \mathbb{L}$, with $x = h(l)$.

Claim 3: $h(\mathbf{0}) = 0$.

$\mathbf{0} = \lambda(a, a)$. So, $h(\mathbf{0}) = h(\lambda(a, a)) = \Delta_\Phi(a, a) = 0$

Claim 4: $h(\lambda(a, b) \oplus \lambda(c, d)) = h(\lambda(a, b)) + h(\lambda(c, d))$.

Let $\lambda(e, g) = \lambda(a, b) \oplus \lambda(c, d)$. We claim that $h(\lambda(e, g)) = h(\lambda(a, b)) + h(\lambda(c, d))$. By the definition of $\oplus$, there exists $f \in \mathbb{P}$ such that $\mathtt{B}(e, f, g)$, and $ef \equiv ab$, and $fg \equiv cd$. So, $\Delta_\Phi(e, f) = \Delta_\Phi(a, b)$, and $\Delta_\Phi(f, g) = \Delta_\Phi(c, d)$, and $\Delta_\Phi(e, g) = \Delta_\Phi(e, f) + \Delta_\Phi(f, g)$. (We

[9] Also, after writing up this material, I found the articles Whitney (1968a) and Whitney (1968b). I found these two articles a bit hard to understand. But I believe Whitney's formulation of these notions (he calls Quantities *rays*) to be more or less the same as what I sketched above, and what Tao sketches in his remarks.

use Lemma 1.) So, $\Delta_\Phi(e,g) = \Delta_\Phi(a,b) + \Delta_\Phi(c,d)$. So, $h(\lambda(e,g)) = h(\lambda(a,b)) + h(\lambda(c,d))$, as claimed.

Claim 5: $\lambda(a,b) \preceq \lambda(c,d) \leftrightarrow h(\lambda(a,b)) \leq h(\lambda(c,d))$.

DIY.

Claim 6: $h(x \bullet \lambda(a,b)) = x \cdot h(\lambda(a,b))$.

The case $a = b$ is trivial, so we assume $a \neq b$. Let $\lambda(c,d) = x \bullet \lambda(a,b)$. We claim that $h(\lambda(c,d)) = x \cdot h(\lambda(a,b))$. By the definition of $\bullet$, there exists $e \in \mathbb{P}$ such that $\mathtt{B}(a,b,e)$, and $cd \equiv ae$, and $\varphi_{a,b}(e) = x$. We have: $\Delta_\Phi(c,d) = \Delta_\Phi(a,e)$, and by Lemma 2, we have: $x\Delta_\Phi(a,b) = \Delta_\Phi(a,e)$. So: $x\Delta_\Phi(a,b) = \Delta_\Phi(c,d)$, and hence $h(\lambda(c,d)) = x \cdot h(\lambda(a,b))$, as claimed.

□

Definition 12. A *numerical scale* for $\mathbb{L}$ is an isomorphism $h : \mathbb{L} \to \mathcal{Q}$.

Theorem 3 implies:

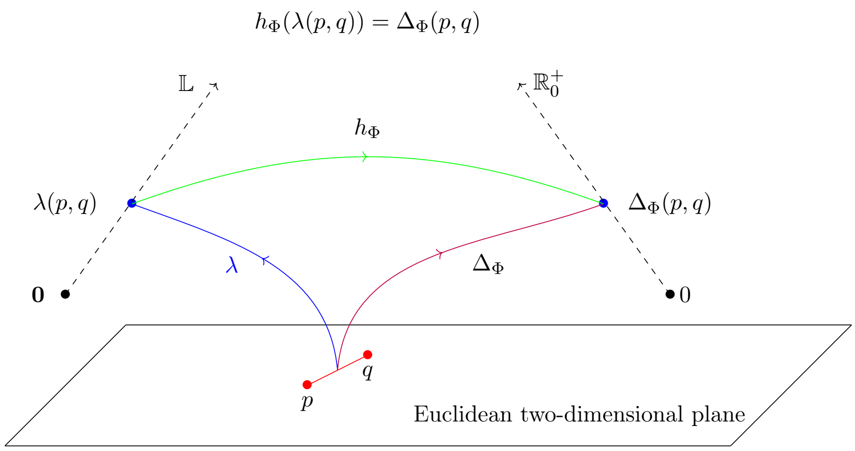
Theorem 4. Let Φ be a Cartesian coordinate system. Then h_Φ is a numerical scale for $\mathbb{L}$.

We can now graphically depict the relation between

- the *Length* function, λ,
- the *coordinate length* function, Δ_Φ,
- the corresponding *numerical* scale, h_Φ,

as follows:

Figure 10: The relation between λ, Δ_Φ and h_Φ



Lemma 9. Let $l \in \mathbb{L}$ be a length. Let $\mathbf{u} \in \mathbb{L}$ be a unit length. There exists a unique $x \in \mathbb{R}_0^+$ such that:

$$l = x \bullet \mathbf{u} \tag{30}$$

Proof. Let $\mathbb{L} \overset{h}{\cong} \mathcal{Q}$ be a numerical scale. Let $w = h(\mathbf{u})$ and let $y = h(l)$. Since $\mathbf{u}$ is a unit, $\mathbf{0} \prec \mathbf{u}$, and hence, by the isomorphism, $0 < h(\mathbf{u})$. So, $0 < w$ and we can divide by w. Let $x := \frac{y}{w} = \frac{h(l)}{h(\mathbf{u})}$. So, $xh(\mathbf{u}) = h(l)$. Since h is an isomorphism, it preserves scalar multiplication $\bullet$. So, $h(x \bullet \mathbf{u}) = h(l)$. And therefore, by injectivity of h, we have: $x \bullet \mathbf{u} = l$, as claimed.

□

Definition 13. Let $l \in \mathbb{L}$ be a length. Let $\mathbf{u} \in \mathbb{L}$ be a unit length. Then we define:

$$\|l\|_{\mathbf{u}} \quad := \quad \text{the unique } x \text{ such that } l = x \bullet \mathbf{u}. \tag{31}$$

The number $\|l\|_{\mathbf{u}}$ is the *magnitude* of the length l relative to the unit $\mathbf{u}$.

Theorem 5 (Maxwell's magnitude formula)**.** Let l be a length. Let $\mathbf{u}$ be a unit. Then:

$$l = \|l\|_{\mathbf{u}} \bullet \mathbf{u}. \tag{32}$$

This provides us with the precise relationship between a length l and the numerical magnitude $\|l\|_{\mathbf{u}}$ of that length relative to any unit $\mathbf{u}$. Theorem 5 may remind the reader of James Clerk Maxwell's analysis of a Quantity:

> EVERY expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity. (Maxwell (1873): 1)

The BIPM (*Bureau International des Poids et Mesures*) summarizes Maxwell's analysis as follows:

> **Defining the unit of a quantity**
> The value of a quantity is generally expressed as the product of a number and a unit. The unit is a particular example of the quantity concerned which is used as a reference, and the number is the ratio of the value of the quantity to the unit. (BIPM (2022): 127)

A standard formulation goes as follows:

> In the scientific literature, especially in the context of metrology, a statement such as physical quantity = pure number × unit is frequently used, sometimes also expressed by the equation
>
> $$Q = \{Q\}[Q],$$
>
> where Q denotes a quantity, $\{Q\}$ a numerical value, and $[Q]$ a unit, both related to the quantity Q. (Krystek (2021): 1)

I refer to

$$Q = \{Q\}[Q],$$

as *Maxwell's magnitude formula*.
Theorem 5 implies:

Theorem 6. Let $\mathbf{u} \in \mathbb{L}$ be a unit length. Then, for any points $a, b \in \mathbb{P}$:

$$\lambda(a,b) = \|\lambda(a,b)\|_{\mathbf{u}} \cdot \mathbf{u} \tag{33}$$

Theorem 7. Let Φ be a Cartesian coordinate system. Let $\mathbf{u}_\Phi \in \mathbb{L}$ be its unit length. Then, for any points $a, b \in \mathbb{P}$:

$$\Delta_\Phi(a,b) = \|\lambda(a,b)\|_{\mathbf{u}_\Phi} \tag{34}$$

Proof. We begin with Definition 10, and proceed by an equation stream:

$$\begin{aligned}
\Delta_\Phi(a,b) &= h_\Phi(\lambda(a,b)) && \text{(Definition 10)} && (35)\\
&= h_\Phi(\|\lambda(a,b)\|_{\mathbf{u}_\Phi} \cdot \mathbf{u}_\Phi) && \text{(Theorem 6)} && (36)\\
&= \|\lambda(a,b)\|_{\mathbf{u}_\Phi} h_\Phi(\mathbf{u}_\Phi) && \text{(Theorem 3(6))} && (37)\\
&= \|\lambda(a,b)\|_{\mathbf{u}_\Phi} && \text{(Lemma 8)} && (38)
\end{aligned}$$

□

Theorem 8. Let Φ be a Cartesian coordinate system. Let $\mathbf{u}_\Phi \in \mathbb{L}$ be its unit length. Then, for any points $a, b \in \mathbb{P}$:

$$\overbrace{\lambda(a,b)}^{\text{length}} = \overbrace{\Delta_\Phi(a,b)}^{\text{numerical magnitude}} \cdot \overbrace{\mathbf{u}_\Phi}^{\text{unit}} \tag{39}$$

Proof. Immediate from Theorem 6 and Theorem 7. □

This provides us with, for any Cartesian coordinate system Φ, the precise relationship between a segment's *length*, $\lambda(a, b)$, the *numerical magnitude* $\Delta_\Phi(a, b)$ of that length in the system Φ, and the *unit length* $\mathbf{u}_\Phi$ of that system Φ.

The final theorem characterizes the symmetry group of the Length structure $\mathbb{L}$:

Theorem 9. $\text{Aut}(\mathbb{L}) \cong (\mathbb{R}^+, 1, \times)$.

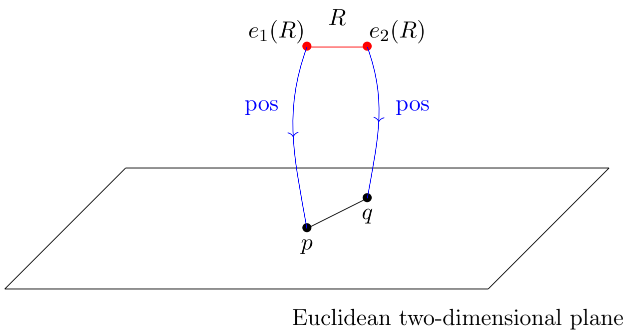
Proof. The length structure $\mathbb{L}$ is isomorphic to $\mathcal{Q}$, and the automorphism group of $\mathcal{Q}$ is well-known to be the multiplicative group $(\mathbb{R}^+, 1, \times)$ of positive reals. □

5 Rods

We have so far only defined length $\lambda(a, b)$ for a spatial line segment ab. But not for material objects. Material objects lie in space, and occupy some region. This occupation relationship is given by a *position* function, we'll call pos. A material object which is to a good approximation thin, and also reasonably straight, has two endpoints (i.e., two material points). It always lies along some line segment (in space), so that the material endpoints of this object match the geometric endpoints of that segment. This sort of material object is traditionally called a *rod*. Usually it is also required that the rod be *rigid*: it maintains its length over time. Here, though, we ignore time and motion.

To illustrate, let a material rod R have material endpoints $e_1(R)$ and $e_2(R)$. Let $\text{pos}(e_1(R)) = p$ and let $\text{pos}(e_2(R)) = q$. So, R "occupies" the spatial segment pq:

Figure 11: Rod R occupies line segment pq



Let's now apply the theory of length. Imagine benchmarks O, I_1, I_2, with associated coordinate system Φ. We use these benchmarks to define a unit length:

$$\mathbf{u}_\Phi := \lambda(O, I_1) \ (= \lambda(O, I_2)). \tag{40}$$

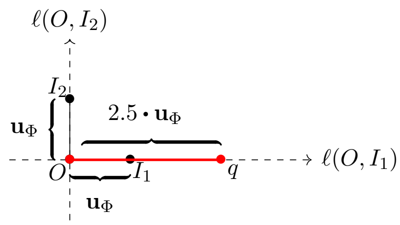
Imagine a small thin metal rod R, with endpoints $e_1(R)$ and $e_2(R)$. Suppose R happens to lie along the $\ell(O, I_1)$-axis as depicted below, with the position of one endpoint

$e_1(R)$ at the origin O, with coordinates (0,0), and the position of the other endpoint $e_2(R)$ at some point q on $\ell(O, I_1)$, with coordinates (0,2.5). Then *the length of the rod* is defined to the *length of the line segment whose endpoints are occupied by the material endpoints of the rod*:

$$\text{the length of } R = \lambda(O, q) = 2.5 \bullet \mathbf{u}_\Phi \tag{41}$$

We depict this:

Figure 12: Length of rod R



We can, in general, for any rod R, define "the length of rod R" as follows:

Definition 14. Let R be a rod with endpoints $e_1(R)$ and $e_2(R)$. Then, the length of R, written $\text{len}(R)$, is defined:

$$\text{len}(R) \quad := \quad \lambda(\text{pos}(e_1(R)), \text{pos}(e_2(R))) \tag{42}$$

Notice that this does not involve units at all.

6 Nelson's Column

We can now resolve the problem we started with in §1, "Introduction". We highlighted the following quantitative data statement ("measurement report"):

$$\text{The length (height) of Nelson's Column} = 5159\,\text{cm}. \tag{43}$$

We asked: what does (43) mean?

First, we implicitly switch from two dimensions, to three dimensions, and assume Euclidean geometry holds for three-dimensional space. The geometrical theory of length abstraction proceeds quite analogously. We are working at a fixed time instant, of course. Second, we are going to approximate Nelson's Column as a *rod*. I do know that Nelson's Column is not a rod, thank you very much. That said, one can *approximate* it as one. Suppose $e_1(NC)$ is bottom material point of Nelson's Column, and suppose $e_2(NC)$ is the top material point of Nelson's Column. These are approximate idealizations, to be sure; but I see no difficulty with such modelling assumptions. Suppose that the position of $e_1(NC)$ is the point O and the position of $e_2(NC)$ is the point t. We are interested in $\lambda(O, t)$. That is, using Definition 14,

$$\text{The length of Nelson's Column} = \lambda(\text{pos}(e_1(NC)), \text{pos}(e_2(NC))) = \lambda(O, t). \tag{44}$$

And $\lambda(O, t)$ is certainly a *length*, as per Definition 7.

Now consider a 10 cm ruler (also a rod). Let the point marked "0" be a, and the point marked "1" be b. We define:

$$1\,\text{cm} \quad := \quad \lambda(a, b) \tag{45}$$

Since $a \neq b$, it follows that 1 cm is a *unit length*, as per Definition 8.

I claim that (43) means:

$$\lambda(O, t) = 5159 \bullet 1\text{ cm} \tag{46}$$

where $\bullet$ is the multiplication operation defined in Definition 5.

7 Concluding Remarks

The geometrical theory of length explained in this paper generalizes to *curves*, as opposed to straight lines: by the usual approximation, taking short straight line segments and summing. The methods of this paper also generalize very easily to synthetic Euclidean geometry of any dimension D (the Upper and Lower Dimension axioms of Definition 1 are modified as appropriate). The methods generalize also to *spacetime* too, and yield an analogous account of *time*. More exactly, instead of Euclidean geometry, we study axiomatic *synthetic Galilean spacetime* (Field (1980), Ketland (2023)). With suitable definitions, we can define an analogous *Time* structure, $\mathbb{T}$ (of finite durations); this is analogous to—and indeed isomorphic to—the length structure $\mathbb{L}$:

$$\mathbb{L} \cong \mathbb{T} \tag{47}$$

The abstraction is now based on *chronological* congruence, $\equiv^{\text{tim}}$. Though trickier, this generalizes too to synthetic Minkowski spacetime (Robb (1911), Robb (1936), Mundy (1986), Goldblatt (1987), Cocco & Babic (2021)). I believe it generalizes also to Riemannian geometries, and relativistic spacetimes, (M, g).[10]

[10]Here there is no synthetic axiomatization or representation theorem, of course. However, we modify the theory of length as follows: the *numerical magnitude* $L_g(\gamma)$ of the length of a finite curve $\gamma : [0, 1] \to$

8 Acknowledgements

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M wrt metric g is defined, as usual, as $\int_0^1 d\lambda \sqrt{g(\dot{\gamma}, \dot{\gamma})}$. This is a number, not a length. How do we get lengths? First, we simulate *congruence* in Riemannian geometry as follows: *congruence of curves* γ, γ', relative to the metric tensor g, is written $\gamma \equiv_g \gamma'$, and is defined by: $\gamma \equiv_g \gamma' \leftrightarrow L_g(\gamma) = L_g(\gamma')$. And then length is defined by taking equivalence classes under congruence, just as we did above. I.e., $\lambda_g(\gamma) = \{\gamma' \mid L_g(\gamma) = L_g(\gamma')\}$. This depends on the metric g, of course. Interestingly, it is easy to show that, if we consider (M, g), then $\lambda_{\alpha g}(\gamma) = \lambda_g(\gamma)$, for any fixed non-negative real α, and curve γ in M.

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