Why $\sqrt{-1}$? The Role of Complex Structure in Quantum Physics

John Earman
Dept. HPS
University of Pittsburgh

Standard textbooks on quantum mechanics present the theory in terms of Hilbert spaces over the field of complex numbers and complex linear operator algebras acting on these spaces. What would be lost (or gained) if a different scalar field, e.g. the real numbers or the quaternions, were used? This issue arose with the birthing of the new quantum theory, and over the decades it has been raised over and over again, drawing a variety of different opinions. Here I attempt to identify and to clarify some of the key points of contention, focusing especially on procedures for complexifying real Hilbert spaces and real algebras of observables.

1 The issue

"Schrödinger put the square root of minus one into the equation, and suddenly it made sense. Suddenly it became a wave equation instead of a heat conduction equation. And Schrödinger found to his delight that the equation has solutions corresponding to the quantized orbits in the Bohr model of the atom. It turns out that the Schrödinger equation describes correctly everything we know about the behavior of atoms. It is the basis of all of chemistry and most of physics. And that square root of minus one means that nature works with complex numbers and not with real numbers ... [Riemann and Weierstrass] always thought of complex numbers as an artificial construction, invented by human mathematicians as a useful and elegant abstraction from real life. It never entered their heads that this artificial number-system that they had invented was in fact the ground on which atoms move. They never imagined that Nature had got there first." (Freeman Dyson 2010, p. 827)[1]

The pioneers of the new quantum theory were led, some of them reluctantly, to a conclusion similar to Dyson's. But in subsequent years both physicists and philosophers concerned with the foundations of quantum mechanics (QM) have felt the need to find a convincing argument for the conclusion, or else a convincing reason to reject it. There are two issues. First, is there a good argument for the conclusion that complex structure is an indispensable ingredient in the empirical success of QM? Second, if so what can be inferred from the indispensability of complex structure about the nature of physical reality? Trying to tackle the second issue involves engaging the debate about scientific realism, a swamp-of-no-return, so it is wise to decline to tread on this treacherous ground. But we should be able to make some headway on the first issue. Let's try.

Conventional textbook quantum mechanics uses the machinery of Hilbert spaces over the complex numbers. But there is a huge array of number fields that could conceivably serve as the "scalars" for a Hilbert space.[2] Due to a remarkable theorem of Solèr (1995) there are only three feasible choices for infinite dimensional Hilbert spaces: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$.

> *Solèr's theorem*[3]: Let $\mathcal{F}$ be a $*$-field.[4] Let V be a vector space over $\mathcal{F}$, and let $(\bullet|\bullet)$ be Hermitian form for V, i.e. a mapping V x V to $\mathcal{F}$ with the properties of an expectation value functional.[5]

[1] It should be noted that either Dyson forgot about his earlier opinion (Dyson 1962) or else he changed his mind in the intervening years.

[2] "Field" is not the technically correct term since the general (but not universal) understanding is that a field involves commutative multiplication, whereas some of the candidates—the quaternions in particular—are noncommutative. Sometimes the term "skew field" is used to cover the noncommutative cases (but this has the flavor of a field that is not really a field). "Division ring" is perhaps the best terminology since it covers both commutative and noncommutative cases. I am not much concerned with terminology as long as we are clear what we are talking about.

[3] See Holland (1995) for an exposition and assessment of the theorem.

[4] Here $*$ is an involution of $\mathcal{F}$, a map $f \mapsto f$ of $\mathcal{F}$ onto itself such that $f^{**} = f$, $(f+g)^* = f^* + g^*$, and $(fg)^* = g^* f^*$ for all $f, g \in \mathcal{F}$. $*$ is the identity for $\mathbb{R}$; for $\mathbb{C}$, $(a+ib)^* = \overline{(a+ib)} = a - ib$, $a, b \in \mathbb{R}$; for quaternions $a + ib + jc + kd$, $a, b, c, d \in \mathbb{R}$, $i^2 = j^2 = -1$, $ij = -ji = k$, $(a + ib + jc + kd)^* = a - ib - jc - kd$.)

[5] (i) $(a\varphi + b\psi|\xi) = a^*(\varphi|\xi) + b^*(\psi|\xi)$
$(\varphi|a\psi + b\xi) = a(\varphi|\psi) + b(\varphi|\xi)$
for all $a, b \in \mathcal{K}$ and $\varphi, \psi, \xi \in V$
(ii) $(\varphi|\psi)^* = (\psi|\varphi)$ for all $\varphi, \psi \in V$

> Suppose that there is an infinite sequence of vectors in V that are ON with respect to $(\bullet|\bullet)$. Then $\mathcal{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The corresponding Hilbert space is the completion of the vector space V with respect to the norm induced by $(\bullet|\bullet)$.[6]

Cassinelli and Lahti (2017) argue that, because of the noncommutative nature of quaternions, trying to do QM over $\mathbb{H}$ leads to problems in describing composite systems in terms of tensor products of component systems (see also Finkelstein et al. 1962 and Araki 1980). No attempt will be made here to evaluate their argument, and in what follows it will simply be assumed that $\mathbb{H}$ can be left aside, and the concentration will be on the issue why $\mathbb{C}$ rather than $\mathbb{R}$.

2 A little potted history: i in wave mechanics and matrix mechanics

In his initial investigation of wave mechanics Schrödinger was willing to use complex numbers to simplify calculations, but when the calculations were done, only the real component of the wave function was to be assigned a physical meaning. As he put it to Lorentz in 1926, the wave function "is surely fundamentally a real function."[7] His desire to implement this attitude can be seen in his attempt to derive a time-dependent wave equation from his time-independent equation

$$\nabla^2\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0. \tag{1}$$

E is eliminated from (1) by assuming $\psi \sim \text{real}(e^{(2\pi i Et)/h})$, differentiating ψ twice with respect to t, and substituting into (1) to obtain an equation that is second order in time and fourth order in spatial coordinates:

(iii) If either $(\varphi|\xi) = 0$ or $(\xi|\varphi) = 0$ for all $\xi \in V$ then $\varphi = 0$.

In some treatments the Hermitian form is required to be conjugate linear in the second argument rather than the first.

[6] In what follows attention will be restricted to separable Hilbert spaces, i.e. those with a countable ON basis..

[7] Quoted in Karam (2020, p. 433). Schrödinger's struggles with a complex wave function are discussed in Yang (1987), Chen (1989, 1990, 1993), and Karam (2020).

$$\left(\nabla^2 - \frac{8\pi^2}{h^2}V\right)^2 + \frac{16\pi^2}{h^2}\frac{\partial^2\psi}{\partial^2 t} = 0 \tag{2}$$

which is analogous to the equation describing a vibrating plate (see Chen 1993). By 1927 he seems to have reconciled to complex wave functions (see Schrödinger 1927). In Schrödinger (1928) what we now call the Schrödinger equation is derived by assuming a periodic time dependence represented by a complex exponential $\psi \sim e^{(2\pi i E t)/h}$, differentiating once with respect to t, and substituting the result into (1) to produce an equation first order in time and second order in spatial coordinates

$$i\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V]\psi,\ \hbar := h/2\pi. \tag{3}$$

The change in Schrödinger's attitude seems to have been driven by the realization that a complex ψ is needed to account for Bohr's relation for the emission frequencies of light (Karam 2020, p. 436).[8]

As for matrix mechanics, the square root of -1 makes an appearance in Heisenberg's pioneering work which involved calculations using complex Fourier transforms. Soon thereafter Born and Jordan (1925) attempted to provide a consistent mathematical framework for Heisenberg's heuristic considerations, and in this framework i appears in important formulas, most notably in what we now call the Heisenberg form of the canonical commutation relations $pq - qp = -i\hbar$. Needless to say, the proponents of matrix mechanics did not have to confront Schrödinger's worry about what significance to assign to the imaginary part of the wave function. It may well be, as Yang (1987) opines, that Heisenberg, Born, and Jordan did not appreciate at the time that making $\mathbb{C}$ play an essential role in QM was a major development.

[8] In Bohmian mechanics the imaginary part of the wave function is, arguably, the most important part. The "guiding equation" for the position Q_k, $k = x, y, z$, of a single spinless particle of mass m is $\frac{dQ_k}{dt} = \frac{\hbar}{m}\,\mathrm{Im}\left(\frac{\psi^*\partial_k\psi}{\psi^*\psi}\right)(Q_k)$. If $\mathrm{Im}(\psi) = 0$ the position of the particle is unchanging.

3 Why $\mathbb{C}$ rather than $\mathbb{R}$?

After the publication of von Neumann's *Mathematische Grundlagen der Quantenmechanik* (1932b) it was generally agreed that wave mechanics and matrix mechanics are not different, competing physical theories but two different forms of the same theory, and that the appropriate mathematical arena for this theory is Hilbert space. von Neumann used Hilbert spaces over a complex field, but in retrospect we can ask whether this was necessary and, if so, why.

First question: What familiar results for Hilbert spaces over $\mathbb{C}$ that seem essential to QM as we know it also hold for Hilbert spaces over $\mathbb{R}$? Spectral theorem—yes. Stone's theorem—in a suitably modified form, yes (see below). Gleason's theorem—Yes. If $\mathbb{R}$ can't be faulted at this very general level, what else can we appeal to that will justify the move to $\mathbb{C}$?

Stueckelberg (1959, 1960) claimed that a non-trivial Heisenberg uncertainty relation for a Hilbert space over $\mathbb{R}$ requires smuggling in a complex structure in the form of an operator J that commutes with all observables. However, at some points he assumed that observables have a pure point spectrum, and his arguments sometimes use questionable heuristic inferences (see Moretti and Oppio 2017). Nevertheless, Stueckelberg's insights will be crucial in what follows.

Subsequently Lahti and Maczynski (1987) argued that complex numbers are required for a proof from equational axioms of a Heisenberg inequality $\Delta A \Delta B \geq h$ for observables A and B, if ΔX for a state ω is interpreted as $Var(X, \omega)$[9] and if $Var(A, \omega)Var(B, \omega) \geq h$ must hold for any state ω of the system for a positive number h. The analogy they give is that the use of Cardano's formulas to calculate the real roots of cubic equations requires the use of complex numbers. I leave it to someone who knows proof theory to evaluate the analogy. And assuming that the analogy holds, what is the payoff for the issue of why a Hilbert space over $\mathbb{C}$ rather than $\mathbb{R}$? Showing that a proof that $Var(A, \omega)Var(B, \omega) \geq h$ holds in any state ω of the system must go through the complex numbers seems to fall short of showing that a non-trivial Heisenberg uncertainty relation is not possible for a Hilbert space over $\mathbb{R}$.

Moretti and Oppio (2017) showed that complex structure, or at least its

[9] $Var(X, \omega) := \omega(X^2) - \omega(X)^2$. Here "state" is being used in he algebraic sense, as explained below.

simulacrum in the form Stueckelberg's operator J, emerges from the Poincaré symmetry. Require that there is a faithful, irreducible, and strongly continuous unitary representation U of the Poincaré group in which $M_U^2 \geq 0$. Then there is a unique (up to sign) complex structure J that commutes with the algebra of observables generated by the unitary representation.[10] Is there some analogous result for the Galilean group in ordinary non-relativistic QM? Or did quantum mechanics have to wait for Minkowski spacetime and relativistic quantum theory to have a reason to move from $\mathbb{R}$ to $\mathbb{C}$?

Recent work by Renou et al. (2021a, 2021b) promises to give a decisive reason for using complex Hilbert spaces, the claim being that QM done on real Hilbert space is incapable of accounting for experimentally verifiable (and, indeed, experimentally verified[11]) predictions of correlations between component systems of a composite system. Some skepticism has been expressed about the effectiveness of their argument (see Finkelstein 2021, Chi and Pan 2022, and Vedral 2023).

In the present work I will invite the reader to step back from the high powered investigations mentioned above in favor of a more humble exploration of what is involved in the choice between QM on a complex Hilbert space vs. QM on a real Hilbert space. Before proceeding it is well to emphasize that the real issue is not complex Hilbert space vs. a real Hilbert space but *QM on a complex Hilbert space* (*complex QM for short*) vs. *QM on a real Hilbert space (real QM for short)*. For present purposes accept the algebraic approach to QM, wherein a quantum system is characterized by three objects: von Neumann algebra of observables acting on a Hilbert space, either a complex algebra $\mathfrak{M}$ acting on a complex $\mathcal{H}$ or a real M acting on a real H, together with a set of admissible states. Here "state" means an expectation value functional ω on the algebra, i.e. $\omega : \mathfrak{M} \rightarrow \mathbb{C}$ (respectively, $\omega : M \rightarrow \mathbb{R}$) that is positive, complex linear on $\mathfrak{M}$ (respectively, real linear on M), and taking the value $\omega(I_{\mathcal{H}}) = 1$ (respectively, $\omega(I_H) = 1$). It is typically assumed that the admissible states are normal, meaning that there is a density operator ρ whereby expectation values are calculated via the trace prescription $\omega(A) = Tr(\rho A)$, $A \in \mathfrak{M}$ or $A \in M$ as the case may be.[12] This assumption will be adopted here. The discussion here will focus on vector states, where

[10] Uniqueness up to sign is the best that can be done since if J simulates a complex structure then so does $\widetilde{J} := -J$.

[11] For the experiments see Chen et al. (2022) and Li et al. (2022).

[12] Equivalently, a normal ω is countably additive on the projection lattice of the algebra. A density operator ρ (aka statistical operator) is a positive selfadjoint operator such that

ω is a vector state means that there is a unit vector $\psi \in \mathcal{H}$ (respectively, $\psi \in H$) such that $\omega(A) = (\psi|A\psi)_{\mathcal{H}}$ (respectively, $\omega(A) = (\psi|A\psi)_H$ for all A in the algebra). In the case of a complex $\mathcal{H}$ the physical/algebraic vector states correspond to unit vectors up to phase since $\omega(A) = (\psi|A\psi)_{\mathcal{H}} = (\psi'|A\psi')_{\mathcal{H}}$ for $\psi' = e^{i\theta}\psi$. In the case of a real H the physical/algebraic vector states correspond to unit vectors up to sign (Moretti and Oppio 2017). This difference makes for problem for real QM as will be discussed presently.

Before embarking on a journey into the bowels of QM it is appropriate to be clear about the substantive issue at stake; namely, is the complex structure of the Hilbert spaces and the algebras of observables essential to the empirical success of conventional complex QM, as is assumed in Dyson's declaration that $\mathbb{C}$ provides the ground on which atoms move? If this issue is kept in focus then the doubling-up bromide that a complex number can be replaced or represented by two real numbers is seen to be beside the point. If real QM cannot duplicate the empirical success of complex QM then repeating the bromide any numbers of times will not change the verdict. On the other hand, if real QM can duplicate the success of complex QM then Dyson's declaration is discredited without help of the bromide.

However, one might think that more sophisticated versions of the bromide can be used to raise doubts about whether the issue of complex vs. real QM is really a substantive one. First version: double-up on the dimension of the Hilbert space; that is, in place of an n-dim complex Hilbert space use a $2n$-dim real Hilbert space where a vector with $2n$ real components is used to represent its complex counterpart with n complex components, and observables are symmetric real linear operators acting on this $2n$-dim space. It turns out that the resulting real QM admits too many observables with too much eigenvalue degeneracy. Second version: a real Hilbert space, unless it has a finite and odd dimension, contains a simulacrum of i that can can be used to complexify the Hilbert space ("internal complexification"), and the real Hilbert space theory is (often claimed to be) empirically equivalent to its complexified version. The trouble here is that real Hilbert space theory used in internal complexification circumscribes the observables in such a way that not all instances of complex QM can be reached by internal complexification of real QM. With these misdirections set aside, the way is open to study another form of complexification ("external complexification") that avoids the circumscription of observables that troubles internal complexifi-

for an ON basis $\xi_j \in \mathcal{H}$, $Tr(\rho) = \sum_j (\xi_j|\rho\xi_j)_{\mathcal{H}} = 1$.

cation. But it too fails to establish the empirical equivalence of real and complex QM because, for different reasons, not every instance of complex QM can be reached by external complexification of an instance of real QM. To understand these issues we need to go on our journey into the bowels of QM. Let's get underway.

4 Why I can't do without you (or someone very much like you)[13]

4.1 Dynamics

For reasons that will not be reviewed here, in conventional QM on a complex Hilbert space $\mathcal{H}$ time translation symmetry is implemented by a one-parameter unitary group $U(t)$, where $U(t+s) = U(t)U(s)$ for all $t, s \in \mathbb{R}$. $U(t)$ is strongly continuous just in case for all $\psi \in \mathcal{H}$ if $t \to t_0$ then $U(t)\psi \to U(t_0)\psi$.[14] In the "Schrödinger picture" the dynamics for vector states takes the form $\psi(t) = U(t)\psi(0)$. Physicists want more than this. Given $U(t)$ they want to extract an equation of motion which, in the case of vector states, is the infinitesimal version of $\psi(t) = U(t)\psi(0)$ obtained by taking the limit as $t \to 0$. And in the other direction they want to be able to construct the unitary dynamics by identifying the appropriate generator for time translation, and they want this generator to be an observable.

These desires are satisfied by appealing to Stone's theorem and a converse. First the converse (proved in Reed and Simon (1980, Theorem VIII.7) and then Stone's theorem for complex $\mathcal{H}$.

> *Theorem.* Let S be a selfadjoint operator acting on a complex $\mathcal{H}$ and let $U(t) := e^{-itS}$, $t \in \mathbb{R}$. Then $U(t)$ is a strongly continuous unitary group and
>
> (i) if $\psi \in \mathcal{H}$ and if $t \to t_0$ then $U(t)\psi \to U(t_0)\psi$
>
> (ii) the limit $lim_{t\to 0} \dfrac{U(t)\psi - \psi}{t}$ exists iff $\psi \in D(S)$, in which case $lim_{t\to 0} \dfrac{U(t)\psi - \psi}{t} = -iS$.

[13] To be read with The Once's version of "I Can't Live without You" playing in the background.

[14] For unitary groups weak and strong continuity coincide.

When $U(t)$ is time translation, then for $\psi(0) \in D(A)$, $\psi(t) = e^{-itS}\psi(0)$ is a solution the Schrödinger equation $\dfrac{d\psi(t)}{dt} = -iS\psi(t)$, or $i\dfrac{d\psi(t)}{dt} = S\psi(t)$.[15]

> *Stone's Theorem*[16]. Let $U(t)$ be a strongly continuous one parameter, $t \in \mathbb{R}$, unitary group acting on $\mathcal{H}$. Then there is a selfadjoint S acting on $\mathcal{H}$ such that $U(t) := e^{-itS}$.

A neutral version of Stone's theorem, applying equally to a complex Hilbert space $\mathcal{H}$ or a real Hilbert space H, asserts that if $U(t)$ is a strongly continuous unitary group acting on $\mathcal{H}$ or H, then there is an anti-selfadjoint (aka skew adjoint) operator A on (i.e., $A^\dagger = -A$) such that $U(t) = e^{tA}$. When $U(t)$ is time translation, for $\psi(0) \in D(A)$, $\psi(t) = e^{tA}\psi(0)$ is a solution the equation $\dfrac{d\psi(t)}{dt} = A\psi(t)$. This equation serves as the Schrödinger equation for a real H.

This real version of the Schrödinger equation is physically awkward. The eigenvalue equation for an anti-selfadjoint operator has solutions only in the purely imaginary numbers so that the spectrum of A cannot represent the energy spectrum of a system of interest unless 0 is regarded as an imaginary number and the system only admits 0 energy. And in any case an anti-selfadjoint A cannot represent an observable if we hew to the idea that observables are represented by selfadjoint operators. If the Hilbert space is over the complex numbers we can simply use i to define a selfadjoint operator $S := iA$, and substituting $-iS$ for A we recover $U(t) = e^{-itS}$ and the standard Schrödinger equation $i\dfrac{d\psi(t)}{dt} = S\psi(t)$. Following the path of canonical quantization S is obtained by writing the classical equations of motion for the system of interest in Hamiltonian form and then substituting appropriate selfadjoint Hilbert space operators for the classical position and momentum variables to obtain the quantum Hamiltonian operator, from which the energy spectrum of the system is obtained.[17]

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[15] The "Schrödinger equation" in this context is not a wave equation but a Hilbert space equation for the time rate of change of the state vector $\psi(t)$. Since $\psi(t)$ is here considered only as a function of t, d/dt is used rather than the partial derivative $\partial/\partial t$ of eq. (3).

[16] For a proof see Reed and Simon (1980, Theorem VIII.8). von Neumann (1932a) showed that when the Hilbert space is separable, the hypothesis of strong continuity of $U(t)$ can be replaced by weak measurability.

[17] When the Stone-von Neumann uniqueness theorem applies one is justified in using

These considerations need not force one out of a real Hilbert space H and into a complex $\mathcal{H}$, but they do indicate that to reach satisfactory physical results H needs to be equipped with a simulacrum of i. Mathematicians refer to this simulacrum as "complex structure." Here a distinction needs to be drawn. "Complex structure" may denote an operator J on the real Hilbert space H that mimics properties of the operator $iI_{\mathcal{H}}$ for a complex $\mathcal{H}$. In this sense this "complex structure" is not some exogenous element super-added to real H—either H admits such an operator or it doesn't. If it doesn't then that's an end to it; if it does then the structure doesn't need to be added. New physics enters if one adds the Stueckelberg (1959, 1960) requirement that J commutes with all observables. This posit has substantive consequences, some of which may be thought unpalatable, as subsequent developments will reveal. But this is getting ahead of the game. Let's start with basics.

For real H we seek an analog of the operator $iI_{\mathcal{H}}$ for a complex $\mathcal{H}$. This would be a bounded (real) linear operator J acting on H with the properties $J^2 = -I_H$ and $J^\dagger = -J$.[18] If A is an anti-selfadjoint operator acting on H then in analogy with the complex case we can define $S := JA$. From the properties of J we have $S^\dagger = (JA)^\dagger = A^\dagger J^\dagger = (-A)(-J) = AJ$, and if J commutes with all of the elements of the algebra of observables (as does $iI_{\mathcal{H}}$) then $S^\dagger = S$ and S is selfadjoint. Questions of existence and uniqueness naturally arise: Does every real H admit such a J and, if so, to what extent is it unique? More on this below.

4.2 Phases and interference

The need for the simulacrum J is also indicated by another consideration. In the standard form of ordinary non-relativistic QM (sans superselection rules) where $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$, the von Neumann algebra of bounded operators acting on a complex $\mathcal{H}$, the pure physical states correspond to unit vectors up to phase. Thus, *overall* phase does not matter. But in superpositions *relative* phases do matter. If ψ_1 and ψ_2 are unit orthogonal vectors then $\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ and

the Schrödinger representation for position and momentum. For some of the pitfalls in applying this theorem see Earman (2023).

[18]Alternatively we can ask for an operator $J : H \to H$ giving a (real) linear surjection that preserves the norm $|| \bullet ||_H$ and having the property that $J^2 = -I_H$. From norm preservation we can infer by polarization that $(Jx, Jy) = (x, y)$ for all $x, y \in H$. So $(J^\dagger Jx, y) = (x, y)$ for all $x, y \in H$, implying that $J^\dagger J = I_H$. So $J^\dagger JJ = -J^\dagger = J$ or $J^\dagger = -J$.

$\frac{1}{\sqrt{2}}(e^{i\theta}\psi_1 + \psi_2)$ correspond to different states if $\theta \neq n\pi$, $n \in \mathbb{N}$, and the difference shows up in interference effects.[19]

In real QM without i, where physical vector states correspond to unit vectors up to sign rather than up to phase (Moretti and Oppio 2017), there doesn't seem to be enough apparatus to express phase relations. Here again J comes to the rescue. Instead of writing $e^{i\theta}$ we can write $e^{J\theta}$. Expanding $e^{J\theta}$ in a power series (which converges because we are dealing with bounded operators) and using the properties of J we find, as expected, that $e^{J\theta} = \cos\theta + J\sin\theta$.[20] If J commutes with all the the elements $A \in M$ of the algebra of observables M then physical vector states correspond to unit vectors up to J-phase, for then $(e^{J\theta}\psi|Ae^{J\theta}\psi)_H = (\psi|e^{-J\theta}Ae^{J\theta}\psi)_H = (\psi|Ae^{-J\theta}e^{J\theta}\psi)_H = (\psi|A\psi)_H$ for all $\psi \in H$ and all $A \in M$.

By the same token if J commutes with the algebra of observables M then the algebra cannot comprise all of $B(H)$, the von Neumann algebra of all bounded operators acting on H. For if M were equal to $B(H)$ the Stueckelberg commutation condition would imply that J commutes with $B(H)$. However, the the only non-zero operators on H that commute with $B(H)$ are real multiples of I_H, but obviously $J \neq rI_H$ for any $r \in \mathbb{R}$ since such a J cannot yield $J^2 = -I_H$. Thus, if we accept Strocchi's and Wightman's (1974) dictum that a superselection rule in the broadest sense for a quantum mechanical theory "can be defined as any restriction on what is observable in the theory" (p. 2198) then real non-relativistic QM M, H has a superselection rule when $JM = MJ$.[21] This humble observation marks the start of a road that leads to important consequences for the subsequent discussion.

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[19] More accurately, they correspond to different physical states if the superpositions are coherent in the sense that ψ_1 and ψ_2 do not belong to different superselection sectors of the Hlbert space. Stueckelberg's commutation condition on J engenders a superselection rule; see below.

[20] To make $e^{i\theta}$ and $e^{J\theta}$ more analogous write $e^{iI_{\mathcal{H}}\theta}$ instead of $e^{i\theta}$, and have the operators $e^{iI_{\mathcal{H}}\theta}$ and $e^{J\theta}$ act on a vector of their respective Hilbert spaces. But there is still an important disanalogy. $e^{iI_{\mathcal{H}}\theta}\psi = (\cos\theta + i\sin\theta I_{\mathcal{H}})\psi = \cos\theta\psi + \sin\theta i\psi$.
And $e^{J\theta}\psi = \cos\theta\psi + \sin\theta J\psi$. $J\psi$ is a vector orthogonal to ψ whereas $i\psi$ is not orthogonal to ψ. If this difference makes for a measurable difference in interference effects then using J to try to account for interference effects in real QM is in real trouble.

[21] Since M is proper subalgebra of $B(H)$ it acts reducibly on H, leaving invariant a non-null proper subspace $\widetilde{H} \subset H$ as well as its orthogonal complement $\widetilde{H}^{\perp}$; thus, H resolves into the direct sum $\widetilde{H} \oplus \widetilde{H}^{\perp}$, and M also resolves into a direct sum of algebras $M_{\widetilde{H}} \oplus M_{\widetilde{H}^{\perp}}$ acting on $\widetilde{H} \oplus \widetilde{H}^{\perp}$.

4.3 Existence and uniqueness

A finite dimensional real Hilbert space H admits a complex structure J iff $\dim(H)$ is even. The only if part can be seen from the facts that for $\dim(H) = n < \infty$ real linear operators can be realized as n x n real matrices, and that for such matrices A, B, $\det(AB) = \det(A)\det(B)$ so that $\det(J^2) = \det(-I_H) = (-1)^n = \det(J)^2 \geq 0$, which is satisfied only for even n. If H is infinite dimensional and separable it admits a complex structure. As for uniqueness, the best that can be hoped is uniqueness up to sign since $J' := -J$ is a complex structure if J is. Depending on features of the algebra of observables M acting on H uniqueness up to sign may be achievable for complex structures J that commute with the algebra.

5 Masquerade? Confection?[22]

Suppose H is a real Hilbert space admitting a complex structure J. Then, if $JM = MJ$, there is a natural way to complexify H (dubbed "internal complexification" by Moretti and Oppio 2017) to produce a complex Hilbert space H_J. The complexification of H is achieved by defining multiplication by complex scalars by $(a+ib)\xi := a\xi + bJ\xi$, $\xi \in H$ and $a, b \in \mathbb{R}$, and defining an inner product on H_J by

$$(\xi|\eta)_{H_J} := (\xi|\eta)_H - i(\xi|J\eta)_H,\ \xi, \eta \in H \tag{4}$$

where $(\bullet|\bullet)_H$, the inner product for H. Evidently $||\xi||_{H_J} = ||\xi||_H$, $\xi \in H$, and H_J is complete in the norm, ensuring that it is a Hilbert space.

Moretti and Oppio (2017, Prop. 2.23) prove that a real linear operator A on H is a complex linear operator on H_J iff $AJ = JA$. If $JM = MJ$ then Moretti and Oppio (2017, p. 9) opine that H is really complex Hilbert space masquerading as a real Hilbert space; or more carefully,

> [A] *quantum theory formulated in a real Hilbert space H may actually be a standard theory, formulated in a corresponding complex Hilbert space H_J. It happens if there is a complex structure J which commutes with every observable of the theory.*[23]

[22] To be read with the Carpenters' version of "This Masquerade" playing in the background, alternating with the Archies' version of "Sugar, Sugar."

[23] Italics in the original.

But equally, if quantum theory formulated in a real Hilbert space actually is a standard theory formulated in a corresponding complex Hilbert space, then one might be tempted to say that standard theory formulated in a complex Hilbert actually is (or is a confection of) quantum theory formulated in a real Hilbert space. Our initial inquiry, "Why i?", is in danger of being buried under epithets. So I will hastily move on to consider a related but different spin that is directed towards the substantive issue of whether or not complex structure is essential to the empirical success of conventional QM.

According to Stueckelberg the "r.h.s. [real Hilbert space] theory may be shown to be *equivalent to conventional theory in complex h.s. (c.h.s.)*" (Stueckelberg 1959, p. 254; see also Stueckelberg 1960 and Stueckelberg and Guenin 1961). This is a claim endorsed by other researchers (see, for example, Finkelstein et al. 1962, p. 208; Aleksandrova et al. 2013, p. 2; and Wooters 2021, p. 608). A more cautious and vaguer claim would be that "complex structure is somehow to be located or embedded in the real theory" (Aleksandrova et al. 2013, p. 2).[24] And if philosophers were to get their hands on the topic one or more of them would surely use the language of emergence with complex structure said to emerge from real structure.

The basis of such claims initially seems rather slender and questionable. Stueckelberg's "proof" of the equivalence of real and complex Hilbert space theory consists of pointing to the inner product (4) on H_J and noting that the relation "exists between the one complex number [$(\xi|\eta)_{H_J}$], formed in c.h.s., and the two real numbers [$(\xi|\eta)_H$ and $(\xi|J\eta)_H$] formed in r.h.s." (Stueckelberg 1959, p. 254, note $^{***)}$). More accurately the relation exists between the one complex number $(\xi|\eta)_{H_J}$, formed in c.h.s., and the two real numbers $(\xi|\eta)_H$ and $(\xi|J\eta)_H$ in r.h.s, *plus* the square root of minus one which is inserted by hand as the coefficient of the second real number.

What is more plausible and still interesting is the claim that real QM and complexified QM are *empirically equivalent*, at least on one understanding of what constitutes the empirical content of a quantum theory. Under Stueckelberg's condition that J commutes with all observables, the expectation value $(\xi|A\xi)_{H_J}$ of any for any selfadjoint $A \in M$ (which is a complex linear operator on H_J) as computed in the complexified Hilbert space H_J is the same as the expectation value $(\xi|A\xi)_H$ computed in real H. Start by noting that for any $\xi \in H$, $(\xi|J\xi)_H = 0$. $((\xi|J\xi)_H = (J\xi|JJ\xi)_H = -(J\xi|\xi)_H = -\overline{(\xi|J\xi)}_H =$

[24] The goal of Aleksandrova et al. (2013) is to present a model in which complex structure emerges dynamically without using Stueckelberg's J that commutes with all observables.

$-(\xi|J\xi)_H \Rightarrow (\xi|J\xi)_H = 0$.) Using the fact that $JA = AJ$ similar manipulations show that $(\xi|JA\xi)_H = (\xi|AJ\xi)_H = 0$. ($(\xi|AJ\xi)_H = \underline{(A^\dagger\xi|J\xi)_H} = (A\xi|J\xi)_H = (JA\xi|JJ\xi)_H = -(JA\xi|\xi)_H = -(AJ\xi|\xi)_H = -\overline{(\xi|AJ\xi)_H} = -(\xi|AJ\xi)_H \Rightarrow (\xi|AJ\xi)_H = (\xi|AJ\xi)_H = 0$, and the imaginary part of $(\xi|A\xi)_{H_J}$ vanishes.)[25]

Now add the premise that the empirical content of a quantum theory is encapsulated (largely? entirely?) in the expectation values it delivers for observables as represented by selfadjoint operators in the algebra of observables. Now the claim that real Hilbert space theory is empirically equivalent to conventional quantum theory in complex Hilbert space becomes plausible, at least if the complex space arises from internal complexification via a J that commutes with all observables. For real QM reproduces the expectation values predicted by complex QM at any given time, and the evolution of the expectation values predicted by complex QM is reproduced by real QM with the help of Stueckelberg's J. Though the claim of empirical equivalence may seem plausible, it is wrong.

6 More on internal complexification[26]

The preceding section focused on the internal complexification of a real Hilbert space H to produce a complex H_J using Stueckelberg's J. But real QM (respectively, complex QM) isn't just real H (respectively, complex $\mathcal{H}$) but H and a real von Neumann algebra of observables M (respectively, a complex von Neumann algebra $\mathfrak{M}$) acting on H (respectively, on $\mathcal{H}$), as well as a set of physically realizable states in the form of normal states on M (respectively, $\mathfrak{M}$). This prompts a pair of questions. First, given the internal complexification H_J of H, what is the corresponding complexification M_J of M? The elements $A \in M$ are bounded real linear operators acting on

[25] The result generalizes to cover expectation values computed from density operators. A density operator ρ acting on a separable Hilbert space $\mathcal{H}$ has a pure point spectrum w_k where the w_k are positive real numbers such that $\sum_k w_k = 1$. If $A \in \mathfrak{M}$ is selfadjoint and ξ_k is an ON basis for $\mathcal{H}$ then $Tr(\rho A) = \sum_k w_k(\xi_k|A\xi_k)_{\mathcal{H}}$, which by the above is equal to $\sum_k w_k(\xi_k|A\xi_k)_H$.

[26] To be read with Avril Lavigne's version of "Complicated" playing in the background, alternating with Weird Al Yankovic's "A Complicated Song (Parody of "Complicated" by Avril Lavigne).

H. Recall that under Stueckelberg's condition that $JM = MJ$ a real linear operator $A \in M$ is a complex linear operator on H_J, so that M is a subalgebra of $\mathfrak{B}(H_J)$. All that remains is to make M into a complex von Neumann algebra M_J acting on H_J. The natural way to do this is to take M_J to be the von Neumann algebra given by the weak closure of M in $\mathfrak{B}(H_J)$ or, equivalently, by the double commutant M'' of M in $\mathfrak{B}(H_J)$, formed by first taking the commutant M' of M (consisting of all elements of $\mathfrak{B}(H_J)$ that commute with M), and then taking the commutant $(M')' := M''$ of M' (consisting of all elements of $\mathfrak{B}(H_J)$ that commute with M').

With this settled, we can say that a case of complex QM $\mathfrak{M}$, $\mathcal{H}$ arises from the internal complexification of a case of real QM iff there is real M, H and a complex structure J of H such that $JM = MJ$ and such that $\mathcal{H}$ is $H_J = \mathcal{H}$ and $M_J = \mathfrak{M}$.[27] The question that naturally arises is then:

> Q1: Does every case of complex QM $\mathfrak{M}$, $\mathcal{H}$ arise from internal J-complexification of some real QM M, H whereby $\mathcal{H} = H_J$ and $\mathfrak{M} = M_J$?

A possible obstruction to internal complexification is the dimension of H—we saw that H admits a complex structure iff $\dim(H)$ is finite and even or else infinite dimensional and separable. But since $N \subset H_J$ is an ON basis for H_J iff $\{x, Jx | x \in N\}$ is a basis for H (Moretti and Oppio 2017, Prop. 2.21), whatever the dimension of $\mathcal{H}$ and thus, of H_J, $\dim(H)$ will be even if $\dim(H_J)$ is finite, and will be infinite dimensional and countable if $\dim(H_J)$ is countably infinite. So no obstruction here. But this is far from proving that the answer to Q1 is positive.

A serious obstruction to a positive answer to Q1 is that the real algebras satisfying Stueckelberg's condition may not contain enough observables

[27] $H_J = \mathcal{H}$ meaning that they are isometrically isomorphic, and $M_J = \mathfrak{M}$ meaning that they are $*$-isomorphic. The "$*$" symbol here is potentially confusing. It does not stand for complex conjugation but the Hermitian conjugation operation, typically denoted in the physics literature by †. (It does involve complex conjugation since for cA, $c \in \mathbb{C}$, $(cA)^\dagger = \overline{c}A^\dagger$.) A $*$-isomorphism $\Theta : \mathfrak{M} \to \mathfrak{N}$ of complex von Neumann algebras is a bijection such that for $A, B \in \mathfrak{M}$ and $\lambda, \mu \in \mathbb{C}$
(i) $\Theta(\lambda A + \mu B) = \lambda\Theta(A) + \mu\Theta(B)$
(ii) $\Theta(A^*) = \Theta(A)^*$
(iii) $\Theta(AB) = \Theta(A)\Theta(B)$.
For a $*$-anti-isomorphism (iii) is replaced by
(iii') $\Theta(AB) = \Theta(B)\Theta(A)$.

so that its complexification results in the desired $\mathfrak{M}$. We saw above that Stueckelberg's commutation condition implies that M is a proper subalgebra of $B(H)$. But how impoverished is M? Stueckelberg's commutation condition implies that M does not contain rank one projections (projections whose ranges are one-dimensional subspaces). Let $\xi \in H$ be a unit vector, and let $E_{[\xi]}$ be the projection onto the ray $[\xi]$ spanned by ξ. Suppose for reductio that $E_{[\xi]} \in M$. Then we can compute $1 = (\xi|\xi) = (E_{[\xi]}\xi|E_{[\xi]}\xi) = (JE_{[\xi]}\xi|JE_{[\xi]}\xi) = (E_{[\xi]}J\xi|E_{[\xi]}J\xi)$. But we have seen that $J\xi$ is orthogonal to ξ and, therefore, $E_{[\xi]}J\xi$ is the zero vector, producing $1 = 0$. So $E_{[\xi]} \notin M$.

The limitation on observables results in a limitation on the admissible (= normal) pure states on a real M satisfying Stueckelberg's commutation condition.[28] It is known that the support projection for a normal pure state on a von Neumann algebra must be minimal in the algebra.[29] If minimality for a projection always implies that the projection is rank one rank one then we could conclude that there are no normal pure states on M.[30] However, a minimal projection is not necessarily rank one. A lengthier argument, sketched in the Appendix, establishes the non-existence of normal pure states on M.[31]

Interpreting the transition probability from a normal pure state ξ to a normal pure state η as the expectation value, computed in state ξ, of the support projection of η, transition probabilities are undefined in the version of real QM under study. In itself there is nothing sinister here. Similar situ-

[28] A state ω on a von Neumann algebra is mixed if there are distinct states ω_1 and ω_2 and a $\lambda \in \mathbb{R}$, $0 < \lambda < 1$ such that $\omega(A) = \lambda\omega_1(A) + (1-\lambda)\omega_2(A)$ for every A in the algebra; otherwise ω is said to be pure.

[29] For von Neumann algebra M acting on H a minimal projection is a projection $E \neq 0$ such that for any projection $0 \neq F \in M$ if $F \leq E$ then $F = E$. $F \leq E$ means that $FH \subseteq EH$.

The support projection (aka carrier) of a normal state ω on the algebra is the orthogonal complement of the largest projection P in the algebra such that $\omega(P) = 0$. More formally, the support projection a projection S_ω in the algebra is defined by $I - S_\omega = \sum_j E_j$ where $\{E_j\}$ is a family of orthogonal projections in the algebra maximal with respect to the property that $\omega(E_j) = 0$ for all j. See Kadision and Ringrose (1991, Sec. 7.1).

[30] There certainly are non-normal pure states—pure states that are not represented by density operators—on M. Admitting non-normal states introduces problems; but this is not the place to discuss them. When the Hilbert space on which the algebra acts is finite all states are normal.

[31] And as a corollary that minimal projections in M satisfying Stueckelberg's commutation condition are rank one.

ations are encountered in relativistic QFT where the local algebras are typically Type III, meaning that not only are there no minimal and no rank one projections in these algebras but also there are no finite rank projections in these algebras. But here we are simply trying to do ordinary non-relativistic QM, and what we are finding is that version of real QM with which we are saddled by internal complexification is certainly not a disguised version of the ordinary non-relativistic complex QM we have been taught to know and love from our textbooks on QM. And we have a tentative answer to Q1, pending a verification of the equality $M_J = M + iM$. For then M_J does not contain any rank one projections and so a $\mathfrak{M}$, $\mathcal{H}$ does not arise from internal J-complexification of a real M, H if $\mathfrak{M}$ does contain rank one projections.

There is something of a damned-if-you-do and damned-if-you-don't situation here. Real QM without Stueckelberg's commutation condition seems to have too many observables, i.e. those that don't correspond to complex linear operators on complex Hilbert space[32] (and correspondingly too many states qua expectation valued functionals on the algebra of observables); but with the commutation condition imposed there are too few observables (and correspondingly too few states) to accommodate various phenomena of ordinary non-relativistic QM.

Fans of $\mathbb{C}$ may feel like celebration. But the celebration should be tempered, for whatever the victory here for i, it depends on accepting internal complexification as the correct way to view the relation between real and complex QM. Other approaches to complexification are available.

7 External complexification[33]

Leaving aside internal complexification, there is a more general complexification procedure that can be applied to any M, H without resort to Stueckelberg's J (see Kadison and Ringrose 1983, pp. 161-162; Meise and Vogt 1997, p. 232; and Li 2003, Ch. 1). In external complexification real H is turned into a complex Hilbert space $H_c = H \times H$; and one writes $x + iy \in H_c = H + iH$ for $(x, y) \in H \times H$. Scalar multiplication is defined by $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x)$, $x, y \in H$, $\alpha, \beta \in \mathbb{R}$. The set $\{(x, 0) : x \in H\}$ is a closed real-linear subspace H_r of H_c and $H \ni x \mapsto (x, 0)$

[32] And in addition the observables in real QM have too much eigenvalue degeneracy. See Mryheim (1999).

[33] To be read with Marc Gulli's "It's Very Complicated" playing in the background.

is an isometric isomorphism onto H_r (Kadison and Ringrose 1983, p. 162). For an A acting on H there is a counterpart A_c acting on H_c defined by $A_c(x+iy) := Ax + iAy$, and if A is selfadjoint for H then A_c is selfadjoint for H_c (Meise and Vogt 1997, p. 232). The inner product for H_c is defined by

$$((x,y),(\xi,\eta))_{H_c} := (x|\xi)_H + (y|\eta)_H - i((y|\xi)_H - (x|\eta)_H) \tag{6}$$

for all $x, y, \xi, \eta \in H$, where $(\bullet|\bullet)_H$ is the inner product for H.

Li (2003, Ch. 4) proposes a companion procedure for complexifying a real von Neumann algebra M acting on H into a complex von Neumann algebra $M_c = M + iM$ acting on $H_c = H + iH$.[34] Say that a case of complex QM $\mathfrak{M}$, $\mathcal{H}$ arises from external complexification of some real M, H if $\mathcal{H} = H_c$ and $\mathfrak{M} = M_c$.

For external complexification the companion question to Q1 for internal complexification is:

> Q2: Does every case of complex QM $\mathfrak{M}$, $\mathcal{H}$ arise from external complexification of some real M, H?

A folk theorem asserts that the answer to Q2 is positive iff $\mathfrak{M}$ admits a $*$-anti-automorphism.[35] For example, Li (2003, p. viii) writes that "A complex operator algebra can be expressed as the complexification of some real operator algebra if and only if it has a $*$-anti-automorphism." I have not been able to find a proof in the literature.

For present purposes an implication of a result reported by Størmer (1967, p. 357) is of central importance: if $\mathfrak{M}$ is a von Neumann algebra, M a real von Neumann subalgebra of $\mathfrak{M}$ such that $\mathfrak{M} = M + iM$, $M \cap iM = \{0\}$, then the map $A + iB \to A^* + iB^*$, $A, B \in M$, is a $*$-anti-automorphism of $\mathfrak{M}$ of order 2.[36] This result together with with negative results on the existence

[34]Is it the case that M_c is a von Neumann algebra, i.e. is it the case that $M_c'' = M_c$, where the commutant and double commutant of M_c are taken in $\mathfrak{B}(H_c)$? Li remarks: "Let M be a $*$ subalgebra of $B(H)$, and $M_c = M + iM$. Clearly, $M_c' = M' + iM'$, and $M_c'' = M'' + iM''$. Therefore, M is a real von Neumann algebra on H, if and only if M_c is a von Neumann algebra on $H_c = H + iH$" (p. 63). Here the commutant and double commutant of M are taken in $B(H)$.

[35]The reader is reminded again that in this context the $*$ symbol does not stand for the operation of complex conjugation but for the Hermetian conjugation operation.

[36]That an anti-automorphism Θ is order 2 means that it is involutory, i.e. $\Theta^2 = I$.

of $*$-anti-automorphisms entail a No answer to Q2. Connes (1975a, 1975b) gave examples of a Type III factor and a Type II_1 factor where there are no $*$-anti-automorphisms at all.[37] Subsequently Jones (1980) gave an example of Type II_1 factor which admits $*$-anti-automorphisms but none are of order 2.

Cases where complex QM $\mathfrak{M}$, $\mathcal{H}$ arises from external complexification of some real M, H would provide a clear sense in which complex structure is to be located in, or embedded in, or emergent from the real QM. More importantly, in such cases the way is open for showing that real QM can duplicate the empirical success of complex QM, as codified in the expectation values of a selfadjoint operator A_c in M_c acting on H_c, using the real inner product of H to compute expectation value of counterpart selfadjoint operator A in M acting on H.[38] And unlike internal complexification where the real algebra M is circumscribed by Stueckelberg's commutation condition, here M is free to be whatever it needs to be. The downside is that, without Stueckelberg's J, real QM has no answer to the issues of dynamics and phase relations/interference effects discussed in Sections 4.1-4.2.

On the opposite side of the coin are cases of complex QM $\mathfrak{M}$, $\mathcal{H}$ where $\mathfrak{M}$ does not admit a $*$-anti-automorphism of order 2 and, consequently, $\mathfrak{M}$, $\mathcal{H}$ does not arise from external complexification of some real M, H. Then the complex structure of the complex QM $\mathfrak{M}$, $\mathcal{H}$, and its contribution to empirical success of $\mathfrak{M}$, $\mathcal{H}$, cannot be located in or grounded in any real QM M, H, at least if the grounding is via external complexification. This qualification puts a damper on any celebrating the fans of $\mathbb{C}$ might want to do. And further, even if external complexification is the correct lens though which to view the relation between real and complex QM, the victory for i will be narrow to nil if the only complex $\mathfrak{M}$s that fail to admit $*$-anti-automorphisms of order 2 are exotic algebras with few if any physical applications. Thus, the importance of expanding existing knowledge of which von Neumann algebras admit $*$-anti-automorphisms.

I close with a point that has emerged from our discussion and, though

[37] That $\mathfrak{M}$ is a factor algebra means that the center $\mathfrak{M} \cap \mathfrak{M}'$ of $\mathfrak{M}$ is $\mathbb{C}I$.

[38] A consistency check shows that the imaginary part of the expectation value of a selfadjoint A_c acting on H_c vanishes, as it must, for any state $(x, y) \in H_c$. For selfadjoint A acting on H, $(y|Ax)_H = (A^\dagger y|x)_H = (Ay|x)_H = \overline{(x|Ay)}_H = (x|Ay)_H$. Thus, for selfadjoint A_c corresponding to A, the expectation value A_c for any $(x, y) \in H_c$ is $((x, y)|A_c(x, y)_{H_c} = (x|Ax)_H + (y|Ay)_H$
$-i((y|Ax)_H - (x|Ay)_H) = (x|Ax)_H + (y|Ay)_H$ for all $x, y \in H$.

obvious, is worth underlining. One should be prepared to find that the issue of whether or not complex structure is essential to the empirical success of QM does not admit simple Yes or No but requires a divided verdict: the empirical success of some instances $\mathfrak{M}$, $\mathcal{H}$ of complex QM can be duplicated by a real QM M, H, in other cases not.

8 Conclusion

We began by noting that for real QM to function as we have grown to expect a quantum theory to function, it needs to employ a simulacrum of i in the form of an operator J on the real Hilbert space H with the properties $J^2 = -I_H$ and $J^\dagger = -J$. As first discussed by Stueckelberg in the 1960s, J can be used to "internally complexify" H. This led to the issues of whether or not internally complexified QM actually is complex QM in disguise or, at least, is empirically equivalent to complex QM. In answering these questions it was assumed that a quantum system is characterized by a von Neumann algebra of observables acting on a Hilbert space—a complex algebra $\mathfrak{M}$ acting on a complex Hilbert space $\mathcal{H}$ in the case of complex QM vs. a real algebra M acting on a real Hilbert space H in the case of real QM—and that the admissible states on the algebra are the normal states. For some instances of complex QM $\mathfrak{M}$, $\mathcal{H}$ the answers were in the negative. Stueckelberg's condition that J commute with the real algebra M puts a substantial restriction on M, limiting the applicability of real QM; specifically, M cannot be a Type I algebra, meaning that it does not contain minimal projections and, consequently, that none of the normal states are pure states. And as an additional consequence there are cases of complex QM $\mathfrak{M}$, $\mathcal{H}$ that do not arise by internal complexification from a real QM M, H.

We then turned to an alternative complexification procedure, naturally dubbed external complexification, which does not use Stueckelberg's J. While not subject to the limitations internal complexification places on the real algebra of observables, the reach of external complexification is also limited because complex von Neumann algebras do not arise from external complexification of real algebras if they do not admit $*$-anti-automorphisms.

While the negative side of the ledger for real QM creates some presumption that, at least in a range of cases, complex structure is essential to the empirical success of QM, the line of attack on the question "Why i?" is is subject to the obvious limitation that it requires allegiance to viewing the relation between real and complex QM through the lens of one or another complexification procedure. But by the same token it is hard to see how a strong no-go result can be obtained without the help of some substantive assumptions about the about the relationship between real QM and complex QM. The presumption would be made stronger by combining an argument for a set of necessary conditions for a real QM M, H to be empirically equivalent to a complex QM $\mathfrak{M}$, $\mathcal{H}$ together with a demonstration that satisfaction of said conditions implies that $\mathfrak{M}$, $\mathcal{H}$ is a complexification of M, H. More generally there is a need for an exploration of the substantive assumptions about the relationship between real QM and complex QM necessary for achieving a no-go result on the ability of real QM to duplicate the empirical successes of complex QM.

On the positive side of the ledger for real QM, it was noted that in cases where complex QM $\mathfrak{M}$, $\mathcal{H}$ arises, from complexification of a real QM M, H, the empirical success of complex QM, insofar as the success is codified in expectation values of selfadjoint elements of $\mathfrak{M}$, can be duplicated by real QM. But this success applies only to the statics of QM. The desire for a unitary dynamics creates an apparent need for complex structure, or a simulacrum thereof. In real QM the generator of a strongly continuous one-parameter unitary group is an anti-unitary operator A that serves as the Hamiltonian in the Schrödinger equation of real QM. Such a Hamiltonian is physically unsatisfactory since A has only purely imaginary eigenvalues. Using Stueckelberg's operator J, A can be converted to a self-adjoint $S := JA$ *if* J is required to commute with A. But to repeat, extending this commutation condition to all elements of the algebra M puts a substantial restriction on M, limiting the applicability of real QM.

The work of Renou et al. (2021a, 2021b) promises an alternative demonstration of the indispensability if complex structure. In staking their claim that the empirical success of some instances of complex QM $\mathfrak{M}$, $\mathcal{H}$ in predicting/explaining measurement outcomes outruns the capacity of real QM, Renou et al. assume that composite systems are to be described by a tensor product of component systems and that when $\mathfrak{M}$ takes the form of a tensor product $\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes ... \otimes \mathfrak{N}_n$ of complex von Neumann algebras $\mathfrak{N}_j$ acting on a tensor product $\mathcal{K}_1 \otimes \mathcal{K}_2 \otimes ... \otimes \mathcal{K}_n$ of complex Hilbert spaces $\mathcal{K}_j$, then any competing real QM M, H must mirror this tensor product structure, i.e. M is a tensor product $N_1 \otimes N_2 \otimes ... \otimes N_n$ of real von Neumann algebras N_j acting a tensor product $K_1 \otimes K_2 \otimes ... \otimes K_n$ of real Hilbert spaces K_j. This is a step down from the claim that the empirical success of some instances of complex QM $\mathfrak{M}$, $\mathcal{H}$ in predicting/explaining measurement outcomes outruns the capacity of *any* real QM M, H. Nevertheless, this weaker claim is of considerable interest in itself, especially since it is backed by the further claim of experimental demonstration of instances of the failure of real QM. If correct, we would have a significant case of naturalized metaphysics in action: real QM has been experimentally falsified.

As a cautionary note, one should recall that analogous claims of experimental metaphysics in action, whereby experimental tests of the Bell inequalities were said to falsify local hidden variable theories, were found to require caveats (see Jones and Clifton 1993). To raise one concern, the Renou et al. claim seems to be in tension with the fact that in cases where complex QM $\mathfrak{M}$, $\mathcal{H}$ arises, by complexification, from real QM M, H, its success can be duplicated by real QM. The tension here is resolved if the case at issue is a case where the complexification of real QM does not properly respect the tensor product structure. This way out seems to be closed off for external complexification since taking the external complexifications of the tensor products $N_1 \otimes N_2 \otimes ... \otimes N_n$ and $K_1 \otimes K_2 \otimes ... \otimes K_n$ is equivalent to taking the tensor products of the respective complexifications of the N_j and the K_j, $j = 1, 2, ..., n$ (see Li 2003, Ch. 4). The tension would also be resolved if the case Renou et al. discuss does not arise from complexification of any real QM; but then their proof is not needed to show the inadequacy of real QM.

One issue surfaces in the proof of Proposition 1 (Renou et al. 2021b) which appeals to the notion that the density operator ρ_c of complex QM describing the experiment that is supposed to falsify real QM is "complex separable" but *not* "real separable." First what does it mean for a density operator ρ_c to be separable for, say, a bipartite system described by a tensor product Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$? It means that ρ_c can be written as $\sum_j p_j \rho_c^1 \otimes \rho_c^2$ where the p_j are positive real numbers such that $\sum_j p_j = 1$ and ρ_c^1 and ρ_c^2 are density operators acting respectively on $\mathcal{H}_1$ and $\mathcal{H}_2$. The definition of separability for a real density operator ρ_r acting on the tensor product $H_1 \otimes H_2$ of real Hilbert spaces H_1 and H_2 is exactly similar. Then what does it mean for ρ_c to be real separable, or for ρ_r to be complex separable? That ρ_c is real separable presumably means that the realification or decomplexification of ρ_c into a a real density operator ρ_r acting on a tensor product $H_1 \otimes H_2$ of real Hilbert spaces is separable; and conversely that ρ_r is complex separable means that the complexification of ρ_r into a complex ρ_c acting on a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of complex Hilbert spaces is separable. Finally, what is the realification or decomplexification of ρ_c, and conversely what is the complexification of ρ_r? The obvious way to approach these questions is to relativize them to complexification scheme whereby the complex QM $\mathfrak{M}$, $\mathcal{H}$ arises from complexifying a real M, H, and in such a scheme $\rho_r \in M$ is the realification of $\rho_c \in \mathfrak{M}$ iff ρ_c is the complexification of $\rho_r \in M$. But in such a scheme real M, H reproduces the the successes of complex QM $\mathfrak{M}$, $\mathcal{H}$ as codified in the expectation values of experimental outcomes. This success includes, of course, the probabilities of experimental outcomes, which are expectation values of the relevant projection operators. In sum, it seems at first glance that the proof strategy of Renou et al. assumes a situation that is at odds with the goal of the proof. This and related issues require a careful analysis, beginning with a careful specification of the algebra of observables for the experimental setup that is supposed to demonstrate the empirical inadequacy of real QM. That task will not be undertaken here.

In conclusion, it may well be, as Freeman Dyson says, that in the quantum realm "nature works with complex numbers and not with real numbers" and that the complex numbers provide the "ground on which atoms move." But if this is the message quantum physics is trying to convey to us, it is not a message written in unequivocal, bold headlines, but rather a message that needs decoding and qualifying. But then isn't the same true of any interesting lesson we are supposed to draw about what physics teaches us about some fundamental aspect of physical reality?

Appendix

A von Neumann algebra M satisfying Stueckelberg's commutation condition admits no normal pure states. Suppose for reductio that φ is a normal pure state on M. A contradiction is obtained by showing that the subalgebra $S_\varphi M S_\varphi$ is a Type I factor, which contains all rank one projections, whereas M contains no rank one projections (as noted in Sec. 6 above).

Towards this end let's show that φ is faithful on $S_\varphi M S_\varphi$. (For this it is sufficient to show that if $\varphi(S_\varphi A S_\varphi) = 0$ then $S_\varphi A S_\varphi = 0$.) Since S_φ is minimal it follows that $S_\varphi A S_\varphi = c_A S_\varphi$ for all $A \in M$, where c_A is a scalar that may depend on A (Kadison and Ringrose 1997, Prop. 6.4.3). Since $\varphi(S_\varphi) = 1$, $\varphi(S_\varphi A S_\varphi) = \varphi(A)$. ($A = S_\varphi A + S_\varphi^\perp A$. So $\varphi(A) = \varphi(S_\varphi A) + \varphi(S_\varphi^\perp A)$. But $\varphi(S_\varphi^\perp) = 0$ and if $\varphi(X) = 0$ then $\varphi(XY) = 0$, so $\varphi(A) = \varphi(S_\varphi A)$. Then apply the same argument again to $S_\varphi A = S_\varphi A S_\varphi + S_\varphi A S_\varphi^\perp$.) Next note that

$\varphi(A) = \varphi(c_A S_\varphi) = c_A \varphi(S_\varphi) = c_A$. So $\varphi(S_\varphi A S_\varphi) = 0 \Rightarrow c_A = 0 \Rightarrow S_\varphi A S_\varphi = 0$.

Now let's tease out the implication of the faithfulness of the state φ on $S_\varphi M S_\varphi$ for the GNS representation induced by φ. A state φ on M induces a GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \Phi_\varphi)$, $\Phi_\varphi \in \mathcal{H}_\varphi$, where $\pi_\varphi : M \to \mathfrak{B}(\mathcal{H}_\varphi)$ gives a representation of M as a subalgebra of $\mathfrak{B}(\mathcal{H}_\varphi)$ whereby $\varphi(A) = (\Phi_\varphi | A \Phi_\varphi)$ for all $A \in M$. When φ is a normal state the continuity of the representation π_φ implies that $\pi_\varphi(N)$, $N \subseteq M$, is closed in the weak topology so that $\pi_\varphi(N)'' = \pi_\varphi(N)$, showing that $\pi_\varphi(N)$ i a von Neumann subalgebra. If in addition φ is pure on M then the representation π_φ is irreducible, implying that $\pi_\varphi(M) = \mathfrak{B}(\mathcal{H}_\varphi)$. The purity of φ on M implies that φ restricted to the subalgebra $S_\varphi M S_\varphi$ is pure. (This can be shown by demonstrating the contrapositive. If φ is mixed on $S_\varphi M S_\varphi$ then there are distinct normal states φ_1 and φ_2 on $S_\varphi M S_\varphi$ such that $\varphi(S_\varphi A S_\varphi) = \lambda \varphi_1(S_\varphi A S_\varphi) + (1-\lambda)\varphi_2(S_\varphi A S_\varphi)$, $0 < \lambda < 1$, for all $A \in M$. Since $\varphi(S_\varphi A S_\varphi) = \varphi(A)$, $\varphi(A) = \lambda \widetilde{\varphi}_1(A) + (1-\lambda)\widetilde{\varphi}_2(A)$, for all $A \in M$, where $\widetilde{\varphi}_1(A) := \varphi_1(S_\varphi A S_\varphi)$ and $\widetilde{\varphi}_2 := \varphi_2(S_\varphi A S_\varphi)$ are distinct normal states on M.) The faithfulness of φ on $S_\varphi M S_\varphi$ implies the faithfulness of the representation π_φ. Hence, π_φ gives a $*$-isomorphism between $S_\varphi A S_\varphi$ and the Type I factor $\mathfrak{B}(\mathcal{H}_\varphi)$ which contains all rank one projections.

As a corollary, infer that if a von Neumann algebra admits normal pure states then the support projection for such a state is rank one since this projection must be minimal in the algebra and since a minimal projection must be rank one if the algebra contains all rank one projections.

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References

[1] Aleksandrova, A., Borish, V. and Wootters, W. K. 2013. "Real-Vector-Space Quantum Theory with a Universal Quantum Bit," *Physical Review A* 87: 052106; arXiv: 1210:4535v2.

[2] Araki, H. 1980. "On a Characterization of the State Space of Quantum Mechanics," *Journal of Mathematical Physics* 75: 1-24.

[3] Born, M. and Jordan, P. 1925. "Zur Quantenmechanik," *Zeitschrift für Physik* 34: 858-888.

[4] Cassinelli, G. and Lahti, P. 2017. "Quantum mechanics: why complex Hilbert space?" *Philosophical Transactions of the Royal Society A* 375: 20160393.

[5] Chen, M.-C., et al. 2022. "Ruling Out Real-Valued Standard Formalism of Quantum Theory," *Physical Review Letters* 128: 040403.

[6] Chen, R. L. W. 1989. "Derivation of the real form of Schrödinger's equation for a nonconservative system and the unique relation between Re(ψ) and Im(ψ)," *Journal of Mathematical Physics* 30: 83-86.

[7] _____________ 1990. "The real Wave Function as an Integral Part of Schrödinger's View on Quantum Mechanics," *Physica B* 167: 183-184.

[8] _____________ 1993. "Schrödinger's real wave equation (continued)," *Physica B* 190: 256-258.

[9] Chi, W. and Pan, L. 2022. "Can One R4eally Disprove a Real Quamtum Theory?" *International Journal of Theoretical and Mathematical Physics* 12 (1): 1-6.

[10] Connes, A. 1975a. "A factor not anti-isomorphic to itself," *Annals of Mathematics* 101: 536-554.

[11] Connes, A. 1975b. "Sur la classification des facteurs type II," *Compte Rendus de l'Académie des Sciences* (Paris) Sér. A-B 281: 13-15.

[12] Dyson, F. J. 1962. "The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics," *Journal of Mathematical Physics* 3: 1199–1215.

[13] __________ 2010. "Birds and frogs in mathematics and physics," *Physics–Uspekhi 53*: 825 - 834.

[14] Earman, J. 2023. "As Revealing in the Breach as in the Observance: von Neumann's Uniqueness Theorem," http://philsci-archive.pitt.edu/id/eprint/22115.

[15] Finkelstein, J. 2021. "In defense of real quantum theory," arXiv: 2103.12740v1.

[16] Finkelstein, D., Jauch, J. M., Schiminovich, S., and Speiser, D. 1962. "Foundations of Quaternion Quantum Mechanics," *Journal of Mathematical Physics* 3: 207-220.

[17] Holland, S. S. 1995. "Orthomodularity in Infinite Dimensions: A Theorem of M. Solèr," *Bulletin of the American Mathematical Society* 32: 205-234.

[18] Jones, M. R. and Clifton, R. K. 1993. "Against Experimental Metaphysics," *Midwest Studies in Philosophy* 18: 295-316.

[19] Jones, R. V. 1980. "A Type II_1 Factor Anti-Isomorphic to Itself but without Involutory Antiautomorphisms," *Mathematica Scandinavica* 46: 103-116.

[20] Kadison, R. V. and Ringrose, J. R. 1983. *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*. Providence, R.I.: American Mathematical Society.

[21] ______________________________ 1991. *Fundamentals of the Theory of Operator Algebras. Volume II: Advanced Theory*. Providence, R.I.: American Mathematical Society.

[22] Karam, R. 2020. "Schrödinger's original struggles with a complex wave function," *American Journal of Physics* 88: 433-438.

[23] Lahti, P. J. and Maczynski, M. J. 1987. "Heisenberg inequality and the complex field in quantum mechanics," *Journal of Mathematical Physics* 28: 1764–1769.

[24] Li. B. 2003. *Real Operator Algebras*. Singapore: World Scientific.

[25] Li, Z. D. et al. 2022. "Testing Real Quantum Theory in an Optical Quantum Network," *Physical Review Letters* 128: 040402

[26] Meise, R. and Vogt, D. 1997. *Introduction to Functional Analysis*. Oxford: The Clarendon Press.

[27] Moretti, V. and Oppio, M. 2017. "Quantum theory in real Hilbert space: How the complex Hilbert space structure emerges from Poincaré symmetry," *Reviews in Mathematical Physics* 29 (2017): 1750021; arXiv: 1611.09029v4.

[28] Myrheim, J. 1999. "Quantum Mechanics on a Real Hilbert Space," arXiv: 9905037v1.

[29] Reed, M. and Simon, B. 1980. *Methods of Modern Mathematical Physics I: Functional Analysis*. New York: Academic Press.

[30] Renou, M.-O. et al. 2021a. "Quantum theory based on real numbers can be experimentally falsified," *Nature* 600: 625-629.

[31] ___________ 2021b. "Quantum theory needs complex numbers. Supplementary Information," https://doi.org/10.1038/s41586-021-04160-4.

[32] Schrödinger, E. 1927. "Energieaustrauch nach der Wellenmechanik," *Annalen der Physik* 83: 956-968.

[33] _____________ 1928. *Four Lectures on Wave Mechanics*. London: Blackie and Son.

[34] Solèr, M. P. 1995. "Characterization of Hilbert spaces by orthomodular spaces," *Communications in Algebra* 23:1: 219-243.

[35] Størmer, E. (1967). "On Anti-Automorphisms of von Neumann Algebras," Pacific Journal of Mathematics 21: 349-370.

[36] Strocchi, F. & Wightman, A. S. (1974). Proof of the charge superselection rule in local relativistic quantum field theory. *Journal of Mathematical Physics*, *15*, 2198-2224.

[37] Stueckelberg, E. C. G. 1959. "Field Quantization and Time Reversal in Real Hilbert Space," *Helvetica Physica Acta* 32: 254-256.

[38] ____________________ 1960. "Quantum Theory in Real Hilbert Space," *Helvetica Physica Acta* 33: 727-752.

[39] Stueckelberg, E. C. G. and Guenin, M. 1961. "Quantum Theory in Real Hilbert Space II (Addenda and Errats)," *Helvetica Physica Acta* 34: 621-628.

[40] Vedral, V. 2023. "Quantum mechanics with real numbers: entanglement, superselection rules and gauges"; arXiv: 2308.05473v1.

[41] von Neumann, J. 1932a. "Über einen Satz von Herrn M. H. Stone," *Annals of Mathematics* 33: 567–573,

[42] von Neumann, J. 1932b. *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer-Verlag. English translation *Mathematical Foundations of Quantum Mechanics*. Princeton, NJ: Princeton University Press, 1955.

[43] Wooters, W. K. "Quantum Theory and the square root of minus one," *Nature* 600: 607-608.

[44] Yang, C. N. 1987. "Square root of minus one, complex phases, and Erwin Schrödinger," in K. S. Kilminster (ed.), *Schrödinger: Centenary Celebration of a Polymath*, pp. 53-64. Cambridge: Cambridge University Press.