The Classical Limit of Teleparallel Gravity

Helen Meskhidze

ABSTRACT

I consider the classical (i.e., non-relativistic) limit of Teleparallel Gravity, a relativistic theory of gravity that is empirically equivalent to General Relativity and features torsional forces. I show that as the speed of light is allowed to become infinite, Teleparallel Gravity reduces to Newtonian Gravity without torsion. I compare these results to the torsion-free context and discuss their implications on the purported underdetermination between Teleparallel Gravity and General Relativity. I conclude by considering alternative approaches to the classical limit developed in the literature.

1 Introduction

2 Motivations

- *2.1 Classical Limits*

3 Preliminaries

- *3.1 The Limit Procedure in General Relativity*
- *3.2 Teleparallel Gravity*
- *3.3 A Torsional Classical Theory of Gravity*

4 The classical limit of Teleparallel Gravity.

- *4.1 The behavior of the tetrads*
- *4.2 The Classical Limit of TPG*
- *4.3 A notable consequence*

5 Discussion

- *5.1 Comparison to the torsion-free context*
- *5.2 Alternative limits*

6 Conclusion

1 Introduction

The standard narrative surrounding the development of General Relativity (GR) states that GR taught us massive bodies curve spacetime. Gravitational influence is then the manifestation of that curvature, eliminating the need for gravitational forces to explain gravitational influence. Such forces had been posited by classical (i.e., non-relativistic) theories like Newtonian Gravity (NG). In NG, there is no spacetime curvature; instead, space is flat, and gravitational influences are captured by forces. These forces are mediated by a gravitational potential which is itself related to the distribution of matter. With the benefit of hindsight, we can develop an intermediary theory, Newton-Cartan theory (NCT), that shares some features with GR and some with NG: like GR, it allows some curvature but like NG, it is set on a flat spacetime background. Its "intermediary" status can even be made precise: it arises as the classical limit of GR (Malament [1986b]) and as the "geometrized" version of NG (Malament [2012]).

Let us add to the common narrative one (relatively uncontroversial) claim: there is an empirically equivalent rival to GR that is set on a flat spacetime background—Teleparallel Gravity (TPG). TPG uses gravitational forces but allows these forces to have non-vanishing torsion (i.e., spacetime twisting). It is thus able to reproduce the results of GR but seems to be more akin to a classical theory of gravity.

Now, consider the following series of claims found in the recent literature on these subjects:

- a particular type of torsional Newton-Cartan "geometry is the correct framework to describe General Relativity in the non-relativistic limit" (Hansen *et al.* [2020], 1)
- the teleparallel equivalent of a particular solution of GR "can be null reduced to obtain standard Newtonian gravitation" (Read and Teh [2018], 2)
- a teleparallel version of Newton–Cartan gravity "arises as a formal large-speed-of-light limit of the teleparallel equivalent of general relativity" (Schwartz [2023a], 1).

The claims above appear to either contradict the standard narrative or one another. As presented, NCT ought to be understood as the classical limit of GR. But, as indicated in the first bullet above, some claim that *torsional* Newton-Cartan geometry arises in the limit instead. Read and Teh, meanwhile, claim that the limit of Teleparallel Gravity is standard Newtonian gravitation. Finally, disagreeing with both, Schwartz claims that torsional Newton-Cartan geometry is not actually the limit of GR but, rather, it ought to arise in the limit of Teleparallel Gravity. What are we to make of these claims?

The present paper aims to resolve this puzzling situation. Here, I consider what it means to take the classical limit of a relativistic theory and how one's goals inform one's methodology. This project enriches the existing literature which has mainly focused on deriving the predictions of one theory in another. I take the geometric limit of TPG and find that the torsion gets squeezed out. One recovers a peculiar kind of Newtonian Gravity—one entirely devoid of gravitational influence—in the limit.

The remainder of the paper will proceed as follows. I begin by motivating the project in §2, discussing why one might be interested in the classical limit of a relativistic (and torsional) theory of gravity. I then turn to presenting some preliminaries in §3: on the formalism of TPG, on the possibility of a torsional classical spacetime, and on the methodology for taking the classical limit of GR. I next apply this limit procedure to TPG. I prove the main result of the paper in §4 and discuss the relation of these results with those in the extant literature in §5.

2 Motivations

Above, I have outlined what seem to be conflicting claims in the literature surrounding torsional theories of gravity. Now, I address the question of why one might be interested in torsional theories of gravity, or modified theories of gravity more generally, to begin with. There has been notable philosophical interest recently in modified theories of gravity. Some are interested in evaluating proposals for modified theories as alternatives to dark matter (see, e.g., Martens and Lehmkuhl, De Baerdemaeker and Dawid [2020, 2022]). Others are interested in better understanding the overall theory space and drawing lessons that can be applied to General Relativity (see, e.g., Duerr [2020]). Yet others are interested in considering more traditional questions of theory equivalence, conventionalism, and underdetermination in the context of alternative theories of gravity (see, e.g., Knox, Dürr, Dürr and Read, Wolf *et al.* [2011, 2021, 2023, 2023]).

In this last category, Teleparallel Gravity has received special interest. Although it is empirically equivalent to GR, TPG is set on a flat spacetime background and represents gravitational influence as a force.[1] Thus, some claim that TPG presents an interesting context for questions about underdetermination and conventionalism. Such arguments themselves depend on our understanding of theoretical equivalence, specifically, whether we ought to understand TPG as a distinct theory from GR. Knox, for instance, argues that TPG presents no

[1]Recent work by Wolf and Read [2023] has questioned the empirical equivalence of GR and TPG.

genuine case for underdetermination because “both theories [GR and TPG] posit the ‘same’ spacetime” (Knox [2011], 273). More recently, though, Mulder and Read [2023] call Knox’s verdict into question, defending TPG as a spacetime theory in its own right. They argue that Knox’s claims require one to posit (contentious) background metaphysical commitments. Here, I am interested in the issue of underdetermination from a different perspective: rather than directly comparing GR and TPG, I will compare their limiting behavior and classical limits. I now turn to discussing classical limits to illustrate the potential insights offered by this alternative perspective.

2.1 Classical Limits

As discussed by Fletcher [2019], there are two main ways of thinking about the limiting behavior of spacetime theories. The first, and more common way, is through the reduction of the predictions of the current theory to those of an earlier one. The goal of such a reduction is to better understand why a previous, false theory was successful. By deriving the predictions of the old theory in the current theory, we can, for instance, provide the conditions under which the previous theory applies. The goal is for the new theory to show us how and why the old theory was approximately true but nonetheless false.

The second way of thinking about limiting relations is through the models of the theories in question. On this approach, one aims to show how to recover models of the previous theory from models of the current one. This approach tends to be more general, not as concerned about deriving particular formulas or observable consequences of one theory from another, but more concerned with the relations between the theories themselves. It has thus been referred to as the “geometric approach.” This approach is less suitable for physical explanations, but can nonetheless be explanatory, as I hope to illustrate below.

Let us consider the classical limit (i.e., non-relativistic) of General Relativity. It is often noted that one can recover classical expressions in the low-velocity limit of GR. This involves a limit of the first kind mentioned above: it helps us understand why Newtonian physics was (and still is) a successful approximation. It is because, most of the time, we are interested in macroscopic bodies moving nowhere near the speed of light. Newtonian physics is a suitable theory to understand such bodies and makes accurate predictions for their motion. Thus, we have explained how NG, although false, could nonetheless have been successful.

But now consider how we might take the limit of GR on the second approach. A central feature of relativistic theories, one that differentiates them from classical theories, is the upper bound they place on the speed of light. This approach asks what happens as we relax this condition. Indeed, our goal is to show how models of GR with upper bounds on the speed of light can be related to models of a classical theory that places no such upper bound.[2] The limiting procedure will begin with a model of GR and allow the light cones to "open up." At every step along the limit, we can ensure that we retain a model of GR (albeit with a different upper bound on the speed of light). It turns out that, as the speed of light goes to infinity, we recover a spatially flat, classical spacetime model.

How is such an approach explanatory? Fletcher argues that understanding how this methodology is explanatory relies on considering the collection of all spacetime models (with an appropriate topology on this collection). This enables us to see the classical and relativistic models as "instantiations of a more general 'frame theory' that makes explicit the conceptual and technical continuity between the two" (Fletcher [2019], 3). There is, however, a narrower question we might ask for which the geometric approach is more immediately explanatory: Is it a mere coincidence that Newtonian physics presumes a Euclidean background? This question is what Malament [1986a] took himself to be explaining. He demonstrates how, when the limiting procedure is applied to General Relativity, the procedure itself guarantees the Euclidean (i.e., spatially flat) nature of the resultant space. As Malament puts it, "...the limiting process which effects the transition from general relativity to Newtonian gravitational theory 'squeezes out' all spatial curvature" ([1986a], 406). Malament describes his methodology for addressing the issue of spatial flatness issue as from the inside, i.e., from within physical theory. He takes a question of long-standing philosophical interest—one that some philosophers take to be *a priori* true while others think is a conventional choice—and locates it within physical theory. He shows that the geometric approach allows us to answer why classical spacetimes are set on flat spatial backgrounds: to the extent that they can be thought of as the classical limit of GR, they must.

Having described how understanding the geometric limit of GR can be explanatory, let us now consider what might be gained by studying the classical limit of TPG. First, and most simply, understanding the limiting behavior of TPG will be instructive for understanding TPG

[2] As noted by Fletcher ([2019], 2), this approach dates back to Minkowski [1952]. Minkowski observed that, as we allow the speed of light to become unbounded (i.e., $c \to \infty$), the hyperboloids of constant coordinate time of General Relativity become hyperplanes. The approach has since been expanded on by many others, including Ehlers ([1991]; since published as a "Golden Oldie," see Buchert and Mädler [2019]) and Malament [1986b] Here I focus on Malament's work.

itself. This is because while TPG is typically formulated using the tetrad formalism, non-relativistic theories are typically not. Thus, studying their relation formally will help clarify various features of the TPG formalism.

Second, such a project will help us situate the theories, not only in relation to one another but also in relation to other nearby gravitational theories. We know some aspects of the conceptual landscape. We know, for instance, how to construct models of TPG from models of GR. We also know that the classical limit of GR is Newton-Cartan theory and we know how to recover Newtonian Gravity from it. By studying the classical limit of TPG, I will address whether the theories form a commuting diagram. Specifically, I will address whether, if one starts with GR, taking the classical limit and then allowing torsion commutes with first allowing torsion and then taking the classical limit.

Finally, determining whether the classical limit of GR is the same as that of TPG will provide some perspective on the above discussion about underdetermination. If the classical limits of the two theories end up being the same, this can provide some evidence towards the claim of underdetermination. If they are not, however, it will be instructive to consider why the two theories yield different classical limits and how this difference ought to be considered in evaluating questions of underdetermination.

3 Preliminaries

In what follows, I assume that a model of GR is specified by a pair (M, g_{ab}) where M is a smooth, connected, four-dimensional, paracompact, Hausdorff manifold, and g_{ab} is a smooth, Lorentz-signature metric on M. A model of a classical spacetime, meanwhile, is given as $(M, t_a, h^{ab}, \tilde{\nabla})$ where M is again a smooth, connected, four-dimensional, paracompact, Hausdorff manifold; t_a is a smooth field on M of signature $(1, 0, 0, 0)$; h^{ab} is a smooth symmetric field on M of signature $(0, 1, 1, 1)$ that is orthogonal to t_a (i.e., $t_a h^{bc} = 0$); and, ∇ is a derivative operator compatible with t_a and h^{ab} (i.e., $\nabla_a t_b = \nabla_a h^{bc} = 0$).

3.1 The Limit Procedure in General Relativity

As discussed above, on the geometric approach to taking the classical limit, the goal is to capture the characteristic difference between models of relativistic and classical theories. The characteristic difference here is the upper bound that relativistic theories place on the speed of light, i.e., the light cone structure. Thus, our goal is to describe a limiting procedure that features this lightcone structure. The methodology will be to develop a limiting procedure that allows the speed of light to become unbounded. This can be visualized as "opening up" the lightcones of a relativistic theory. Indeed, allowing the lightcones to "open up" is the method adopted in Malament's proof showing the relation between GR and NCT ([1986b], discussed in more detail below). There, he writes:

> ...the work under discussion provides the means with which to make clear geometric sense of the standard claim that Newtonian gravitational theory is the "classical limit" of general relativity. One considers an appropriate one-parameter family of relativistic models $(M, g_{ab}(\lambda), T_{ab}(\lambda))$ satisfying Einstein's equation, defined for $\lambda > 0$, and the proves that in the limit as $\lambda \rightarrow 0$ a classical model $(M, t_a, h^{ab}, \nabla_a, \rho)$ satisfying (the recast version of) Poisson's equation is defined.[3] Intuitively, as $\lambda \rightarrow 0$, the null cones of the $g_{ab}(\lambda)$ "flatten" until they become degenerate. ([1986b], 182)

The proofs deriving the classical limit of GR proceed in two steps. The first step involves specifying the process of opening up the light cones and showing that, in the limit, the metric and derivative operator of GR converges to those of NCT. The second step involves considering how the matter content behaves in the limit. Let us consider each step in turn below.

Since classical theories have both a spatial metric (h^{bc}) and a temporal metric (t_a), our goal is to have the GR metric (g_{ab}) converge to the temporal and spatial metrics of a classical theory. We will allow $g_{ab}(\lambda)$ to be a one-parameter family of non-degenerate Lorentz metrics where λ ranges over some interval $(0, k)$. In the limit, however, we will require the metric to satisfy

C1 $g_{ab}(\lambda) \rightarrow t_a t_b$ as $\lambda \rightarrow 0$ for some closed field t_a, and

C2 $\lambda g^{ab}(\lambda) \rightarrow -h^{ab}$ as $\lambda \rightarrow 0$ for some field h^{ab} of signature (0,1,1,1),

where λ is $\frac{1}{c^2}$. The scaling is necessary in the second condition because, as the light cones open up, spacelike vectors will begin to diverge. To ensure convergence, we rescale the spatial metric with $\frac{1}{c^2}$.

Malament shows that, if the metric converges in a way satisfying the above conditions, there is a derivative operator that one can use to define a classical spacetime.

Proposition 1 *(Malament [1986b], Proposition on Limits (1)) Suppose $g_{ab}(\lambda)$ is a one-parameter family of Lorentz metrics on a manifold, M. Suppose also that t_a and h^{ab} satisfy C1 and C2. Then*

[3] The recast version of Poisson's equation referred to here is geometrized Poisson's equation: $R_{ab} = 4\pi\rho t_a t_b$.

1. *There is a derivative operator* ∇_a *on* M *satisfying* $\overset{\lambda}{\nabla} \to \nabla_a$ *as* $\lambda \to 0$.

2. $(M, t_a, h^{ab}, \nabla_a)$ *is a classical spacetime model satisfying* $R^{[a}{}_{(b}{}^{c]}{}_{d)} = 0$.

In other words, as long as our metric is properly behaving in the limit (i.e., it satisfies C1 and C2), the derivative operators corresponding to the GR metric "along the way" will converge. Furthermore, the derivative operator they converge to, along with the temporal and spatial metrics, will yield a classical spacetime.

Though I will not reproduce the proof, I will highlight one aspect of it: the role of the connecting fields. Connecting fields, $C^a{}_{bc}$, relate any two derivative operators. In this context, we are considering the derivative operators in the limit, $\overset{\lambda}{\nabla}_a$. We want to show that these derivative operators converge to ∇_a, the derivative operator of a classical spacetime. To do so, the proof considers an intermediary derivative operator, $\tilde{\nabla}_a$, that relates any two derivative operators along the way. These intermediary derivative operators express the difference between the derivative operator of GR and of a classical theory at any point in the limiting process. Specifically, we take $\overset{\lambda}{\nabla}_a = (\tilde{\nabla}_a, \overset{\lambda}{C}{}^a{}_{bc})$ and $\nabla_a = (\tilde{\nabla}_a, C^a{}_{bc})$. Then, it suffices to show that $\overset{\lambda}{C}{}^a{}_{bc} \to C^a{}_{bc}$ in order to demonstrate the convergence of the derivative operators. I highlight the role of these connecting fields here as they will play an important role in the below proof regarding the classical limit of TPG.

Let us now consider how the matter fields behave in the limit. Somewhat surprisingly, it is in this step that we see the curvature being "squeezed out." Up to this point, all we know about the curvature is that it satisfies $R^{[a}{}_{(b}{}^{c]}{}_{d)} = 0$. By considering the limiting behavior of Einstein's equation, aiming to show that it reduces to (some form of) Poisson's equation in the limit, we find that the resultant spacetime is spatially flat.

We begin, like above, by placing conditions on the limiting behavior of Einstein's equation

C3 $\overset{\lambda}{R}_{ab} = 8\pi(T_{ab}(\lambda) - \frac{1}{2}g_{ab}(\lambda)\overset{\lambda}{T})$ holds for all λ, and

C4 $T^{ab}(\lambda) \to T^{ab}$ as $\lambda \to 0$ for some field T^{ab}.

The second condition arises from the requirement that the limiting process assign limiting values to various components of the energy-momentum tensor (i.e., the mass-energy density, three-momentum density, and three-dimensional stress tensor). Using these conditions, Malament proves the following.

Proposition 2 *(Malament [1986b], Proposition on Limits (2)) Suppose* $g_{ab}(\lambda)$ *is a one-parameter family of Lorentz metrics on a manifold, M which, together with the symmetric* *family $T_{ab}(\lambda)$, satisfies conditions C1 - C4. Further suppose $(M, t_a, h^{ab}, \nabla_a)$ is the classical spacetime model described in the previous proposition. Then, there is a function ρ on M satisfying*

1. $T_{ab}(\lambda) \to \rho t_a t_b$ *as* $\lambda \to 0$.
2. $R_{ab} = 4\pi\rho t_a t_b$.

The above states that Einstein's equation reduces to a geometric formulation of Poisson's equation in the limit. This geometrized Poisson's equation is, in turn, what tells us that space is flat (see (Malament [2012], Proposition 4.1.5) for detailed discussion on this point). Reflecting on this point, Malament writes:

> If at every intermediate stage of the collapse process [i.e., the opening up of the lightcones] spacetime structure is in conformity with the dynamic constraints of general relativity (as embodied in Einstein's field equation), then the resulting induced hyperspaces are necessarily flat, i.e., have vanishing Riemann curvature. ([1986a], 406).

This two-step, geometric limiting procedure is what I aim to reproduce in the contexts of torsional theories of gravity.

3.2 Teleparallel Gravity

As mentioned above, Teleparallel Gravity is a relativistic theory (it posits a Lorentz-signature metric) and is empirically equivalent to GR, at least locally.[4] Unlike GR, though, it is set on a flat spacetime background. Instead of the Levi-Civita connection, the unique, torsion-free metric compatible connection, TPG uses the Weitzenböck connection, a flat connection with

[4]The first formulation of teleparallel gravity is often attributed to Einstein. In a paper published in June 1928 ("Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus" or "Riemannian Geometry with Maintaining the Notion of Distant Parallelism"), he begins developing a gravitational theory with torsion using the tetrad approach. The motivation for the project was to unify gravity and electromagnetism, the idea being that the (six) extra degrees of freedom afforded by torsion could be used to represent the electromagnetic field. Just one week later, he publishes "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität" ("New Possibility for a Unified Field Theory of Gravitation and Electricity") which presents the field equations of the new theory. After corresponding with Weitzenböck and Cartan, Einstein abandoned the project, finding himself unable to attribute physical meaning to the structures posited by the theory (see, especially, his 1932 letter to Cartan, reprinted in Debever 2015, 209-10).
The theory of absolute parallelism remained abandoned until it was taken back up nearly 30 years later by Møller in 1961, and, completely independently, by Hayashi and Nakano in 1967. After some further work in this area, Hayashi and Shirafuji bring together these distinct projects in their paper "New general relativity" published in [1979]. The contemporary formulation TPG began to emerge in the 1990s with work by de Andrade, Pereira, Obukhov, and Aldrovandi (see especially de Andrade and Pereira [1997]).

torsion.[5] Thus, (con)torsion is used to represent the (velocity-dependent) forces that characterize gravitational influence.[6]

The TPG literature typically uses the tetrad approach.[7] The tetrad approach generalizes the coordinate basis approach: instead of requiring (holonomic) coordinate bases as the bases for the tangent bundle, one only requires a locally defined set of linearly independent vector fields as the basis for the tangent bundle. When the vector bundle has four-dimensional fibers, it is referred to as a "tetrad" or a "vierbein" (where "veir" means "four"). Though in general, there may not be such bases (also called "frames" or "frame fields") across all the manifold, they do always exist locally.

Suppose we fix an arbitrary manifold, M. Let $E \xrightarrow{\pi} M$ be a vector bundle over the manifold with $n-$dimensional fibers. We denote elements of the bundle with capital Latin indices and define a frame field (and coframe field) as follows.

Definition 1 *A frame field for E on a neighborhood of the manifold ($O \subseteq M$) is a collection of n vector fields, $\{(e_i)^A\}$ (where $i = 1, \ldots, n$), that form a basis for the fiber at E at each point $p \in O$. A* coframe field *on O is a collection of n covector fields $\{(e^i)_A\}$ forming a basis for the dual bundle at each point $p \in O$.*

Note here that the lowercase indices are counting, not abstract, indices. Taking i to range from 1 to $n = 4$ yields the tetrads.

The above definition yields the following proposition.

Proposition 3 *Given any frame field $\{(e_i)^A\}$ for E on $O \subseteq M$, there exists a unique coframe field $\{(e^i)_A\}$ such that $(e^i)_A(e_j)^A$ equals 1 if $i = j$ and $\mathbf{0}$ otherwise.*

We express the metric in terms of the tetrads as

$$g_{ab} = \sum_{i=1}^{4}\sum_{j=1}^{4} \eta_{ij}(e^i)_a(e^j)_b = \sum_{i=1}^{4} \eta_{ii}(e^i)_a(e^i)_b, \tag{3.1}$$

[5]One can visualize torsion as the twisting of the tangent space as it is parallel transported along a curve. One can also imagine parallel transporting two end-to-end vectors along one other. When the torsion is vanishing, this procedure yields a parallelogram. However, in spaces with torsion, the parallelograms break because the vectors do not end up tip-to-tip.

[6]The expression for the acceleration of test bodies in the presence of a gravitation field in TPG involves the contorsion tensor (given below). In brief, the contorsion tensor relates any metric-compatible connection to the unique Levi-Civita connection and, when antisymmetrized on its bottom indices, returns the torsion tensor.

[7]I derive the main result of the present paper in the tetrad formalism in order to be consistent with the TPG literature so it will be important to introduce it in detail here.

where η_{ij} are the Minkowski metric components and the second equality follows because $\eta_{ij} = 0$ when $i \neq j$ (c.f., Aldrovandi and Pereira 2013, Eq. 1.27). The metric with raised indices is given similarly.

Finally, let us also consider the torsional derivative operator. Assume that ∂ is a coordinate derivative operator relative to some local coordinate system on M; we know ∂ is both flat and torsion-free (symmetric). Next, consider any frame field, $\{(e_i)^a\}$, as defined above. It turns out that there always exists a *unique* derivative operator D relative to which all n frame elements are constant: $D_a(e^i)^b = \mathbf{0}$. This derivative operator can be defined relative to ∂ by taking $D = (\partial, C^a{}_{bc})$, where for each $i = 1, \ldots, n$, we have $C^a{}_{bc}(e_i)^c = \partial_b(e_i)^a$, or, equivalently, $C^a{}_{bc} = \sum_{i=1}^n (e^i)_c \partial_b (e_i)^a$.

The torsion of this derivative operator is given in terms of the tetrads as

$$T^a{}_{bc} = \sum_{i=1}^{n} (e^i)_{[c}\partial_{b]}(e_i)^a. \tag{3.2}$$

Finally, as mentioned above, the trajectories of massive test particles are influenced by the presence of mass. Their acceleration is

$$\xi^n \nabla_n \xi^a = K^a{}_{bc}\xi^b\xi^c, \tag{3.3}$$

where ξ^a is tangent to the particle's trajectory, $K^a{}_{bc}$ is the contorsion tensor, and ∇ is the torsional derivative operator.[8] On the conventions adopted here $K^a{}_{bc} = \frac{1}{2}(T^a{}_{bc} + T_{cb}{}^a - T_{bc}{}^a)$.

3.3 A Torsional Classical Theory of Gravity

Before considering the classical limit of TPG, we should ask whether it is possible to formulate a classical theory of gravity with torsion. If not, we will clearly be unable to maintain torsion in the classical limit. But if it is possible, it is not immediately clear what the implications would be to our present project. On the one hand, we might expect such a theory to arise as the classical limit of TPG. On the other hand, in the context of GR, we saw how the curvature gets squeezed out in the limit, so we might expect that torsion will be similarly squeezed out in the limit.

There have been some efforts in the literature to develop a torsional, classical theory (e.g.,

[8]This expression is sometimes given in terms of the torsion tensor instead of the contorsion tensor as in Eq. 28 of Knox [2011]; see Comment 6.4 of Aldrovandi and Pereira [2013] for further discussion.

Bergshoeff *et al.*, Geracie *et al.* [2014, 2015]). Many of these projects, however, drop the requirement that the (classical) temporal metric be closed (see Meskhidze and Weatherall 2023, §3.4 for more). For the purposes of the present paper, I require the temporal metric to be closed. (Indeed, this requirement will be formalized below as a condition on the behavior of the tetrads in the limit.) This means that these previously developed theories will be unsuitable as the limit of TPG on our approach.

Fortunately, a classical theory with torsion and a closed temporal metric has been developed by Meskhidze and Weatherall ([2023], Theorem 1). There, the authors prove a theorem analogous to the Trautman degeometrization theorem which establishes that, for every model of Newton-Cartan theory, there is a corresponding (non-unique) model of classical, torsional gravity with the same mass density and particle trajectories. The model of classical, torsional gravity features a closed temporal metric as well as a derivative operator that is compatible with the spatial and temporal metrics, is flat, and has non-vanishing torsion. Additionally, they provide force and field equations featuring the torsion. Notably, the field equation is a generalization of Poisson's equation (i.e., in the limit of vanishing torsion, they recover Poisson's equation). The result provides a proof of concept: it is possible to formulate a classical spacetime with torsion. It remains to be seen whether this spacetime is the classical limit of GR.

4 The classical limit of Teleparallel Gravity.

4.1 The behavior of the tetrads

To take the classical limit of TPG, we will need to constrain the behavior of the metric in the limit. Recall that, ultimately, we want to derive the temporal and spatial metrics of a classical spacetime theory. Given that the TPG metric we begin with is expressed in terms of tetrads, we will need to express the classical spacetime metrics in terms of tetrads too. The goal will be to mimic the decomposition of the standard metric into the temporal and spatial metrics with tetrads. On the tetrad formalism, we have, at each point, a collection of four orthonormal vector fields. Suppose that we fix the first "leg" of the cotetrad to correspond to the temporal metric of a classical spacetime. As with the standard temporal metric of a classical spacetime, we require it to be closed. The remaining cotetrad elements will vanish for a classical spacetime while the other tetrad elements (besides the first) will compose the spatial metric.

Definition 2 *Call a tetrad $\{(f_i)^a\}$ a "classical tetrad" if and only if an element of its cotetrad is closed.*[9] *If this is the case, then classical metrics t_a and h^{ab} can be defined such that the cotetrad element that is closed corresponds to t_a and the following conditions are satisfied:*

1. $(f_1)^a t_a = 1$,
2. $(f_i)^a t_a = 0$ *for* $i = 2, 3, 4$,
3. $(f^1)_a (f^1)_b h^{ab} = \mathbf{0}$, *and*
4. $(f^i)_a (f^j)_b h^{ab} = 1$ *for* $i = j = 2, 3, 4$ *and* $\mathbf{0}$ *otherwise.*

With this definition of a classical tetrad in hand, we now turn to the desired limiting behavior. In the case of GR, we considered the limit of a family of metrics as the speed of light is allowed to become unbounded (i.e., $c \to \infty$). Here, we similarly consider the behavior of $\{(e_i)^a(\lambda)\}$, a one-parameter family of tetrads where $0 < \lambda < k$. To recover a "classical tetrad," we will consider the convergence of the tetrad components. We need the first "leg" of the cotetrad to yield the temporal metric in the limit, i.e. $(e^1)_a \to t_a$. We will need to appropriately rescale the tetrad components yielding the spatial metric to ensure that they converge in the limit. We rescale these components with $\frac{1}{c} = \sqrt{\lambda}$ so the desired limiting behavior is $\sqrt{\lambda}(e_i)^a \to (f_i)^a$ for $i = 2, 3, 4$.[10]

I formalize these considerations with the following conditions (designed to be analogous to Malament's [1986b] limiting conditions, C1 and C2, discussed in §3.1 above):

C1* $\sum_{i=1}^{4} (\overset{\lambda}{e}{}^i)_a \to \sum_{i=1}^{4} (f^i)_a = (f^1)_a = t_a$ as $\lambda \to 0$ for some closed field t_a, and

C2* $\sqrt{\lambda} \sum_{i=1}^{4} (\overset{\lambda}{e}_i)^a \to \sum_{i=2}^{4} (f_i)^a$ as $\lambda \to 0$.

For C1*, the equality follows because $\{(f^i)_a\}$ is a classical tetrad, which means that only the first "leg" of the cotetrad (i.e., $i = 1$) is non-vanishing. For C2*, we take $\sum_{i=1}^{4} \eta^{ii} (f_i)^a (f_i)^b = -h^{ab}$ for some field h^{ab} of signature $(0, 1, 1, 1)$, ensuring that we recover the spatial metric in the limit.

[9] From the fact that a cotetrad element is closed, we know that it is locally exact. So, taking the first cotetrad element to be $\nabla_{[a}(f^1)_{b]} = 0$ implies $(f^1)_b = \nabla_b t$ for some smooth function t. Contracting this with the spatial metric would yield $\mathbf{0}$ since $(\nabla_a \nabla_b t) h^{ab} = \mathbf{0}$.

[10] Note that, in the case of GR, we had $\lambda g^{ab}(\lambda) \to -h^{ab}$. The proposed rescaling ($\sqrt{\lambda}$) is reasonable since we are rescaling one tetrad leg at a time.

4.2 The Classical Limit of TPG

We are now in a position to consider the classical limit of TPG. The proof will begin by considering the behavior of the tetrads and derivative operators in the limit. I will express the derivative operators at each stage of the limit in terms of a connecting field relating them to a flat, torsion-free derivative operator. I will then consider the limiting behavior of the connecting field. Following the requirement that the derivative operator converges in the limit, I will require that the connecting field converge too. This will lead to the surprising result that the torsion must vanish in the limit. For those interested in the implications but not the details of the proof, I suggest skipping to §4.3.

Theorem 1 *Suppose that* $\{(e_i)^a(\lambda)\}$ *is a one-parameter family of tetrads on a manifold,* M*. Suppose* $\{(f_i)^a\}$ *satisfies conditions C1* and C2* and that there is a derivative operator* ∇_a *on* M *satisfying* $\overset{\lambda}{\nabla}_a \to \nabla_a$ *as* $\lambda \to 0$*. Then, the following holds.*

1. ∇_a *is such that* $(M, \{(f_i)^a\}, \nabla_a)$ *is a classical spacetime model where* $\{(f_i)^a\}$ *is a classical tetrad and* ∇_a *is flat.*
2. *For any derivative operator,* ∇_a*, satisfying the above, the torsion vanishes.*

Proof. Suppose, for the sake of contradiction, that the torsion does not vanish in the limit (i.e., $\overset{\lambda=0}{T}{}^a{}_{bc} \neq 0$). We know that $\{(\overset{\lambda}{e}{}^i)_a\}) \to (f^1)_a$ smoothly. This means that there must exist smooth fields m_a, $n_a(\lambda)$, and n_a satisfying[11]

$$(\overset{\lambda}{e}{}^1)_a \to (f^1)_a - \lambda m_a + \lambda^2 n_a(\lambda), \quad \text{and} \quad n_a(\lambda) \to n_a \text{ as } \lambda \to 0.$$

Similarly, since $\sqrt{\lambda}\{(\overset{\lambda}{e}_i)^a\} \to \{(f_i)^a\}$ for $i = 2, 3, 4$, there must exist some smooth fields r^a, $s^a(\lambda)$, and s^a satisfying

$$\sum_{i=1}^{4} \sqrt{\lambda}(\overset{\lambda}{e}_i)^a \to \Big(\sum_{i=2}^{4}(f_i)^a\Big) - \lambda r^a + \lambda^2 s^a(\lambda), \quad \text{and} \quad s^a(\lambda) \to s^a \text{ as } \lambda \to 0.$$

We begin by fixing some flat, torsion-free ∂ on M. The existence of such a ∂ is always guaranteed locally (see §3.2). Suppose we define $\overset{\lambda}{\nabla}$ in terms of a ∂ and a connecting field

$$\overset{\lambda}{\nabla} = (\partial, C^a{}_{bc}(\lambda))$$

[11] As noted by Malament ([1986b], 194), these conditions follow from the smoothness of the limit and are another way of saying the limit is twice-differentiable.

for all λ. The expression for the torsion in terms of the tetrads and this derivative operator is

$$\overset{\lambda}{T}{}^{a}{}_{bc} = \sum_{i=1}^{n} (\overset{\lambda}{e}{}^{i})_{[c}\partial_{b]}(\overset{\lambda}{e}_{i})^{a}$$

for all λ. (Recall that we have assumed, for contradiction, that the torsion is non-vanishing.) Relative to ∂ and for each λ, the C-fields defining the tetrad derivative operators can be expressed as[12]

$$\begin{aligned} C^{a}{}_{bc}(\lambda) = \frac{1}{2} g^{ad}(\lambda)\Big[&\partial_{d} g_{bc}(\lambda) - \partial_{b} g_{dc}(\lambda) - \partial_{c} g_{db}(\lambda) \\ &- g_{dm}(\lambda) T^{m}{}_{cb}(\lambda) + g_{bm}(\lambda) T^{m}{}_{dc}(\lambda) + g_{cm}(\lambda) T^{m}{}_{db}(\lambda)\Big]. \end{aligned} \tag{4.1}$$

We now consider the tetrad expressions for all the terms starting with the metric, then the first three terms in the square brackets, and then the torsion terms. Using C2*, we rewrite $g^{ad}(\lambda)$ in terms of the tetrads as

$$g^{ad}(\lambda) = \sum_{i=1}^{4} n^{ii} (\overset{\lambda}{e}_{i})^{a} (\overset{\lambda}{e}_{i})^{d}.$$

Then, in the limit, we have

$$\begin{aligned} \sum_{i=1}^{4} n^{ii} (\overset{\lambda}{e}_{i})^{a} (\overset{\lambda}{e}_{i})^{d} \rightarrow & \frac{1}{\lambda}\Big[\Big(\sum_{i=2}^{4} (f_{i})^{a} - \lambda r^{a} + \lambda^{2} s^{a}(\lambda)\Big)\Big(\sum_{i=2}^{4} (f_{i})^{d} - \lambda r^{d} + \lambda^{2} s^{d}(\lambda)\Big)\Big] \\ &= \Big[\frac{1}{\lambda}\Big(\sum_{i=2}^{4} (f_{i})^{a} (f_{i})^{d}\Big) - \Big(\sum_{i=2}^{4} (f_{i})^{a} r^{d} + \sum_{i=2}^{4} (f_{i})^{d} r^{a}\Big) \\ &+ \lambda\Big(r^{a} r^{d} + \sum_{i=2}^{4} (f_{i})^{a} s^{d}(\lambda) + \sum_{i=2}^{4} (f_{i})^{d} s^{a}(\lambda)\Big) \\ &- \lambda^{2}\Big(r^{d} s^{a}(\lambda) + r^{a} s^{d}(\lambda)\Big) + \lambda^{3}\Big(s^{a}(\lambda) s^{d}(\lambda)\Big)\Big]. \end{aligned}$$

[12] C.f. Jensen 2005, Eq. 3.1.27 where $T_{abc}(\lambda) = g_{am}(\lambda) T^{m}{}_{bc}(\lambda)$.

Next, consider the first three terms inside the square brackets of Eq. 4.1.

$$\partial_d g_{bc}(\lambda) - \partial_b g_{dc}(\lambda) - \partial_c g_{db}(\lambda) = \partial_d \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_b (\overset{\lambda}{e}{}^i)_c - \partial_b \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_c - \partial_c \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_b$$

$$\begin{aligned} &\to \partial_d \Big((f^1)_b - \lambda m_b + \lambda^2 n_b(\lambda) \Big) \Big((f^1)_c - \lambda m_c + \lambda^2 n_c(\lambda) \Big) \\ &- \partial_b \Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_c - \lambda m_c + \lambda^2 n_c(\lambda) \Big) \\ &- \partial_c \Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_b - \lambda m_b + \lambda^2 n_b(\lambda) \Big). \end{aligned} \tag{4.2}$$

We now consider the terms arising from the torsion. It will suffice to consider the expansion of the first torsion term

$$g_{dm}(\lambda) T^m{}_{cb} = \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_m \sum_{j=1}^{4} (\overset{\lambda}{e}{}^j)_{[b} \partial_{c]} (\overset{\lambda}{e}{}_j)^m.$$

In the limit, this yields

$$\begin{aligned} g_{dm}(\lambda) T^m{}_{cb} \to &\Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_m - \lambda m_m + \lambda^2 n_m(\lambda) \Big) \\ &\Big((f^1)_{[b} - \lambda m_{[b} + \lambda^2 n_{[b}(\lambda) \Big) \partial_{c]} \frac{1}{\sqrt{\lambda}} \Big(\sum_{i=2}^{4} (f_i)^m - \lambda r^m + \lambda^2 s^m(\lambda) \Big). \end{aligned} \tag{4.3}$$

With the expression for some of the components of the connecting fields expanded, we can now consider the behavior of these components in the limit. Since $\lambda \to 0$, any terms that are multiplied by $\sqrt{\lambda}$, λ, or higher orders of λ will vanish. Any terms divided by $\sqrt{\lambda}$, λ, or higher orders of λ will become unbounded. The first term, the metric, yielded an expression with a $\frac{1}{\lambda}$ term, a term with no λ dependence, and terms with λ and higher-orders of λ dependence. Of the remaining terms in the expression for the connecting field, only the torsion terms have an inverse λ dependence. Thus, when the terms from the metric are multiplied through and the limit is taken, only the $\frac{1}{\lambda}$ term and the term with no λ dependence will remain, yielding $\sum_{i=1}^{4} \Big[\frac{1}{\lambda} \Big((f_i)^a (f_i)^d \Big) - \Big((f_i)^a r^d + (f_i)^d r^a \Big) \Big]$.

Consider the limit of the next three terms when multiplied by the above. Only terms with no λ dependence or those linearly dependent in λ will remain in the limit. Eq. 4.2 simplifies to

$$\begin{aligned} &\partial_d \Big((f^1)_b - \lambda m_b \Big) \Big((f^1)_c - \lambda m_c \Big) - \partial_b \Big((f^1)_d - \lambda m_d \Big) \Big((f^1)_c - \lambda m_c \Big) - \partial_c \Big((f^1)_d - \lambda m_d \Big) \Big((f^1)_b - \lambda m_b \Big). \\ &= \partial_d (f^1)_b (f^1)_c - \partial_b (f^1)_d (f^1)_c - \partial_c (f^1)_d (f^1)_b - 2\lambda \partial_d (f^1)_{(b} m_{c)} + 2\lambda \partial_b (f^1)_{(d} m_{c)} + 2\lambda \partial_c (f^1)_{(d} m_{b)}. \end{aligned}$$

where we have dropped the λ^2 terms in the second line. Take just the first three terms in each expansion above (i.e., all the terms without λ in front). In the limit, they yield

$$
\begin{aligned}
&\partial_d(f^1)_b(f^1)_c - \partial_b(f^1)_d(f^1)_c - \partial_c(f^1)_d(f^1)_b \\
&\quad = (f^1)_b\partial_d(f^1)_c + (f^1)_c\partial_d(f^1)_b - (f^1)_d\partial_b(f^1)_c - (f^1)_c\partial_b(f^1)_d - (f^1)_d\partial_c(f^1)_b - (f^1)_b\partial_c(f^1)_d \\
&\quad = -2(f^1)_d\partial_{(b}(f^1)_{c)} = -2(f^1)_d\partial_b(f^1)_c.
\end{aligned}
$$

We have used the fact that the temporal metric will be closed (i.e., $\partial_{[a}(f^1)_{b]} = 0$) to simplify this expression. To move from line two to three, notice that the first and last terms cancel and the second and fourth terms cancel. This leaves only the third and fifth terms. Then, since $\partial_{[a}(f^1)_{b]} = 0$, we can drop the symmetrization parentheses. (One can see that this is the corresponding tetrad expression to Malament's $-2t_d\tilde{\nabla}_b t_c$.) Combining with the above, we find that Eq. 4.2 simplifies to

$$
-2(f^1)_d\partial_b(f^1)_c - 2\lambda\Big(\partial_d(f^1)_{(b}m_{c)} - \partial_b(f^1)_{(d}m_{c)} - \partial_c(f^1)_{(d}m_{b)}\Big).
$$

Finally, we return to the limiting behavior of the terms corresponding to the torsion, Eq. 4.3. When multiplied by the expression out front (which has a λ^{-1} and a term with no λ dependence), we end up with terms of $\frac{1}{\lambda^{3/2}}$, $\frac{1}{\sqrt{\lambda}}$ dependence, $\sqrt{\lambda}$ dependence, and higher-orders of λ dependence. In order to show that the torsion must vanish in the limit, it will suffice to consider the $\frac{1}{\lambda^{3/2}}$ terms. These are

$$
\begin{aligned}
\sum_{i=2}^{4}\frac{1}{\lambda}(f_i)^a(f_i)^d\Big(&-(f^1)_d(f^1)_m(f^1)_{[b}\partial_{c]}\frac{1}{\sqrt{\lambda}}\sum_{j=2}^{4}(f_j)^m \\
&+(f^1)_b(f^1)_m(f^1)_{[c}\partial_{d]}\frac{1}{\sqrt{\lambda}}\sum_{i=2}^{4}(f_i)^m + (f^1)_c(f^1)_m(f^1)_{[b}\partial_{d]}\frac{1}{\sqrt{\lambda}}\sum_{i=2}^{4}(f_i)^m\Big).
\end{aligned}
$$

The first term in the parentheses will vanish as $\sum_{i=2}^{4}(f_i)^d(f^1)_d$ yields 0. Any terms with $(f^1)_d$ from anti-symmetrization will also vanish. However, we have no way of constraining the remainder: the $\frac{1}{\lambda^{3/2}}$ dependence means that they will become unbounded in the limit. Indeed, we can collect together some of these terms into a tensor, $\overset{\lambda}{Z}{}^a{}_{bc}$, that we define as follows.

$$
\overset{\lambda}{Z}{}^a{}_{bc} := \frac{1}{\lambda^{3/2}}\sum_{i=2}^{4}(f_i)^a(f_i)^d(f^1)_b(f^1)_m(f^1)_c\partial_d\sum_{i=2}^{4}(f_i)^m.
$$

From the above discussion, we know $\overset{\lambda}{Z}{}^a{}_{bc}$ will become unbounded in the limit. What is the significance of this? Recall that our theorem statement requires that the derivative operators converge in the limit. The connecting fields relate the derivative operators and we have been expanding the expression for the connecting fields in the limit. We have, in this expansion, found an unbounded quantity, $\overset{\lambda}{Z}{}^a{}_{bc}$, and derived a contradiction.

Indeed, for the connecting fields to converge in the limit, $\overset{\lambda}{Z}{}^a{}_{bc}$ must be bounded. *However, the only way for it to be bounded is if it vanishes.* If $\overset{\lambda}{Z}{}^a{}_{bc} = 0$, $C^a{}_{bc}(\lambda) \to C^a{}_{bc}$ as $\lambda \to 0$ where

$$C^a{}_{bc} = \sum_{i=1}^{4} \Big[\frac{1}{\lambda}\Big((f_i)^a(f_i)^d\Big) - \Big((f_i)^a r^d + (f_i)^d r^a\Big)\Big]$$
$$\Big[- 2(f^1)_d\partial_b(f^1)_c - 2\lambda\Big(\partial_d(f^1)_{(b}m_{c)} - \partial_b(f^1)_{(d}m_{c)} - \partial_c(f^1)_{(d}m_{b)}\Big)\Big].$$

Note, though, that this expression is just the expression for the connecting fields in the torsion-free context. In other words, if $\overset{\lambda}{Z}{}^a{}_{bc}$ vanishes, then the torsion will also vanish since $\overset{\lambda}{Z}{}^a{}_{bc}$ captures the only remaining contribution to the torsion. Consequently, *torsion must vanish in the limit.* □

4.3 A notable consequence

One striking aspect of the reduced theory is regarding the behavior of test bodies. In TPG, recall that the acceleration of test particles is given as

$$\xi^n\nabla_n\xi^a = K^a{}_{bc}\xi^b\xi^c, \tag{4.4}$$

where ξ^a is tangent to the particle's trajectory, $K^a{}_{cd}$ is the contorsion tensor which relates any metric-compatible connection to the unique Levi-Civita connection, and ∇ is the torsional derivative operator. Recall that in Newtonian gravity, the trajectories of massive test particles are also influenced by the presence of matter. The acceleration is governed by

$$\xi^n\nabla_n\xi^a = -\nabla^a\phi,$$

where ξ^a is tangent to the particle's trajectory, ϕ is the gravitational potential, and ∇ is flat.

The torsion vanishing in the classical theory means that the contorsion, too, will vanish. This means that Eq. 4.4 will reduce to $\xi^n\nabla_n\xi^a = 0$. One can think of this reduced theory as a special case of Newtonian gravity where the scalar field ϕ representing the gravitational potential vanishes. This means that in the reduced theory, particles are not accelerated in the presence of matter. Put simply, in this reduced theory, *there are no gravitational effects.*

From the above discussion, one might conclude that the theory derived in the limit cannot be considered a gravitational theory. Indeed, one might argue that Teleparallel Gravity has no classical limit at all.[13] I am sympathetic to this argument but worry it might be too quick. Namely, one might want to think about classical spacetimes as providing arenas for gravitational forces/effects. On such an interpretation, we do still have a spacetime theory at the classical limit of TPG, though perhaps not a gravitational one. In any case, I do not think it is of great importance whether we say TPG has no classical limit or it has a trivial one.

5 Discussion

We now turn to contextualizing these results. I first compare the results presented here to the results obtained in the torsion-free context. Then, I move to considering the methodology used here in comparison to other methods proposed in the recent literature. This discussion will clarify the seemingly inconsistent claims found in the recent literature.

5.1 Comparison to the torsion-free context

Recall (from §3.1) that GR reduces to NCT theory in the limit and that spatial flatness is proven by considering Einstein's equation in the limit. From the perspective of the reduction of GR to NCT, one might find the results proven here unsurprising. One might argue that since the classical limit squeezes out curvature, we should have expected it to squeeze out the torsion as well. Put differently, the two results seem consistent. When the limiting procedure is applied to General Relativity, it returns a spatially flat theory; meanwhile, when it is applied to Teleparallel Gravity, it returns a torsion-free theory.[14]

However, there is an important disanalogy between the results: the failure to derive a classical torsional spacetime was not a result of the behavior of matter fields as in the case of GR. Rather, it arose from the requirement that the derivative operators converge, a requirement

[13] I thank [redacted] as well as the audience of my presentation of this work at the [redacted] for pressing this interpretation.

[14] Notably, though, in the context of the reduction of GR to NCT, the space*time* is not flat; while the spatial curvature is squeezed out, one does still find temporal curvature. In the classical limit of TPG, the torsion vanishes entirely.

that had to be made independently.[15] It is not clear what to conclude from this disanalogy. On the one hand, one might want to argue that the link between the matter fields and geometry is, in some sense, more tightly constrained in TPG. On the other hand, this disanalogy may be more simply interpreted as a consequence of the limiting behavior of the torsion tensor.

5.2 Alternative limits

What is the relation between the results presented here and the many seemingly contradictory claims regarding limits found in the literature? As mentioned in §2.1, there are multiple means of taking the classical limit of a relativistic theory. One's goals may dictate which methodology is most appropriate. This makes the fact that there is a diversity of viewpoints quite natural. Indeed, we are now in a position to evaluate the merits of some of these approaches and highlight inconsistencies. Let us consider the different approaches in turn, comparing them to the geometric approach taken here.[16]

5.2.1 $1/c^2$ expansion

We begin by considering the claim that torsional Newton-Cartan "geometry is the correct framework to describe General Relativity in the non-relativistic limit" (Hansen *et al.* 2020, 1; see also Van den Bleeken, Hansen *et al.*, Hartong *et al.* [2017, 2020, 2023]). Considering the results from Malament discussed above—that standard NCT is the non-relativistic limit of GR—, how can such a claim be substantiated? This claim is based on Van den Bleeken's [2017] paper. There, he uses $\frac{1}{c^2}$ expansion to derive what he takes as the non-relativistic limit of GR, recovering a torsional theory in the limit. On the $\frac{1}{c^2}$ expansion approach, one typically allows the metric to diverge in the large c limit but the associated Levi-Cevita connection remains finite. However, Van den Bleeken takes this approach further, allowing the connection to diverge as well. As he puts it,

> [In previous work,] it is assumed that the relativistic metric is such that the associated Levi–Cevita connection remains finite in the large c limit. Although this might appear a natural assumption at first, one should keep in mind that the metric is allowed to diverge as $c \to \infty$. So why not the connection one could ask. ([2017], 2)

[15] Another way of understanding the disanalogy between the two situations is in terms of the strength of the requirement that the connecting fields be symmetric. The symmetric components of the connecting fields have a well-behaved $\lambda \to 0$ limit while the anti-symmetric components turn out not to. Therefore, taking the limit requires that those anti-symmetric components—here, the torsion—vanish entirely.

[16] This section has benefited immensely from discussions with [redacted]. Needless to say, any errors in the below presentation are my own.

Allowing the connection to diverge, he argues that the standard connection of NCT is not "the most natural connection for the expanded theory, as it is not compatible with the structure provided by the [spatial and temporal metrics]" ([2017], 6). Instead, he advocates for a torsional connection, that of twistless-torsional Newton-Cartan theory (TTNC). TTNC is a classical theory of gravity with torsion that does not require the temporal metric to be closed; it instead uses Frobenius's theorem and the so-called "hypersurface orthogonality condition" (i.e., $t_{[a}\partial_b t_{c]} = 0$) to derive flat, spacelike hypersurfaces.[17]

The first and more pressing issue with Van den Bleenken's methodology concerns its viability. Previous work, including Malament's [1986b] proofs discussed above, demonstrate that if one has a sequence of metrics of GR parameterized by λ and these converge, their derivative operators converge as well. In the above quotation, Van den Bleenken suggests that one can consider allowing the connections to diverge. However, given Malament's results, if the metrics converge, then their connections must as well. Thus, it seems that Van den Bleenken is entertaining a contradictory methodology.

Though critical, let us set aside this issue and consider the methodology generally. When taking the classical limit using the geometric approach, one considers a sequence of models, parameterized by λ, but all of one theory. In other words, at each step along the limit, one has a model of GR. Insofar as the sequence of models are models of GR, their connections—and, correspondingly, the connection of the recovered spacetime—are all required to be symmetric. Thus, on the geometric approach, if you begin with GR, you cannot derive a spacetime with torsion in the limit. However, the $1/c^2$ expansion approach places no such constraints—one simply expands the relevant equations or quantities of interest in powers of c. This means you may be able to derive a limiting spacetime that has a very different structure than the original.

What are the implications of this difference in approach? Typically, the goal of projects that expand in powers of the speed of light is to show how a previously successful theory was successful, i.e., to derive the empirical consequences of an earlier theory in the limit of a later one. If this is Van den Bleenken's goal, it is not clear what we should say based on the above since our previous theory did not involve torsion. Indeed, if we want to explain the success of the previous theory, it seems like we would also need to answer why is it that the temporal curvature of Newton-Cartan theory can be traded for the torsion of TTNC. At best, we could use them to argue for the conventionality of geometry—that the choice between torsion and

[17] For more on TTNC, see (Meskhidze and Weatherall [2023], §3.4).

curvature here is conventional as one can derive either in the limit. However, given the above-described inconsistency between these results and previous work, such an argument does not seem viable.

5.2.2 Null Reduction

Another limiting procedure, one that has received philosophical treatment, is null reduction. In their [2018] paper, Read and Teh develop a method for "teleparallelizing" in the classical context and show the relation between teleparallel gravity and their classical, teleparallelized theory using null reduction. Their argument draws on a notion of "extended torsion" which I describe next.

The notion of "extended torsion" was introduced by Geracie, Prahbu, and Roberts [2015] but the general approach dates back to the 1980s (see, e.g., Duval and Künzle [1984]). One standardly takes the symmetries of Newton-Cartan theory to be those described by the Galilean group. Recent projects have argued that, properly considered, the symmetry group of NCT is not the Galilean group, but rather the Bargmann group. The (inhomogenous) Galilei group (**IGal**) includes space and time translations and rotations as well as Galilei boosts. The Bargmann group is the one-dimensional central extension of inhomogenous Galilei group

$$\mathbf{Barg} = \mathbf{Gal} \ltimes (\mathbb{R}^4 \times \mathbf{U}(1)).^{18}$$

Using the Bargmann group as the symmetry group of NCT yields an "extended vielbein": ${e_\mu}^I = (\tau_\mu, {e_\mu}^a, m_\mu)$ where τ_μ is dubbed the clock torsion, ${e_\mu}^a$ is the spatial torsion, and m_μ is the mass torsion (see, e.g., Geracie *et al.* 2015, Eq. 2.14). As explained by Geracie, Prabhu, and Roberts, the mass torsion introduces "an additional gauge-field which couples to the mass of matter fields" ([2015], 4). Importantly, the mass torsion cannot be converted into spacetime torsion, as noted by Read and Teh ([2018], 2).

With this broadened notion of torsion, Read and Teh construct the teleparallel equivalent of Newton-Cartan theory and consider its relation to TPG using null reduction. Null reduction is a limiting procedure outlined in Duval [1985] wherein one considers the reduction of a $(D + 1)$-dimensional gravitational wave solution of a relativistic theory. Read and Teh show that the null reduction of TPG is their teleparallelized Newtonian Gravity.

[18]The unitary extension corresponds to "translations along a 'mass dimension"' (Read and Teh [2018], 2).

These results offer another way of investigating the relationships among these theories. By broadening the notion of torsion that is at play to include "mass torsion," Read and Teh show that a commuting diagram can be constructed. Namely, either one can start with GR, teleparallelize to get TPG, and null reduce to get NG; or one can start with GR, null reduce to get NCT, and teleparallelize to get NG.

The methods adopted in the present paper are more continuous with those typically used in the torsion-free context (e.g., those of Wald, Malament [2010, 2012]). With such methods, I have shown that a commuting diagram cannot be constructed: if one only allows spacetime torsion and considers the classical limit as the speed of light becomes unbounded, the addition of torsion in the classical context does not commute with the classical limit of TPG. Taking the classical limit of GR yields NCT and allowing torsion yields the classical theory described by Meskhidze and Weatherall [2023]. However, as shown here, starting with GR, allowing torsion to get TPG, and then considering the classical limit yields Newtonian Gravity.

5.2.3 $1/c$ expansion

The next comparison is to a recent paper by Philip Schwartz [2023b]. Schwartz is interested in investigating the limit of Teleparallel Gravity but his project combines the two methods discussed above: he constructs the classical limit by performing a $1/c$ expansion while also considering the 'gauge-theoretic' description of Newton-Cartan gravity in terms of the Bargmann group. In the end, Schwartz claims that teleparallel Newton-Cartan gravity is the large speed-of-light limit of TPG. His is the final approach I consider.

The classical, torsional theory developed by Schwartz closely resembles that developed by Read and Teh but is intended to be more general. As Schwartz puts it

> [Read and Teh's theory] is constructed only in a restricted 'gauge-fixed' situation; in the present paper, we develop instead a completely general teleparallel description of Newton–Cartan gravity, without introducing arbitrary assumptions on the connection or the frame. ([2023b], 2)

The 'gauge-fixing' indicated by Schwartz is regarding the so-called spatial torsion of the extended vielbein. Whereas Read and Teh assume that this quantity vanishes for the torsional

Newton-Cartan gravity they construct, Schwartz does not.[19,20]

Then, with this torsional equivalent to Newton-Cartan theory in hand, Schwartz turns to the $1/c$ expansion of TPG. The tetrads are expanded as follows (Schwartz [2023b], Equations 3.2a and 3.2b). $\mathrm{E}^0_\mu = c\tau_\mu + c^{-1}a_\mu + O(c^{-3}), \quad E^a_\mu = e^a_\mu + O(c^{-2})$, and

$$E^\mu_0 = c^{-1}v^\mu + O(c^{-3}), \quad E^\mu_a = e^\mu_a + O(c^{-2}).$$

Here, a_μ is eventually related to the mass torsion, τ_μ corresponds to the temporal metric, and the spatial metric is defined as $h := \delta^{ab} e_a \otimes e_b$.

With these expansions in hand, Schwartz claims to recover a torsional classical theory from TPG. As he writes

> This means that *as the formal $c \to \infty$ limit of the Lorentzian manifold we started with, we obtain a Galilei manifold with a Bargmann structure*. We stress again that the only assumption that is needed for this result is an expansion of the Lorentzian tetrad and dual tetrad as in [the equations above], with a nowhere vanishing τ. ([2023b], 15)

He then shows how one might derive the field and force equations of standard Newtonian gravity in the recovered theory.

One can understand his project as another way of capturing the results of Read and Teh. While Read and Teh use null reduction to show how to recover a torsional classical theory from TPG, Schwartz shows how this theory arises as the large speed-of-light limit of TPG. In a sense, then, the two methods converge on similar results. Importantly, however, both proposals require generalizing the notion of torsion with the extended vielbein formalism. Indeed, the results in the present paper indicate that one *must* generalize the notion of torsion to recover a classical spacetime with torsion from TPG. Without this more general notion, the torsion is proven to vanish in the limit. That said, once you do generalize the torsion, the limiting methods (null reduction and $1/c$ expansion) seem to agree. This helps make sense of

[19]Though he does not assume it at the outset, Schwartz does ultimately take the spatial torsion to vanish:

> Let us stress here again that this 'gauge-fixing' assumption of vanishing purely spatial torsion is, differently to the situation considered in [Read and Teh], not part of the formulation of the theory, but only added afterwards for the recovery of standard Newtonian gravity. ([2023b], 20)

Given that the spatial torsion ultimately does not seem to play any meaningful role in the theory, it is not clear what the significance of this generalization to the theory is supposed to be.

[20]Notably, the form for the connection that Schwartz adopts is more general than that typically adopted in this literature. This allows him to incorporate both torsion and a notion of absolute time expressed by means of a closed temporal metric (see his discussion in §2, especially his expression for the connection in Eq. 2.13).

the seemingly surprising claim that torsional Newton-Cartan theory is the large speed-of-light limit of the TPG. Only once allows this extended notion of torsion is such a claim plausible.

6 Conclusion

In this paper, I have proven that the classical limit of Teleparallel Gravity is Newtonian gravity devoid of any gravitational influence. The classical limit squeezes out any torsion in the original theory. I discussed the relation of these results to recent, related results in the literature, results that seemed to either contradict some standard claims made about GR or contradict one another. I found that many of these inconsistencies stemmed from differences in methodology and objectives regarding the classical limit of a theory. However, some inconsistencies remain that require further investigation to resolve fully.

References

Aldrovandi, R. and Pereira, J. G. [2013]: *Teleparallel Gravity: An Introduction*, Fundamental Theories of Physics. Springer.

Bergshoeff, E. A., Hartong, J. and Rosseel, J. [2014]: 'Torsional Newton-Cartan Geometry and the Schrödinger Algebra', *arXiv e-prints*, p. arXiv:1409.5555.

Buchert, T. and Mädler, T. [2019]: 'Editorial note to: On the Newtonian limit of Einstein's theory of gravitation (by Jürgen Ehlers)', *General Relativity and Gravitation*, **51**(12). <http://dx.doi.org/10.1007/s10714-019-2623-1>

de Andrade, V. C. and Pereira, J. G. [1997]: 'Gravitational Lorentz force and the description of the gravitational interaction', *Phys. Rev. D*, **56**, pp. 4689–4695. <https://link.aps.org/doi/10.1103/PhysRevD.56.4689>

De Baerdemaeker, S. and Dawid, R. [2022]: 'MOND and meta-empirical theory assessment', *Synthese*, **200**(5), pp. 344.

Debever, R. [2015]: *Elie Cartan and Albert Einstein: Letters on Absolute Parallelism, 1929-1932*, vol. 1252 Princeton University Press.

Duerr, P. M. [2020]: 'Unweyling Three Mysteries of Nordström Gravity', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*,

69, pp. 32–49.
<https://www.sciencedirect.com/science/article/pii/S1355219819300978>

Dürr, P. and Read, J. [2023]: 'Reconsidering Conventionalism: An Invitation to a Sophisticated Philosophy for Modern (Space-) Times', .

Dürr, P. M. [2021]: 'Theory (In-)Equivalence and conventionalism in $f(R)$ gravity', *Studies in History and Philosophy of Science*, **88**, pp. 10–29.
<https://www.sciencedirect.com/science/article/pii/S0039368121000510>

Duval, C., Burdet, G., Künzle, H. P. and Perrin, M. [1985]: 'Bargmann structures and Newton-Cartan theory', *Phys. Rev. D*, **31**, pp. 1841–1853.
<https://link.aps.org/doi/10.1103/PhysRevD.31.1841>

Duval, C. and Künzle, H. [1984]: 'Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation', *General Relativity and Gravitation*, **16**(4), pp. 333–347.

Ehlers, J. [1991]: 'The Newtonian limit of general relativity', in *Classical Mechanics and Relativity: Relationship and Consistency : International Conference in Memory of Carlo Cattaneo, Elba, June, 1989*, Monographs and textbooks in physical science: Lecture notes, Bibliopolis.
<https://books.google.com/books?id=jD5-PQAACAAJ>

Fletcher, S. C. [2019]: 'On the reduction of general relativity to Newtonian gravitation', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, **68**, pp. 1–15.
<https://www.sciencedirect.com/science/article/pii/S1355219818301308>

Geracie, M., Prabhu, K. and Roberts, M. M. [2015]: 'Curved non-relativistic spacetimes, Newtonian gravitation and massive matter', *Journal of Mathematical Physics*, **56**(10).
<https://doi.org/10.1063

Geracie, M., Son, D. T., Wu, C. and Wu, S.-F. [2015]: 'Spacetime symmetries of the quantum Hall effect', *Phys. Rev. D*, **91**(4), pp. 045030.

Hansen, D., Hartong, J. and Obers, N. A. [2020]: 'Non-relativistic gravity and its coupling to matter', *Journal of High Energy Physics*, **2020**(6).

Hartong, J., Obers, N. A. and Oling, G. [2023]: ‘Review on non-relativistic gravity’, *Frontiers in Physics*, **11**.

Hayashi, K. and Shirafuji, T. [1979]: ‘New general relativity’, *Phys. Rev. D*, **19**, pp. 3524–3553.
<https://link.aps.org/doi/10.1103/PhysRevD.19.3524>

Jensen, S. [2005]: ‘General relativity with torsion: Extending Wald’s chapter on curvature’, .
<http://www.slimy.com/ steuard/teaching/tutorials/GRtorsion.pdf>

Knox, E. [2011]: ‘Newton–Cartan Theory and Teleparallel Gravity: The Force of a Formulation’, *Studies in History and Philosophy of Science Part B*, **42**(4), pp. 264–275.

Malament, D. [1986a]: ‘Gravity and spatial geometry’, in *Studies in Logic and the Foundations of Mathematics*, vol. 114, Elsevier, pp. 405–411.

Malament, D. [1986b]: ‘Newtonian Gravity, Limits, and the Geometry of Space’, in R. Colodny (*ed.*), *From Quarks to Quasars*, Pittsburgh University Press Pittsburgh.

Malament, D. [2012]: *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*, Chicago Lectures in Physics. University of Chicago Press.
<https://books.google.com/books?id=otV9S1BSB1sC>

Martens, N. C. M. and Lehmkuhl, D. [2020]: ‘Dark Matter = Modified Gravity? Scrutinising the Spacetime–Matter Distinction Through the Modified Gravity/ Dark Matter Lens’, *Studies in History and Philosophy of Modern Physics*, **72**, pp. 237–250.

Meskhidze, H. and Weatherall, J. O. [2023]: ‘Torsion in the Classical Spacetime Context’, *Philosophy of Science*, p. 1–12.

Minkowski, H. [1952]: ‘Space and time’, *The Principle of Relativity. Dover Books on Physics. June 1*, pp. 73–91.

Mulder, R. A. and Read, J. [2023]: ‘Is spacetime curved? Assessing the underdetermination of general relativity and teleparallel gravity’, .

Read, J. and Teh, N. J. [2018]: ‘The Teleparallel Equivalent of Newton–Cartan Gravity’, *Classical and Quantum Gravity*, **35**(18), pp. 18LT01.

Schwartz, P. K. [2023a]: 'Teleparallel Newton–Cartan gravity', *Classical and Quantum Gravity*, **40**(10), pp. 105008.
<https://dx.doi.org/10.1088/1361-6382/accc02>

Schwartz, P. K. [2023b]: 'Teleparallel Newton–Cartan gravity', *Classical and Quantum Gravity*, **40**(10), pp. 105008.
<https://dx.doi.org/10.1088/1361-6382/accc02>

Van den Bleeken, D. [2017]: 'Torsional Newton–Cartan gravity from the large c expansion of general relativity', *Classical and Quantum Gravity*, **34**(18), pp. 185004.

Wald, R. [2010]: *General Relativity*, University of Chicago Press.
<https://books.google.com/books?id=9S-hzg6-moYC>

Wolf, W. J. and Read, J. [2023]: 'Respecting Boundaries: Theoretical Equivalence and Structure Beyond Dynamics', *arXiv preprint arXiv:2302.07180*.

Wolf, W. J., Sanchioni, M. and Read, J. [2023]: 'Underdetermination in classic and modern tests of general relativity', *arXiv preprint arXiv:2307.10074*.