A frame-bundle formulation of quantum reference frames:
from superposition of perspectives to superposition of geometries

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Recent experimental advances suggest we may soon be able to probe the gravitational field of a mass in a coherent superposition of position states—a system which is widely believed to lie outside the scope of classical and semiclassical gravity. The recent theoretical literature has applied the idea of *quantum reference frames* (QRFs), originally introduced for non-gravitational contexts, to such a scenario.

Here, we provide a possible fully geometric formulation of the core idea of QRFs as it has been applied in the context of gravity, freeing its definition from unnecessary (though convenient) ingredients, such as coordinate systems. Our formulation is based on two main ideas. First, a QRF encodes uncertainty about what is the observer's (and, hence, the measuring apparatus's) perception of time and space at each spacetime point (i.e., *event*). For this, an observer at an event p is modeled, as usual, as a tetrad in the tangent space T_p. So a QRF at an event p is a complex function on the tetrads at p. Second, we use the result that one can specify a metric on a given manifold by stipulating that a basis one assigns at each tangent space is to be a tetrad in the metric one wants to specify. Hence a spacetime, i.e. manifold plus metric, together with a choice of "point of view" on it, is represented by a section of the bundle of bases, understood as taking the basis assigned to each point to be a tetrad. Thus a superposition of spacetimes gets represented as, roughly speaking, an assignment of complex amplitudes to sections of this bundle. A QRF, defined here as the collection of complex amplitudes assigned to bases at events—i.e., a complex function defined on the bundle of bases of the manifold—can describe, in a local way (i.e., attributing the amplitudes to bases at events instead of to whole sections), these superpositions.

We believe that this formulation sheds some light on some conceptual aspects and possible extensions of current ideas about QRFs. For instance, thinking in geometric terms makes it clear that the idea of QRFs applied to the gravitational scenarios treated in the literature (beyond linear approximation) *lacks* predictive power due to arbitrariness which, we argue, can only be resolved by some further input from physics.

I. INTRODUCTION

Full comprehension of the interplay between gravity and the principles which rule the quantum world remains one of the most elusive and challenging enterprises in physics. In contrast to the situation in the early decades of the twentieth century, when numerous puzzling experimental results slowly, but steadily, paved the way towards an understanding of the quantum aspects of matter and electromagnetism, the arid landscape of observations involving possible quantum aspects of gravity has led to the development of several different theoretical approaches to the subject but with almost no guidance for us about what is the right direction. To make things worse, our best current description of (classical) gravity, viz. general relativity (GR), intermingles it with the most fundamental notions of space and time, which apparently suggests the need for a radical change in the way we understand these fundamental concepts if we are to succeed in this endeavor.

Recent advances in table-top experiments which are sensitive to the gravitational field of tiny macroscopic masses (of order $\lesssim 10^{-1}$ g [1]) have fostered the hope that in the not-too-distant future we may begin probing the gravitational field of masses in quantum (coherent) superposition of position states. Although there is little doubt about what to expect in these experiments in the linear regime of gravity—which is the regime we have some hope to probe—it is important to explore the full implications of having these coherent superpositions. Strictly speaking, the exact analysis of such a system is already beyond our current understanding of gravity and quantum physics[1]. Is it possible to have a (quantum) su-

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[1] Unless it is eventually observed that the gravitational field of the coherent superposition is the same as the one generated by a

perposition of geometries? What does it mean and what are its effects? These are some open questions whose answers (even if tentative) may take us a step closer to a full understanding of the problem.

In an attempt to deal with situations such as the one described above—quantum superpositions of well defined spacetimes—the idea of *quantum reference frames* (QRFs) has been applied to these scenarios. It is important to clarify, at this point, that the term 'QRF' has a long history in the literature (see, e.g., Refs. [2–5]), not necessarily corresponding exactly to the same idea—although some of them may well be equivalent. Here, we take on some of its more recent uses as initiated by Ref. [6]: which was then applied to the gravitational scenario described above, such as in Refs. [7–10]. Instead of focusing on what a superposition of geometries *is* in general, QRFs are used by these authors to address the much more pragmatic question of what are the *effects* of such a superposition on a (possibly quantum) probe, in particular situations. In order to accomplish this, these authors use QRF *transformations* and then invoke a principle of "quantum covariance."

Using these ingredients, the authors of Ref. [10] calculate the evolution of a "test particle" (i.e., a particle whose own gravitational field is neglected in the analysis) under the influence of a mass in a macroscopic superposition of position states. In summary, this application of QRFs formalizes, in a manifold with an undefined metric, the prediction that the evolution of the test particle should be a superposition of its "histories," each given by its evolution subject to one well-defined position state of the source mass—at least as long as the test particle does not disturb the superposed mass state.

Our primary goal, here, is to recast this idea of QRFs and their transformations into a geometric language, avoiding the use of auxiliary but unnecessary (and possibly confusing) ingredients—such as coordinate systems—in their constructions. But in pursuing this goal, we are naturally led to a framework where *arbitrary* superpositions of *arbitrary* geometries (on a given manifold) can be described in a local geometric (i.e., coordinate independent) manner: a framework which may have some value, apart from our initial goal. But this extension beyond our initial goal is left largely unexplored in this paper. In particular, we will not consider how our framework could treat theories of gravity other than GR. In this sense, we make no claim to give a complete, "all-set" framework, which leaves no open questions or loose ends. Rather, this paper is an attempt to put the interesting idea of QRFs in a new language which may highlight its merits (and perhaps, shortcomings), and may suggest directions for further investigation.

As mentioned above, our first goal is to eliminate the use of coordinate systems in the *definition* of QRFs. In fact, making non-uniquely-determined coordinate systems play important physical roles seems common to much[2] of the recent literature about QRFs: thus treating coordinate systems as if they were synonymous with "points of view" or "perspectives" with respect to which *observables* of physical system are described—which they are not. To make this point clear: although coordinate systems are enough to determine components of tensors which characterize the physical system, they are *not* enough (and, in fact, not necessary) to characterize observables of the same system, since, in general, components have no direct physical meaning.[3] If observables are to play an important role, then the notion of *observers* or *reference frames*—which are rigorously formalized as *tetrads* or *tetrad fields*—must be introduced and we stress that the freedom to do so in a given spacetime is *not* the same as the freedom to choose coordinate systems in that same spacetime—the latter being immensely larger. Moreover, once observers or reference frames are introduced, nothing physical can depend on the choice of coordinates.

But having made this statement of intent, we should stress again (so as to avoid unrealistic expectations and being unfair to the existing literature) that our purpose here is merely to recast the QRF idea in geometric terms. We do *not* claim that analyses carried out using our approach could not be performed using the coordinate-based one. In fact, since the freedom in choosing coordinate systems in any given spacetime is larger than that in choosing "points of view" (i.e., tetrad fields) in that same spacetime, it is natural to expect that by a judicious use of coordinate systems (e.g., anchoring its definition on some physical criteria) one could reach the same conclusions as the ones drawn using the geometric approach. In an analogy, this would correspond to formulating (and working with) GR using only multivariable calculus on $\mathbb{R}^n$, without ever mentioning manifolds, tangent spaces, tensors as multilinear mappings, connections, and the whole *abstract* framework of differential geometry. So, if there is a merit in our approach, it is to make a clear distinction between the choices which *can* have physical consequences and the ones which cannot—something which we believe gets blurred in the coordinate-based approach.

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classical mass distribution given by the average mass distribution of the superposition, as described by the semiclassical Einstein equation.

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[2] But noteworthy exceptions include approaches whose key idea is that a QRF is a subsystem of the total system, together with a state of it with respect to which the remaining subsystems get described (in broadly "relational" terms). These 'perspective-neutral' approaches have been richly developed using elegant group-theoretic ideas, cf. Refs. [11, 12]; and recently, using tetrad fields, cf. Ref. [13]. But these approaches' relation to our approach, though interesting, is unclear; and must be postponed to future work.

[3] For a concrete example of the dangers of attributing direct physical meaning to tensor components, see, e.g., Refs. [14, 15].

Our geometric formulation of QRFs will be based on two main ideas. First, we think of a QRF as encoding quantum uncertainty about what is the observer's or apparatus's perception of time and space in a given spacetime. In Section II, we characterize the apparatus's frame by a tetrad: i.e. in physical terms, by an infinitesimal clock and three infinitesimal orthogonal rulers, located at a spacetime point. So a QRF is a (suitably normalizable) "wave-function" assigning a complex amplitude to such tetrads. Though simple, this formulation is sufficient to reproduce one of the key themes of the literature on QRFs: that a superposition in the state of the measured system can be "transferred", by a transformation between QRFs, to a superposition of QRFs. (This is shown at the end of Section II.)

Second: in Section III, we invoke the result that an assignment of a *generic* basis of the tangent space, at each point of a given manifold, *defines* a metric on the manifold—simply by stipulating that at each point, the assigned basis is to be orthonormal, i.e. a tetrad, for the metric to be defined. As a result, we can interpret a section of the frame bundle (i.e. the bundle over spacetime whose fibre over each point is the set of all bases of the tangent space at that point) as a specification of:

(i) a metric field, i.e. a spacetime, since the basis within the section that lies above the point p is stipulated to be orthonormal in that spacetime's geometry; together with:

(ii) a tetrad field, viz. the very elements of the section, on that spacetime.
Accordingly, a *pair* of spacetimes, for instance (each with its own tetrad field), is specified by a pair of sections. Thus, now invoking the first idea above: by assigning complex amplitudes to generic bases at *each* point, one can describe, in a local manner, not only a superposition of a pair of spacetimes (when only two bases at each point are singled out), but also an *arbitrary* superposition of geometries.

So much for the two main ideas. Then, in the last main Section (Sec. V) we apply this fibre bundle formulation of QRFs to the "simple" scenario analyzed in the literature, which we recalled above: viz. a test particle probing the gravitational field sourced by a macroscopic superposition of position states of a large mass. Section VI is a brief discussion Section. In particular, it compares our formulation of QRFs with *another* fibre bundle framework for QRFs, viz. that of Ref. [16]. Section VI also gives a brief outlook on possible lines of further inquiry.

It will also help to have a prospectus of the paper, that does not use the jargon of fibre bundles. In these terms, the paper is organized as follows. In Section II, we introduce the notion of QRFs in a given, well-defined spacetime (which we call *restricted* in such a context). In Section III, by demanding that the notion of QRFs be diffeomorphism invariant, we relate QRFs in isometric spacetimes and, eventually, extend their definition to generic superpositions of geometries. This extension prompts the discussion of Section IV, emphasiz-

ing the different roles that our mathematical description of QRFs can play: one of merely describing (superpositions of) points of view in (superpositions of) arbitrary geometries—the *perspectival* conception—and another of describing the arbitrary superpositions of geometries in themselves—the *basic* conception. We make it clear that in this work we focus on the perspectival conception of QRFs. In Section V, we revisit the scenario which has been used in the recent literature to motivate the introduction of QRFs. We argue that if one intends to apply the QRF idea beyond the linear-gravity regime—which has been the goal in the recent literature—then one must face the subtlety of the non-uniqueness of mapping between diffeomorphically-related spacetimes. In particular, this mapping may be related to the *dynamics* of the evolution of the source mass to the superposition of position states—a dynamics whose treatment lies beyond current abilities. Finally, in Section VI we present our final remarks. We also sketch the general idea behind our construction in Fig. 2—which the reader is invited to look at even before proceeding to the next Section, and to revisit while reading.

Finally, note that throughout the paper, we use units in which $c = 1$ and the abstract-index notation for tensorial quantities (see, e.g., Ref. [17]). According to this notation, Greek letters are used for "concrete" indices—i.e., those which assume numerical values—from 0 to 3. Latin letters $i, j, ...$ are used for "concrete" indices from 1 to 3. Finally, Latin letters from the beginning of the alphabet, $a, b, ...$ (up to h), are used for "abstract" indices—i.e., indices which do *not* assume numerical values but instead indicate, by their position and number, the type of tensor which the indexed object is.

II. QUANTUM REFERENCE FRAMES ON A GIVEN SPACETIME: SUPERPOSITION OF PERSPECTIVES

In order to motivate our construction by step-by-step reasoning, let us begin by considering a generic spacetime $(\mathcal{M}, g_{ab})$, where g_{ab} is the metric tensor defined on a differentiable manifold $\mathcal{M}$. Although this gives complete information about the stage on which all systems evolve, our fragmented perception (and measurements) of space and time as separate entities usually leads us to introduce the notion of a *reference frame* (RF), which basically tells how *absolute* (infinitesimal) spacetime intervals are to be decomposed into *relative* (infinitesimal) space distances and time lapses in the neighborhood of each event. Obviously, each decomposition only makes sense for specific observers/apparatuses and is only necessary when talking about *observables*—i.e., outcomes of measurements performed by these specific observers/apparatuses.

Geometrically, this classical notion of a reference frame in region $\mathcal{O} \subseteq \mathcal{M}$ is implemented by means of an assignment (which we shall take to be smooth) $\mathcal{M} \supseteq \mathcal{O} \ni p \mapsto \{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$, which to every event p in

$\mathcal{O} \subseteq \mathcal{M}$ assigns a tetrad $\{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$. Here, using abstract-index notation: a simply says that each element of $\{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$ is a tensor of type $(1,0)$—i.e., a 4-vector—and μ labels the four elements of this set of 4-vectors.[4] So, the fact that $\{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$ is a tetrad means that $g_{ab}\mathbf{e}_\mu^a(p)\mathbf{e}_\nu^b(p) = \eta_{\mu\nu} := \mathrm{diag}(-1,1,1,1)$. At p, $\mathbf{e}_0^a(p)$ sets *both* direction *and* unit for what is to be considered as "time"—i.e., $\mathbf{e}_0^a(p)$ *is* the 4-velocity of the observer/apparatus which the tetrad is supposed to describe at p—while $\{\mathbf{e}_j^a(p)\}_{j=1,2,3}$ then gives the space directions and units in such a way that $c = 1$ is ensured.

Despite the simplicity of the notion of tetrad, it can be argued that it is adequate to encode whatever the observer's apparatus happens to be, essentially because all measurements reduce, at bottom, to measurements of lengths and time-intervals; cf. for example Ref. [18]. For whatever distinguishes different apparatuses carried by the *same observer*—hence, associated to the same tetrad—is modelled by the way the tetrad couples to the system being observed so as to give the *observable* quantity. For instance, an apparatus described by the tetrad $\{\mathbf{e}_\mu^a\}_{\mu=0,1,2,3}$ which measures electric field, along the spatial direction $\mathbf{e}_j^a$, is modeled by the coupling $F_{ab}\mathbf{e}_0^a\mathbf{e}_j^b$, where F_{ab} is the Faraday tensor which encodes all the information about the electromagnetic field of the system being observed. Were we interested in measuring the energy density, then the measuring procedure would be modeled by the coupling $T_{ab}\mathbf{e}_0^a\mathbf{e}_0^b$, with T_{ab} being the stress-energy-momentum tensor of the system. Note an important fact: given the system and the observer/apparatus, the *observables*—$E_j := F_{ab}\mathbf{e}_0^a\mathbf{e}_j^b$ and $\rho := T_{ab}\mathbf{e}_0^a\mathbf{e}_0^b$ in the examples above—must be *scalars*, in the sense of being insensitive to choice of coordinate systems; (indeed, coordinates have not even been mentioned thus far).

In the more elegant and concise language of fibre bundles, one can think of a reference frame as a smooth local section of the bundle $\mathcal{F}_{\mathrm{o}}[g_{ab}]$ of (pseudo-)orthonormal frames of $(\mathcal{M}, g_{ab})$. (So here, 'o' stands for orthonormal.) This is the language we shall adopt.

As a first attempt at implementing the idea of a QRF, we want to introduce uncertainties related to the state of motion of observers/apparatuses at each event. This could be easily implemented by defining a probability measure on each fibre of $\mathcal{F}_{\mathrm{o}}[g_{ab}]$—which represents the different choices of tetrad at p.

However, since we ultimately want this uncertainty to be quantum in nature—with all its linear, complex structure—we shall here *define* a **QRF** to be an as-

signment $\Psi : \mathcal{F}_{\mathrm{o}}[g_{ab}] \to \mathbb{C}$, which to every tetrad $\{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$ at $p \in \mathcal{M}$ assigns a complex number (a "probability amplitude") $\Psi(p; \{\mathbf{e}_\mu^a(p)\})$, such that

$$\|\Psi(p)\|^2 := \int_{O(3,1)} d\mu_L(\Lambda) \left| \Psi(p; \{\Lambda_\mu^\nu \mathbf{e}_\nu^a(p)\}) \right|^2 < +\infty \ (1)$$

for *each* p and *any fixed* $\{\mathbf{e}_\mu^a(p)\}_{\mu=0,1,2,3}$; where $\Lambda \in O(3,1)$ (the Lorentz group; i.e., the entries Λ_μ^ν of matrix Λ satisfy $\Lambda_\mu^\alpha \Lambda_\nu^\beta \eta_{\alpha\beta} = \eta_{\mu\nu}$) and μ_L is the (unique up to a multiplicative factor) invariant Haar measure defined on $O(3,1)$. (Note that Eq. (1) *defines* the symbol $\|\Psi(p)\|^2$, *not* an object $\Psi(p)$; in fact, we will have no use for the latter here.)

The definition Eq. (1) is basic to the rest of this paper; and in the rest of this Section, we develop ideas based on it. First, we make five general comments, numbered (1), (2) etc.; and then we give some examples (beginning with Eq. (2) below) and some discussion of transformations.

(1): As the first general comment, note that since $O(3,1)$ is a noncompact group, this integrability condition implies that any particular choice of a QRF Ψ necessarily breaks Lorentz invariance: which is expected when adopting a "point of view." But note that the result of the integration in Eq. (1) is independent of the choice of $\{\mathbf{e}_\mu^a(p)\}$ held fixed. (From now on, we will write simply $\{\mathbf{e}_\mu^a\}$ instead of $\{\mathbf{e}_\mu^a(p)\}$ where it is understood that it stands for the tetrad at an event—which should be clear by context. Moreover, when figuring in arguments of functions, we simplify it further and write merely $\{\mathbf{e}\}$, so as not to proliferate unnecessary indices—unless needed by context.)

(2): We shall refer to this notion of QRF as *restricted*, and write 'rQRF'; since we will later generalise it. The reason for the generalisation will be clear in Sec. V. But in short, we will treat superpositions of geometries by using the idea that given one geometry, a non-orthonormal basis at a point p is a tetrad, i.e. is orthonormal, for another geometry.

(3): The concept of rQRF introduces uncertainties on measurements of space distances and time intervals (and, from them, in measurements of any other observable, cf. Ref. [18]) without, however, making the metric tensor uncertain. That is: it allows for "superpositions" of "points of view"—e.g., due to the possible states of motion of the observers/apparatuses at each event—on a well defined spacetime geometry. Note, also, that one can consider linear combinations of rQRFs in the obvious (point-wise in $\mathcal{F}_{\mathrm{o}}[g_{ab}]$) way. They are like collections of wave functions, each of them being defined on the fibres $\pi_{\mathrm{o}}^{-1}(p)$ of $\mathcal{F}_{\mathrm{o}}[g_{ab}]$; where $\pi_{\mathrm{o}} : \mathcal{F}_{\mathrm{o}}[g_{ab}] \to \mathcal{M}$ is the canonical projection of the bundle.

(4): In addition to Eq. (1), one might impose further (reasonable) constraints, such as that $(\mathcal{M}, g_{ab})$ is time orientable and that Ψ has support on tetrads whose $\mathbf{e}_0^a(p)$ is future directed—although violation of this latter condition may find application in indefinite causal order analyses [19]. Moreover, if Ψ is meant to repre-

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[4] This concrete index μ in $\mathbf{e}_\mu^a$ must not be confused with the concrete indices which label components of tensors. Some references avoid this confusion by using Latin capital letters in the former case, writing, e.g., $\mathbf{e}_A^a$. On the other hand, Latin capital letters are also often used to represent algebra-valued quantities. So, here, we stick to the simpler index notation which only distinguishes between concrete and abstract indices.

sent (superposition of) "points of view" of *physical* observers/apparatuses, it seems reasonable that it must satisfy some sort of transport ("Boltzmann-like") differential equation, describing "diffusion" in $\mathcal{F}_o[g_{ab}]$—in addition, perhaps, to some other "dynamical" equation—constraining its local behavior, particularly for nearby events p, q related by curves having the vectors $\mathbf{e}_0^a(p)$ as tangents (for $\Psi(p; \{\mathbf{e}\}) \neq 0$). But here, we leave the discussion of these further constraints aside, since they are not essential to this paper's presentation of the general ideas.

(5): Note that we are not imposing integrability of $\|\Psi(p)\|^2$ on $\mathcal{M}$ (with respect to the preferred 4-volume element ϵ, or ϵ_{abcd} in abstract-index notation, selected by the metric g_{ab}), since this can be too restrictive. For instance, it would exclude most globally defined (Q)RFs, common even in the classical context to represent families of observers covering the whole spacetime. And even though such an integrability condition for $\|\Psi(p)\|^2$ may be requested, if convenient—hence obtaining a finite $\|\Psi\|^2 := \int_{\mathcal{M}} \epsilon \|\Psi(p)\|^2$, perhaps representing a (spacetime-)localized "observation"—one must be careful about interpreting $\|\Psi(p)\|^2 / \|\Psi\|^2$ as any sort of

"probability density" on $\mathcal{M}$. In this case, it seems more reasonable to codify in $\|\Psi(p)\|^2$ a similar role as the one played by the smearing functions $f \in C_0^\infty(\mathcal{M})$ which are used to define observables $\hat{A}[f] = \int_{\mathcal{M}} \epsilon \, f(p)\hat{A}(p)$ in quantum theory, since, in general, point-wise observables $\hat{A}(p)$ only exist as operator-valued *distributions*. We will return to this issue, including smearing, in Subsection A below.

So much by way of general comments on Eq. (1). In order to illustrate the effect of rQRFs, first on classical observables, let us consider a specific example: the electric field observed/measured at an event p, w.r.t. a given rQRF Ψ. Let us assume that the electromagnetic field is completely described, at p, by the Faraday tensor F_{ab}. If a classical observer (or particle) were passing through p with 4-velocity u^a, the electric field he/she would measure (or be subject to) along the spatial direction characterized by the unit vector n^a (with $g_{ab}u^a n^b = 0$) would be $E_n = F_{ab}n^a u^b$. So, it is only natural to define the expectation value for the j component of the electric field in the rQRF Ψ by

$$\langle E_j \rangle := \|\Psi(p)\|^{-2} \int_{O(3,1)} d\mu_L(\Lambda) F_{ab} \mathbf{e}_\alpha^a \mathbf{e}_\beta^b \Lambda_j^\alpha \Lambda_0^\beta \, |\Psi(p; \{\Lambda\mathbf{e}\})|^2 \,, \tag{2}$$

where $\{\mathbf{e}_\mu^a\}$ is any (fixed) tetrad at p and $\{\Lambda\mathbf{e}\}$ stands for $\{\Lambda_\mu^\nu \mathbf{e}_\nu^a\}$ when in arguments of functions (following the convention set above).

Similarly, *any* classical observable A can be expressed as a *scalar* (in the sense of being *coordinate-independent*) quantity constructed out of the tensors characterizing the system *and* the coupling between the system and the tetrad characterizing the observer/apparatus performing the measurement. Hence, by averaging over the tetrads with the probability distribution given by $|\Psi(p; \{\mathbf{e}\})|^2$, one obtains the expectation value $\langle A \rangle$ at p.

A similar rationale applies to quantum observables. Any quantum observable can be represented by a *scalar* operator-valued distribution $\hat{A}$ which depends on the

physical system and on the observer/apparatus performing the measurement (i.e., on the tetrad which characterizes the latter and its coupling to the system). Let us consider, again, a concrete example: the operator-valued distribution $\hat{\rho} = \mathbf{e}_0^a \mathbf{e}_0^b \hat{T}_{ab}$ which describes the energy density, with respect to a given observer/apparatus with 4-velocity $\mathbf{e}_0^a$, of a system whose (renormalized) energy-momentum tensor is described by the operator-valued distribution $\hat{T}_{ab}$. Thus, if the system is in a state $|s\rangle$—at this point supposed to be an observer-independent statement—then the expectation value of its energy density w.r.t. Ψ is given by (recalling that ϵ is the preferred 4-volume element on $(\mathcal{M}, g_{ab})$ and using $\|\Psi\|^{-2} = 1/\|\Psi\|^2 = 1/\int_{\mathcal{M}} \epsilon \|\Psi(p)\|^2$)

$$\langle \rho \rangle := \|\Psi\|^{-2} \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda) \, |\Psi(p; \{\Lambda\mathbf{e}\})|^2 \, \Lambda_0^\alpha \Lambda_0^\beta \mathbf{e}_\alpha^a \mathbf{e}_\beta^b \langle s|\hat{T}_{ab}|s\rangle, \tag{3}$$

again for any fixed tetrad field $\{\mathbf{e}_\mu^a\}$. Note that, now, we considered a square-integrable (in fact, a compactly-supported) $\|\Psi(p)\|^2$, due to the distributional nature of quantum observables.

Now, let us consider changing the rQRF. Different rQRFs can lead to different values for the observable quantities, since they describe different "superpositions of perspectives." An exception to this may occur when

the rQRFs are related by an *isometry* of $(\mathcal{M}, g_{ab})$, $\iota : \mathcal{M} \to \mathcal{M}$, via[5]

$$\Psi \mapsto \widetilde{\Psi}(p; \{\mathbf{e}\}) := \Psi(\iota^{-1}(p); \{\iota_* \mathbf{e}\}), \qquad (4)$$

where ι_* is the associated pull-back mapping between tangent vectors at p and at $\iota^{-1}(p)$. It follows directly from the definition that any observable of a system in a configuration/state *which respects this isometry*—in particular, any geometric observable—will be invariant under this particular transformation.

Note, also, that one can consider linear combinations of Ψ and $\widetilde{\Psi}$ to get other rQRFs on $(\mathcal{M}, g_{ab})$. So, the set of rQRFs on a *given* spacetime is a complex vector space where the isometries of the spacetime (if there are any) can be naturally represented.

Although our definition of QRFs has relied totally on geometric objects, it may be convenient to introduce coordinate systems (CSs) when carrying out explicit calculations. Given a rQRF Ψ and a CS $\chi : \mathcal{O} \to U \subseteq \mathbb{R}^4$ defined on $\mathcal{O} \subseteq \mathcal{M}$, we obtain a *representation* $[\Psi]_\chi$ of Ψ, by requiring that, at the event with coordinates x^μ, and at the tetrad with *components* $e^\alpha_{(\beta)}$ in the coordinate basis $\{\partial^a_\alpha\}$ induced by χ—i.e., $\mathbf{e}^a_\beta = e^\alpha_{(\beta)} \partial^a_\alpha$—$[\Psi]_\chi$ is to assume the value that Ψ takes there. (The use of parentheses in the lower index of $e^\alpha_{(\beta)}$ is merely to differentiate the labeling of each tetrad element from the labeling of the components of these elements.) Thus we define:

$$[\Psi]_\chi(x^\mu; \{e^\alpha_{(\beta)}\}) := \Psi(\chi^{-1}(x^\mu); \{e^\alpha_{(\beta)} \partial^a_\alpha\}). \qquad (5)$$

Note that the indices in this definition belong to the arguments of the functions, not to the functions themselves, which are *scalars*—hence, there is no need for matching indices on the two sides.

It is important to note that changing the coordinate system only changes the representation $[\Psi]_\chi$, but *not* Ψ itself—hence, it has no effect on expectation values of observables. These two concepts, reference frames and coordinate systems, are often confused with one another, even in classical physics. As we have explained before, the former prescribes how spacetime is to be described, by observers/apparatuses, as space and time separately. So it *can* have effects on the value of observables/measurements performed on a given physical system; (although of course, the physical system, itself, is oblivious to the adoption, or not, of a reference frame—at least in classical physics). The latter, on the other hand, is a mere (arbitrary) numerical labelling and, as such, has no consequences for physical results. In the context

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above, this means that given a rQRF Ψ, no physical observable can depend on which representation $[\Psi]_\chi$ is used to perform the calculations. Changes in the coordinate system (to be interpreted as *passive* diffeomorphisms) lead to different representations which, nonetheless, *must* lead to the same physical observations.

Before ending this Section, it is important to emphasize, once more, that, as far as the spacetime geometry is concerned, different rQRFs only represent different "perspectives" of the *same* physical situation. In the same way that a tetrad $\{\mathbf{e}^a_\mu\}$ can stand for an idealization of a classical observer/apparatus performing time and distance measurements (from which, strictly speaking, any other measurement is obtained [18]), a rQRF Ψ can be thought of as an idealization (perhaps overly generalized) of observers/apparatuses which can exhibit quantum aspects, such as spacetime delocalization and velocity uncertainty.

One particularly simple (and idealized) example of a rQRF, defined with the help of a coordinate system, is one completely concentrated on a time-like curve described by $x^\mu(\tau)$—representing the perspective of a point-like observer/apparatus or test particle having the curve as its worldline:

$$supp[\Psi]_\chi = \left\{ \left(x^\mu(\tau), \{u^\alpha(\tau), e^\alpha_{(j)}(\tau)\} \right); \ \tau \in I \subseteq \mathbb{R} \right\}, (6)$$

with $u^\alpha(\tau) = dx^\alpha(\tau)/d\tau$ being the components of its 4-velocity and τ its proper time ($e^\alpha_{(j)}(\tau)$ are the components of the other elements of the tetrad, which are left unspecified here). From this, less trivial examples—which play a significant role, e.g., in Ref. [10]—can be constructed by superposition, possibly representing "probes" whose location may not be well defined. We shall come back to this later.

A. An application to flat spacetime

Before considering more general scenarios (involving possible superposition of different geometries), let us illustrate this geometric approach in flat spacetime. In flat spacetime, there are preferred families of inertial observers, namely, the ones characterized by *uniform* tetrad fields. It is common to use these families to interpret/characterize the states of the system. Here, insisting on the use of tetrads may seem pedantic, since there is for each of these families a natural choice of coordinates which faithfully characterizes time and space measurements, viz. the usual inertial Cartesian coordinates. However, our purpose here is to illustrate the use of the tetrad-based notion of QRF—and this faithful characterization in terms of inertial Cartesian coordinates is restricted to *this* class of inertial families of observers in *flat* spacetime.

First, if we want to consider superposing different "points of view," we have to make explicit the dependence of observables on observers. Indeed, there are two

aspects to this. First, we recall the need, familiar in quantum field theory, to smear observables with test functions f defined on the spacetime $(\mathcal{M}, g_{ab})$. (But what follows will not depend on the details of quantum field theory.) This is done, as usual, by having for a given observable A, a smearing map, $C_0^\infty(\mathcal{M}) \ni f \mapsto A_f \in \mathcal{L}(\mathcal{H})$, which takes compactly supported, smooth functions f on $\mathcal{M}$ to linear operators A_f acting on the Hilbert space $\mathcal{H}$.

But now, we extend the idea of this definition, so that the domain is the compactly supported, smooth functions F defined on the entire bundle of (pseudo-)orthonormal frames, $\mathcal{F}_o[g_{ab}]$. This extension is not *ad hoc*, but accords with our philosophy of talking about observables only once observers are explicitly mentioned.

Thus we have a smearing function $\hat{A} : C_0^\infty(\mathcal{F}_o[g_{ab}]) \to \mathcal{L}(\mathcal{H})$; by which compactly supported, smooth functions F defined on $\mathcal{F}_o[g_{ab}]$ are (linearly) mapped to operators $\hat{A}[F]$. Formally, we write:

$$\hat{A}[F] = \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda) F(p; \{\Lambda \mathbf{e}\}) \hat{A}(p; \{\Lambda \mathbf{e}\}). \quad (7)$$

Note that in Eq. (7), $\hat{A}(p, \{\mathbf{e}\})$ is an operator-valued distribution—for instance, in the example of Eq. (3), $\hat{A}(p, \{\mathbf{e}\}) = \hat{T}_{ab}(p) \mathbf{e}_0^a(p) \mathbf{e}_0^b(p)$.

Consider, now, a quantum *mechanical* system and let $|s\rangle$ be its (observer-independent) state. (Here, we say 'mechanical' because we shall not here consider the subtleties of quantum field theory, such as the non-localizability of its states and the non-existence of finite-rank spectral projectors.) An inertial family of observers, O, described by the *uniform* tetrad field $\{\mathbf{o}_\mu^a\}$, can characterize the system (in a compact region) by the expectation values of self-adjoint operators $\hat{A}_O[f]$ which, for them, represent observables $A_O(p)$ smeared with test functions $f \in C_0^\infty(\mathcal{M})$.

We thus envisage a rQRF Ψ_O which describes the *classical* family of observers O—i.e., "peaked" at $\{\mathbf{o}_\mu^a\}$—and with $\|\Psi_O(p)\|^2 = f(p)$. Thus we write:

$$\begin{aligned}
\hat{A}_O[f] &= \int_{\mathcal{M}} \epsilon f(p) \hat{A}_O(p) \\
&= \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda) |\Psi_O(p; \{\Lambda \mathbf{e}\})|^2 \, \hat{A}(p; \{\Lambda \mathbf{e}\}) \\
&= \hat{A}[|\Psi_O|^2].
\end{aligned} \quad (8)$$

Note that this equation is only defined for nonnegative test functions f, since we set $f(p) = \|\Psi_O(p)\|^2$. Nonetheless, this is enough to define, by linearity, the mappings $\hat{A}$

on arbitrary functions since an arbitrary real function F can be (nonuniquely) written as $F = F_+ - F_-$, with both F_+ and F_- being nonnegative functions. (The decomposition becomes unique if we further impose $F_+ F_- = 0$ pointwise.) In what follows, we suppose that Eq. (8) is also applicable for a generic QRF Ψ—as we have already done when proposing Eq. (3).

To close this Section, we now suppose that the state $|s\rangle$ can be written as a coherent superposition, $|s\rangle = \sum_{I \in \mathcal{I}} c_I |s_I\rangle$ ($\mathcal{I}$ being some index set), with the particular property that there are spacetime isometries $\{\iota_I\}_{I \in \mathcal{I}}$ whose unitary representations $\{\hat{U}_I\}_{I \in \mathcal{I}}$ on $\mathcal{H}$ relate some fiducial state, say $|s_0\rangle$, to each state $|s_I\rangle$: $\hat{U}_I |s_0\rangle = |s_I\rangle$, $I \in \mathcal{I}$.

We will now show that our framework of QRFs, using maps $\Psi : \mathcal{F}_o[g_{ab}] \to \mathbb{C}$, is able to "transfer the superposition" in the state $|s\rangle = \sum_{I \in \mathcal{I}} c_I |s_I\rangle$ "onto" the set of quantum reference frames that are related [as in Eq. (4)] to a given QRF Ψ by the isometries ι_I. Here "transferring a superposition" will amount to an equality of expectation values. That is: we will show that the expectation value of a quantity A in a given QRF, i.e. smeared as in Eq. (8), for the state $|s\rangle = \sum_{I \in \mathcal{I}} c_I |s_I\rangle$, can equal the expectation value of the quantity A, as smeared according to some linear combination of isometrically defined QRFs, for the fiducial state $|s_0\rangle$.

To be vivid, we can suppose, for example, that: (i) according to the given QRF Ψ, the state $|s_0\rangle$ is well-localised spatially (though not necessarily at any particular point which one chooses to call "the origin"—by, e.g., assigning a particular value of Ψ at it); and (ii) the isometries ι_I are spatial translations (not necessarily in the same spatial direction). For such a case, we will show that it is possible to have equality between: (a) the expectation value of A attributed by the QRF Ψ, as smeared by some (not necessarily well-localised) Ψ [using Eq. (8)], for the spatially superposed state $|s\rangle = \sum_{I \in \mathcal{I}} c_I |s_I\rangle$; and (b) the expectation value of A attributed by a "weighted" set of QRFs $\widetilde{\Psi}_I$, $I \in \mathcal{I}$, each related to Ψ by the spatial translation ι_I^{-1}—i.e., smeared instead by the corresponding linear combination of isometrically-defined QRFs $\widetilde{\Psi}_I$ [Eq. (4) with $\iota = \iota_I^{-1}$]—now for the single well-localised state $|s_0\rangle$.

But we stress that we mention this spatial example only for vividness. The calculation below is valid for any fiducial state $|s_0\rangle$ that is related by spacetime isometries $\{\iota_I\}_{I \in \mathcal{I}}$ to the states $|s_I\rangle$ in the expansion of $|s\rangle$ as $\sum_{I \in \mathcal{I}} c_I |s_I\rangle$. Thus we calculate, in general:

$$\langle s|\hat{A}[|\Psi|^2]|s\rangle = \sum_{I,J\in\mathcal{I}} c_J^* c_I \langle s_J|\hat{A}[|\Psi|^2]|s_I\rangle = \sum_{I\in\mathcal{I}} |c_I|^2 \langle s_0|\hat{U}_I^\dagger \hat{A}[|\Psi|^2]\hat{U}_I|s_0\rangle + \sum_{I\neq J\in\mathcal{I}} c_J^* c_I \langle s_J|\hat{A}[|\Psi|^2]|s_I\rangle$$

$$= \sum_{I\in\mathcal{I}} |c_I|^2 \langle s_0|\hat{A}[|\widetilde{\Psi}_I|^2]|s_0\rangle + \sum_{I\neq J\in\mathcal{I}} c_J^* c_I \langle s_J|\hat{A}[|\Psi|^2]|s_I\rangle$$

$$= \left\langle s_0 \left| \hat{A}\left[\left|\sum_{I\in\mathcal{I}} c_I \widetilde{\Psi}_I\right|^2\right]\right| s_0\right\rangle + \sum_{I\neq J\in\mathcal{I}} c_J^* c_I \left[\langle s_J|\hat{A}[|\Psi|^2]|s_I\rangle - \langle s_0|\hat{A}[\widetilde{\Psi}_J^* \widetilde{\Psi}_I]|s_0\rangle\right], \qquad (9)$$

where (i) as anticipated, in accordance with Eq. (4) (with $\iota = \iota_I^{-1}$), we have defined $\widetilde{\Psi}_I(p;\{\mathbf{e}\}) := \Psi(\iota_I(p);\{\iota_I^*\mathbf{e}\})$ and (ii) we have used the invariance of the measure/volume elements whenever needed to demonstrate the identity

$$\langle s_0|\hat{U}_I^\dagger \hat{A}[|\Psi|^2]\hat{U}_I|s_0\rangle = \left\langle s_0 \left| \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda)\, |\Psi(p;\{\Lambda\mathbf{e}\})|^2\, \hat{A}(\iota_I^{-1}(p);\{\iota_{I*}(\Lambda\mathbf{e})\}) \right| s_0 \right\rangle$$

$$= \left\langle s_0 \left| \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda)\, |\Psi(p;\{\Lambda\mathbf{e}\})|^2\, \hat{A}(\iota_I^{-1}(p);\{(\Lambda\Lambda_I^{-1})\mathbf{e}\}) \right| s_0 \right\rangle$$

$$= \left\langle s_0 \left| \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\tilde{\Lambda}\Lambda_I)\, |\Psi(\iota_I(p);\{(\tilde{\Lambda}\Lambda_I)\mathbf{e}\})|^2 \hat{A}(p;\{\tilde{\Lambda}\mathbf{e}\}) \right| s_0 \right\rangle$$

$$= \left\langle s_0 \left| \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\tilde{\Lambda})\, |\Psi(\iota_I(p);\{\iota_I^*(\tilde{\Lambda}\mathbf{e})\})|^2 \hat{A}(p;\{\tilde{\Lambda}\mathbf{e}\}) \right| s_0 \right\rangle$$

$$= \left\langle s_0 \left| \int_{\mathcal{M}} \epsilon \int_{O(3,1)} d\mu_L(\Lambda)\, |\widetilde{\Psi}_I(p;\{\Lambda\mathbf{e}\})|^2 \hat{A}(p;\{\Lambda\mathbf{e}\}) \right| s_0 \right\rangle$$

$$= \langle s_0|\hat{A}[|\widetilde{\Psi}_I|^2]|s_0\rangle \qquad (10)$$

which we used passing from the first to the second line of Eq. (9).

The result shown in Eq. (9), whose validity is quite general, is very reasonable: it means that it is *possible* to transfer the coherent superposition of states in $\mathcal{H}$ to a superposition of QRFs, in what concerns the observable A, provided the term in square brackets in the last line vanishes—i.e., the $\mathcal{H}$-off-diagonal terms $\langle s_J|\hat{A}[|\Psi|^2]|s_I\rangle$ match the QRF-off-diagonal terms $\langle s_0|\hat{A}[\widetilde{\Psi}_J^* \widetilde{\Psi}_I]|s_0\rangle$. Obviously, this will not be true for an arbitrary QRF Ψ—although the freedom mentioned in footnote 5 can be put to some use here—and it will, in general, depend on the observable A. As we see matters, this dependence may be taken to point to: (i) to a limitation of our proposed implementation of QRFs (or of the generalization of Eq. (8) to a generic Ψ); or (ii) a limitation of the QRF idea itself; or (iii) the need to impose constraints on Ψ. We postpone to future work the investigation of these three alternatives. Here, we just note that, according to our understanding, a similar dependence on the observable A is also present in the coordinate-based formulation of QRFs. In applications of QRFs to non-relativistic quantum mechanics, for instance, one has to choose whether to privilege the position or the momentum representation before deciding which are the relevant QRF transformations (see, e.g., Ref. [6]).

III. GENERALIZED QRFs AND SUPERPOSITION OF GEOMETRIES

In the context of dynamical theories for the metric field g_{ab} (such as GR), it is widely assumed that the field equations are *diffeomorphism invariant*. This means that given *any* diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, the spacetimes $(\mathcal{M}, g_{ab})$ and $(\mathcal{M}, \phi^* g_{ab})$ are *physically indistinguishable*—where ϕ^* is the push-forward mapping between tensors defined at p and at $\phi(p)$ (i.e., ϕ here is seen as an *active* diffeomorphism). According to this view, points p of the underlying manifold $\mathcal{M}$ have no ab-

solute meaning by themselves, acquiring significance only to the extent that they can be characterized by physical quantities evaluated at them. This view is of course the legacy of Einstein's hole argument. For discussion, cf. e.g. Ref. [20]; and for a recent perspective which is close to this paper's fibre bundle approach to QRFs, and which we further discuss in Section VI, cf. Refs. [21–23].[6]

Now, let us analyze how this invariance of the classical theory fits into the scope of QRFs. It is *not* true that any given diffeomorphism ϕ induces a transformation between rQRFs on $(\mathcal{M}, g_{ab})$ via Eq. (4) (with ι replaced by ϕ). For given a tetrad $\{\mathbf{e}_\mu^a\}$ at p, in general $\{\phi_* \mathbf{e}_\mu^a\}$ will fail to be a tetrad, according to g_{ab} at $\phi^{-1}(p)$ (unless ϕ is an isometry).

Notwithstanding this, we can consider that ϕ induces transformations *between* rQRFs defined on $(\mathcal{M}, g_{ab})$ *and* rQRFs defined on $(\mathcal{M}, \phi^* g_{ab})$, since these spacetimes are, by their very definition, isometric. However, now, a superposition of Ψ and $\widetilde{\Psi}$ is no longer technically possible, since their domains (the bundles of orthonormal frames $\mathcal{F}_\mathrm{o}[g_{ab}]$ and $\mathcal{F}_\mathrm{o}[\phi^* g_{ab}]$, respectively) are, in general, distinct.

In order to circumvent this difficulty, we extend the definition of a QRF to be an assignment defined on the *bundle of frames* $\mathcal{F}(\mathcal{M})$ on the manifold $\mathcal{M}$. Beware of ambiguity: the 'frame-bundle' or 'bundle of frames' $\mathcal{F}(\mathcal{M})$ has as the fibre above a point $p \in \mathcal{M}$ the set of all (so: not necessarily (pseudo-)orthonormal) *bases* of T_p—*not* the set of all tetrads, i.e. 'frames' in our physical sense. Hence our use, since the start of Sec. II, of the subscript 'o' for 'orthonormal' for the bundle $\mathcal{F}_\mathrm{o}[g_{ab}]$, whose fibres *do* consist of tetrads according to the given g_{ab}.

That is: we now define a QRF to be a map, $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$, that to every (not necessarily orthonormal in the aforementioned metric g_{ab}) *basis* $\{\mathbf{x}_\mu^a\}$ of the tangent space at $p \in \mathcal{M}$ assigns a complex value $\Psi(p; \{\mathbf{x}\})$. More precisely, we "work locally" in $\mathcal{M}$, and so define Ψ on the bundle's points lying above some region $\mathcal{O} \subseteq \mathcal{M}$. That is: Ψ is defined on some $\pi^{-1}(O)$, where $\pi : \mathcal{F}(\mathcal{M}) \to \mathcal{M}$ is the canonical projection of the frame bundle.

The reader may wonder why we went through the trouble of restricting, at first, the definition of Ψ to the bundle of orthonormal frames (obtaining the rQRFs) only to later extend it to the bundle of frames. Why not consider this extended conception of a QRF from the beginning?

The reason is for the sake of clarity. For although now Ψ is defined for any basis $\{\mathbf{x}_\mu^a\}$ at p, its *meaning* continues to be the same as before, i.e., that of a "complex amplitude" assigned to the basis $\{\mathbf{x}_\mu^a\}$ *seen as a tetrad* at p—and hence referring to a spacetime whose metric g_{ab} at p satisfies $g_{ab} \mathbf{x}_\mu^a \mathbf{x}_\nu^b = \eta_{\mu\nu}$.

Thus the idea is that *any* section of $\mathcal{F}(\mathcal{M})$ (i.e., a basis field) can be seen as a section of $\mathcal{F}_\mathrm{o}[^{(\mathbf{x})}g_{ab}]$ (i.e., a tetrad field) for *some* (uniquely defined) metric field $^{(\mathbf{x})}g_{ab}$, namely

$$^{(\mathbf{x})}g_{ab} := \eta_{\mu\nu} \mathbf{X}_a^\mu \mathbf{X}_b^\nu,$$

where $\{\mathbf{X}_a^\mu\}$ is the dual-basis field related to $\{\mathbf{x}_\mu^a\}$ (i.e., satisfying $\mathbf{X}_a^\mu \mathbf{x}_\nu^a = \delta_\nu^\mu$ at each manifold point).[7]

Thus a spacetime with a specific metric is given uniquely by a section of $\mathcal{F}(\mathcal{M})$, the bundle of all frames (although different sections can lead to the same spacetime). *More precisely: a section specifies a spacetime, together with a tetrad field (i.e., a point of view) on it, since the elements of the section are bases at the spacetime points, that are orthonormal in that spacetime.*

As a result of this, a generic QRF $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$ can represent *not only* a superposition of "points of view" on a given classical spacetime—if $supp(\Psi) \subset \mathcal{F}_\mathrm{o}[g_{ab}]$ for some metric field g_{ab}—*but also* a superposition of spacetime metrics in the region $\pi(supp(\Psi)) \subseteq \mathcal{M}$; (provided of course that these metrics are defined on the same base manifold $\mathcal{M}$). Thus the idea of stipulating a basis-field to be a tetrad-field is the device by which Sec. II's simple and restricted conception of a QRF can be applied so as to describe a superposition of spacetime metrics.[8]

In Sec. V, we will apply the above ideas to describe the gravitational field due to a macroscopic spatial superposition of a large mass. So to prepare for the details of that, we end this Section by stating how the normalisation condition, Eq. (1), and the definition of a rQRF transformation by an isometry, Eq. (4), get generalised in the setting of generic QRFs.

Thus the normalisation condition (1) is now replaced by

$$\|\Psi(p)\|^2 := \int_{GL(4)} d\mu_G(M) \, |\Psi(p; \{M\mathbf{x}\})|^2 < +\infty \quad (11)$$

for all p and *any fixed* $\{\mathbf{x}_\mu^a\}$, where $M \in GL(4)$ (i.e., M is an arbitrary, invertible 4×4 real matrix) and μ_G is the (unique up to a multiplicative factor) invariant Haar

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[6] There are of course meanings of 'general covariance' that are logically stronger than diffeomorphism invariance. We will not need to discuss them. But we note, for example, that Anderson in Ref. [25] argued that GR is distinctive in that all fields, even the metric and connection, are dynamical, rather than "absolute" or a "fixed canvas": so that the traditional requirement that the symmetry group of a theory formulated on a manifold $\mathcal{M}$ should preserve absolute geometric objects is vacuously satisfied by *any* diffeomorphism of the manifold $\mathcal{M}$ just because there are no such absolute objects, i.e. $\mathrm{Diff}(\mathcal{M})$ is the symmetry group of GR (taken as using only one manifold $\mathcal{M}$). But as subsequent literature showed, it is hard to define 'absolute' precisely so as to get the intuitively right verdicts for all theories; cf. e.g. Ref. [26].

[7] Here, again, the concrete index μ in $\mathbf{X}_a^\mu$ labels the different basis elements (as in $\mathbf{e}_\mu^a$ and $\mathbf{x}_\mu^a$).

[8] Our invoking the frame bundle prompts a comparison with a different use of fibre bundles as a framework for QRFs, recently proposed by Ref. [16]; (cf. also Refs. [21, 22], and Ref. [24] Sections 3, 7 and Appendix A). We will make this comparison in Sec. VI.

measure defined on $GL(4)$. (Note that, again, the result of the integration in Eq. (11) does *not* depend on the choice of the fiducial $\{\mathbf{x}^a_\mu\}$ held fixed.)

Also, now, any diffeomorphism ϕ induces a QRF transformation $\Psi \mapsto \widetilde{\Psi}$ via [cf. Eq. (4)]:

$$\widetilde{\Psi}(p; \{\mathbf{x}\}) := \Psi(\phi^{-1}(p); \{\phi_* \mathbf{x}\}), \qquad (12)$$

which, in particular, maps rQRFs on $(\mathcal{M}, g_{ab})$ into rQRFs on $(\mathcal{M}, \phi^* g_{ab})$. It is important to stress, though, that a generic QRF *cannot* be interpreted as a rQRF on some spacetime; for instance, a linear complex combination of Ψ and $\widetilde{\Psi}$ given above (which is now possible) is not, in general, a rQRF even if Ψ and (consequently) $\widetilde{\Psi}$ are.

In this new framework, *classical* diffeomorphism invariance—i.e. the physical equivalence of $(\mathcal{M}, g_{ab})$ and $(\mathcal{M}, \phi^* g_{ab})$—is ensured by imposing invariance of all (geometric) observables under the map $\Psi \mapsto \widetilde{\Psi}$ given by Eq. (12).

It is important to point out that, in contrast to rQRFs defined on isometric spacetimes—to which the second-to-last paragraph before Subsec. II A still applies—two generic QRFs, or even two rQRFs defined on nonisometric spacetimes, describe completely different physical situations (not only different "perspectives" on a given, classical spacetime). In fact, as pointed out above, since *any* spacetime (with base manifold $\mathcal{M}$) can be (uniquely) characterized by a section of $\mathcal{F}(\mathcal{M})$—viz. it is that spacetime for which the given section is also a section of $\mathcal{F}_\mathrm{o}[g_{ab}]$—a QRF may describe an arbitrary superposition of spacetimes (with the same base manifold). This generality of QRFs in comparison to rQRFs prompts the discussion in the next Section.

IV. PERSPECTIVAL vs. BASIC QRF CONCEPTIONS

Notice that when we defined a rQRF in Sec. II, say $\Psi_r : \mathcal{F}_\mathrm{o}[g_{ab}] \to \mathbb{C}$, we did not attribute to Ψ_r the role of *determining* the geometry of the spacetime. Ψ_r merely represented possible superpositions of the "points of view" (i.e., tetrads) consistent with the *given* classical geometry. As such, Ψ_r did not need to be defined globally—i.e., there was no need for $\pi_\mathrm{o}(supp(\Psi_r)) = \mathcal{M}$. One can consider, for instance, a Ψ_r that represents the perspective of a particle in a superposition of worldlines evolving in the given spacetime, in which case a natural choice for Ψ_r would be supported on curves in $\mathcal{F}_\mathrm{o}[g_{ab}]$ whose projection to $\mathcal{M}$ would be concentrated on the possible worldlines of the particle [see Eq. (6)].

Then, diffeomorphism invariance together with imposition of a complex vector space structure led us to consider general QRFs, $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$: which we have argued can represent not only superpositions of perspectives but also superpositions of geometries.

It is important to emphasize, however, that this flexibility does *not* necessarily mean that Ψ must be interpreted as *defining* the superposed geometry on $\mathcal{M}$. For it could be that the superposed geometry is already somehow given by some mathematical structure added to $\mathcal{M}$, and that Ψ merely describes the superposition of "points of view" which are permissible in the given superposed geometry—which obviously must respect the fact that these possible perspectives do not belong to a single orthonormal-frame bundle $\mathcal{F}_\mathrm{o}[g_{ab}]$. In this case too, Ψ does not need to be globally defined, like in the rQRF case.

In order to avoid confusion and to make it clear when we are dealing with QRFs which merely describe possible superposition of "points of view" or "perspectives" on a *given*—superposed or not—geometry, we refer to such QRFs as *perspectival*. So perspectival QRFs do not need to be defined globally, since they are not interpreted as defining the (superposed) geometry. The "passive" character of perspectival QRFs—i.e., complying with, but not determining nor influencing the background—makes them appropriate to describe superposition of systems whose own gravitational influence (i.e., back-reaction) is neglected: the so-called "test" systems.[9]

But agreed: one can envisage, in contrast to this perspectival conception of QRFs, a different, more "basic" conception of $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$ which *does* take Ψ to define and describe the superposition of geometries. On this conception, it *is* natural to impose that $\pi(supp(\Psi)) = \mathcal{M}$—unless one were willing to allow pathological scenarios where regions of $\mathcal{M}$ would be left without any geometric structure. Although the expression "reference frame" usually carries the connotation of being passive—which is perfectly compatible with the perspectival conception of QRFs—we shall also use the expression to refer to this latter, more fundamental (and possibly dynamical) conception of Ψ. So we shall talk of *basic* QRFs.

But in this paper, we will be exclusively interested in perspectival QRFs, because these are the ones needed to describe probes (like test particles) evolving in superposition of geometries, which motivated this work in the first place. However, the fact that our framework accommodates, and even suggests, this basic conception of QRFs is something which, we believe, should be further explored. This would inevitably involve formulating dynamical equations for Ψ, in the same way that GR involves dynamical equations for the metric field—and, in fact, Einstein's equation should somehow be recovered in the context of *basic, restricted QRFs*. But from now on, we leave such questions aside, and focus on the use of perspectival QRFs to describe the scenario that has been a focus in the recent literature on QRFs: the field

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[9] But returning to comment (4), early in Section II: we do allow that a perspectival QRF could obey a continuity-like or Boltzmann-like equation describing "diffusion" in $\mathcal{F}(\mathcal{M})$.

due to a macroscopic superposition of a large mass.

V. A STEP BEYOND SEMICLASSICAL GRAVITY?

In the previous Sections, we have proposed a framework where the idea of QRFs, used in the gravitational context in previous works [7–10], is formulated in a fully geometric way. Although setting a stage where possible new phenomena can be described—such as *arbitrary* geometry superpositions—no "new physics" has been introduced. However, as is often the case, the motivation for introducing new concepts, such as QRFs, is the prospect of dealing with (at least some) situations which cannot be (or are doubtfully) treated with other available approaches, in the hope that new insights can be obtained.

In fact, in Ref. [8], in order to take such a further step, and propose a new principle, the authors consider a paradigmatic situation which is widely believed to lie beyond the scope of semiclassical gravity (the latter taken to mean a theory where a classical, well-defined spacetime only "feels" the average energy-momentum tensor of the quantum matter). Namely: the gravitational field engendered by a mass in a superposition of two position states—let us vaguely call them $|L\rangle$ and $|R\rangle$, with the latter representing a "spatial translation" of the former, from the 'Left' to the 'Right', by a distance, say, d—and the effect of this field on a test particle.

In this Section, we describe how this scenario is treated by our framework. We will give a simple description; so this will be a very special case of superposed spacetimes, with each represented by a section of the frame bundle. For we will take the two spacetimes, labelled by L and R, to be isometric, i.e. the L and R peaks of the mass' spatial wave-function will be idealised as being "the exact same shape" as each other, and so related by a "spatial translation." (Section VI will return to the general case of superposed spacetimes.)

This idealization accords with the recent literature's treatment of the scenario. The difference here, in addition to our representing spacetimes by sections of the frame bundle, is that we will try to state the idealization and its limitations precisely, in terms of reference frames (in our sense, i.e. contrasted with coordinate systems). As we warned the reader in the Introduction (concerning unrealistic expectations), we stress here that we do not go beyond what has been done in the literature using coordinate systems. However, as we also said in the Introduction: by distinguishing physical choices from arbitrary ones, the geometric approach makes it clear that the treatment of this simple gravitational scenario (coherent superposition of $|L\rangle$ and $|R\rangle$) using QRFs lacks predictive power, in the current state of knowledge—difficulty which we will also discuss in the coordinate-based approach in what follows (especially in relation to Fig. 1), but which, we believe, has not been duly appreciated in the literature.

If we were to restrict ourselves to the linear regime of gravity—which we shall not, to stay faithful to the main motivation of applying QRFs to gravitational scenarios—the set-up would be simple to describe. For there would be an underlying (flat) geometry where the idea of "rigid translation by a distance d" would make clear (i.e. unambiguous) sense. However, the classical spacetimes associated with localized mass distributions—say $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$, corresponding to states $|L\rangle$ and $|R\rangle$, respectively—are *not* translationally invariant. This means that, in general, one cannot change the spatial location of a "test system" of $N(>4)$ particles w.r.t. the source mass in, say, $(\mathcal{M}, {}^{(L)}g_{ab})$, while satisfying the condition that their *relative physical distances* (i.e., the physical distances among the particles) remain unchanged—unless the initial and final positions of the particles are related by an isometry of $(\mathcal{M}, {}^{(L)}g_{ab})$. Therefore, in general, there is no natural notion of a "rigid spatial translation" of an extended system $\mathcal{S}$ in $(\mathcal{M}, {}^{(L)}g_{ab})$. From the perspective of the system $\mathcal{S}$, this is equivalent to saying that there is no natural "rigid spatial translation" of the source mass (together with the geometry it engenders on $\mathcal{M}$).

The non-existence of a preferred notion of "rigid spatial translation" in $(\mathcal{M}, {}^{(L)}g_{ab})$ [and $(\mathcal{M}, {}^{(R)}g_{ab})$] causes difficulties for defining a superposition of $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$. For if such a notion existed, it could be used to naturally identify points in $(\mathcal{M}, {}^{(L)}g_{ab})$ with points in $(\mathcal{M}, {}^{(R)}g_{ab})$ in a nontrivial way (i.e., not through the isometric identification which maps source to source). Namely, in such a way that a given point of the manifold with the superposition would correspond to points in both $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$ with possibly different local properties. In fact, the protocol for such identification would be: (i) since $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$ are isometric, the hypothetical *displaced-to-the-left* $\mathcal{S}$ in $(\mathcal{M}, {}^{(L)}g_{ab})$ could be isometrically mapped to points of $(\mathcal{M}, {}^{(R)}g_{ab})$; (ii) then, the identification is made between the original (undisplaced) $\mathcal{S}$ in $(\mathcal{M}, {}^{(L)}g_{ab})$ and the image, in $(\mathcal{M}, {}^{(R)}g_{ab})$, of the displaced-to-the-left $\mathcal{S}$; (iii) finally, by considering arbitrary $\mathcal{S}$—for instance, filling in each entire spatial section with constituents of the system $\mathcal{S}$—one would get the desired identification between $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$. With such an identification, $\mathcal{S}$ would be seen to be "the same" in both spacetimes—hence, each constituent of $\mathcal{S}$ would be subject to two different local geometries: a *superposition.*

This is an important point which must be treated carefully. In Refs. [8, 10], the authors try to address this point by fixing a spatial coordinate system in the superposed configuration in such a way that a *coordinate* translation maps the mass distribution associated to $|R\rangle$ into the mass distribution associated to $|L\rangle$—hence, making the background geometry well defined after this "matching" translation is performed. This is just the same identification protocol mentioned above for an everywhere-defined "system" $\mathcal{S}$; but now with a coordinate system playing the role of $\mathcal{S}$, and with the constraint of preserv-

ing relative *physical* distances—which, in general, cannot be satisfied—weakened to requiring just preservation of relative "coordinate distances": which, by construction, is trivially satisfied for *any* coordinate system globally-defined first on $(\mathcal{M}, {}^{(L)}g_{ab})$. As a result, there are infinitely many *different* coordinate systems defined on the superposed configuration which can be equally well adopted for carrying out this protocol (if no further conditions are imposed). And different choices would lead to different mappings, eventually leading to different *physical* conclusions—which would be unacceptable: see Fig. 1.

A possible way out of this conundrum would be to select the coordinate system according to a clear physical prescription. However, this cannot be done in the superposed mass configuration without risking circular reasoning. For we do not know, beforehand, the "physics" of the background associated to such a superposition; and there is no reason to privilege one or the other "branch" of the geometry.

One might try, for example, fixing a physical coordinate system *before* the superposition is prepared and then consider its "evolution." In Ref. [10], this is attempted by saying that a Euclidean coordinate system is "fixed" in the laboratory before the superposition is prepared— perhaps even before the mass is brought in for the experiment, since the spacetime is assumed to be flat then. However, in a spacetime which is non-stationary—which it must be, otherwise it would continue to be flat—there is no natural identification which allows one to say which are "the same points in space" before and after the mass is brought in (and then superposed): in a non-stationary geometry, the idea of "holding things in place" so as to set the spatial coordinates is meaningless. That is: in the non-linear regime, the spatial coordinate system defined in such a way that the nearby objects in the "laboratory" are "spatially fixed" would strongly depend on the details of the stresses and strains in the physical objects constituting the "laboratory". (This is just the physical counterpart of our discussion above that there is no natural notion of spatial translation in a background geometry which is not translationally invariant.) And, again, we risk incurring circular reasoning since we do not know the physics of the geometry's *evolution* from an initially well-defined spatial geometry up to the geometry associated with the mass in superposition.

This point is, indeed, intricate. And agreed: here we shall not be able to do better than Refs. [8, 10], simply by recasting the problem in geometric terms—except, perhaps, in recognizing its subtleties and pointing to its possible *dynamical* nature: as hinted at above, with our discussion of the evolution of physically-constructed coordinate systems, and further discussed in what follows.

In our geometric set-up, we may take $\mathcal{M}$ to be the arena where the superposition is to be defined—hence points of the manifold are naturally identified in both "branches" of the geometry—and the challenge is to ascribe the metric fields ${}^{(L)}g_{ab}$ and ${}^{(R)}g_{ab}$ to $\mathcal{M}$ in such a way that each represents the spacetime of each position configuration of the source mass. For a start, we want $(\mathcal{M}, {}^{(L)}g_{ab})$ and $(\mathcal{M}, {}^{(R)}g_{ab})$ to be isometric: ${}^{(R)}g_{ab} = \phi_d^{*\,(L)}g_{ab}$, with ϕ_d being a diffeomorphism on $\mathcal{M}$ which should satisfy some conditions in addition to mapping the mass distribution of $|L\rangle$ into that of $|R\rangle$. Since the "relative" separation of the positions associated to the states $|L\rangle$ and $|R\rangle$ is supposed to be fixed, ϕ_d should preserve the static worldlines: $\phi_d^* \xi_{(L)}^a \propto \xi_{(L)}^a$, where $\xi_{(L)}^a$ is the Killing field representing the static symmetry of $(\mathcal{M}, {}^{(L)}g_{ab})$. Moreover, since $(\mathcal{M}, {}^{(L)}g_{ab})$ is asymptotically flat at spatial infinity, we can demand ϕ_d to "tend" (in a sense which must be made precise) to rigid spatial translations as we consider points arbitrarily far away from the mass distributions.[10] We may also impose, for symmetry purposes, that the dependence of ϕ_d on d (as a one-parameter family of diffeomorphisms) is such that $\phi_d^{-1} = \phi_{-d}$.

All these conditions, however, do not fix a unique ϕ_d; in fact, they still leave an infinite number of possibilities— which is the geometric counterpart of the infinitely many coordinate systems mentioned above. And it is relevant to note that, even though different possibilities of ϕ_d lead to spacetimes $(\mathcal{M}, \phi_d^{*\,(L)}g_{ab})$ which are physically indistinguishable among themselves, superpositions of ${}^{(L)}g_{ab}$ and ${}^{(R)}g_{ab} = \phi_d^{*\,(L)}g_{ab}$ *are sensitive* to ϕ_d, in accordance with our previous discussion that different choices of coordinate systems (for which a coordinate translation maps $|R\rangle$ to $|L\rangle$), which determine different choices of ϕ_d, lead to different physical conclusions. In other words, determining *the* diffeomorphism ϕ_d is a *physical* question.

Here, we pragmatically assume that one such diffeomorphism has been privileged. Based on our previous discussion about the evolution of physically-constructed coordinate systems, it is natural to conjecture that such a diffeomorphism should be selected by details of the background *dynamics* which evolved the total system so as to prepare the superposition. So, whatever perspectival QRFs we use to describe quantum probes on this superposition, they should all be consistent with the basic QRF which, we presume, describes the superposed geometry.[11] This dynamics could perhaps be characterized by one-parameter families of diffeomorphisms, ${}^{(L)}\phi_t$ and ${}^{(R)}\phi_t$, each describing the evolution of each "branch," with $\phi_d := {}^{(R)}\phi_t \circ {}^{(L)}\phi_t^{-1}$ for t sufficiently large to relate the asymptotic static configurations. But at this point, we have no detailed suggestion about this dynamics, or the diffeomorphisms ${}^{(L)}\phi_t$ and ${}^{(R)}\phi_t$.

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[10] In a more realistic situation, where the superposition is obtained from a previously well-defined mass distribution, one should impose that ϕ_d is trivial outside the causal future of the region where the superposition was prepared. But here we consider the idealized case where the superposition has always existed.

[11] We say that a perspectival QRF Ψ_P is *consistent* with a basic QRF Ψ_B if for each $(p; \{\mathbf{x}_\mu^a\}) \in supp(\Psi_P)$, there is at least one $\Lambda \in O(3,1)$ such that $(p; \{(\Lambda\mathbf{x})_\mu^a\}) \in supp(\Psi_B)$.
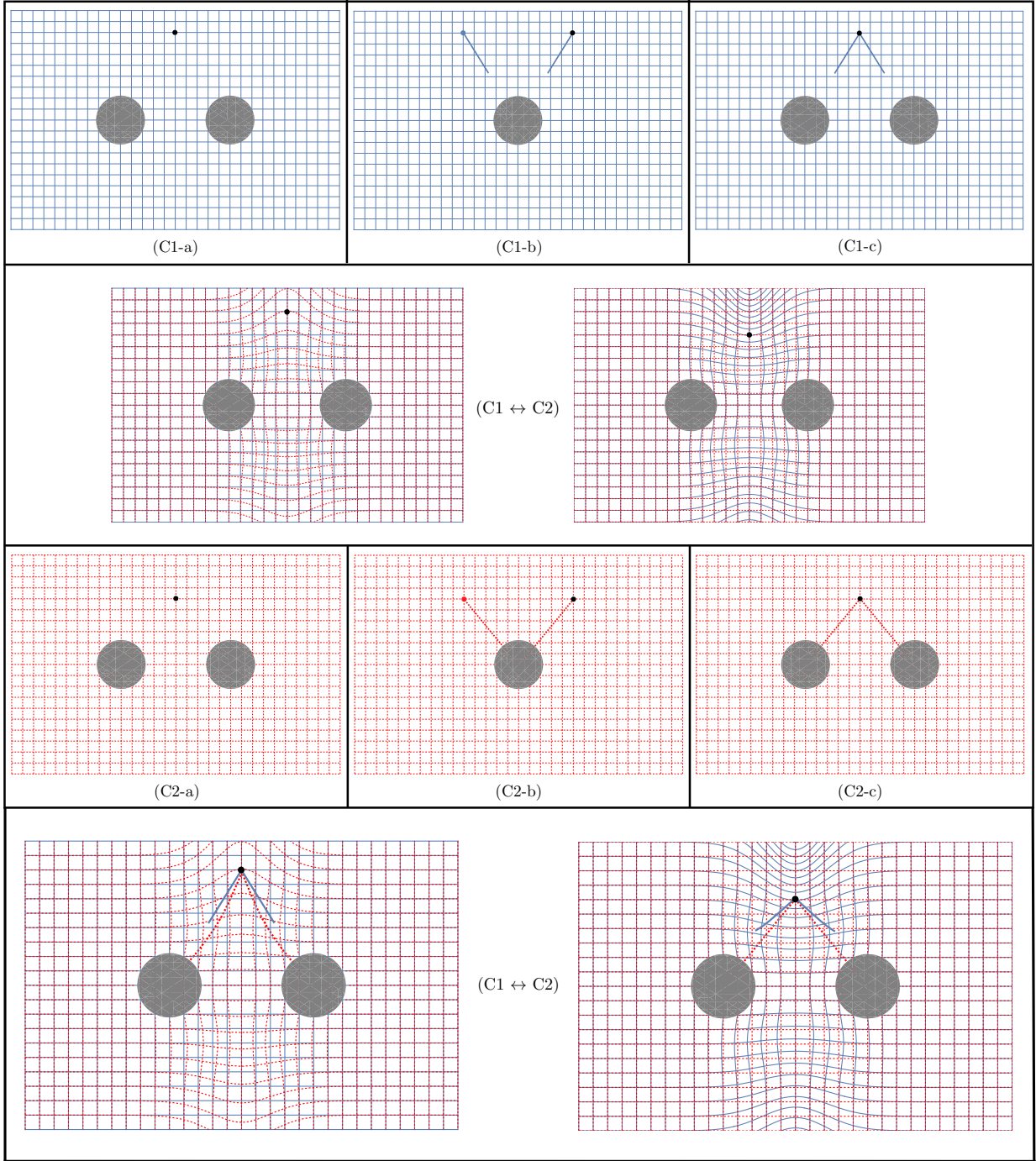
FIG. 1: We schematically illustrate, here, the prescription given in Refs. [8, 10] for determining the evolution of a test particle (the black dot) under the influence of a mass in a macroscopic superposition of position states (the gray balls). (C1-a) A coordinate system C1 (solid, blue grid) is defined on $\mathcal{M}$, in such a way that a "rigid" *coordinate* translation maps one source-mass configuration state into the other (gray balls); (C1-b) After performing this coordinate translation, which maps the position state of the test particle into a superposition of position states (blue and black dots), the geodesic equation is solved (for a certain coordinate- or proper-time interval) for each position state in the background geometry associated to the now well-defined source-mass distribution; (C1-c) The inverse coordinate translation is applied to the system (including the solution of the geodesic equation), hence obtaining a superposition of trajectories (solid, blue curves). (C1↔C2–second row) If no well-defined background metric is assumed for the superposition of source-mass configuration states, there is no way to privilege coordinate system C1 in comparison to another one, such as C2 (dashed, red grid). (C2-a,b,c) By applying the same steps above, but now using the coordinate system C2 (with the red dot in C2-b playing the same role as the blue dot in C1-b), *different* trajectories would be obtained for the same source-mass superposition state and same test particle (dotted, red curves). The relation between these trajectories is represented in the fourth row. (This figure is to be seen as a mere qualitative illustration, since a more precise depiction should also allow for different choices of coordinate-time "slices.")

Given one preferred ϕ_d, the spirit of our approach to the QRF idea is that the *geometric effects* of the superposition $\alpha|L\rangle + \beta|R\rangle$—with each of $|L\rangle$ and $|R\rangle$ associated to $(\mathcal{M},{}^{(L)}g_{ab})$ and $(\mathcal{M},{}^{(R)}g_{ab})$, respectively, with ${}^{(R)}g_{ab} = \phi_d^{*}{}^{(L)}g_{ab}$—on a "test system" (i.e., one which can be neglected as a source of changes to the geometry) can be described by adopting the perspectival QRF $\Psi = \alpha\Psi_L + \beta\Psi_R$, where Ψ_L and Ψ_R are *arbitrary* rQRFs defined on $(\mathcal{M},{}^{(L)}g_{ab})$ and $(\mathcal{M},{}^{(R)}g_{ab})$, respectively.

Note that, on their own, Ψ_L and Ψ_R represent (possibly different) "points of view" (i.e. frames) on isometric spacetimes. However, Ψ itself is not a rQRF on any classical spacetime. And as such, it represents, in principle, a completely different situation, beyond the scope of our current (classical- and semiclassical-gravity) understanding.

Here is where a new *invariance* principle is introduced. According to our reading of the literature, it is the implementation of the so-called "quantum covariance" proposed in the coordinate-based approach—see, e.g., Sec. IIA of Ref. [10]; but we now rephrase in terms of the geometric language presented here. Although, as pointed out above, $\Psi = \alpha\Psi_L + \beta\Psi_R$ describes a situation beyond classical or semiclassical gravity, it is *imposed* that it should lead to the same observables as (i.e., be "equivalent" to) the *restricted QRF* $\widetilde{\Psi}$ obtained using the pullback of Ψ_R to $(\mathcal{M},{}^{(L)}g_{ab})$:

$$\widetilde{\Psi}(p;\{\mathbf{x}\}) := \alpha\Psi_L(p;\{\mathbf{x}\}) + \beta\Psi_R(\phi_d(p);\{\phi_d^{*}\mathbf{x}\}) : \quad (13)$$

the values of observable quantities should be *invariant* under $\Psi \mapsto \widetilde{\Psi}$. With $\widetilde{\Psi}$ representing a mere "point of view" on a well-defined background geometry—namely, $(\mathcal{M},{}^{(L)}g_{ab})$—this brings the description of the evolution of the test system within the jurisdiction of classical or semiclassical gravity.

This is the analogue, in the gravitational scenario, of the "transfer" of the superposition of the state of the system to a superposition of "points of view" which we illustrated in Subsec. II A in the flat-spacetime context. Here, the superposition of geometries (which is the "physical system"), described by Ψ, is mapped to a superposition of "points of view" in a non-superposed geometry, described by $\widetilde{\Psi}$. Since Ψ_L and Ψ_R are arbitrary, they can, in particular, *each* correspond to a "point of view" of a localized test particle; cf. again row 1 i.e. C1 of Fig. 1.

For instance, in the case of a free (point-like) test particle with a given initial condition—say, passing through the manifold point p_0 with tangent vector $v_0^a \propto \xi_{(L)}^a$, at its *proper time* τ_0—as considered in Ref. [10], we may consider Ψ to be the "point of view" of the particle itself, in the sense of Eq. (6) (but without our needing here to adopt any coordinate system); i.e., both Ψ_L and Ψ_R have supports whose projection to $\mathcal{M}$ are worldlines (to be determined) "starting" at p_0 with tangent vector v_0^a. Hence, $\widetilde{\Psi}$ given by Eq. (13) has support whose projection to $\mathcal{M}$ are worldlines (still to be determined)

"starting" at p_0 and $\phi_d^{-1}(p_0)$ with tangent vectors v_0^a and $\phi_{d*}v_0^a$, respectively. But now $\widetilde{\Psi}$ is a *restricted* QRF on $(\mathcal{M},{}^{(L)}g_{ab})$; so, the condition that the particle is free means that its possible worldlines are the geodesics of $(\mathcal{M},{}^{(L)}g_{ab})$ determined by the initial conditions (p_0,v_0^a) and $(\phi_d^{-1}(p_0),\phi_{d*}v_0^a)$—call them γ_1 and γ_2, respectively. Then, going back to the initial QRF Ψ, we have, in this case, $\pi\left(supp(\Psi)\right) = \pi\left(supp(\Psi_L)\right) \cup \pi\left(supp(\Psi_R)\right) = \gamma_1 \cup \phi_d[\gamma_2]$, where $\pi : \mathcal{F}(\mathcal{M}) \to \mathcal{M}$ is as before the canonical projection defined on the bundle of frames.

This is just the expected result obtained in the coordinate-based formulation of QRFs: the particle evolves to a superposition of geodesics, referring to each $(\mathcal{M},{}^{(L)}g_{ab})$ and $(\mathcal{M},{}^{(R)}g_{ab})$—bearing explicit dependence on the diffeomorphism ϕ_d. A figure representing the procedure described above (projected to the static spatial sections) would be quite similar to Fig. 1 with *one* given preferred choice of coordinate system.

Note that Eq. (12)—which is just *classical* diffeomorphism invariance expressed in the context of QRFs—ensures that the principle expressed by Eq. (13) is actually independent of the "representative" $(\mathcal{M},{}^{(L)}g_{ab})$ with which we start our construction. We could instead have started with $(\mathcal{M},{}^{(R)}g_{ab})$ or any other isometric spacetime. A more symmetric—but equivalent—description, for instance, would be obtained by defining $\widetilde{\Psi}$ using the pullback of both Ψ_L and Ψ_R through the hypothetical ${}^{(L)}\phi_t$ and ${}^{(R)}\phi_t$ mentioned above, respectively, for t sufficiently large to relate the asymptotic static configurations.

The same rationale can be applied to analyze a static (point-like) "clock"—as also considered in Ref. [10]—whose proper time will then evolve as a superposition of two proper times.

These examples about worldlines and proper times illustrate that—at least as far as geometrical observables are concerned (of which geodesics and proper times are examples)—any conclusion one might reach using the original coordinate-based formulation of QRFs applied to the simple gravitational scenario of superposition of well-defined spacetimes, as in Refs. [7–10], can be obtained using the geometric formulation we have presented here. At least, this is true, provided that the subtlety which we highlighted above, about the non-uniqueness of the so-called "spatial translation" relating $|L\rangle$ and $|R\rangle$, is properly addressed in *both* formulations.

VI. DISCUSSION

We have presented a geometric formulation of the idea of a QRF, which had already been applied, in a coordinate-based way, to describe the effects of gravitational fields engendered by masses in coherent superpositions of position states [10]. Our approach has been to take a QRF as a mapping $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$ (seen as "wave functions" on the fibres of $\mathcal{F}(\mathcal{M})$); and thus to rely heavily on the conceptual distinction between "points of

view" (on which values of observable quantities *can* depend) and coordinate systems (whose different choices can make no difference to observed values).

In a nutshell, the idea of QRFs applied to the gravitational scenario is simple. A "test" system (i.e., one whose own gravitational effects can be completely neglected) would supposedly evolve as a coherent superposition of "histories," one for each well-defined position state of the source mass—at least as long as it does not disturb the very state of the source mass. Although some may consider that this "trivializes" the effect of a superposition of geometries, the idea of QRFs formalizes calculations without the need for a well-defined background metric— once a preferred coordinate system or, equivalently, a preferred diffeomorphism relating both source-mass configurations is selected.

We emphasize that our purpose here has not been to assess the merits of the general idea of a QRF, but merely to recast it in a geometric language. By doing so, it is our view that some conceptual aspects of its implementation, particularly when going beyond the linear-gravity regime, get highlighted—such as the possible relation of the preferred diffeomorphism, relating the source-mass configurations, with the (unknown, quantum) dynamics which prepares the superposition.

We also make no claim that the geometric formulation presented here is "minimalist" in any sense. In fact, since "points of view" may affect values of observable quantities but (we presume) *not* the evolution of the system itself, one might try to devise a simpler or "more economic" geometric formulation by, e.g., doing away with (or "tracing out") different points of view and focusing solely on spacetime geometries—i.e., basing the construction on the "space" of possible geometries instead of on the bundle of frames.

This last comment prompts a comparison between this paper's fibre-bundle framework and that of Ref. [16] (cf. also Refs. [21, 22], and Ref. [24] Sections 3, 7 and Appendix A).[12]

It will be clearest to begin with an obvious contrast, about the dimensions of the fibre bundles. This will help us to spell out how the two frameworks differ in their treatments of how to identify spacetime points across two different spacetimes.[13]

Our fibre bundle $\mathcal{F}(\mathcal{M})$ is finite-dimensional, with the spacetime manifold $\mathcal{M}$ as its 4-dimensional base-space and with the bases at each spacetime point p forming the 16-dimensional fibre above p. (And similarly, for Section II's fibre bundle $\mathcal{F}_o[g_{ab}]$ of orthonormal frames of $(\mathcal{M}, g_{ab})$, whose fibres are 6 dimensional.) But in the fibre bundle adopted by Ref. [16], both (i) the base-space, and (ii) each fibre, are infinite-dimensional. Namely, for vacuum general relativity on the manifold $\mathcal{M}$, they are: (i) the set of Lorentzian geometries, where each geometry is determined by an isometry-class of Lorentzian manifolds[14] $(\mathcal{M}, g_{ab})$, and (ii) such an isometry-class, i.e. an orbit of the diffeomorphism group $\text{Diff}(\mathcal{M})$ of $\mathcal{M}$. (Cf. the discussion at the start of Sec. III.)

This difference implies immediately that: (a) for us, a spacetime with a specific metric (and equipped with a tetrad field) is given by a section of $\mathcal{F}(\mathcal{M})$, so that two such spacetimes in superposition are represented by a (basic) QRF Ψ with support on two such sections; (more precisely: on two such sections that are not related by an application of an element of $O(3,1)$ at each spacetime point p of the manifold $\mathcal{M}$). On the other hand: (b) for Ref. [16], a spacetime with a specific metric (and *not* equipped with a tetrad field) is a point, i.e. element of the bundle, so that two spacetimes in superposition are to be represented by two such points (each with a complex amplitude).

This difference also means that: (a) for us, identifying a point across two spacetimes in superposition is simply a matter of "looking up or down along the fibre, from one section to the other", since the fibre is labelled by the point p. So for us, the self-same point has different metrical (and material) properties and relations to other points (as encoded by the metric and matter-field tensors) on the different sections.

On the other hand, (b): for Ref. [16], identifying a point across two spacetimes in superposition is a matter of "looking inside" the points of the fibre bundle (each of which is an entire spacetime, endowed with a configuration of four coordinate scalar fields—the so-called χ-fields), then focusing on the *spacetime points* therein, and then defining what it is for two points p and q in non-isometric spacetimes, or in isometric spacetimes with different χ-fields configurations, to be identified with each other, so as to move from one fibre to another.

In Refs. [16, 21, 22], two points that are thus identified are said to be *threaded*. In Ref. [22], and Ref. [24] (Sections 3, 7 and Appendix A), the relations between threading and the topic of connections on fibre bundles is discussed. In Ref. [16], this threading is done by: (i) making a choice of four scalar fields (the χ-fields) that take, in a single spacetime, suitably non-repeating values, so

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[12] The two frameworks were developed independently; (this paper's framework, mostly by D.V.) We thank the authors of these papers, with whom later discussions have helped us understand the relation between the frameworks.

[13] Recall Section V's discussion of how to make rigorous sense of "which point is which" when comparing two spacetimes. There, we first saw that if one spacetime admitted a notion of rigid spatial translation by distance d, this notion naturally defined a protocol for identifying points. But we then stressed that such a notion is *not* in general available, and thus we discussed what features a diffeomorphism ϕ_d might be hoped or required to have, for it to effect such an identification.

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[14] Strictly speaking, this brief statement needs to be qualified so as to register the role of four scalar fields that one has to choose; cf. the discussion below.

that any two points of $\mathcal{M}$ (or working locally: of a region $\mathcal{O} \subseteq \mathcal{M}$) have distinct quadruples of values; and then (ii) saying that two points p and q in "distinct" spacetimes are to be threaded (i.e. are to be identified: "are physically the same") iff their quadruples of values for the four scalar fields match. (Obviously, (i) and (ii) here are a formal analogue or model of Section V's considerations about what features a diffeomorphism ϕ_d might be hoped or required to have, for it to effect such an identification. Cf footnote 13. Hence also our scare-quotes around the word 'distinct': for the spacetimes do not need to be non-isometric—they are not, in Section V's case of the mass in a macroscopic superposition of position states.)

These different treatments of how to non-isometrically identify points across spacetimes prompt a brief philosophical comment. Despite the difference, we think *both* treatments are compatible with the moral drawn from the hole argument at the start of Sec. III, viz. that points only acquire significance through the physical quantities evaluated at them.

As to (a): this paper's treatment in effect makes points' identity *fiducial*. For nothing turns on whether the point at the base of (and so labelling) a fibre is p or some other point, say q—it is just that in order to define the bundle, a choice must be made.[15]

And (b): we of course agree that Ref. [16]'s framework of identification-by-matching-field-values is also compatible with this moral from the hole argument. This compatibility is also reinforced by:

(1) this framework's admission that the scalar fields in (i) above involve a choice, i.e. that other quartets of suitably non-repeating scalar fields are equally legitimate; and

(2) this framework's having a motivation (cf. Ref. [21]) given by a philosophical theory about identity, called *counterpart theory*.

So much by way summarising the contrast between the two frameworks about dimensions, and their ensuing differences about how to identify points across spacetimes. Finally, we note an obvious but important way in which the frameworks are concordant. The underlying point is that—as all must agree—the set of spacetimes is infinite-dimensional. This is of course explicit in the fibre bundle adopted in the framework of Refs. [16, 21, 22]. But of course, it is also true, though implicit, in our framework. In fact, if we were to build the classical "space of models" (as the bundle of Ref. [16] is called) based on our framework (but without any reference to the complex amplitudes Ψ yet), we would have to work with the space $Sec[\mathcal{F}(\mathcal{M})]$ of local sections of $\mathcal{F}(\mathcal{M})$—which is of course infinite-dimensional as well.

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[15] This is reminiscent of the philosophical doctrine called 'haecceitism', whose tenability as a response to the hole argument was first formulated by Ref. [27] (p. 21, Sec. 5).

Thus we stress that in this way, the dimension contrast which we have expounded does not involve any conflict between the two frameworks. Agreed: in this way it could seem a bit odd that we have compared our finite-dimensional bundle $\mathcal{F}(\mathcal{M})$ with the infinite-dimensional bundle of Ref. [16]. But our motivation for making this comparison is clear: in each framework, these are the bundles which function as the stage on which to define QRFs—with the infinite-dimensional space $Sec[\mathcal{F}(\mathcal{M})]$ playing no role here.

But we must leave further comparisons of these frameworks to future work. We end this Section with a summary, using a Figure, of how this paper's framework arises from a handful of ideas; and this summary will lead to a brief final list of open questions.

We believe that this paper's framework follows naturally from combining the core idea of QRFs with some well-established general ideas, namely linearity, diffeomorphism invariance, and what we will call 'locality of geometry'. The latter means that there is no way to determine whether two spacetimes (defined on the same manifold $\mathcal{M}$) are isometric by only "looking at" an open region $\mathcal{O} \subsetneq \mathcal{M}$. Hence, if a QRF Ψ is to be locally constructed, there is no reason to consider only superpositions of globally isometric spacetimes; superposition of arbitrary geometries must be allowed. In Fig. 2, we depict, in a schematic way, the line of reasoning which motivated our framework.

As with any mathematical framework in physics, the merit of our geometric formulation of QRFs must be assessed by how useful it can be for portraying known scenarios and/or enabling the description and understanding of new ones. We note in particular that this paper has *not* explored the generality enabled by taking a QRF as a map $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$: which allows *arbitrary* superpositions of spacetime geometries (on $\mathcal{M}$). For in Sec. V, our reading of the (new) invariance principle which underpins the use of QRFs in the gravitational scenario analyzed in Refs. [8, 10]—viz. the equivalence of $\Psi = \alpha\Psi_L + \beta\Psi_R$ and $\widetilde{\Psi}$ given by Eq. (13), for $supp(\Psi_L) \subseteq \mathcal{F}_o[^{(L)}g_{ab}]$ and $supp(\Psi_R) \subseteq \mathcal{F}_o[^{(R)}g_{ab}]$, with $^{(R)}g_{ab} = \phi_d^{*(L)}g_{ab}$—was applied only to the very special cases where Ψ describes the superposition of two (but easily generalizable to a finite number of) *isometric* spacetimes.

Thus we arrive at some natural questions for future analyses. Can this invariance principle be extended to a more generic Ψ? More importantly, can such generality be put to some use beyond mere formal description? At a higher degree of speculation: can this framework be used to say anything about the *evolution* ϕ_t hypothesised in Sec. V?

There are also interesting questions about the *meaning* of the QRF assignments $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$, and even of the rQRFs $\Psi : \mathcal{F}_o[g_{ab}] \to \mathbb{C}$. Here, we recall the Section IV's distinction between perspectival and basic conceptions of a QRF. For an rQRF $\Psi : \mathcal{F}_o[g_{ab}] \to \mathbb{C}$, where the spacetime metric is well defined, the classical
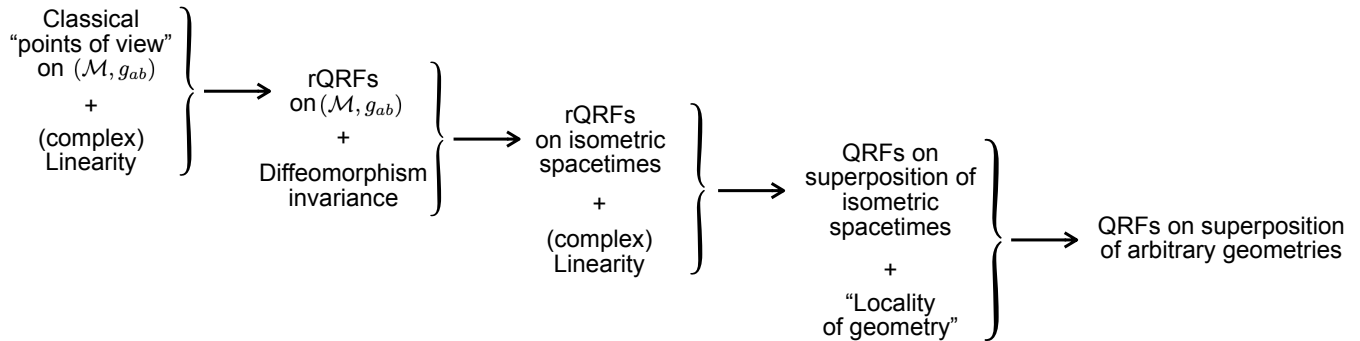
FIG. 2: Schematic rationale for our geometric formulation of QRFs (presented in Secs. II and III). It starts (in Sec. II) as a way to superpose (using assignments of complex amplitudes) different "points of view" on a given spacetime (rQRF). Then, imposing diffeomorphism invariance (in Sec. III) naturally leads to maps between rQRFs on isometric spacetimes. Requiring (also in Sec. III) superposition to again make sense (via assignments of complex amplitudes), we are "forced" to consider superposition of "equivalent" (global) geometries (i.e., isometric spacetimes). However, given an open neighborhood $\mathcal{O} \subsetneq \mathcal{M}$, it is *not* possible, in general, to distinguish isometric from non-isometric *global* geometries by their restrictions to $\mathcal{O}$. Therefore, if we do *not* want to impose constraints on $\Psi|_{\mathcal{O}}$ which would depend on global structures (a sort of "locality" principle), then we are naturally led (still in Sec. III) to consider superpositions of *arbitrary* geometries.

principle of *general covariance* demands that the *physics* of the systems being described with respect to a particular $\Psi : \mathcal{F}_o[g_{ab}] \to \mathbb{C}$ cannot depend on the choice of Ψ. This demand is not to be confused with values of observables: which *do* depend on Ψ, but in such a way that the physical system being "observed" has an absolute, frame-independent underlying evolution. For this reason, the question of whether the rQRF Ψ represents a physical apparatus or a mere fictitious idealisation was irrelevant.

However, in the general case, different choices of $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$ *can* represent different physical situations; and so it is more reasonable that Ψ itself should satisfy further physical constraints, such as continuity-like and/or Boltzmann-like transport equations on $\mathcal{F}(\mathcal{M})$.

Again at a higher degree of speculation: could time-interval and distance uncertainties introduced by such a Ψ be (at least partially) responsible for the uncertainty relations which are usually obtained from the non-commutativity of observables? And more generally, can the formalism of describing arbitrary superpositions of geometries by $\Psi : \mathcal{F}(\mathcal{M}) \to \mathbb{C}$ have applications other than for the QRF idea?

The answers to these questions are unclear. But we hope that our fibre bundle formulation may help attract the attention of researchers with different backgrounds.

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