Conditionally inaccessible decisions

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Abstract

We define a notion of inaccessibility of a decision between two options represented by utility functions, where the decision is based on the order of the expected values of the two utility functions. The inaccessibility expresses that the decision cannot be obtained if the expectation values of the utility functions are calculated using the conditional probability defined by a prior and by partial evidence about the probability that determines the decision. Examples of inaccessible decisions are given in finite probability spaces. Open questions and conjectures about inaccessibility of decisions are formulated. The results are interpreted as showing the crucial role of priors in Bayesian taming of epistemic uncertainties about probabilities that determine decisions based on utility maximizing.

Keywords: Utility maximizing, conditional probability, inaccessible decisions, Bayesianism

1 The main idea of decision making in terms of utility maximization

The core idea of decision theory based on utility maximization is that an Agent models the world in terms of probability measure spaces and decisions are identified with utility functions represented by random variables. Decisions are preferred that have the maximal expected utility with respect to the probability measure.

To be more explicit, let $(X, \mathcal{S}, p)$ be a probability measure space, with X a set, $\mathcal{S}$ a σ-field of some subsets of X and p a probability measure on $\mathcal{S}$. Real valued random variables $f_i \colon X \to \mathbb{R}$ $(i \in I)$ represent decisions, and the prescription of utility theory is: The Agent should choose decision f_j for which

$$\langle f_j \rangle_p > \langle f_i \rangle_p \quad \text{for all } i \neq j \tag{1}$$

where $\langle f \rangle_p \doteq \int f dp$ is the expectation value of f with respect to p (see Briggs (2023), Buchak (2022), Bradley (2017) and Chakrabarty and Kanaujiya (2023) for a more detailed review of the main ideas and some history of utility theory).

In the decision theoretic context elements in $\mathcal{S}$ are interpreted as states/properties of the world (equivalently: as propositions stating some properties of the world). The value $f_j(x)$ is the utility (degree of preference) of the action f_j from the Agent's perspective if the state of the world is x (the larger $f_j(x)$ the more the Agent prefers action f_j if x obtains). The probability measure p can be viewed either as

(i) representing subjective degrees of belief in the truth of the propositions in $\mathcal{S}$; or as

(ii) expressing objective features of the world (e.g. as frequencies with which the features of the world obtain).

If p is viewed subjectively, the expectation values $\langle f_j \rangle_p$ express the Agent's subjective degrees of expectations of the value of the decision f_j, and the Agent behaves rationally if it prefers f_j because this choice is in harmony with his subjective expectation. If p is interpreted objectively, say as relative frequency, then the Agent behaves rationally if prefers f_j because (on the average) the value of f_j will be objectively higher than the average value of f_i. In this paper we deal only with this latter kind of *objectively rational* decision based on utility maximization. From now on, we use the notation p^* to indicate that the probability measure is viewed as an objective probability.

2 A potential difficulty for decision making based on utility maximization using probabilities inferred via conditioning

One difficulty of rational decision making based on utility maximization is that the Agent might not know the objective probability p^*; or it might not know p^* in full, possibly knowing the values $p^*(A)$ only for elements A in a Boolean *sub*algebra $\mathcal{A}$ of $\mathcal{S}$. In such situations the Agent can try to infer the values of p^* for elements that are in $\mathcal{S}$ but not in $\mathcal{A}$ via conditioning, and calculate the expectation values of the utility functions using the so-inferred conditional probability measure. This procedure, in more details, is the following (we refer to Chapter 6. in Billingsley (1995) or Rosenthal (2006) for the theory of conditioning with respect to σ-fields):

The Agent specifies a probability measure p on $\mathcal{S}$ as his prior in such a way that the restriction $p^*_{\mathcal{A}}$ of p^* to $\mathcal{A}$ is absolutely continuous with respect to the restriction

$p_{\mathcal{A}}$ of p to $\mathcal{A}$. Then by the Radon-Nikodym theorem there exists the Radon-Nikodym derivative $\frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}}$ of $p^*_{\mathcal{A}}$ with respect to $p_{\mathcal{A}}$: the function $\frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}}$ is the density function of $p^*_{\mathcal{A}}$ with respect to $p_{\mathcal{A}}$, which gives $p^*_{\mathcal{A}}$ on $\mathcal{A}$ as

$$p^*_{\mathcal{A}}(A) = \int \chi_A \frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}} dp_{\mathcal{A}} \qquad A \in \mathcal{A} \tag{2}$$

(χ_A above denotes the characteristic (indicator) function of the set A; more generally below we will use χ_Z to denote the characteristic function of a set Z in $\mathcal{S}$). The formula (2) allows one to extend $p^*_{\mathcal{A}}$ from $\mathcal{A}$ to a probability measure $p^*_{p,\mathcal{A}}$ on $\mathcal{S}$ by defining

$$p^*_{p,\mathcal{A}}(B) \doteq \int \chi_B \frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}} dp \qquad B \in \mathcal{S} \tag{3}$$

Remark 1. If $\mathcal{A}$ is generated by a countable partition $\{A_i\}_{i\in\mathbb{N}}$, then (a version of) $\frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}}$ is

$$\frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}}(x) = \sum_{i\in\mathbb{N}} \frac{p^*_{\mathcal{A}}(A_i)}{p_{\mathcal{A}}(A_i)} \chi_{A_i}(x) \qquad x \in X \tag{4}$$

The probability $p^*_{p,\mathcal{A}}(B)$ defined by (3) with the density function $\frac{dp^*_{\mathcal{A}}}{dp_{\mathcal{A}}}$ in (4) has the form:

$$\begin{aligned}
p^*_{p,\mathcal{A}}(B) &\doteq \int \left[\chi_B \sum_{i\in\mathbb{N}} \frac{p^*_{\mathcal{A}}(A_i)}{p_{\mathcal{A}}(A_i)} \chi_{A_i}\right] dp \\
&= \sum_{i\in\mathbb{N}} \frac{p^*_{\mathcal{A}}(A_i)}{p_{\mathcal{A}}(A_i)} \int \chi_B \chi_{A_i} dp \\
&= \sum_{i\in\mathbb{N}} \frac{p^*_{\mathcal{A}}(A_i)}{p_{\mathcal{A}}(A_i)} p(B\cap A_i) \\
&= \sum_{i\in\mathbb{N}} \frac{p(B\cap A_i)}{p(A_i)} p^*_{\mathcal{A}}(A_i) \\
&= \sum_{i\in\mathbb{N}} p(B|A_i) p^*_{\mathcal{A}}(A_i) \qquad (5)
\end{aligned}$$

The formula (5) is known in the philosophical literature as "Jeffrey conditioning" Jeffrey (1983), Jeffrey (1992) (the terminology "probability kinematics" also is used to refer to Jeffrey conditioning, see Diaconis and Zabell (1982)). If X in the probability space $(X, \mathcal{S}, p^*)$ has a finite number of elements, then $p^*_{p,\mathcal{A}}(B)$ is always of the form (5). From now on we assume that the set of elementary events is finite, having n elements: X_n. See Remark 3 on the case of infinite probability spaces.

Having inferred $p^*_{p,\mathcal{A}}(B)$ from the values of p^* on the elements of $\mathcal{A}$ as evidence, the Agent can use $p^*_{p,\mathcal{A}}(B)$ to calculate the expectation values $\langle f_i \rangle_{p^*_{p,\mathcal{A}}}$ of the utility functions f_i and can base the decision on the relation of the expectation values $\langle f_i \rangle_{p^*_{p,\mathcal{A}}}$. One potential difficulty such a decision making has to face is that the inferred probability $p^*_{p,\mathcal{A}}$ *might* not be equal to the objective probability: $p^* \neq p^*_{p,\mathcal{A}}$. Thus it *might* happen that $\langle f_i \rangle_{p^*_{p,\mathcal{A}}} \neq \langle f_i \rangle_{p^*}$. Consequently, it is not obvious that the decision between f_i and f_j based on considering the expectation values of f_i and f_j calculated using the inferred probability $p^*_{p,\mathcal{A}}$ is objectively correct in the sense that it coincides with the decision based on considering the expectation values of f_i and f_j calculated using the objective probability p^*. That is to say, it is not obvious that

$$\langle f_i \rangle_{p^*} > \langle f_j \rangle_{p^*} \quad \text{entails} \quad \langle f_i \rangle_{p^*_{p,\mathcal{A}}} > \langle f_j \rangle_{p^*_{p,\mathcal{A}}} \tag{6}$$

The next section formulates the possibility of violation of the entailment in (6) in terms of two definitions, restricted to the case of two utility functions.

3 Inaccessible decisions – definition

Given a probability space $(X_n, \mathcal{S}, p^*)$ and two random variables $f_1, f_2 \colon X_n \to \mathbb{R}$, we call

$$\big\langle (X_n, \mathcal{S}, p^*), f_1, f_2 \big\rangle \tag{7}$$

a *decision context* and the inequality $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ a *decision*. A Boolean subalgebra $\mathcal{A}$ of $\mathcal{S}$ is called *proper and non-trivial* if $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{A} \neq \{\emptyset, X_n\}$.

Definition 1. Given a decision context $\big\langle (X_n, \mathcal{S}, p^*), f_1, f_2 \big\rangle$, we say that the decision

$$\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*} \tag{8}$$

is $(p, \mathcal{A})$-*inaccessible* if

- $\mathcal{A}$ is a proper, non-trivial Boolean subalgebra of $\mathcal{S}$;
- p is a probability measure on $\mathcal{S}$ (prior) such that the restriction $p^*_{\mathcal{A}}$ of p^* to $\mathcal{A}$ is absolutely continuous with respect to the restriction $p_{\mathcal{A}}$ of p to $\mathcal{A}$;
- and we have:
$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}}} \leq \langle f_2 \rangle_{p^*_{p,\mathcal{A}}} \tag{9}$$
where $p^*_{p,\mathcal{A}}$ is the $(p, \mathcal{A})$-conditional probability measure defined by (3).

We call the decision $(p, \mathcal{A})$-*accessible* if it is *not* $(p, \mathcal{A})$-*inaccessible*.

The content of $(p, \mathcal{A})$-inaccessibility is that the Agent cannot reach the right decision determined by the objective probability if the information available for the Agent are the values of the objective probability on the subalgebra $\mathcal{A}$, and the Agent calculates the expectation values of utility functions using conditional probabilities obtained from the partial information available and using prior p: either the Agent cannot make a decision between f_1 and f_2 (if there is equality in (9)), or the decision made this way by the Agent is objectively wrong (when there is strict inequality in (9)).

A natural strengthening of Definition 1 is:

Definition 2. Given a decision context $\big\langle (X_n, \mathcal{S}, p^*), f_1, f_2 \big\rangle$, we say that the decision $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ is p-*inaccessible* if it is $(p, \mathcal{A})$-*inaccessible* (in the sense of Definition 1) for *all* (proper, non-trivial) subalgebras $\mathcal{A}$ of $\mathcal{S}$.

The content of Definition 2 is that the prior p chosen by the Agent makes it impossible to reach the objectively good decision in the following sense: if the Agent does not know the full objective probability and calculates the expectation values of the utilities using probabilities inferred via conditionalizing his prior p on partial information about the objective probability, then either the Agent cannot make a decision between f_1 and f_2 , or the decision the Agent makes will be objectively wrong – *no matter what partial information the Agent has about the objective probability.*

Given the definition of p-inaccessibility of decisions, one is led to

Problem 1. Under what conditions on the decision context $\langle (X_n, \mathcal{S}, p^*), f_1, f_2 \rangle$, and for which p does it happen that the decision $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ is p-inaccessible?

Clearly, p-inaccessibility of the decision $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ is a much stronger property than $(p, \mathcal{A})$-inaccessibility for some $\mathcal{A}$, and p-inaccessibility can only occur if p^* belongs to the set of probability measures that form what is called the "Bayes Blind Spot" of p: The Bayes Blind Spot of p is the set of probability measures on $\mathcal{S}$ that are absolutely continuous with respect to p, yet they cannot be obtained as conditional probabilities from incomplete evidence using p as prior Gyenis and Rédei (2017), Rédei and Gyenis (2021). That is to say, probability measure p^* is in the p-Bayes Blind Spot if it is absolutely continuous with respect to p but cannot be written in the form of (3) for any *proper, non-trivial* subalgebra $\mathcal{A}$.

It is known that the p-Bayes Blind Spot is not empty for a lot of p in typical probability spaces; in fact the p-Bayes Blind Spot is known to be a very large set for *every* p in *all* finite probability spaces: the Bayes Blind Spot

> "... has the same cardinality as the set of all probability measures (continuum); it has the same measure as the measure of the set of all probability measures (in the natural measure on the set of all probability measures); and is a 'fat' (second Baire category) set in topological sense in the set of all probability measures taken with its natural topology." Rédei and Gyenis (2021)[p. 3801].

The p-Bayes Blind Spot is known to be large also in probability spaces where X is countably generated Shattuck and Wagner (2024).

Thus, a necessary condition for the existence of p-inaccessible decisions does hold in typical probability spaces. But this necessary condition is not sufficient: $p^* \neq p^*_{p,\mathcal{A}}$ does not entail that the decision $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ is p-inaccessible: One can have a situation in which p^* is in the p-Bayes Blind Spot but the decision is $(p, \mathcal{A})$-accessible for *some* $\mathcal{A}$ and not $(p, \mathcal{A}')$-accessible for some other $\mathcal{A}' \neq \mathcal{A}$ (see Remark 2). It can even happen that p^* is in the p-Bayes Blind Spot but the decision is $(p, \mathcal{A})$-accessible for *all* $\mathcal{A}$ (see the Example 4.4). That p^* lies in the p Bayes Blind Spot is not sufficient for p-inaccessibility of a decision is not surprizing because the p-inaccessibility depends sensitively not only on p^* and on the prior p but also on the utility functions f_1, f_2. For this reason, finding a compact and useful general sufficient condition for p- and $(p, \mathcal{A})$-inaccessibility seems a difficult problem. At any rate we are not able to give such a condition.

Given lack of a general sufficient condition for p- and $(p, \mathcal{A})$-inaccessibility, it is not even obvious that decision contexts displaying $(p, \mathcal{A})$- (and especially) p-inaccessibility exist. In the next section we give examples of $(p, \mathcal{A})$- and p-accessibility in probability spaces having three and four elementary events.

4 Inaccessible decisions – examples in finite probability spaces

4.1 Example of a $(p, \mathcal{A})$-inaccessible decision in a probability space having three elementary events

Consider the decision context described by Table 1 (with p^* as objective probability and p as prior probability):

X_3	x_1	x_2	x_3
p^*	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
f_1	-2	-2	9
f_2	-2	8	-2
p	$\frac{3}{40}$	$\frac{32}{40}$	$\frac{5}{40}$

Table 1
$(p, \mathcal{A})$-inaccessibility, $n = 3$

We have

$$\langle f_1 \rangle_{p^*} = 2\frac{1}{8} \tag{10}$$

$$\langle f_2 \rangle_{p^*} = \frac{1}{2} \tag{11}$$

So the objectively good decision is

$$\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*} \tag{12}$$

In the case of a three-element set X_3 there are three non-trivial proper sub-Boolean algebras of the power set of X_3, they are generated by three partitions:

$$\mathcal{A}_1 \text{ generated by } \{\{x_1\}, \{x_2, x_3\}\} \tag{13}$$

$$\mathcal{A}_2 \text{ generated by } \{\{x_2\}, \{x_1, x_3\}\} \tag{14}$$

$$\mathcal{A}_3 \text{ generated by } \{\{x_3\}, \{x_1, x_2\}\} \tag{15}$$

The Radon-Nikodym derivatives $\frac{dp^*}{dp_{\mathcal{A}_i}}$ $(i = 1, 2, 3)$ (see Remark 1, especially eq. (5)) are given by Table 2:

So one can calculate the expectation values of f_1, f_2 with respect to the inferred probabilities $p^*_{p,\mathcal{A}_i}$ $(i = 1, 2, 3)$:

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_i}} = \sum_{j=1}^{3} f_1(x_j) \frac{dp^*}{dp_{\mathcal{A}_i}}(x_j) p(x_j) \quad i = 1, 2, 3 \tag{16}$$

X_3	x_1	x_2	x_3
$\frac{dp^*}{dp_{\mathcal{A}_1}}$	5	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{dp^*}{dp_{\mathcal{A}_2}}$	$3\ \frac{3}{4}$	$\frac{1}{3}$	$3\ \frac{3}{4}$
$\frac{dp^*}{dp_{\mathcal{A}_3}}$	$\frac{5}{7}$	$\frac{5}{7}$	3

Table 2 Radon-Nikodym derivatives

$$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_i}} = \sum_{j=1}^{3} f_2(x_j) \frac{dp^*}{dp_{\mathcal{A}_i}}(x_j) p(x_j) \quad i = 1, 2, 3 \tag{17}$$

The results are given in Table 3.

$\mathcal{A}_1$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} = -1$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} = 3\frac{2}{5}$
$\mathcal{A}_2$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} = 3\frac{1}{6}$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} = -12\frac{1}{2}$
$\mathcal{A}_3$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} = 2\frac{1}{8}$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} = 3\frac{5}{7}$

Table 3 Expectation values of utility functions

Thus we have the following ordering of the expected utilities calculated using the inferred probabilities:

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} \tag{18}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} > \langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} \tag{19}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} \tag{20}$$

The inequality (18) means that the decision (12) is $(p, \mathcal{A}_1)$-inaccessible; the inequality (20) means that the decision (12) is $(p, \mathcal{A}_3)$-inaccessible, and the inequality (19) means that the decision (12) is $(p, \mathcal{A}_2)$-accessible.

Remark 2. The decision (12) is not p-inaccessible because the decision is *not* $(p, \mathcal{A}_2)$-inaccessible: the decision between f_1 and f_2 based on the relation of their expectation values expressed by inequality (19) is the same as the decision (12) based on p^*. Yet, the objective probability p^* lies in the Bayes Blind Spot of p: It is known (Proposition 3.1 in Rédei and Gyenis (2021)) that p^* is in the p-Bayes Blind Spot if and only if the Radon Nikodym derivative $\frac{dp^*}{dp}$ is an injective function. This holds for p^* and p in this example:

$$\frac{dp^*}{dp}(x_1) = \frac{p^*(\{x_1\})}{p(\{x_1\})} = 5 \tag{21}$$

$$\frac{dp^*}{dp}(x_2) = \frac{p^*(\{x_2\})}{p(\{x_2\})} = \frac{5}{16} \tag{22}$$

$$\frac{dp^*}{dp}(x_3) = \frac{p^*(\{x_3\})}{p(\{x_3\})} = 3 \tag{23}$$

4.2 Example of a p-inaccessible decision with the uniform p in a probability space having three elementary events

Consider the decision context described in Table 4.

X_3	x_1	x_2	x_3
p^*	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
f_1	1	-1	2
f_2	2	3	-2
p	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 4
p-inaccessible decision, $n = 3$, uniform prior

An explicit calculation carried out exactly along the steps outlined in the Example in section 4.1 yields:

$$\langle f_1 \rangle_{p^*} = 1 \tag{24}$$

$$\langle f_2 \rangle_{p^*} = \frac{5}{6} \tag{25}$$

So the objectively good decision is

$$\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*} \tag{26}$$

But we have the expectation values of utility functions given by Table 5.

$\mathcal{A}_1$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} = \frac{3}{4}$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} = \frac{11}{4}$
$\mathcal{A}_2$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} = 1$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} = \frac{11}{2}$
$\mathcal{A}_3$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} = \frac{2}{3}$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} = 1$

Table 5 Expectation values of utility functions

Table 5 shows that he relation of the expectation values is:

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} \tag{27}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} \tag{28}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} \tag{29}$$

The inequalities (27)-(29) mean that the decision (26) is p-inaccessible.

To illustrate p-inaccessibility described in this section, imagine the following decision situation: A random generator produces the numbers $1, 2$ and 3 randomly according to the distribution p^* in Table 4: in an N-long series of the numbers $1, 2, 3$ produced by the generator, the probabilities give the relative frequencies of how many times the numbers $1, 2, 3$ come up in the N-long sequence. The Agent is offered the following two lotteries:

1. Lottery #1:

Pay \$ 5
and get
\$ 6 if 1 is the outcome, \$ 4 if 2 is the outcome, \$ 7 if 3 is the outcome.

2. Lottery #2:

Pay \$ 6
and get
\$ 8 if 1 is the outcome, \$ 9 if 2 is the outcome, \$ 4 if 3 is the outcome.

The Agent is then told that the probability that the generator generates 1 is equal to $\frac{1}{2}$, and that the probability that the generator generates either 2 or 3 is also equal to $\frac{1}{2}$. Then the Agent is asked which lottery he prefers. Assuming that the Agent wishes to maximize gain, the Agent first defines the gain function (utility function) for each of the two lotteries:

$$\text{Lottery \#1} \qquad \begin{aligned} f_1(1) &= \$\ 6 - \$\ 5 = \$\ 1 & (30)\\ f_1(2) &= \$\ 4 - \$\ 5 = -\$\ 1 & (31)\\ f_1(3) &= \$\ 7 - \$\ 5 = \$\ 2 & (32) \end{aligned}$$

$$\text{Lottery \#2} \qquad \begin{aligned} f_2(1) &= \$\ 8 - \$\ 6 = \$\ 2 & (33)\\ f_2(2) &= \$\ 9 - \$\ 6 = \$\ 3 & (34)\\ f_2(3) &= \$\ 4 - \$\ 6 = -\$\ 2 & (35) \end{aligned}$$

These functions are exactly the ones in Table 4. Then the Agent wishes to find out what the expectation values of these two utility functions are. Since the Agent does not know the objective probability p^* in Table 4, only the restriction of p^* to $\mathcal{A}_1$, the values of p^* must be inferred from this information. If the Agent chooses the uniform probability p in Table 4 (unbiased prior), and infers the values of p^* via conditioning, then the values will be given by the conditional probability measure $p^*_{p,\mathcal{A}_1}$. Calculating the expectation values of the gain functions f_1, f_2 with respect to $p^*_{p,\mathcal{A}_1}$, the Agent obtains the values in "row $\mathcal{A}_1$" of Table 4, and concludes that, because of the relation (27), Lottery #2 is the preferred one. But, by (26), Lottery #1 has objectively higher expectation value. The same reasoning holds if the information given to the Agent is the value of the objective probabilities on $\mathcal{A}_2$ or on $\mathcal{A}_3$. The Agent, with the unbiased prior, cannot make an objectively good decision about the two Lotteries on the basis of partial information and conditioning.

One might think that the p-inaccessibility of the decision (26) with respect to the uniform prior is due to the very special nature of this prior and that with a non-uniform prior there might not exist p-inaccessible decisions. The next example shows that this intuition is wrong: One can have p-inaccessible decisions with respect to a non-uniform prior p.

4.3 Example of a p-inaccessible decision with a non-uniform prior p in a probability space having three elementary events

Consider the decision context described by Table 6.

X_3	x_1	x_2	x_3
p^*	0.31	0.34	0.35
f_1	-1	-1	1
f_2	-1	1	-1
p	0.05	0.9	0.05

Table 6 p-inaccessible decision, $n = 3$, non-uniform prior

An explicit calculation carried out exactly along the steps outlined in the Example in section 4.1 yields:

$$\langle f_1 \rangle_{p^*} = -0.3 \tag{36}$$

$$\langle f_2 \rangle_{p^*} = -0.32 \tag{37}$$

So the objectively good decision is

$$\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*} \tag{38}$$

But we have the expectation values described by Table 7.

$\mathcal{A}_1$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} = -0.92737$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} = 0.307368$
$\mathcal{A}_2$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} = -0.34$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} = -0.32$
$\mathcal{A}_3$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} = -0.3$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} = 0.231579$

Table 7 Expectation values of utility functions

Table 7 shows that we have the following relations:

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} \tag{39}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} \tag{40}$$
$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} < \langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} \tag{41}$$

The inequalities (39)-(41) show that the decision (38) is p-inaccessible.

4.4 Example of a decision in a probability space having three elementary events that are $(p, \mathcal{A})$-accessible for all (proper, non-trivial) $\mathcal{A}$; yet the objective probability is in the p-Bayes Blind Spot

Consider the decision context described by Table 8.

X_3	x_1	x_2	x_3
p^*	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
f_1	-1	-1	1
f_2	-1	1	-1
p	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 8 Accessible decision with objective probability in Blind Spot

We have

$$\langle f_1 \rangle_{p^*} = 6\frac{5}{6} \tag{42}$$
$$\langle f_2 \rangle_{p^*} = -1 \tag{43}$$

So the objectively good decision is

$$\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*} \tag{44}$$

And we have the expectation values described by Table 9.

$\mathcal{A}_1$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} = 3\frac{1}{4}$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} = 0$
$\mathcal{A}_2$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} = 10$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} = -3\frac{1}{2}$
$\mathcal{A}_3$	$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} = 6$	$\langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} = \frac{2}{3}$

Table 9 Expectation values of utility functions

Table 9 shows:

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_1}} > \langle f_2 \rangle_{p^*_{p,\mathcal{A}_1}} \tag{45}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_2}} > \langle f_2 \rangle_{p^*_{p,\mathcal{A}_2}} \tag{46}$$

$$\langle f_1 \rangle_{p^*_{p,\mathcal{A}_3}} > \langle f_2 \rangle_{p^*_{p,\mathcal{A}_3}} \tag{47}$$

The inequalities (45)-(47) show that the decision (44) is $(p, \mathcal{A}_i)$-accessible for all $i = 1, 2, 3$. But p^* is in the p-Bayes Blind Spot because the Radon-Nikodym derivative of p^* with respect to p is an injective function.

4.5 Example of a p-inaccessible decision in a probability space having four elementary events

All the above examples are in a probability space having *three* elementary events. In such probability spaces there are *three* non-trivial partitions defining *three* non-trivial proper sub-Boolean algebras. The utility functions have a domain that also has *three* elements. One might think that the existence of p-inaccessible decisions in such spaces might thus be due to the peculiar circumstance that there are exactly as many elementary events as non-trivial subalgebras. But this is not so: Here we give an example of a p-inaccessible decision in a probability space having four elementary events.

When the number of elementary events increases, the number of non-trivial subalgebras is growing exponentially: the number of non-trivial proper subalgebras is equal to the number of all non-trivial partitions, which is the number of all partitions minus 2, since the finest partition does not define a proper subalgebra and the trivial partition defines a trivial subalgebra. The number of all partitions of a finite set having n elements is called the "n-th Bell number" $Bell(n)$ Conway and Guy (1996). In case of $n = 4$, the Bell number is $Bell(4) = 15$; hence in case of a probability space having four elementary events, the number of proper, non-trivial subalgebras is 13. Thus, checking whether a decision is p-inaccessible in this situation requires checking $(p, \mathcal{A})$-inaccessibility with respect to 13 subalgebras $\mathcal{A}$. We will not present the detailed calculations here (they are exactly along the lines of the calculation in case of the example in section 4.1). We just claim that the decision context described in Table 10 represents a p-inaccessible decision when the probability space has 4 elementary events:

X_4	x_1	x_2	x_3	x_4
p^*	0.05	0.08	0.86	0.01
f_1	-0.99	-1	0.1	1.0001
f_2	6	1.98	-1	36.58
p	0.25	0.25	0.25	0.25

Table 10 p-inaccessible decision, $n = 4$

5 Degree of inaccessibility

In the example in section 4.1 displaying a $(p, \mathcal{A})$-inaccessible decision, the decision is $(p, \mathcal{A})$-inaccessible for two (of the altogether three) subalgebras – the decision in this example is "one-subalgebra-close" to being p-inaccessible. The decision in the example in section 4.4 showing that a decision can be $(p, \mathcal{A})$-accessible for all $\mathcal{A}$ (and p^* still be in the p-Bayes Blind Spot) is "three-subalgebra-close" to being p-inaccessible. Numerical calculations also show that the following is true in the situation when the set of elementary events is 4: Given any number $k \in \{0, 1, 2, \ldots 12\}$, one can change the value of $f_1(x_3)$ in Table 10 to obtain decisions that are $(p, \mathcal{A})$-inaccessible for k number of non-trivial subalgebras $\mathcal{A}$. Table 11 below shows these values of $f_1(x_3)$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(x_3)$	39	37	36	12	10	8	6	5	4	2	1.6	0.8	0.3

Table 11 Values of $f(x_3)$ to obtain $(p, \mathcal{A})$-inaccessible decisions for k number of subalgebras $\mathcal{A}$ in the decision context described in Table 10

These observations motivate the following

Definition 3. Let $\langle (X_n, \mathcal{S}, p^*), f_1, f_2 \rangle$ be a decision context and let p be a prior. We call the number of Boolean subalgebras $\mathcal{A}$ for which the decision $\langle f_1 \rangle_{p^*} > \langle f_2 \rangle_{p^*}$ is $(p, \mathcal{A})$-inaccessible *the degree of p-inaccessibility* of the decision.

The degree of inaccessibility of a decision is a measure of how suitable a prior is in connection with a decision situation in the case when no complete information about the objective probability is available. If the degree of p-inaccessibility is maximal (in this case the degree is equal to the Bell number minus 2), then this is the situation of p-inaccessibility. If the degree is zero, then the prior suits the decision situation well: one can obtain the objectively good decision on the basis of partial information about the objective probability – no matter what the partial information is. In the intermediate cases, the prior is the more suitable the lower the degree of p-inaccessibility.

Having the notion of degree of p-inaccessibility, one can ask several questions about it:

- Given a decision context, what are the properties of the set of probability measures p having a *fixed* degree of p-inaccessibility?
- Given a decision context and a fixed number k, does there exist a prior for which the decision is p-accessible to degree k?
- Are there some compact sufficient conditions that entail p-inaccessibility to degree k?

We do not have answers to the above questions. But on the basis of the examples of $(p, \mathcal{A})$- and p-inaccessibility presented in this paper we make a general conjecture:

Conjecture 1. In any finite probability space having $n \geq 3$ number of elementary events there exist decisions and for any k priors p_k such that the decisions are p_k-inaccessible to degree k, where $0 \leq k \leq Bell(n) - 2$.

The truth of this conjecture entails that the following weaker conjecture is true, yet we formulate it explicitly:

Conjecture 2. For any decision context (with a finite probability space) there exist priors p such that the decision is p-inaccessible.

Remark 3. The notions of $(p, \mathcal{A})$- and p-inaccessibility are meaningful also in probability spaces with an infinite number of elementary events. The examples presented in section 4. can be "embedded" into an infinite probability space in a natural way to see that $(p, \mathcal{A})$-inaccessible decisions exist in infinite probability spaces as well. But p-inaccessible decisions do not exist in infinite probability spaces in general: if the σ-field $\mathcal{S}$ is such that there exists a filtration $\mathcal{A}_i$ ($i \in \mathbb{N}$) in $\mathcal{S}$ that generates $\mathcal{S}$ (Billingsley (1995)[p. 458]), then martingale convergence theorems (Billingsley (1995) Theorems 35.6 and 35.7) entail that the conditional probabilities $p^*_{p,\mathcal{A}_i}$ tend to p^* as $i \to \infty$; thus, for all large enough i, the expectation values $\langle f_j \rangle_{p^*_{p,\mathcal{A}_i}}$ and $\langle f_j \rangle_{p^*}$ ($j = 1, 2$) will be close enough to yield the objectively correct order of the expectation values of the utility functions f_j ($j = 1, 2$). In such situations the concept of the degree of p-inaccessibility should be modified appropriately in order for it to express the degree of suitability of a prior for making the right decision – we leave this problem for further study.

6 Closing Comments

The possibility of decision situations with p- and $(p, \mathcal{A})$-inaccessible decisions is not to be interpreted as an argument against decision making based on utility maximization. Rather, they should be viewed as caution about how to treat epistemic uncertainty about objective probabilities in such decision contexts.

There are in principle two sorts of epistemic uncertainties about the objective probability: One can know the probabilities of *all* events with *some imprecision* (call this "*Type I uncertainty*"); or one can know with *precision* the probabilities of *some* events but not of all (call this "*Type II uncertainty*").

Reducing the *Type I uncertainty* by learning more about the probability of every event is clearly a very safe strategy to produce good decisions based on utility maximization: as one gets closer and closer to the objective probability p^*, the expectation values of utility functions calculated using the more and more objectively correct probability tend to the expectation values calculated using p^*. At some point the calculated expectation values of the utilities can be so close to the one calculated using the objective one that the correct order of the expectation values is obtained.

It is the *Type II*, epistemic uncertainty that one can try to mitigate by inferring the unknown objective probabilities from the known ones via conditioning. The phenomenon of p- and $(p, \mathcal{A})$-inaccessible decisions shows that dealing with *Type II uncertainty* by inferring unknown probabilities via conditionalization is very risky: it can result in probabilities that lead to objectively wrong decisions. This risk does not seem to be eliminable: in order to avoid choosing a prior for which a decision is p-inaccessible one would need a condition that tests p-inaccessibility of a decision, and such a condition needs to involve the precise values of the objective probability – which is precisely what is not known.

Thus in decision theory based on utility maximization in the context of finite probability spaces it seems better to know something about everything than knowing everything about something and inferring what one does not know via conditioning.

The reason for this (which is the cause of p-inaccessibility) is that the inference via conditioning is a content-increasing inference, not a deductive one (the logic of this kind of inference is not even finitely axiomatizable Brown et al (2019)). The material, objective correctness of the inductively inferred probabilities is contingent on the chosen prior. The existence of decision contexts with p- and $(p, \mathcal{A})$-inaccessible decisions thus displays another aspect of the crucial role of the priors in Bayesian taming of epistemic uncertainties.

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