**Identification of numbers with operators to construct cardinals**

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**Abstract**

This article describes confirmation of two propositions: (1) a set of operators to construct cardinals satisfies Peano Axioms, hence the set is identified with the natural numbers, and (2) these operators can be extended to form three kinds of operators that are identified with the integers, the fractions, and the complex numbers with fractions as their coefficients, respectively. These four kinds of operators stand in a sequential inclusion relationship, in contrast to the embedding relationship between numbers that are identified with sets.

Keywords: numbers; operators; cardinals; structures of sets; iteration; activation.

1. **Introduction**

Our intuitive conception of a natural number would be the number of elements of a finite set, such as the number of the people in a room. Then, the number is a common property to the sets with the same number of elements, where the same number is defined using a bijection between the sets. The set of the sets related by a bijection is a cardinal as a property of the sets. Thus, the number is defined as the cardinal of the sets. This definition of numbers is given by Russell (Russell 1919, Section 2, pp. 9-15).

There is another viewpoint on cardinals. Since the sets with the same number of elements are defined using a bijection between them, it is the disjunction of the elements in each set that defines the set. For example. for a bijection between the set {The sun, Socrates} and the set {3, USA}, these sets are treated as the connection of the elements, ‘The sun or Socrates’ and ‘3 or USA’ respectively. Then, these sets have a common structure that keeps constant regardless of the variation of their elements. This structure is obtained by replacing the elements of the sets with variables, e. g. ‘*x* or *y*’ or {*x*, *y*} in the above example. Such a connection of variables is a common structure of those sets related by a bijection, and is a function from isolated elements to a set. Then, there is a constraint on substitution of elements for the variables: different variables must be substituted with different elements. For example, a classroom with ten chairs (ten variables) is a structure of classes, where any group of ten students (ten elements) sit on the chairs to form a class; every student can sit on any chair but on only one chair and different students must sit on different chairs. This kind of set structure can also be cardinals. I adopt this conception of cardinals in this article. The set structures as functions are denoted by *{x, y}*, *{x, y, z},* etc.

To define the natural numbers, Russell constructs the natural numbers from the null set and a successor function. The successor function is the operator to add a new element to each of the sets with a cardinal, say *n*, to construct the sets with the cardinal *n+1*. Then, the set of cardinals satisfies Peano Axioms (Russell, 1919, Section 3, pp. 16-22). Thus, the natural numbers can be defined as the set of the cardinals thus related by the successor function with the set whose only member is the null set as the beginning of the relation.

For cardinals as set structures, the successor function is the operator to add a new variable to a set structure with, say *n* variables, to construct the set structure with *n+1* variables. As the beginning of the construction of cardinals, let the set structure with no variable introduce into the set of set structures, which defines the cardinal 0. This is denoted by *{}* or *φ*. The successor function ‘addition of a new variable *x* to a set structure *η*‘ is denoted by *P(η)*. Thus, for example, P(*φ*)={*x1*}, P(P(*φ*))=P({*x1*})={*x1*, *x2*}. The set of all set structures constructed in this way is supposed to satisfy Peano Axioms. If this is the case, the natural numbers can also be defined by the set of the set structures thus constructed.

Corresponding to the set of these set structures, the operators to construct them, which are iterations of P operating on *φ*, form a set. If the set of the set structures satisfies Peano Axioms, this set of operators should also satisfies Peano Axioms. Then, the set of the operators can be identified with the natural numbers. If this is the case, we can further suppose that certain extension of the operators may exist that satisfy the conditions for them to be identified with the integers, the fractions, or others. These anticipations are examined in the following sections.

**2. Natural numbers**

The set of all the set structures constructed from iterations of P on *φ*isdenoted by *[N]*. For this construction, it is postulated that whenever P operates on a set structure, a new variable not in the set structure exists, and substitution of elements for the variables is constrained in the manner as stated earlier. Then, it can be shown that [N] with P as a successor function satisfies following Peano Axioms:

1. *0*∈N ;

2. For any *n*∈N, there is only one *n’*∈N ;

3. For any *n*, *m*∈N, if *n*≠*m*, then, *n’*≠*m’* ;

4. There is no *n*∈N such that *n’*=*0* ;

5. If S⊆N such that *0*∈S and for all *n*∈S, *n’*∈S, then S=N,

Where ‘ is a successor function,

The proof is as follows.

1. *φ*∈[N], by the definition of *φ*.

2. There is only one P(*η*)∈[N] for any *η*∈[N]. This is obvious by the definition of [N].

3. Let *α*, *β*∈[N] and *α*≠*β*, then, P(*α*)={variables of *α* and *x*}, where *x*∉*α*, and P(*β*)={variables of *β* and *x’*}, where *x’*∉*β*. Since there is no bijection between *α* and *β*, so between P(*α*) and P(*β*). Thus, P(*α*)≠P(*β*).

4. There is no such *η* that P(*η*)=*φ*. This is obvious by the definition of *φ*.

5. Let S⊆[N], *φ*∈S, and for any *η*∈S, P(*η*)∈S. If S≠[N] there exists *χ*∈[N] and *χ*∉S. Let the first element with respect to P within *χ*∉S be *β*. Then there exists *α*∈S such that P(*α*)=*β*, which is followed by P(*α*)∈S. Hence, there is no such *β*.

Thus, N is identical with [N].

P(P(P…(*η*)…)) is denoted by *P\*P\*P\*…(η)*. Iteration of P, P\*P\*P\*…P, is called a *connection of Ps*, and the iteration times of P in the connection is called *it-times*, and P is called *it-unit* of the connection.

[N] is constructed by operating connections of P on *φ* without limit. Then, [N] do not include the set structure with infinitely many elements, which is the limit that the construction of operators cannot reach.

Since construction of [N] follows corresponding construction of operators, i.e. connections of Ps, these operators also form a set. This set plus the operator that adds no variable to a set structure, denoted by *P0*, is denoted by *[No]*. Then, it is obvious that [No] also satisfies Peano Axioms with P\* as the successor function.

1. Po∈[No] ;

2. If *a*∈[No], then P\**a*∈[No] ;

3. If *a*≠*b* for *a*, *b*∈[No], then, P\**a*≠P\**b* ;

4. There is no *b*∈[No] such that P\**b*=Po ;

5. If S⊆[No], Po∈S and for all *η*∈S, P\**η*∈S, then S=[N0].

The definition of addition between elements of [N] follows addition between elements of [No]. It would be natural that the latter addition is connection of the elements with \* to form a connection of Ps.

• Definition of addition on [No].

For *a*, *b*∈[No], *a*+*b*=*a*\**b*.

This addition satisfies Peano Axioms for addition:

1. ∀*x*(*x*+*0*=*x*) ;

2. ∀*x*∀*y*(*x*+*y’*=(*x*+*y*)’).

By the definition of addition stated above,

1. For any *x*∈[No], *x*+Po= *x*\*Po=*x* ;

2. For any *x*, *y*∈[No], *x*+P\**y*=*x*\*(P\**y*)=P\**x*\**y*=P\*(*x*\**y*), because *x* is a connection of P.

Thus, this addition satisfies those axioms.

Since every element of [No] is a simple iteration of Ps by \*, there is no order of Ps in the connection. That is, addition of elements of [No] is irrelevant to the order of connection of Ps in the elements. Therefore, commutative law and associative law hold for this addition. Then, Po is the additive identity for this addition.

An operator *a*, an iteration of P it-times in *a,* is denoted as *ΣaP*. Then,

*a+b=a*\**b*=Σ*a*P\*Σ*b*P=Σ*a*+*b*P.

This is the connection of Ps it-times of P in *a* plus those in *b*.

Addition between elements of [N] follows from addition in [No].

For *α*, *β*∈[N], there are *a*, *b*∈[No] such that

*α*=*a*(*φ*) and *β*=*b*(*φ*).

Then, addition of *α* and *β* is defined as

*α*+*β*=*a*\**b*(*φ*).

Naturally, Peano Axioms on addition hold for [N] with this addition. Associative law and commutative law also hold for [N], where *φ* is the additive identity.

• The conception of multiplication *a*×*b*, where *a*, *b*∈[No], is iteration of *a* it-times of P in *b*, i.e. *a* is the it-unit of *a*×*b*. Then, multiplication on [No] is defined as

*a*×*b*=Σ*ba*.

Then, Peano Axioms on multiplication

1. ∀*x*(*x*×*0*)=*0* ;

2. ∀*x*∀*y*(*x*×*y*’=*xy*+*x*) ;

are satisfied by this definition in the following way.

1. For any *a*∈[No], *a*×Po=ΣPo*a*=Po ;

2. *a*×(P+*b*)=ΣP\**ba* = ΣP*a*\*Σ*ba*=*a*\*Σ*ba*=Σ*ba*\**a*=*a*×*b*+*a*.

Since

*a*×P=ΣP*a*=*a*,

P is the unit element for this multiplication.

Associative law and commutative law of this multiplication and distributive law for [No] can be proved by simply comparing the it-times of P in the right side and the left side of each of the following equations.

1. *a*×*b*=*b*×*a* ;

2. *a*×(*b*×*c*)=(*a*×*b*)×*c* ;

3. *a*×(*b*+*c*)=(*a*×*b*)+(*a*×*c*).

I omit the proofs of these equations.

As a rule, multiplication precedes addition in calculation.

For *α*, *β*∈[N], let *α*=*a*(*φ*) and *β*=*b*(*φ*), where *a*, *b*∈[No]. Then, multiplication of *α* and *β* is defined as

*α*×*β*=(*a*×*b*)(*φ*).

Naturally this multiplication satisfies Peano Axioms on multiplication. Commutative law, associative law and distributive law also hold for [N], according to those for [No].

The order relation between the elements of [N] is also formed according to the construction of the set structures. Therefore, this order is a inclusion relation, i.e. {}⊂{*x1*}⊂{*x1*, *x2*}⊂{*x1*, *x2*, *x3*}…. The elements of [N0] is also ordered by their it-times of P, which corresponds to set structures.

**3. Integers**

There is an operator that operates on P to reverse the direction of its operation, that is, from addition of a variable to a set structure to subtraction of a variable from a set structure. This operator is denoted by −. For *a*∈[No], −*a*(*η*) is the operator that subtract variables one by one the it-times of P in *a* from the set structure *η*. Since

*a*\*−*a*=P0,

−*a* is the inverse element of *a*. The operator that does not vary the direction of P is denoted by *+*: +*a*=*a*. The variable that range over the set {+, −} is denoted by *Δ*. Then the set of Δ*a*, for *a*∈[No], is an extension of [No], which is denoted by *[ΔNo]*.

• Addition on [ΔNo] is extension of that on [No].

For *a*, *b* ∈[ΔNo],

*a*+*b*=*a*\**b*.

The additive identity is P0: *a*+P0=*a*\*P0=*a*

The inverse element of *a*∈[ΔNo] is −*a*, as stated above.

An operator, a connection of it-unit operators ΔP, is irrelevant to its order of the connection. Hence, this addition satisfies associative law and commutative law. There are cases where the results of the operations by operators of [ΔNo] on set structures do not exist in [N], e.g. −P(*φ*). However, operators themselves can be constructed from operators that exist on the basis of existence of cases where the operators are effective, and indeed there exist the cases such as, −P({*x1*, *x2*})={*x*}; *x1*, *x2* and *x* are equivalent as a variable to construct the set structure {*x1*}, {*x2*} and {*x*}, which are the same set structures: the function from an element to a singleton. The differentiation of *x1* and *x2* in {*x1*, *x2*} is necessary to designate that this set structure consists of two different variables.

• Multiplication on [ΔNo] is also extension of that on [No].

For *a*, *b*∈[No],

Δ*a*×Δ*b*=ΣΔbΔ*a*.

Then, the following equations hold.

Δ*a*×(+P)= ΣpΔ*a*=Δ*a*

(i.e. The operator +P=P is the unit element for this multiplication.) ;

Δ*a*×(−P)=Σ−pΔ*a*=−Δ*a* ;

ΔP0=P0 ;

Δ*a*×P0=P0 ;

*a*×*b*=(+*a*)×(+*b*)=(−*a*)×(−*b*)=+(*a*×*b*) ;

(−*a*)×(+*b*)=(+*a*)×(−*b*)==−(*a*×*b*).

• The associative law of this multiplication is derived using mathematical induction. For *a*, *b*∈[ΔNo],

1. *a*×(*b*×ΔP)=*a*×Δ*b*=Δ(*a*×*b*)=(*a*×*b*)×ΔP … (1)

2. To the next step of the proof, the distributive law

*b*×(*c*+*d*)=(*b*×*c*)+(*b*×*d*) for *c*, *d*∈[ΔNo],

is necessary. Let *d*=ΔP, then,

*b*×(*c*+ΔP)=Σ*c*+ΔP*b*=(*b*×*c*)+(*b*×ΔP) … (2).

By the induction hypothesis,

*b*×(*c*+(*d*+ΔP))=*b*×((*c*+*d*)+ΔP)=*b*×(*c*+*d*)+Δ*b*=(*b*×*c*)+(*b*×*d*)+Δ*b*

=(*b*×*c*)+(*b*×(*d*+ΔP) … (3).

Thus, distributive law follows from the equations (2) and (3). Hence,

*a*×(*b*×(*c*+ΔP))=*a*×((*b*×*c*)+Δ*b*)=*a*×(*b*×*c*)+Δ(*a*×*b*) … (4).

On the other hand.

(*a*×*b*)×(*c*+ΔP)=(*a*×*b*)×*c*+Δ(*a*×*b*) … (5).

By the induction hypothesis, the formula (5) is equivalent with the formula (4). Hence, the associative law follows from the equations (1), (4), and (5); (*a*×*b*)×*c* can be denoted by *a*×*b*×*c*.

• Commutative law

a×b=b×a

is proved in the similar way as associative law.

1. *a*×ΔP=ΔP×*a* … (6).

2. *a*×(*b*+ΔP)=*a*×*b*+Δ*a* … (7).

3. To prove (*b*+ΔP)×*a*=*b*×*a*+Δ*a* … (8),

distributive law, (*a*+*b*)×*c*=*a*×*c*+*b*×*c*, is necessary in advance.

In the first place,

(*a*+*b*)×ΔP=*a*×ΔP+*b*×ΔP.

By the induction hypothesis,

(*a*+*b*)×(*c*+ΔP)=((*a*+*b*)×*c*)+Δ(*a*+*b*)=*a*×*c*+*b*×*c*+Δ*a*+Δ*b*.

On the other hand,

*a*×(*c*+ΔP)+*b*×(*c*+ΔP)=*a*×*c*+Δ*a*+*b*×*c*+Δ*b*. Thus,

(*a*+*b*)×(*c*+ΔP)=*a*×(*c*+ΔP)+*b*×(*c*+ΔP).

Accordingly, distributive law, hence, equation (8), holds. Therefore, commutative law follows from the equations (6), (7), and (8).

Distributive law has been proved in the course of the above two proofs.

The order relation in [No] can be extended to [ΔNo] by ordering −a, a∈ [No], as inverse order of a∈[No] and combining this order with that of [No].

Thus, the set of integers can be identified with [ΔNo], accordingly, [ΔNo] is also written as [Z].

**4. Fractions**

When *a*=*b*×*c* for *a*, c∈[Z], an operator to construct *b* (the it-unit for the it-times of P in *c*) from *a* and *c* can be defined as *a* divided equally it-times of P in *c*. This operator is denoted by *÷(a, c)* or *a÷c*. When P=*b*×*c*, P÷*c* is the inverse element of *c*, which is denoted by *Pc*: P divided equally it-times of P in *c*, so is the same kind of operator with P. Thus, P*c*is associative and commutative with P. Then,

P÷ P*c*=*c* ;

P÷P=Pp=P by definition ;

*a*÷*c*=(*a*×(P*c*×*c*))÷*c*=((a×P*c*)×*c*)÷*c*=Σc(*a*×Pc)÷*c*=*a*×Pc.

Except for the case *x*=P, the inverse element of *x*∈[Z], P*x*, is not a member of [Z]. The set of operators ÷(*x*, *y*) (=P*y*×*x*, for *x*, *y*∈[Z]) is written as [F]. It should be noted that ÷(*x*, Po) cannot be constructed. By the definition,

÷(*a*, P)=*a,* for *a*∈[Z].

That is, ÷(*a*, P) and *a* are the same operators. Therefore,

[Z]⊂[F].

• Since P*x* is the same kind of operator as P, addition of P*y*×*x* is defined by \*.

P*a*×*m*+P*b*×*n*=P*a*×*m*\*P*b*×*n*=P*a*×*b*×(*m*×*b*)\*P*b*×*a*×(*n*×*a*)=Σ*m*×*b*\**n*×*a* P*a*×*b*

= P*a*×*b*×(*m*×*b*+*n*×*a*), for *a*, *b*, *m*, *n*∈[Z].

For *x*∈[F], *x*+Po=*x*, and *x*−*x*=Po.

Thus, additive identity and the inverse element of *x* for the addition are Po and −*x* respectively.

• Associative law

P*a*×*m*+(P*b*×*n*+P*c*×*k*)=P*a*×*m*+P*b*×*c*×(*n*×*c*+*k*×*b*)

= P*a*×*b*×*c*×(*m*×*b*×*c*+*a*×(*n*×*c*+*k*×*b*))=P*a*×*b*×*c*×(*m*×*b*×*c*+*a*×*n*×*c*+*a*×*k*×*b*) … (1).

On the other hand,

(P*a*×*m*+P*b*×*n*)+P*c*×*k*=P*a*×*b*×(*m*×*b*+*n*×*a*)+P*c*×*k*=P*a*×*b*×*c*×((*m*×*b*+*n*×*a*)×*c*+*a*×*b*×*k*)

= P*a*×*b*×*c*×(*m*×*b*×*c*+*n*×*a*×*c*+*a*×*b*×*k*) … (2).

Associative law follows from equations (1) and (2).

• commutative law

BY the communicative law for [Z],

P*a*×*m*+P*b*×*n*=P*a*×*b*×(*m*×*b*+*n*×*a*)=P*b*×*a*×(*n*×*a*+*m*×*b*)=P*b*×*n*+P*a*×*m*.

• Multiplication on [F] is also an extension of that on [Z].

(P*a*×*m*)×(P*b*×*n*)=(P*a*×*b*×*m*×*b*)×(P*b*×*a*×*n*×*a*)=P*a*×*b*×P*a*×*b*×*m*×*b*×*n*×*a*=P*a*×*b*×(*m*×*n*)

Then,

(P*a*×*m*)×(P*m*×*a*)=P*a*×*m*×(*m*×*a*)=P, and

(P*a*×*m*)×P=P*a*×*m*.

Thus,

P*m*×*a* is the inverse element of P*a*×*m*, and P is the identity element for the multiplication.

• Associative law follows from the equations:

(P*a*×*m*)×((P*b*×*n*)×(P*c*×*k*))=(P*a*×*m*)×(P*b*×*c*×(*n*×*k*))=P*a*×*b*×*c*×(*m*×(*n*×*k*)).

((P*a*×*m*)×(P*b*×*n*))×(P*c*×*k*)=(P*a*×*b*×(*m*×*n*))×(P*c*×*k*)=P*a*×*b*×*c*×((*m*×*n*)×*k*).

• Commutative law

(P*a*×*m*)×(P*b*×*n*)=P*a*×*b*×(*m*×*n*)=P*b*×a×(*n*×*m*)=(P*b*×*n*)×(P*a*×*m*).

•Distributive law

(P*a*×*m*)×((P*b*×*n*)+(P*c*×*k*))=(P*a*×*m*)×(P*b*×*c*×(*n*×*c*+*k*×*b*))=P*a*×*b*×*c*×(*m*×(*n*×*c*+*k*×*b*))= P*a*×*b*×*c*×(*m*×*n*×*c*+*m*×*k*×*b*) … (3).

On the other hand,

(P*a*×*m*×P*b*×*n*)+(P*a*×*m*×P*c*×*k*)=(P*a*×*b*×(*m*×*n*))+(P*a*×*c*×(*m*×*k*))= P*a*×*b*×*a*×*c*×(*m*×*n*×*a*×*c*)+P*a*×*b*×a×*c*×(*m*×*k*×*a*×*b*)=P*a*×*b*×*c*×(*m*×*n*×*c*+*m*×*k*×*b*) … (4).

Distributive law follows from the equations (3) and (4).

• Addition of *m*∈[Z] and P*a*×*n*∈[F], for *n*∈[Z].

Since *m*=P*a*×*a*× P*P*×*m*=P*a*×(*a*×*m*),

*m*+P*a*×*n*=P*a*×(*a*×*m*)+Pa×*n*=P*a*×(a×*m*+*n*).

**5. complex numbers**

I postulate that the operator − has a structure with a capacity that causes its operation. That is, there exist stages to reach the activation of the operator − or to cause its operation, such as stages to charge an electrode up to atmospheric discharge. Three stages are set up: The stage of full satisfaction of the condition for activation of −, the stage of null satisfaction of the condition for the activation of −, which inhibits − from activating, the stage of half satisfaction of the condition for the activation, which reaches its activation when more half condition is satisfied. I try to construct complex numbers by extending − to − with these three stages. The operator − with a stage θ is denoted by *−θ*, where *θ* is the variable ranging over {0, 1, 1/2}: −1*a*=−*a* for *a*∈[F], −0*a*=*a*, −1/2(−1/2)*a*=(−1/2\*−1/2)*a*=−1*a*=−*a*. Thus, −1/2 is identical with imaginary unit.

Because *a* and −1/2*b* for *a*, *b*∈[F] are both certain stages of operators, which have a capacity or potential to operate on set structures, they can be connected with \* to form an operator that can operate on set structures,

*a*\*(−1/2)*b*= *a*+(−1/2)*b*.

Naturally, they are associative and commutative for \*.

The set of operators, {*x*+(−1/2)*y* | *x*, *y*∈[F]} is written as [CF].

• Addition on [CF] is defined naturally as

(*a*+(−1/2)*b*)+(*a’*+(−1/2)*b’*)=(*a*+(−1/2)*b*)\*(*a’*+(−1/2)*b’*)=(*a*+*a’*)+(−1/2)(*b*+*b’*), for *a*, *b*, *a’*, *b’*∈[F].

Since connection of operators does not depend on its order, associative law and commutative law hold for [CF].

(*a*+(−1/2)*b*)+(Po+(−1/2)Po)=*a*+(−1/2)*b*,

hence, Po+(−1/2)Po is the additive identity.

(*a*+(−1/2)*b*)+(−(*a*+(−1/2)*b*))=Po+(−1/2)Po,

hence, −(*a*+(−1/2)*b*) is the inverse element of *a*+(−1/2)*b*.

• Multiplication on [CF] is defined as extension of that on [F].

At first, by the definition of −1/2,

*a*×−1/2*b*=Σ−1/2*ba*=−1/2(*a*×*b*). Then,

(*a*+(−1/2)*b*)×(*a’*+(−1/2)*b’*)=Σ *a’*+(−1/2)*b’*(*a*+(−1/2)*b*)=

Σa’(*a*+(−1/2)*b*)+Σ−1/2b’(*a*+(−1/2)*b*)=

(*a*+(−1/2)*b*)×*a’*+(*a*+(−1/2)*b*)×(−1/2)*b’*=

(*a*×*a’*)+(−1/2)*b*×*a’* +(−1/2)*a*×*b’*+(−*b*×*b’*)=(*a*×*a’*)+(−*b*×*b’*)+( −1/2)(*b*×*a’*+*a*×*b’*).

Because of associative law and commutative law of multiplication on [F], they also hold for [CF].

Since

(a+(−1/2)b)×P=a+(−1/2)b,

P is the unit element for the multiplication.

• Distributive law

((*a*+(−1/2)*b*)+(*a’*+(−1/2)*b’*))×(*c*+(−1/2)*c’*)=((*a*+*a’*)+(−1/2)(*b*+*b’*))×(*c*+(−1/2)*c’*)=

Σc+(−1/2)c’((*a*+*a’*)+(−1/2)(*b*+*b’*))=

((*a*+*a’*)+ (−1/2)(*b*+*b’*))×*c*+((*a*+*a’*)+(−1/2)(*b*+*b’*))×(−1/2)*c’*=

(*a*+*a’*)×*c*+(−1/2)(*b*+*b’*)×*c*+(−1/2)((*a*+*a’*)×*c’*)+(−(*b*+*b’*)×*c’*)=

(*a*+*a’*)×*c*+(−(*b*+*b’*)×*c’*)+(−1/2)((*b*+*b’*)×*c*+(*a*+*a’*)×*c’*). … (1),

On the other hand,

((*a*+(−1/2)*b*)×(*c*+(−1/2)*c’*)+((*a’*+(−1/2)*b’*)×(*c*+(−1/2)*c’*))=

*a*×*c*+(−*b*×*c’*)+(−1/2)(*a*×*c’*+*b*×*c*)+*a’*×*c*+(−*b’*×*c’*)+(−1/2)(*a’*×*c’*+*b’*×*c)*=

*a*×*c*+(−*b*×*c’*)+*a’*×*c*+(−*b’*×*c’*)+(−1/2)(*a*×*c’*+*b*×*c*+*a’*×*c’*+*b’*×*c*)=

(*a*+*a’*)×*c*+−(*b*+*b’*)×*c’*+(−1/2)((*b*+*b’*)×*c*+(*a*+*a’*)×*c’*) … (2).

Thus, distributive law follows from the equations (1) and (2).

• Inverse element for the multiplication

Let (*a*+(−1/2)*b*)×*x*=P . Then,

*x*=P÷(*a*+(−1/2)*b*)=P*a*+(−1/2)*b*=((*a*+−(−1/2)*b*)÷(*a*+−(−1/2)*b*))×P*a*+(−1/2)*b*

=P*a*+−(−1/2)*b*×(*a*+−(−1/2)*b*)×P*a*+(−1/2)*b*=P(*a*+−(−1/2)*b*)×(a+(−1/2)*b*)×(*a*+−(−1/2)*b*)

=P*a×a*+*b×b*×(*a*+−(−1/2)*b*).

This is the inverse element of *a*+(−1/2)*b*.

As stated above, [CF] has a unit element, inverse elements, and satisfies associative law, commutative law and distributive law with respect to addition and multiplication respectively. Therefore, [CF] is identified with complex number with fraction as its coefficients.

**Summary and Discussion**

In this article I have identified numbers with operators to construct cardinals in the following steps.

1. Finite cardinals are defined by structures of sets based on a bijection, which are functions from isolated elements to sets that consists of the elements.

2. The operator P to construct the set structure of a cardinal from the set structure of one smaller cardinal than the former is introduced. The iteration of the operation of P made by connections of Ps relates the set structures of the finite cardinals in an order.

3. Furthermore, addition and multiplication are defined on the set structures on the basis of connections of Ps. Then, it has been shown that the set of set structures with the addition and the multiplication satisfies Peano Axioms. Accordingly, the set is identified with the natural numbers.

4. At the same time, it has been shown that the set of connections of Ps satisfies Peano Axioms, hence is also identified with the natural numbers.

5. Extensions of P in three ways are made: 1. To reverse the direction of the operation of P, 2. To divide P into finite number of sub-operators, 3. To set up the stages for the operator − to reach its activation or to cause its operation. Addition and multiplication are defined on these three sets of the extended operators, respectively. Then, the integers, the fractions, and the complex numbers with fractions as their coefficients are identified with the sets of the operators extended in these three ways, respectively.

Since these operators are extensions of P and the same kinds of objects with P, they are related in inclusion relationship. Naturally, these three sets of operators can be connected by addition or multiplication to form the same kind of operators as stated in this article. On the contrary, numbers identified with sets, e.g., one that proposed by Russell (Russell, 1919, Sections 2-7, pp. 9-61) are related in embedding relationship. The natural numbers, the integers, the fractions, and the complex numbers are all different objects. In this case, numbers form a complex system that consists of many different kinds of objects. Moreover, it seems unnatural that complex numbers have the form: addition of real part and imaginary part, that is, addition of different kinds of objects or sets.

Three problems concerning this construction of numbers are left untouched.

1. Operators [No], [Z], [F], and [CF] exist on the basis of the existence of P, \*, −, Pn, and −θ, respectively, regardless of existence of the set structures that should be the result of the operations. [No] has [N] as the set structures constructed by [No]. I wonder if the set structures constructed by [Z], [F], or [CF] exist. This would be the traditional problem of existence of numbers.

2. Construction of [N] and [No] is limitless. Accordingly, the result of the limit is not included in [N] and [No]. This limit is necessary to include infinity in numbers. What is the limit of the construction, i.e. limit of operators and set structures?

3. Construction of the real numbers, which needs infinity for their construction.

These problems will be the next steps to construct numbers.

Finally, it will be natural to expect that provided that the operator − has the structure with stages to reach its activation, P or + also has the same kind of structure. I wonder whether there exists such a structure of P (+) or not, and whether this is a meaningful problem or not.

**Reference**

Russell, B. (1919). *Introduction to mathematical philosophy.* George Allen & Unwin, Ltd. London.