What Do Privileged Coordinates Tell Us About Structure?*

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Abstract

The aim of this paper is to examine the extent to which the 'privileged coordinates' of a physical theory provide a window into how much structure it posits. We first isolate a problem for this idea. We show that there are geometric spaces that admit the same privileged coordinates, but have different amounts of structure. We then compare this 'coordinate approach' to comparing amounts of structure to the familiar 'automorphism approach,' and we conclude with some brief remarks about implicit definability.

1 Introduction

It is sometimes the case that one theory posits less structure than another. For example, Newtonian spacetime posits all of the structure that Galilean spacetime does, but in addition it comes equipped with *absolute rest* structure. It allows one to distinguish between trajectories that are at rest and those that are moving at a constant (non-zero) velocity. Galilean spacetime does not have the conceptual resources to draw such a distinction, so the move from the Newtonian to the Galilean theory represents a move to a less structured spacetime.

The standard method of comparing amounts of structure has been called the "automorphism approach" (Barrett, 2021b). It appeals to the automorphisms or 'symmetries' of the object under consideration. In brief, an automorphism of an object is a structure-preserving map from the object to itself. If an object admits more automorphisms, that suggests that the object has less structure that the automorphisms are being required to preserve. Conversely, fewer automorphisms indicates that the object has more structure that they must preserve. All symmetries of Newtonian spacetime are symmetries of Galilean spacetime, but *Galilean boosts* are symmetries of the latter but not the former. This is taken to be an indication that Newtonian spacetime has more structure than Galilean spacetime.

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The automorphism approach goes back at least to Earman's famous remark that "as the space-time structure becomes richer, the symmetries become narrower" (Earman, 1989, p. 36). North (2009, p. 87) echoes this thought when she writes that "stronger structure [. . .] admits a smaller group of symmetries" and again when she says that one indication of more structure on an object is that the "associated group of structure-preserving transformations becomes narrower" (North, 2021, p. 50). The automorphism approach has been fruitfully applied in many cases; for example, see Barrett (2015a,b), Bradley (2020), and Barrett (2021b). But another way to compare amounts of structure has recently been proposed. Instead of looking to symmetries, one looks to the 'privileged coordinates' that the space admits. The idea is that the more privileged coordinates a space admits, the less structure it must have. This is best illustrated by an example (North, 2021, p. 17–26). Consider the smooth manifold $\mathbb{R}^2$. This geometric space admits many global coordinate charts. But suppose that one were to add to $\mathbb{R}^2$ the standard Euclidean metric g_{ab}. The metric g_{ab} ascribes 'distance structure' to $\mathbb{R}^2$; it determines the distances between points and the angles between lines. Some global coordinate charts on $\mathbb{R}^2$ will not adequately respect this new structure. Some, for example, will have coordinate axes that are not orthogonal to one another. The 'rectilinear coordinates' — those obtained by rotating, translating, and reflecting the standard x-y coordinates — are the ones in which g_{ab} is most perspicuously presented. In this sense, laying down a metric on $\mathbb{R}^2$ reduces the class of 'privileged coordinates' on our geometric space.

Cases like this give us reason to think that the privileged coordinates of a geometric space provide a window into the amount of structure that it has. When discussing the Euclidean plane, North (2021, p. 26) puts the idea as follows:

> the features or quantities that are agreed upon by all the different [privileged] coordinate systems we can use for the plane, the coordinate-independent, invariant features, correspond to the intrinsic nature of the plane, to aspects of the plane itself, apart from our descriptions of it — that is, to what I have been calling its *structure*.

If this idea is right, then the privileged coordinates of a geometric space are a good guide to its amount of structure. More privileged coordinates will mean fewer "features or quantities that are agreed upon" by them, and hence less structure. North (2021, Ch. 4) employs this reasoning in a concrete case. She argues that standard Newtonian mechanics admits fewer privileged coordinates than Lagrangian mechanics does. The former must therefore posit more structure, and hence the two theories must be inequivalent, a conclusion that dissents from the standard view. (See Barrett (2022) and Jacobs (2024) for further discussion.) Others have also stressed the significance of 'privileged coordinates'. Fock (1964, p. 374) writes that "the existence of a preferred set of coordinates [. . .] reflects intrinsic properties of spacetime". And Wallace (2019) shows that one can present many geometric structures by singling out their privileged coordinates.

The aim of this paper is to investigate the coordinate approach to comparing amounts of structure. We suggest that there are geometric spaces that admit the same privileged coordinates, but have different amounts of structure, and hence privileged coordinates do not provide a perfect guide to amounts of structure. We will then step back and compare the coordinate approach to the automorphism approach. The coordinate approach to comparing amounts of structure employs the same core mechanism — implicit definability — as the automorphism approach does. We will therefore conclude with a few brief remarks on implicit definability.

2 What Are Privileged Coordinates?

In order to discuss the coordinate approach, we need to provide an account of what the privileged coordinates of a geometric space might be. In order to do so, we will employ the framework of locally G-structured spaces, which was recently discussed at length by Wallace (2019). We will review this framework here, but the reader is invited to consult Barrett and Manchak (2024) for technical details.

Locally G-structured spaces

We begin with some preliminaries. The automorphism group $\mathrm{Aut}(X)$ of a mathematical object X is the collection of bijective structure-preserving maps from X to itself. For example, if M is a smooth manifold with tensor fields $\alpha_1, \ldots, \alpha_n$ on it, the automorphism group of the geometric space $(M, \alpha_1, \ldots, \alpha_n)$ is the collection of diffeomorphisms $f : M \to M$ such that $f^*(\alpha_i) = \alpha_i$ for each $i = 1, \ldots, n$. A **pseudogroup** is the 'local analogue' of the automorphism group of a geometric space. It is a collection of bijective structure-preserving maps between open subsets of a topological space that satisfy some basic conditions (Kobayashi and Nomizu, 1996, p. 1). The simplest example of a pseudogroup is the **diffeomorphism pseudogroup** of a smooth manifold M, i.e. the class of diffeomorphisms $f : U \to V$ between open sets U and V of M. Recall that a **relativistic spacetime** is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth Lorentzian metric on M. The **isometry pseudogroup** of a relativistic spacetime (M, g_{ab}) is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(g_{ab}) = g_{ab}$. In general, if M is a smooth manifold with $\alpha_1, \ldots, \alpha_n$ smooth tensors of arbitrary index structure on M, then we will call the collection of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(\alpha_i) = \alpha_i$ for each i the **automorphism pseudogroup** of the geometric space $(M, \alpha_1, \ldots, \alpha_n)$.

Let G be a pseudogroup on $\mathbb{R}^n$ that is contained in the diffeomorphism pseudogroup of $\mathbb{R}^n$. A **locally G-structured space** is a pair (S, C), where S is a set, C is a collection of injective partial functions $c : S \to \mathbb{R}^n$, and the following hold:

Cover condition. For every point $p \in S$ there is a map $c \in C$ such that $p \in \text{dom}(c)$.

Range condition. For every map $c \in C$ there is a map $g \in G$ such that $\text{ran}(c) = \text{dom}(g)$.

Compatibility condition. For any partial function $f : S \to \mathbb{R}^n$ whose range is the domain of an element of G, $f \in C$ if and only if for every $f' \in C$ such that $\text{dom}(f) \cap \text{dom}(f')$ is non-empty, $f \circ f'^{-1} \in G$.

We can think of the maps in C as the 'privileged local coordinates' on our space S.

There is a natural way to recover a geometric space from a locally G-structured space (S, C). We begin by showing how (S, C) inherits smooth manifold structure. It is easy to build an atlas on S. For each $f \in C$, $(\text{dom}(f), f)$ is an n-chart on S. Let C^+ be the collection of all n-charts on S that are compatible with all these n-charts in C. One then shows that (S, C^+) is a smooth n-dimensional manifold (Barrett and Manchak, 2024, Proposition 2.2.1). Various levels of geometric structure are then recovered on the manifold (S, C^+) in the following manner. The maps in C suffice to induce a pseudogroup Γ on (S, C^+). Intuitively, this **coordinate transformation pseudogroup** contains all of the maps between open subsets of S that 'transform' from one of our privileged coordinate systems in C to another one of them. Γ contains those homeomorphisms between open sets of S generated by functions of the form $f^{-1} \circ g$, where f and g are in C. (See Barrett and Manchak (2024, Definition 2.2.2) for a precise definition.) The coordinate transformation pseudogroup Γ now allows one to recover geometric structures on (S, C^+). We will say that a smooth tensor field α (of arbitrary index structure) on a smooth manifold M is **implicitly defined** by a pseudogroup G on M just in case $h^*(\alpha) = \alpha$ for all $h : U \to V$ in G. We now simply equip (S, C^+) with those smooth tensor fields α that are implicitly defined by the coordinate transformation pseudogroup Γ. We can therefore recover a geometric space — in the form of a smooth manifold with tensor fields on it — from a locally G-structured space.

It is important to say when two locally G-structured spaces are 'the same'. Let (S, C) and (S', C') be locally G- and G'-structured spaces, respectively. An **isomorphism** $f : (S, C) \to (S', C')$ is a bijection $f : S \to S'$ such that

1. f is a diffeomorphism between (S, C^+) and (S', C'^+) and
2. the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection between Γ and Γ', the pseudogroups associated with (S, C) and (S', C').

This is a natural notion of isomorphism between locally G-structured spaces. It requires that isomorphisms preserve the smooth manifold structure (condition 1) and the pseudogroups (condition 2) that the spaces inherit. One can verify that if (S, C) and (S', C') are isomorphic, then the geometric spaces recovered from them are isomorphic too (Barrett and Manchak, 2024, Proposition 3.2.2).

Privileged Coordinates

We will now present an account, due to Barrett and Manchak (2024), of what the privileged coordinates of a relativistic spacetime (M, g_{ab}) are. We do this by showing how one builds a locally G-structured space (S, C) from (M, g_{ab}); the maps in C will then be the privileged coordinates on (M, g_{ab}).

The case of the Euclidean plane suggests a natural first attempt at defining the privileged coordinates of a relativistic spacetime. In that case, it is natural to say that the privileged coordinates are the 'rectilinear coordinates'. If we let $(\mathbb{R}^2, g_{ab})$ be the Euclidean plane, then one can verify that these are the coordinate charts $(\mathbb{R}^2, \phi)$ such that $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and $\phi^*(g_{ab}) = g_{ab}$. Now let (M, g_{ab}) be a relativistic spacetime, and consider the coordinate charts (U, ϕ) on M such that $\phi : U \to \mathbb{R}^n$ satisfies $\phi^*(\eta_{ab}) = g_{ab}$, where here η_{ab} is the Minkowski metric on $\mathbb{R}^n$. Recall that Minkowski spacetime is the pair $(\mathbb{R}^n, \eta_{ab})$, where η_{ab} is flat and geodesically complete. This proposal is perfectly analogous to the rectilinear coordinates on the Euclidean plane, with the only differences being that we are not requiring the coordinate charts to be global, and we are considering the Minkowski metric instead of the Euclidean metric. The problem with this proposal, as North (2021, p. 22) and Barrett (2022) suggest, is that the existence of privileged coordinates of this kind implies that the spacetime is flat. If there is a diffeomorphism $\phi : U \to \mathbb{R}^n$ such that $\phi^*(\eta_{ab}) = g_{ab}$, then g_{ab} must be flat on U. Any spacetime that is nowhere flat will not admit privileged coordinates in this sense.

This first proposal therefore does not define privileged coordinates for arbitrary spacetimes, but it suggests another proposal that will. Instead of restricting our attention to smooth maps to $\mathbb{R}^n$ that preserve the Minkowski metric, we can strategically pick another spacetime with underlying manifold $\mathbb{R}^n$ that allows the construction to work. We will say that a relativistic spacetime $(\mathbb{R}^n, g'_{ab})$ is a **representation of** (M, g_{ab}) if for every point $p \in M$, there are open sets $O \subset M$ and $O' \subset \mathbb{R}^n$ such that $p \in O$ and (O, g_{ab}) is isometric to (O', g'_{ab}). Intuitively, a representation of (M, g_{ab}) is just a spacetime with underlying manifold $\mathbb{R}^n$ that 'reflects' the structure of (M, g_{ab}) in the sense that around each point $p \in M$, there is an open set that is isometric to some open set in the representation. One can show that every relativistic spacetime has a representation (Barrett and Manchak, 2024, Lemma 3.2.2).

This fact provides us with a method of constructing a locally G-structured space from a relativistic spacetime (M, g_{ab}). Let (M, g_{ab}) be a relativistic spacetime with $(\mathbb{R}^n, g'_{ab})$ a representation of it. We then define the following:

- Let $S = M$.
- Let G be the isometry pseudogroup of $(\mathbb{R}^n, g'_{ab})$.
- Let C be the collection of isometries between open subsets of (M, g_{ab}) and open subsets of $(\mathbb{R}^n, g'_{ab})$, i.e. diffeomorphisms $c : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $c^*(g'_{ab}) = g_{ab}|_O$.

This (S, C) is indeed a locally G-structured space (Barrett and Manchak, 2024, Lemma 3.2.3). (In particular, the fact that $(\mathbb{R}^n, g'_{ab})$ is a representation of (M, g_{ab}) guarantees that cover condition holds.) We will call (S, C) the **locally G-structured space determined by** (M, g_{ab}). This terminology is justified, for one can show that different choices of representation in our construction of (S, C) result in isomorphic locally G-structured spaces (Barrett and Manchak, 2024, Proposition 3.2.3).

This locally G-structured space (S, C) provides one particularly natural way of saying what the 'privileged coordinates' of (M, g_{ab}) are. The privileged coordinates are just those maps in C. It is particularly natural because there is a sense in which these coordinates allow one to recover the structure of (M, g_{ab}). We require the following result to see this (Barrett and Manchak, 2024, Proposition 3.2.1).

Theorem 1. *Let (S, C) be a locally G-structured space determined by (M, g_{ab}). Then both of the following hold:*

1. *The identity map 1_M is a diffeomorphism between the manifold (S, C^+) and M.*
2. *The coordinate transformation pseudogroup Γ on S is the isometry pseudogroup of (M, g_{ab}), i.e. the collection of diffeomorphisms $f : U \to V$ between open subsets $U, V \subset M$ such that $f^*(g_{ab}) = g_{ab}$.*

Theorem 1 captures a sense in which the locally G-structured space (S, C) determined by (M, g_{ab}) recovers the relativistic spacetime (M, g_{ab}). The manifold structure (S, C^+) that (S, C) naturally inherits is the same as that of M; we know this since 1_M is a diffeomorphism between the two manifolds. And moreover, (S, C) recovers the metric structure g_{ab}, since its coordinate transformation pseudogroup Γ implicitly defines g_{ab}. This is because Theorem 1 guarantees that Γ is the isometry pseudogroup of (M, g_{ab}). There is thus a sensible way to make sense of the 'privileged coordinates' of a relativistic spacetime. And on this understanding, Theorem 1 implies that the "intrinsic nature" of the spacetime — in the form of its underlying manifold and its metric — is among "the coordinate-independent, invariant features" (North, 2021, p. 26).

3 Do privileged coordinates determine amounts of structure?

This account of the privileged coordinates of relativistic spacetimes seems to be the best one can do, and without such an account, the coordinate approach to comparing amounts of structure will not get off the ground. Other accounts might be possible (as Barrett and Manchak (2024, Revision 1) mention), but on this account one can recover (in a weak sense) the structures of (M, g_{ab}) from its privileged coordinates. This sense is weak for the following reason, which is discussed in detail by Barrett and Manchak (2024). Some spacetimes will admit particularly small isometry pseudogroups, and in such cases more than one metric on M will be implicitly defined by Γ. The locally G-structured space (S, C) will then recover more than one metric. This means that it is not perfectly clear what relativistic spacetime one recovers from (S, C), and hence one cannot in general use (S, C) to present the entire structure of a relativistic spacetime.

The privileged coordinates of a relativistic spacetime therefore do not completely determine its structure. One might wonder, however, whether the privileged coordinates of a geometric space tell us something weaker about the space. In particular, one wonders whether they encode the space's 'amount of structure'. We begin with a question posed by Barrett and Manchak (2024, Reservation 1): Are there geometric spaces with different amounts of structure that determine the same locally G-structured space? Barrett and Manchak (2024) conjecture that the answer is "yes". If so, this would mean that the amount of structure that a geometric space has is not encoded by the locally G-structured space that it determines. Our next aim is to capture a sense in which this affirmative answer is correct.

One might be able to extend the account of privileged coordinates that we provided for relativistic spacetimes to arbitrary geometric spaces. But we will not pursue the details here. Rather, we will make some natural assumptions about what such an account will look like. Let M be a smooth manifold with $\alpha_1, \ldots, \alpha_n$ smooth tensors of arbitrary index structure on M. We consider the geometric space $(M, \alpha_1, \ldots, \alpha_n)$ and make the following two assumptions about the locally G-structured space (S, C) that it determines.

P1. (S, C^+) and M are diffeomorphic.

P2. The coordinate transformation pseudogroup Γ on (S, C) is the same as the automorphism pseudogroup of $(M, \alpha_1, \ldots, \alpha_n)$.

In order to get the coordinate approach off the ground one needs an account of privileged coordinates that satisfies P1 and P2. This is tantamount to the requirement that one be able to define the locally G-structured space (S, C) determined by an arbitrary geometric space in such a way that (S, C) recovers $(M, \alpha_1, \ldots, \alpha_n)$ in (at least) the weak sense described above. P1 guarantees that (S, C) recovers the manifold structure of M. P2 guarantees that all of the tensor fields α_i on M are implicitly defined by Γ; this is because if $h \in \Gamma$, it must be that $h^*(\alpha_i) = \alpha_i$ since h is in the automorphism pseudogroup of $(M, \alpha_1, \ldots, \alpha_n)$. Theorem 1 says that these two assumptions hold of the locally G-structured space determined by a relativistic spacetime. P1 and P2 together comprise the claim that analogues of Theorem 1 will go through for other geometric spaces. We will show that even granting P1 and P2, the coordinate approach appears unsatisfactory.

We will say that a pseudogroup on a manifold M is **trivial** if it only contains identity maps. A relativistic spacetime (M, g_{ab}) is **Heraclitus** if, for any open subsets $U, V \subset M$ and any isometry $\psi : U \to V$, it follows that (i) $U = V$ and (ii) ψ is the identity map. Manchak and Barrett (2024b) show that a

Heraclitus spacetime exists. One can easily verify that the isometry pseudogroup of (M, g_{ab}) is trivial if and only if (M, g_{ab}) is Heraclitus.

Let (M, g_{ab}) be a Heraclitus spacetime. The two geometric spaces that we will consider are the relativistic spacetime (M, g_{ab}) with trivial automorphism pseudogroup, and the geometric space (M, g_{ab}, λ), where λ is an arbitrary tensor field on M that is not 'constructible' in terms of the metric g_{ab}. So, for example, λ is not some scalar multiple of g_{ab}, the Riemannian curvature tensor associated with g_{ab}, etc. Note that (M, g_{ab}, λ) has a trivial automorphism pseudogroup since it must be contained the isometry pseudogroup of (M, g_{ab}). We now put forward the following natural claim:

P3. (M, g_{ab}, λ) has more structure than (M, g_{ab}).

(M, g_{ab}, λ) results from adding the structure λ to (M, g_{ab}). Since λ is not constructible from g_{ab} it is a genuinely new level of structure on the space.There is thus a compelling sense in which P3 holds.

We now have the following result.

Theorem 2. *If P1, P2, and P3, then there are geometric spaces with different amounts of structure that determine isomorphic locally G-structured spaces.*

Proof. We consider the two geometric spaces (M, g_{ab}) and (M, g_{ab}, λ). P3 implies that they have different amounts of structure. We need only show that they determine isomorphic locally G-structured spaces. Let (S, C) be the locally G-structured space determined by (M, g_{ab}) and (S', C') the locally G-structure space determined by (M, g_{ab}, λ). P1 and condition 1 of Theorem 1 together imply that there is a diffeomorphism $f : (S, C^+) \to (S', C'^+)$, since both of those manifolds must be diffeomorphic to M. Since the automorphism pseudogroups of (M, g_{ab}) and (M, g_{ab}, λ) are trivial, P2 and condition 2 of Theorem 2 imply that the coordinate transformation pseudogroups Γ and Γ' are trivial too.

We now show that $f : S \to S'$ must be an isomorphism between (S, C) and (S', C'). We know immediately that f satisfies condition 1 of the definition of an isomorphism. We show that f also satisfies condition 2, in that it preserves the coordinate transformation pseudogroups on (S, C) and (S', C'). We need to show that the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection from Γ to Γ'. Let $s, s' \in \Gamma$ and suppose that $f \circ s \circ f^{-1} = f \circ s' \circ f^{-1}$. Since $f : S \to S'$ is a bijection, it must be that $s = s'$. Hence our map $s \mapsto f \circ s \circ f^{-1}$ is injective. Now let $s' \in \Gamma'$, so s' is the identity map 1_O on some open set $O \subset S'$. We see that $f^{-1} \circ 1_O \circ f = 1_{f^{-1}[O]}$. Since f is a diffeomorphism, $f^{-1}[O]$ is an open subset of S, and hence $1_{f^{-1}[O]}$ must be in Γ. (This is because a pseudogroup must contain the identity map for every open subset (Kobayashi and Nomizu, 1996, p. 1).) Since $f \circ 1_{f^{-1}[O]} \circ f^{-1} = 1_O$, our map is bijective, f satisfies condition 2, and hence f is an isomorphism between (S, C) and (S', C'). □

Theorem 2 tells us that there are geometric spaces with different amounts of structure that nonetheless determine the same locally G-structured space, and we have in this sense answered in the affirmative the question that Barrett and Manchak (2024) ask. It seems that the privileged coordinates of a geometric space do not provide a perfect guide to its amount of structure. One can know the privileged coordinates of a geometric space — in the form of the locally G-structured space that it determines — but not be able to assess how much structure it has. Insofar as P1, P2, and P3 are correct, the coordinate approach to comparing amounts of structure does not always work. We note that the problem isolated here for the coordinate approach parallels the one (mentioned briefly above) that Barrett and Manchak (2024) isolate for attempts to present a geometric space by appealing to its privileged coordinates. Both problems are generated by Heraclitus spacetimes, and both point to a central issue with implicit definability. We return to this point later.

4 Symmetries, Coordinates, and Definability

Our next aim is to compare the coordinate and the automorphism approaches. Doing so will allow us to diagnose where the difficulties faced by the coordinate approach come from, and it will allow us to make precise the close relationship between the two approaches.

Coordinates or Automorphisms?

We begin by briefly reviewing the automorphism approach. The following criterion is representative; see Barrett (2021b) and the references therein for discussion.

SYM*. A mathematical object X at least as much structure as a mathematical object Y if (and only if) $\text{Aut}(X) \subset \text{Aut}(Y)$.

The condition $\text{Aut}(X) \subset \text{Aut}(Y)$, that the automorphism group of X is contained in that of Y, is one way to make precise the idea that X admits 'no more' automorphisms than Y does. SYM* works well in easy cases. Newtonian spacetime has more structure than Galilean spacetime according to SYM* (or more precisely, Newtonian spacetime has at least as much structure as Galilean spacetime, and not vice versa). An inner product space $(V, \langle\ ,\ \rangle)$ has more structure than its underlying vector space V, a topological space (X, τ) has more structure than its underlying set X, a relativistic spacetime (M, g_{ab}) has more structure than its underlying smooth manifold M, and so on.

But it has been noticed that SYM* makes unsatisfactory verdicts in cases where the objects under consideration admit *few* symmetries (Barrett, 2021b; Barrett et al., 2023). While it is usually the case that adding structure to a mathematical object results in fewer automorphisms, if the automorphism group of the object is already trivial — that is, it only contains the identity map and is thus as small as can be — adding structure cannot result in fewer automorphisms. This point can be made precise by employing the same idea from the proof of Theorem 2. Following Barrett et al. (2023) and Manchak and Barrett (2024b), we will call a relativistic spacetime (M, g_{ab}) **giraffe** if it has a trivial isometry group, i.e. the only diffeomorphism $f : M \to M$ such that

$f^*(g_{ab}) = g_{ab}$ is the identity map. Since every Heraclitus spacetime is giraffe, it follows that a giraffe spacetime (M, g_{ab}) exists. Consider an arbitrary tensor field λ on M that is not constructible in terms of g_{ab}. According to SYM*, the geometric space (M, g_{ab}) has at least as much structure as (M, g_{ab}, λ) because the automorphism group of (M, g_{ab}) is already as small as can be. Indeed, the automorphism group of (M, g_{ab}, λ) is the same trivial group. This again strikes one as a bad verdict, and hence SYM* is not ideal.

We need to make the coordinate approach precise in order to see whether it improves upon SYM*. Recall that the coordinate approach is based upon the idea that fewer privileged coordinates should indicate more structure. The most natural first attempt at making this precise is to say that a locally G-structured space (S, C) has at least as much structure as (S', C') if (and only if) $C \subset C'$. If this is right, one could then compare the structure of two geometric spaces by asking whether the locally G-structured spaces that they determine stand in this relationship. But the following example illustrates that this first attempt runs into difficulty.

Example 1. Consider Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ and let (S, C) be the locally G-structured space that it determines. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a diffeomorphism such that $f^*(\eta_{ab}) \neq \eta_{ab}$. We define a locally G-structured space (S, C'), where C' is the collection of isometries between open subsets of $(\mathbb{R}^4, \eta_{ab})$ and open subsets of $(\mathbb{R}^4, f^*(\eta_{ab}))$, i.e. diffeomorphisms $c : U \to V$ where $U \subset \mathbb{R}^4$ and $V \subset \mathbb{R}^4$ are open and $c^*(f^*(\eta_{ab})) = \eta_{ab}$. It is now easy to see that neither C nor C' is a subset of the other. The identity map is contained in C (but not C'), and f is contained in C' (but not C). One can, on the other hand, verify that $\Gamma = \Gamma'$. For if $c, d \in C'$ then

$$(c^{-1} \circ d)^*(\eta_{ab}) = d^* \circ c_*(\eta_{ab}) = \eta_{ab}$$

The first equality follows from properties of the pullback, and the second from the fact that $c^*(f^*(\eta_{ab})) = \eta_{ab}$ and hence $f^*(\eta_{ab}) = c_*(\eta_{ab})$. Hence $c^{-1} \circ d \in \Gamma$, since Γ is the isometry pseudogroup of Minkowski spacetime, and so $\Gamma' \subset \Gamma$. Conversely, if $g \in \Gamma$, this means that $g^*(\eta_{ab}) = \eta_{ab}$. One can now verify that $c \circ g \in C'$ for every $c \in C$, since $(c \circ g)^*(f^*(\eta_{ab})) = \eta_{ab}$. That means that $c^{-1} \circ c \circ g \in \Gamma'$, which implies that $g \in \Gamma'$, and so $\Gamma \subset \Gamma'$. Since $\Gamma = \Gamma'$, both (S, C) and (S, C') recover all of the same the same tensor fields. One therefore wants to say that (S, C) and (S, C') have the same amount of structure, despite the fact that neither C nor C' is a subset of the other. ⌟

This example shows that there can be collections of privileged coordinates C and C' that induce the same coordinate transformation group — and therefore recover precisely the same structures — despite neither being a subset of the other. So while this first attempt fails, it suggests another promising way to make the coordinate approach precise. The amount of structure that a geometric space has is correlated with the size of the coordinate transformation group Γ that its privileged coordinates determine. After all, the coordinate transformation group is what one uses to recover tensor fields on the geometric space. The following criterion makes this idea precise.

COORD. X has at least as much structure as Y if (and only if) the coordinate transformation pseudogroup Γ that X determines is a subset of the coordinate transformation pseudogroup Γ' that Y determines.

COORD satisfies the basic idea behind the coordinate approach; more privileged coordinates will mean a larger coordinate transformation group, and that will mean less structure. And the motivation for COORD is closely related to the motivation for SYM*. If Γ is contained in Γ', then all of the tensor fields invariant under the maps in Γ' will also be invariant under the maps in Γ, and hence — insofar as we equip X and Y with those tensor fields 'invariant under coordinate transformations' — X will have at least as much structure as Y. There is a sense in which COORD is worse than SYM*, and another sense in which it is better. It is worse because it is only applicable to geometric spaces, not arbitrary mathematical objects that one might use to formulate a physical theory. It makes sense to discuss the automorphisms of any mathematical object; it does not always make sense to discuss an object's privileged coordinates or its coordinate transformation group. At best, that will only make sense for geometric spaces. And even then, we do not yet have a full account of what the privileged coordinates of a geometric space in general are; recall the two assumptions P1 and P2 that we made to get the coordinate approach off the ground.

On the other hand, COORD is better than SYM* because it does not run into difficulty with mere giraffe spacetimes. Compare again the giraffe spacetime (M, g_{ab}) with (M, g_{ab}, λ). Crucially, the fact that (M, g_{ab}) and (M, g_{ab}, λ) have the same trivial automorphism group does not imply that they have the same coordinate transformation *pseudogroups*. Coordinate transformation pseudogroups do not merely admit diffeomorphisms from the entire space to itself. They also admit diffeomorphisms between open subsets of the space, and hence they contain more information than mere automorphism groups do. For this reason, COORD does not necessarily run into problems with giraffe spacetimes. Not every giraffe spacetime is Heraclitus (Manchak and Barrett, 2024b). And so if (M, g_{ab}) is giraffe but not Heraclitus, then while its automorphism group is 'as small as can be', its isometry pseudogroup is not. This implies that the addition of the tensor field λ may further reduce the isometry pseudogroup, and hence Γ' can be properly contained in Γ. Thus it might be that (M, g_{ab}) does not have at least as much structure as (M, g_{ab}, λ) according to COORD. Of course, Theorem 2 shows that Heraclitus spacetimes — since their coordinate transformation pseudgroups are trivial — generate problems for COORD. But giraffe spacetimes do not, and in this sense COORD represents an improvement upon SYM*.

Altogether, these considerations actually suggest a way to improve the automorphism approach:

SYM*2. A mathematical object X has at least as much structure as a mathematical object Y if and only the automorphism pseudogroup of X is contained in the automorphism pseudogroup of Y.

This kind of revision of SYM* is a step toward what Barrett (2021b) calls the "category approach" to comparing amounts of structure, which is motivated by the idea that when comparing structure one should take into account *all* of the structure-preserving maps between objects of the same type as X, not merely the automorphisms of X. In particular, SYM*2 takes into account maps between objects that can be 'embedded in' X. For if X_1 and X_2 can be embedded in X, in the sense that there is a structure-preserving map between X_1 and some open subset of X and likewise for X_2, then the more (or conversely, fewer) structure-preserving maps there are between X_1 and X_2, the more (or fewer) maps there will be in the automorphism pseudogroup of X.

Heraclitus spacetimes pose a difficulty for SYM*2 in the same manner as they do for COORD (Manchak and Barrett, 2024b). For if (M, g_{ab}) is Heraclitus, then it will have the same trivial automorphism pseudogroup as (M, g_{ab}, λ) for any tensor field λ on M, and hence the former will have at least as much structure as the latter according to SYM*2. Moreover, SYM*2 is less widely applicable than SYM*. Not all mathematical objects have automorphism pseudogroups. One can only define a pseudogroup on objects that have at least topological structure, and so SYM*2 does not allow one to compare arbitrary mathematical objects X and Y; it only works for geometric spaces. But SYM*2 also has benefits over both SYM* and COORD. Mere giraffe spacetimes do not pose a problem for SYM*2 as they did for SYM*; this is for the same reason that they do not pose a problem for COORD. And in order to apply COORD in cases of interest, one needs P1 and P2. Without assuming P2, in particular, one has no idea what the coordinate transformation pseudogroup of an arbitrary geometric space is. SYM*2 dodges this worry. Indeed, one can define (as we did above) the automorphism pseudogroup of an arbitrary geometric space.

If one assumes P2, then SYM*2 and COORD are actually the same criterion. This is because P2 implies that the coordinate transformation pseudogroups of the spaces X and Y are equal to the automorphism pseudogroups of X and Y, respectively. Recall that P2 represents a best case scenario for the coordinate approach. It will hold if one can define privileged coordinates for arbitrary geometric spaces and prove analogues of Theorem 1. While assuming P2 does allow one to apply COORD to various geometric spaces, it seems infelicitous to apply that criterion before having a precise account of what the privileged coordinates of an arbitrary geometric space are. We have given an account for relativistic spacetimes, but in order to put the coordinate approach on firm footing, one would need to extend this treatment to arbitrary geometric spaces. Altogether, this means that the best case scenario for the coordinate approach (as represented by COORD) is just a manifestation of the automorphism approach (as represented by SYM*2).

In brief, the automorphism and coordinate approaches are so closely related to one another because singling out a collection of privileged coordinates is just another way of singling out a collection of symmetries. So at best, the privileged coordinates of a space tell us the same information about its underlying structure as its automorphism pseudogroup does. And it seems that this is not everything.

The Argument from Definability

It is worth dwelling on this point. The automorphism approach and the coordinate approach both rely upon implicit definability. Suppose that we have an object X and a collection of maps from X to itself. A structure is implicitly defined on X by this collection of maps if they all 'preserve' that structure. There are a number of ways to make this precise (Barrett, 2018; Winnie, 1986). We will here discuss two and show how they lead to these criteria for comparing amounts of structure that we have been discussing.

One natural understanding of implicit definability looks to those structures that are preserved by all of the maps in the automorphism group of X. In the context of geometric spaces, one can make this precise as follows. Let M be a smooth manifold with G a group of diffeomorphisms $f : M \to M$. We will say that a smooth tensor field λ on M is **globally implicitly defined** by G if $f^*(\lambda) = \lambda$ for every $f \in G$. And following our discussion in section 2, if Γ is a pseudogroup on M, we will say that a smooth tensor field λ on M is **locally implicitly defined** by Γ if $f^*(\lambda) = \lambda$ for every $f \in \Gamma$. If $(M, \alpha_1, \dots, \alpha_n)$ is a smooth manifold with tensor fields on it, then there are two natural choices of G and Γ. If G is the automorphism group of $(M, \alpha_1, \dots, \alpha_n)$ and G globally implicitly defines λ, we will say simply that λ is globally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$. Similarly, if Γ is the automorphism pseudogroup of $(M, \alpha_1, \dots, \alpha_n)$ and Γ locally implicitly defines λ, we will say simply that λ is locally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$.

These two varieties of implicit definability are related to one another exactly as one would expect. Because local implicit definability is requiring that λ be preserved by strictly more maps than global implicit definability requires, local implicit definability is a strictly stronger variety of definability. Let $(M, \alpha_1, \dots, \alpha_n)$ be a geometric space.

Proposition 1. *If λ is locally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$, then it is globally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$; the converse does not hold.*

Proof. It follows easily from definitions that if λ is locally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$, then it is globally implicitly defined by $(M, \alpha_1, \dots, \alpha_n)$. Let $(\mathbb{R}^2, \eta_{ab})$ be Minkowski spacetime, and consider the spacetime (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, 0 < x, x^2 < t^2\}$. Manchak and Barrett (2024b, Example 6) show that this spacetime is giraffe but not Heraclitus. Since (M, η_{ab}) is giraffe, every tensor field λ on M is globally implicitly defined by (M, η_{ab}). But consider the smooth vector field $(\frac{\partial}{\partial t})^a$ on M. Let $O = \{(t, x) \in M : t + x < 1\}$ and consider the diffeomorphism $\psi : O \to O$ defined by $\psi(t, x) = (-t + 1, x)$. Now one can easily verify that $\psi^*(\eta_{ab}) = \eta_{ab}$, but $\psi^*(\frac{\partial}{\partial t})^a \neq (\frac{\partial}{\partial t})^a$. Hence $(\frac{\partial}{\partial t})^a$ is not locally implicitly defined on (M, η_{ab}), despite the fact that it is globally implicitly defined by (M, η_{ab}). □

The basic idea behind this result is easy to appreciate. There are geometric spaces with trivial automorphism groups that do not have trivial automorphism pseudogroups. Every tensor field on such a space will be globally implicitly defined, despite some of those fields not being preserved by the richer collection of maps in the automorphism pseudogroup.

Now that we have these two varieties of implicit definability, we can show how they provide the core mechanism by which COORD, SYM*, and SYM*2 function. These criteria are all tracking facts about implicit definability. The following results are simply rehearsing the idea behind what Barrett et al. (2023) call 'the implicit definability conception' and Barrett (2018, 2021b) calls the 'argument from definability'.

We begin the cases for SYM* and SYM*2.

Proposition 2. *Let $(M, \alpha_1, \ldots, \alpha_m)$ and $(M, \beta_1, \ldots, \beta_n)$ be geometric spaces. The following are equivalent.*

1. *The automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ is a subset of the automorphism group of $(M, \beta_1, \ldots, \beta_n)$.*

2. *The space $(M, \alpha_1, \ldots, \alpha_m)$ globally implicitly defines all of the tensors that $(M, \beta_1, \ldots, \beta_n)$ globally implicitly defines.*

Proof. Assume 1 and let λ be a tensor that $(M, \beta_1, \ldots, \beta_n)$ globally implicitly defines. Since the automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ is contained in the automorphism group of $(M, \beta_1, \ldots, \beta_n)$, the former globally implicitly defines λ too. Now assume 2 and suppose for contradiction that f is in the automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ but not in the automorphism group of $(M, \beta_1, \ldots, \beta_n)$. This means that $f : M \to M$ is a diffeomorphism but that there is some β_j such that $f^*(\beta_j) \neq \beta_j$. This means that $(M, \alpha_1, \ldots, \alpha_m)$ does not globally implicitly define β_j. This contradicts 2 since $(M, \beta_1, \ldots, \beta_n)$ clearly does globally implicitly define β_j. □

An analogous result holds about SYM*2. We leave the proof to the reader since it is essentially the same as that of Proposition 2.

Proposition 3. *Let $(M, \alpha_1, \ldots, \alpha_m)$ and $(M, \beta_1, \ldots, \beta_n)$ be geometric spaces. The following are equivalent.*

1. *The automorphism pseudogroup of $(M, \alpha_1, \ldots, \alpha_m)$ is a subset of the automorphism pseudogroup of $(M, \beta_1, \ldots, \beta_n)$.*

2. *The space $(M, \alpha_1, \ldots, \alpha_m)$ locally implicitly defines all of the tensors that $(M, \beta_1, \ldots, \beta_n)$ locally implicitly defines.*

Propositions 2 and 3 illustrate that SYM* and SYM*2 are tracking implicit definability. The first condition of Proposition 2 says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to SYM*. Hence SYM* says that X has at least as much structure as Y just in case X globally implicitly defines all of the structures of Y. The first condition of Proposition 3 says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to SYM*2. Hence SYM*2 says that X has at least as much structure as Y just in case X locally implicitly defines all of the structures of Y.

The case of COORD is analogous. The reason why 'more privileged coordinates' indicates less structure is that more privileged coordinates leads to a larger coordinate transformation pseudogroup, which then locally implicitly defines fewer structures on our space. We again leave the straightforward proof to the reader.

Proposition 4. *Let $(M, \alpha_1, \ldots, \alpha_m)$ and $(M, \beta_1, \ldots, \beta_n)$ be geometric spaces with coordinate transformation pseudogroups Γ and Γ', respectively. If A2 holds, then the following are equivalent.*

1. $\Gamma \subset \Gamma'$.
2. *The space $(M, \alpha_1, \ldots, \alpha_m)$ locally implicitly defines all of the tensors that $(M, \beta_1, \ldots, \beta_n)$ locally implicitly defines.*

Since the first condition says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to COORD, we see that COORD is also tracking facts about implicit definability. (We note that a version of the result would still go through without assuming P2, so long as one replaces the second condition with the claim that Γ locally implicitly defines all of the tensors that Γ' locally implicitly defines.)

It is simple to now parlay these results into arguments for these criteria. In order to do so, we need to isolate one further thought about implicit definability. The basic idea is that a mathematical object comes equipped with all and only the structures that it implicitly defines. In other words, it is common to take those structures that are 'invariant under symmetries' of a mathematical object to be part of the genuine structure of that object. This is often taken to indicate that, in some sense, the structure 'comes for free' given the basic structures on the object. For example, a metric space (X, d) comes equipped with its metric topology τ, despite the fact that τ is not *explicitly* mentioned in the notation we use to present (X, d). One way of accounting for this is to notice that every symmetry of (X, d) — that is, every distance-preserving bijection from X to itself — preserves τ in the sense that it is a homeomorphism with respect to τ. Hence τ is invariant under the symmetries of (X, d); (X, d) 'implicitly defines' τ. If implicit definability tracks which structures an object comes equipped with, then we have an account of why (X, d) comes equipped with its metric topology.

This basic idea about implicit definability is often employed in discussions of the significance of symmetry. For example, the 'Kleinian method' of presenting a geometric space turns on exactly this idea. Norton (2002, p. 259) describes this method as one in which a "geometric theory would be associated with a class of admissible coordinate systems and a group of transformations that would carry us between them. The cardinal rule was that physical significance can be assigned just to those features that were invariants of this group." Similarly, North (2021, p. 48) writes that "Klein suggested that any geometry can be identified by means of the transformations that preserve the structure, likewise by the quantities that are invariant under the group of those transformations." The Kleinian method therefore employs exactly this basic idea about implicit definability, invariance under symmetry, and structure. (See Barrett (2018) and the references therein for discussion and additional examples.)

For our purposes, there are two ways of making this idea about implicit definability precise in the context of geometric spaces. Each corresponds to one of our varieties of implicit definability and hence a different understanding of 'invariance under symmetry'.

Global P4. A geometric space $(M, \alpha_1, \dots, \alpha_n)$ comes equipped with all and only the structures that it globally implicitly defines.

Local P4. A geometric space $(M, \alpha_1, \dots, \alpha_n)$ comes equipped with all and only the structures that it locally implicitly defines.

Local P4 implies that a geometric space will (in general) come equipped with fewer structures than Global P4 implies it will. This is because by Proposition 1 fewer tensor fields will be locally implicitly defined than will be globally implicitly defined.

In conjunction with Propositions 2, 3, and 4, these principles allow one to provide arguments for SYM*, SYM*2, and COORD. Global P4 implies that the second condition of Proposition 2 is saying that $(M, \alpha_1, \dots, \alpha_m)$ comes equipped with all of the structures that $(M, \beta_1, \dots, \beta_n)$ comes equipped with. This is a particularly natural way in which the former might have at least as much structure as the latter. Similarly, Local P4 implies that the second conditions of Propositions 3 and 4 are saying that $(M, \alpha_1, \dots, \alpha_m)$ comes equipped with all of the structures that $(M, \beta_1, \dots, \beta_n)$ comes equipped with. We have therefore seen how a basic idea about implicit definability, captured by Global P4 or Local P4, leads to these criteria for comparing amounts of structure.

The triviality problem

All of these criteria for comparing amounts of structure rely upon the idea that an object comes equipped with those structures it implicitly defines, and the problems they face stem from precisely this. In brief, the issue is that both Global P4 and Local P4 seem false. The problem that Global P4 faces is just what Barrett (2021b) has called the 'triviality problem' for SYM*; it is discussed in detail by Barrett et al. (2023). North (2021, p. 117) points to it when she writes that there are geometric spaces that "lie beyond the scope of Klein's program," and Torretti (2016) explicitly mentions the problem of trivial isometry groups faced by Kleinian methods. The problem that Local P4 faces is the 'local analogue' of this triviality problem; it has been briefly discussed by Manchak and Barrett (2024b). Unsurprisingly, these arguments perfectly parallel the problems faced by COORD, SYM*, and SYM*2 that have already been discussed.

The case of Global P4 is straightforward. Let (M, g_{ab}) be a giraffe spacetime. Since it is giraffe, every smooth tensor field λ on M is globally implicitly defined by (M, g_{ab}). But most of these fields are not in any sense 'constructible' from g_{ab}; they are simply arbitrary tensor fields on M. So one does not want to say that (M, g_{ab}) comes equipped with them. Indeed, in many cases they will not even be locally implicitly definable by (M, g_{ab}). To put the point acutely, an arbitrary *metric* on M is globally implicitly defined by (M, g_{ab}), and since most of these metrics will not be related to g_{ab} in any interesting sense, one certainly does not want to say that (M, g_{ab}) comes equipped with them in the same sense as it comes equipped with g_{ab}. It therefore seems that Global P4 cannot be right.

One might expect that the prospects are better for Local P4. Indeed, by Proposition 1, Local P4 will imply that (M, g_{ab}) comes equipped with fewer structures than Global P4 implies it does, and so perhaps the triviality problem can be avoided. Unfortunately, an analogous argument can be put forward against Local P4. We now let (M, g_{ab}) be a Heraclitus spacetime. Every smooth tensor field λ on M — including, of course, all of the metrics on M, most of which one conjectures will be unrelated to g_{ab} — is locally implicitly defined by (M, g_{ab}) since it has a trivial automorphism pseudogroup. One again has the strong feeling that the vast majority of these fields are not in any sense 'constructible' from g_{ab}, and so one does not want to say that (M, g_{ab}) comes equipped with them. This means that Local P4 also cannot be right.

5 Conclusion

We conclude with a meditation on definability and some suggestions for future work. In particular, there is one subtlety concerning Local P4 that is worth examining. One might ask the following question (Barrett and Manchak, 2024, Revision 2).

Q1. Is there an interesting account of 'explicit definability' in the context of geometric spaces?

Small steps in the direction of investigating explicit definability in spacetime theories have been taken. See, for example, the 'maximally structured' spacetimes of Manchak and Barrett (2024a), the suggestive concept of 'covariant definability' of Glymour (1977), the category theoretic methods discussed by Halvorson (2019) and Weatherall (2019), and the formulations of spacetime theories provided by Andréka and Németi (2014) and Cocco and Babic (2020). It is easy to discuss explicit definability when one has a clear understanding of what the 'language' of the objects under consideration is. But the geometric spaces that we are considering are not usually formulated within a formal language, and so challenges remain for answering Q1 in the affirmative.

Suppose that one could answer Q1 in the affirmative and formulate an interesting variety of explicit definability. Such an account would come to bear on the issues discussed here. It would be natural to then consider the following revision of Global P4 and Local P4.

P5. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with all and only the structures that it explicitly defines.

The same kinds of examples that motivated Global P4 and Local P4 could be used to motivate P5. Suppose, for example, that one has a vector space with inner product $(V, \langle , \rangle)$. It is natural to think that this object comes equipped with a norm $|| \cdot ||$, which assigns to a vector $v \in V$ its 'length' $||v||$. Not only is the norm invariant under the symmetries of V, it is directly constructible from the inner product. One 'explicitly defines' the norm in terms of the inner product by letting $||v|| = \langle v, v \rangle$. So P5 would provide us with an account of why we are inclined to say that $(V, \langle , \rangle)$ comes equipped with a norm $|| \cdot ||$. One can tell the same kind of story in the case of the metric space and metric topology, along with other examples (Barrett, 2018).

One might, however, have better conceptual reasons to adopt P5 than Local P4 or Global P4. In particular, if some structure is explicitly definable on X, that would capture a sense in which the basic structures of X suffice to 'construct' or 'build' that new structure. And this would perhaps provide a more compelling reason to think this new structure 'comes for free' given the basic structures on X that mere implicit definability provides.

We will assume that if an interesting variety of explicit definability for geometric spaces could be made precise, then it would entail local implicit definability. (This parallels the state of affairs in first-order logic where these notions are well understood (Barrett, 2018; Winnie, 1986).) It is then natural to ask the following question.

Q2. Does local implicit definability entail explicit definability?

It is well known that in the first-order context, there are some (particularly strong) varieties of implicit definability that entail explicit definability (Barrett, 2018). Beth's Theorem, for example, is a particularly famous example of this. An affirmative answer to Q2 would therefore not be entirely without precedent.

If the answer to Q2 is "yes" and P5 holds, then one would have an argument for Local P4 and against P3. The argument for Local P4 would be precisely the same as the argument that one provides for P5. Indeed, the principles P5 and Local P4 would be equivalent, since local implicit definability and explicit definability would themselves be equivalent. The argument against P3 would then note that since (M, g_{ab}) is Heraclitus, it locally implicitly defines every tensor field λ on M. The affirmative answer to Q2 would then imply that λ is explicitly definable by (M, g_{ab}). P5 would entail that (M, g_{ab}) comes equipped with λ, and hence (M, g_{ab}, λ) would not have more structure than (M, g_{ab}).

On the other hand, if the answer to Q2 is "no", then insofar as P5 holds, Local P4 would be false. The negative answer to Q2 would imply that local implicit definability and explicit definability are inequivalent, and hence one cannot both adopt P5 and Local P4. In particular, there would be some geometric space $(M, \alpha_1, \ldots, \alpha_n)$ and tensor λ on M such that $(M, \alpha_1, \ldots, \alpha_n)$ locally implicitly defines λ but does not explicitly define λ. P5 would imply that $(M, \alpha_1, \ldots, \alpha_n)$ does not come equipped with λ, contradicting Local P4. And moreover, one would have a correspondingly more robust argument for P3. The addition of a tensor field λ to a geometric space that does not explicitly define λ will result in a space that comes equipped with more structure.

It is therefore important to examine Q2 and P5 further. We will leave careful investigation to another time, but it is worth mentioning one suggestive example here. Suppose that (M, g_{ab}) is a Heraclitus spacetime. We know that every derivative operator ∇ on M is locally implicitly defined by (M, g_{ab}). (Of course, strictly speaking ∇ is not a tensor field on M, but by slightly extending our terminology, we can still speak of it being locally implicitly defined by (M, g_{ab}), in the sense that all maps in the isometry pseudogroup of (M, g_{ab}) preserve ∇. See Weatherall (2016) or Barrett (2015b, 2021a) for a precise account.) Despite the fact that all of these derivative operators are locally implicitly defined by (M, g_{ab}), one is tempted to say that (M, g_{ab}) only genuinely comes equipped with one of them: the unique derivative operator that is compatible with g_{ab}, i.e. the Levi-Civita derivative operator of (M, g_{ab}). If this is right, then the answer to Q2 will be "no" for any variety of explicit definability for which P5 holds. For if P5 holds and explicit definability entails local implicit definability, then a Heraclitus spacetime (M, g_{ab}) will come equipped with all of the derivative operators on M.

The central question at play here concerns what kind of definability (if any) in the context of geometric spaces best captures which structures the space comes equipped with. We have seen reasons to think that this is neither global nor local implicit definability. It remains to be seen whether there is a better candidate.

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