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COMPLEXITY VALUATIONS: A GENERAL SEMANTIC FRAMEWORK FOR PROPOSITIONAL LANGUAGES

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ABSTRACT. A general mathematical framework, based on countable partitions of Natural Numbers [1], is presented, that allows to provide a Semantics to propositional languages. It has the particularity of allowing both the valuations and the interpretation Sets for the connectives to discriminate *complexity* of the formulas. This allows different *adequacy* criteria to be used to assess formulas associated with the same connective, but that differ in their *complexity*. The presented method can be adapted potentially infinite number of connectives and truth values, therefore, it can be considered a general framework to provide semantics to several of the known logic systems (eg, LC, L3 LP, FDE). The presented semantics allow to converge to different standard semantics if the *separation complexity* procedure is annulled. Therefore, it can be understood as a framework that allows greater precision (in complexity terms) with respect to formula satisfaction. Naturally, because of how it is built, it can be incorporated into non-deterministic semantics. The presented procedure also allows generating valuations that grant a different truth value to each formula of propositional language. As a positive side effect, our method allows a constructive proof of the equipotence between $\mathbb{N}$ and $\mathbb{N}^n$ for all Natural n.

1. INTRODUCTION

We will begin by recalling some basic concepts that will be essential to understand the development of the work. As it can be linked, (among others) with the non-deterministic semantics of Nmatrices, in the first section, we will give a very brief introduction to the topic. We will also give an explanation of what it means to have functional connectives and valuations. As the argumentative line of the entire work is closely linked to the concept of *complexity* of a well-formed formula of the language *wff*, we will also remember some basic issues associated with it. In the section 2.2, we present a method that generates special Naturals Number Partitions, called *doubly numbered partitions* (DNP). Our exposition is a summary of main ideas, presented in [1], necessary for the development of this article. The semantics used in our presentation depend on the generated DNPs. In 2.2 (end of section) we show the algorithm that allows generating DNP for any base greater than 1. Valuations are strictly linked to these special partitions in section 3. Once the semantics have been presented, we show some characteristics that the interpretation sets have for the connectives and we provide *adapted adequacy* criteria in section 4. In section 5 we show how our formalism is applied to the systems $Ł_3$, LP and FDE. Continue studying new fields of application will be future work. Finally, in the last section we draw some conclusions from our article.

1.1. Complexity of a Formula, functional connectives and consequence relations.

We will begin by recalling some basic definitions.

Definition 1.1. The set $Frm_{\mathscr{L}}$ of the well-formed formulas of a propositional language $\mathscr{L}$ is defined, giving [2]

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(1) The symbols:
- Propositional variable symbols: $p_1, p_2, ..., p_n, ...$ (considered countable).
- Connective symbols: $(\odot_i)_{i\in I}$, where each $\odot_i$ corresponds to a n natural, which is its arity: 0-ary or constant, unary, binary, . . .
- Punctuation symbols: (,).

2. Formation rules:
- Each p_i is a formula, for $i = 1,2,...$
- If $A_1,A_2,...,A_n$ are formulas and $\odot$ is a n-ary connective, then $\odot(A_1,A_2,...,A_n)$ is a formula. If $\odot$ is binary, we will write
$A_1 \odot A_2$ rather than $\odot(A_1,A_2)$.
- A string of symbols is a formula if and only if it can be obtained from the above two items in a finite number of steps. The above definition gives us the syntax of the propositional calculus language.

Definition 1.2. *Complexity* of a formula.
We define the *complexity* of a well-formed formula of our language as its number of connectives, i.e.:
- If A is an atomic formula, $compl(A) = 0$.
- $compl(\neg A) = compl(A)+1$.
- $compl(A\odot B) = compl(A)+compl(B)+1$.

Where $\odot$ denotes any of the binary connectives of our language. If we were to work with connectives of arity n greater than 2, then $compl(\odot(A_1,...,A_n)) = compl(A_1)+...+compl(A_n)+1$.

In 1.2.1 we will define the deterministic matrices and the functional valuations. But the truth-functionality can also be defined for the connectives of a language. It can be shown that, under very general conditions (involving, for example, closure under subformulas), asking for functionality of valuations is equivalent to asking for veritative functionality of connectives. Let's start by defining truthful-functionality for the connectives of our language.

Let T be the set of valuations (or maps) $t : Sent(L) \to V$ (see [3]).

Definition 1.3. We say that T truely respects the negation functionality if, for all $t,t' \in T$ and any ϕ, $\phi' \in Sent(L)$,

$$t(\phi) = t'(\phi') \Rightarrow t(\neg\phi) = t'(\neg\phi').$$

Definition 1.4. We say that T respects the veritative functionality of the conjunction if

$$\forall t,t' \in T, \quad \forall \phi,\phi',\psi,\psi' \in Sent(L) \quad (t(\phi) = t'(\phi') \text{ y } t(\psi) = t'(\psi') \\ \Longrightarrow t(\phi\wedge\psi) = t'(\phi'\wedge\psi')).$$

Definition 1.5. We say that T respects truth-functionality of disjunction if

$$\forall t,t' \in T, \quad \forall \phi,\phi',\psi,\psi' \in Sent(L) \quad (t(\phi) = t'(\phi') \text{ y } t(\psi) = t'(\psi') \\ \Longrightarrow t(\phi\vee\psi) = t'(\phi'\vee\psi')).$$

We will now give two definitions of logical consequence.

Definition 1.6. Pure logical consequence.
Let $S \subset V$ be a proper subset of the set of truth values. We say that the sets Γ y Δ are connected by a *pure* relationship of logical consequence, $\Gamma \models_P \Delta$, if and only if for each valuation v and each formula $\gamma \in \Gamma$, if $v(\gamma) \in S$, then there exists $\delta \in \Delta$, such that $v(\delta) \in S$.

Definition 1.7. Mixed logical consequence.
Let $S_1 \subseteq V$ and $S_2 \subseteq V$ be subsets of truth values. A logical consequence relation between sets of formulas Γ y Δ will be called *mixed*, $\Gamma \models_M \Delta$, if and only if for each valuation v and each formula $\gamma \in \Gamma$, if $v(\gamma) \in S_1$, then, there exists $\delta \in \Delta$, such that $v(\delta) \in S_2$.

Pure consequence relations can be considered a proper subset of mixed relations. To see more details and definitions about these relations, we recommend [4, 5]. Each of these relations can be thought of as defining a subset of $\mathscr{P}(Frm_{\mathscr{L}}) \times \mathscr{P}(Frm_{\mathscr{L}})$.

1.2. **Non-deterministic semantics.** Non-deterministic multi-valued matrices (Nmatrices) are a fruitful and rapidly expanding field of research. They were introduced into [6, 7, 8] and, since then, they have developed rapidly as a fundamental logical theory finding numerous applications, that go from automata theory, to quantum mechanics [9, 10], going through various areas of logic, such as modal logics. The novelty of Nmatrices is that this formalism extends the usual multivalued algebraic semantics of logical systems by importing the idea of non-deterministic calculation,allowing the truth value of a formula to be chosen non-deterministically from a given set of options. Nmatrices have proven to be a powerful tool, whose use conserves all the advantages of ordinary multivalued matrices, while being applicable to a much wider range of logics [11]. In fact, there are many non-classical (propositional) logics which, while not having finite multivalued characteristic matrices, do admit finite Nmatrices and are therefore decidable.

1.2.1. *Deterministic Matrices.* In this section we will follow the approach presented in [11]. In what follows, L is a propositional language and Frm_L denotes the set of well-formed formulas of the language. Metavariables φ, ψ,..., traverse L-formulas, while Γ, Δ,..., will be used for sets of L-formulas. Also, all the outfits are classic. The standard general method for defining propositional logic is based on the use of deterministic matrices (possibly many-valued):

Definition 1.8. An matrice for L is a tuple

$$P = \langle V; D; O\rangle$$

where

- V is a non-empty set of truth values.
- D (designated values) is a non-empty proper set of V.
- For each n-ary connective $\diamondsuit$ in L, O includes an interpretation function $\widetilde{\diamondsuit} : V^n \to V$.

A partial valuation in P is a function v, going from V to a subset $\mathscr{W} \subseteq Frm_L$ closed under subformulas, such that for each connective n-ary $\diamondsuit$ from L and for all$\psi_1, \ldots, \psi_n \in \mathscr{W}$:, the following is fulfilled

$$v(\diamondsuit(\psi_1, \ldots, \psi_n)) = \widetilde{\diamondsuit}(v(\psi_1), \ldots, v(\psi_n)) \quad (1)$$

Proposition 1. ***Analyticity***. *Any partial valuation of a matrice P for L, defined over a set of L-formulas closed under sub-formulas, it can be extended to a total valuation in P.*
Due to this property, any finite matrice P will be decidable.

1.2.2. *Non-deterministic matrices (Nmatrices). We now turn to the non-deterministic case. The main difference is that, in contrast to deterministic matrices, nondeterministic matrices, given their input truth values, assign a set of possible values (instead of a single value).*

Definition 1.9. A non-deterministic matrix (Nmatrix) for L is a tuple $M = \langle V, D, O\rangle$, where:

- V is a non-empty set of truth values.
- $D \in \mathscr{P}(V)$ (Designated truth values) is a proper non-empty subset of V.
- For each n-ary $\diamondsuit$ connective in L, O includes the corresponding interpretation function
$$\widetilde{\diamondsuit} : V^n \to \mathscr{P}(V) \setminus \{\emptyset\}$$

Definition 1.10. (1) A dynamic partial valuation in M is a function v on a closed set under subformulas $\mathscr{W} \subseteq Frm_L$ a V, such that for each connective n-ary $\diamondsuit$ of L and for all $\psi_1, ..., \psi_n \in \mathscr{W}$ the next is fulfilled:

$$v(\diamondsuit(\psi_1, ..., \psi_n)) \in \widetilde{\diamondsuit}(v(\psi_1), ..., v(\psi_n))$$

A partial valuation in M is called a (total) valuation if your domain is Frm_L.

(2) A static (partial) valuation in M is a dynamic (partial) valuation that also satisfies the following principle of compositionality (or functionality) (defined in some $\mathscr{W} \subseteq Frm_L$): for each connective n-ary $\diamondsuit$ of L and for each $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n \in \mathscr{W}$, si $v(\psi_i) = v(\varphi_i)$ $(i = 1, \ldots, n)$, then
$$v(\diamondsuit(\psi_1, \ldots, \psi_n)) = v(\diamondsuit(\varphi_1, \ldots, \varphi_n))$$

It is important to note that the classical (deterministic) matrices correspond to the case in which each $\widetilde{\diamondsuit} : V^n \to \mathscr{P}(V)$ is a function that takes singleton values. In this case there is no difference between static and dynamic valuations, we have (functional) determinism.
To understand the difference between ordinary matrices and Nmatrices, we remember that in the deterministic case, the truth value assigned by a valuation v to a complex formula is defined as follows: $v(\diamondsuit(\psi_1, ..., \psi_n)) = \widetilde{\diamondsuit}(v(\psi_1), ..., v(\psi_n))$. The truth value assigned to $\diamondsuit(\psi_1, \ldots, \psi_n)$ is uniquely determined by the truth values of its subformulas: $v(\psi_1), \ldots, v(\psi_n)$. However, this is not the case for Nmatrices: in general, the truth values of $\psi_1, \ldots, \psi_n$ no univocally determine the value assigned to $\diamondsuit(\psi_1, \ldots, \psi_n)$, since different appraisals that have the same truth values for $\psi_1, ..., \psi_n$ can assign different elements of the performance set $\widetilde{\diamondsuit}(v(\psi_1), \ldots, v(\psi_n))$ a $\diamondsuit(\psi_1, \ldots, \psi_n)$. Therefore, the non-deterministic semantics of Nmatrixes do not fulfill truth functionality, as opposed to matrix semantics. In the table (1), some differences between matrices and Nmatrices are shown.

TABLE 1. Matrices deterministas vs Nmatrices.

	Deterministic matrices	**Nmatrices**
Set of truth values	V	V
Designated Value Set	$D \subset V$	$D \subset V$
Connectives $\diamondsuit$	$\widetilde{\diamondsuit} : V^n \to V$	$\widetilde{\diamondsuit} : V^n \to \mathscr{P}(V) \setminus \{\emptyset\}$
Valuations	Not dynamic	Possibly dynamic / not static.
Veritative functional	If	Not necessarily

Now, we will review the standard definitions of logical consequence. [11].

Definition 1.11. (1) A (partial) valuation v in M satisfies a formula ψ ($v \models \psi$) si ($v(\psi)$ is defined and $v(\psi) \in D$. We say that it is a model of Γ ($v \models \Gamma$) if it satisfies each formula of Γ.

(2) We say that ψ is dynamically (statically) valid in M, in symbols $\models^d_M \psi$ ($\models^s_M \psi$), if $v \models \psi$ for each dynamic (static) valuation v in M.

(3) The dynamic (static) consequence relationship induced by M is defined as follows: $\Gamma \vdash^d_M \Delta$ ($\Gamma \vdash^s_M \Delta$) if each dynamic (static) model v in M of Γ satisfies some $\psi \in \Delta$.

Obviously, the relation of static consequence includes the dynamic, that is to say, $\vdash^d_M \subseteq \vdash^s_M$. Furthermore, for ordinary matrices, we have that $\vdash^s_M = \vdash^d_M$.

Proposition 2. *Let M be an Nmatrix of two values that has at least one non-deterministic operation. So there is no finite family of finite ordinary matrices F, such that* $\vdash^d_M \psi$ *sii* $\vdash_F \psi$.

Proposition 3. *For every (finite) Nmatrix M, there is a (finite) family of ordinary matrices F, such that* $\vdash^s_M = \vdash_F$.

Thus, only the expressive power of dynamic semantics based on Nmatrices is stronger than that of ordinary matrices. The following theorem taken from [6] *is a generalization of the proposition 1 for the case of Nmatrices:*

Proposition 4. *(Analyticity) Let* $M = \langle V, D, O \rangle$ *be an Nmatrix for L, and let* v' *be a partial valuation in M. Then* v' *can be extended to a (total) valuation in M.*

2. Partitions of the Naturals

In this section, we show the method that will allow us to partition natural numbers into countable classes. It will be of crucial importance for the rest of the work. The method, together with several of the consequences that arise from it, was originally presented in [1]. *In this section we give a brief summary of it to introduce the results that we need. The basic idea is to separate the set of Naturals into countable sets, each countable and disjoint in pairs. We will call* Doubly Numberable Partitions *(DNP) to partitions that meet the above requirements.*

2.1. Partitions and equivalence relations.

Let's briefly recall the concepts of partition *and* equivalence relation.

Definition 2.1. A relation is said to be an equivalence relation if and only if it is

- *reflexive*. Every element in the domain is related to itself: $\forall x(x\mathscr{R}x)$.
- *symmetric*. If x is related to y, then y is related to x: $\forall x\, \forall y\, (x\mathscr{R}y \Rightarrow y\mathscr{R}x)$.
- *transitive*. If x is related to y and y is related to z, then x is related to z: $\forall x \forall y \forall z$ $(x\mathscr{R}y \wedge y\mathscr{R}z \Rightarrow x\mathscr{R}z)$.

The equipotential relation between sets is an equivalence relation. As an example we can name that both even and odd numbers are equipotent with the set of natural numbers. This is related to the fact that there is a partition of the natural numbers formed by the even and the odd numbers. On the other hand, since the Naturals and the Integers are equipotent, we have that even or odd have the same number of elements as the Integers (using the transitivity of the equipotential relation). When X is equipotent with the set of natural numbers (X *numerable*), its cardinal is said to be *Aleph subzero* and it is denoted by $\aleph_0$. All countable sets have the same cardinal value, $\aleph_0$. When the above is true, we will write

$$|X| = \aleph_0.$$

This is central in our work, since the DNPs fulfill the previous equation, they are all equipotent with the Natural ones. Partitions have a very close link with equivalence relations: every equivalence relation defined on elements of a set X generates a unique *partition* of this set. What do we mean by a *partition*? Intuitively, a partition of a set X is a way of generating subsets of X, such that they are disjoint, that is, their intersection is empty, and their union is also the original set X (no element can be left out).

Definition 2.2. Let $\{X_i\}_{i\in\Delta}$ be a family of non-empty subsets of X, with i belonging to a fixed set of subscripts Δ. We say that this family is a partition of X if

- $\bigcup_{i\in\Delta} X_i = X$.
- $X_i \cap X_j = \emptyset \quad$ for all $\quad i \neq j$.

As we mentioned before, every partition of a domain defines an equivalence relation on it and vice versa. [12, 13]. We will generate the partitions and therefore obtain the associated equivalence relations.

2.2. **Doubly Numberable Partitions.** We will start this section by showing how to generate one of the *at least* countable DNPs. For a partition of the Naturals to be called *doubly countable*, it is necessary and sufficient that each one of the countable subsets that make up the partition of the domain (it cannot be a finite partition) is countable, that is, equipotent with $\mathbb{N}$, that are disjoint two by two (their intersection is the empty set) and that their union equals the natural numbers.

Let us consider the natural numbers with their usual order starting with 1 (but the argument is the same if we consider that 0 is natural). We could also extend this process to Integers. Let's start with the natural series:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 ...

With the colors we simply specify the set to which we will make each natural belong, that is, our process generates successive sets of double length than the previous one starting with the initial length 2. This is,

$$C_1 = \{1,2\} \quad ; \quad C_2 = \{3,4,5,6\} \quad ; \quad C_3 = \{7,8,9,10,11,12,13,14\}$$

$$C_4 = \{15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30\}, \text{etc.}$$

It is clear that

$$\mathbb{N} = \bigcup_{l=1}^{\infty} C_l \quad ; \quad C_l \cap C_{l'} = \emptyset \quad ; \quad l \neq l'.$$

> <u>*Remark*</u>: Note that the C_l form a partition of the Naturals, but such a partition is not doubly countable. This is because, despite there being as many C_l as there are natural numbers (and being disjoint), they are not countable (they are all finite).

The next step is to choose the first element of each of the sets C_l to form the set A_1 . This can be done algorithmically because the Naturals are *well-ordered*. We then have:

$$A_1 = \{1,3,7,15,31,...\}.$$

Where the elements of A_1 maintain the following recursion:

$$a_{1,i+1} = 2.a_{1,i} + 1 \;;\; a_{1,1} = 1.$$

Each element of this set is equal to the successor of the double of its previous one. Furthermore, it is clear that it is a countably infinite set, since there are infinitely many C_l and each has a first element. Now we build the next array, A_2, by taking the second element of each C_l . This is equivalent to taking the first element already used in A_1 out of the original sets and taking the first element back. This is something that can be done without drawback recursively.

$$A_2 = \{2, 4, 8, 16, 32, \ldots\}.$$

Where its elements maintain the following relationship:

$$a_{2,i+1} = 2.a_{2,i} \ ; \ a_{2,1} = 2.$$

We obtain a new countable set where each element is twice its previous one. In forming the third set, we must take the third element from each of the C_l. We directly observe that C_1 has no third element, so we start with the next set, C_2, which is the first set that has a third element.

$$A_3 = \{5, 9, 17, 33, \ldots\}$$
$$a_{3,i+1} = 2.a_{3,i} - 1 \ ; \ a_{3,1} = 5.$$

In this case, each element is generated from the previous one by duplicating it and subtracting 1.
Continuing this way:

$$A_4 = \{6, 10, 18, 34, \ldots\}$$
$$a_{4,i+1} = 2.a_{4,i} - 2 \ ; \ a_{4,1} = 6.$$

For the following case, it should be noted that the first two C_l do not have a fifth element, so the first element of A_5 will belong to C_3, the first of these sets that has a fifth element, 11.

$$A_5 = \{11, 19, 35, 67, \ldots\}$$
$$a_{5,i+1} = 2.a_{5,i} - 3 \ ; \ a_{5,1} = 11.$$

It is clear how to continue our construction and, furthermore, that it can be done taking as base any number greater than 1, instead of base 2 as we have chosen to show the construction process. By construction, it is true that:

$$\mathbb{N} = \bigcup_{n=1}^{\infty} A_n \quad ; \quad A_n \cap A_{n'} = \emptyset \ , \ n \neq n' \quad \text{and} \quad \forall n \quad |A_n| = |\mathbb{N}| = \aleph_0 \tag{2}$$

It is easy to prove that each natural number belongs to only one of the A_n. This is due to the fact that each natural belongs to a unique C_l (they are disjoint) and that within that set it can occupy only one location, the nth. When we show the general case shortly, it will be seen that because of how the n associated with each natural number is computed, having the same n generates a relation that is reflexive, transitive, and symmetric, that is, it generates a relation of *equivalence*. Therefore, we have the natural numbers separated into a countable number of disjoint sets, each one equipotent with $\mathbb{N}$. This partition of the natural numbers is an example of what we call *doubly countable partition* (DNP) and will have its corresponding equivalence relation associated. In addition, they will be absolutely linked to our semantics.

DNPs originally arose in the context of *Hilbert's Hotel*. They present an alternative solution for the case in which the hotel is full and numerous contingents of tourists arrive simultaneously, each with countable individuals. In such a case, we have how to assign a piece to each person. This way is not only a different alternative to the classical solution shown, but also generates infinitely many more equivalent alternatives. We index the contingent in question with n and assign it the class A_n together with an index i that denotes its position within the A_n. Since there will be a single natural associated with each pair (n, i), each individual will have their own piece. Each person belongs to a single contingent,

which univocally determines the n, and has a given order (i) within it, which is also unique. Therefore, each person has a unique pair (n,i). If the hotel were already fully occupied when the infinite contingents of tourists arrive, then we leave all the odd rooms free (with the classic ruse of the problem) and carry out the partition shown on the odd rooms.
We have created a particular example of countable disjoint sets of pairs, each countable, whose union is countable (since their union is the set $\mathbb{N}$). Each constructed DNP represents an example of a countable union of countable sets that results in a new countable set (the Naturals). That is, we have infinite examples (as many as real numbers, see [1]) of a general theorem that can only be proved with the help of the *axiom of choice*: *the countable union of countable sets and disjoint pairs have the cardinality of the Naturals.*

2.2.1. *Formalization of the problem.* We are going to show the recursive function that bi-univocally assigns n and i to each natural number x. The n will determine to which class (A_n) it belongs and the i will designate its position within it. That is, given $x \in \mathbb{N}$, we will assign a single class A_n and within this enumerable class, the i-th position. This will be one possible way, among many, of bijectively assigning a pair (n,i) to each natural x, which shows the relationship between the problem of generating doubly countable partitions, and joining countable disjoint sets, with that of establishing a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$.

Any natural number x greater than 2 can be delimited in the following way (the case $x=1$ and $x=2$ does not cause any problem, since we know how to locate them):

$$\sum_{j=1}^{m} 2^j < x \leq \sum_{j=1}^{m+1} 2^j. \tag{3}$$

The above condition can be equivalently expressed as

$$2^{m+1}-2 < x \leq 2^{m+2}-2.$$

We will use the first form when doing the following calculations because it better demonstrates the sequential and constructive reasoning that we use.
According to the equation (3), each natural univocally defines a m. It can be seen that $l=m+1$, where l is the subscript of the sets C_l at the beginning. We define the number n (class A_n) associated with x as follows:

$$n = x-\sum_{j=1}^{m} 2^j = x-(2^{m+1}-2) = x-2^{m+1}+2, \tag{4}$$

where m is determined by the equation (3).
Let's see an example: si $x=17$, then

$$2+4+8 < 17 \leq 2+4+8+16.$$

Using (3) we have that $m=3$ and $l=4$. With (4) we calculate that $n=17-2^4+2=3$. Therefore, $17\in A_3$, a result that agrees with our definition of A_3. It remains now that we determine what position it occupies within this class. The i-th location within A_n is given by:

$$i = log_2\left(\frac{x-n+2}{a_{n,1}-n+2}\right)+1. \tag{5}$$

Where $a_{n,1}$ is the first element of the set A_n and is given by:

$$a_{n,1} = n+\sum_{j=1}^{k} 2^j = 2^{k+1}+n-2 \tag{6}$$

with k satisfying:

$$2^k < n \leq 2^{k+1}. \tag{7}$$

The number k can be null or negative (-1). If $n = 1$, then $k = -1$:

$$k \in \{-1, 0, 1, 2, 3, 4, 5, ...\}.$$

This does not generate a problem in the equation (6) (when calculating the first element of that set) if we take the convention that the result of the summation is null every time the upper index is less than the lower one. Let us now see the complete operation of this algorithmic procedure. Suppose we have $x = 131$ and see what location it corresponds to within its corresponding A_n class. Primero debemos obtener el n con las ecuaciones (3) y (4):

$$2+4+8+16+32+64 < 131 \leq 2+4+8+16+32+64+128 \implies m = 6 \text{ y } n = 131-126 = 5.$$

That is to say, $131 \in A_5$. Using the equations (5), (6) and (7) we calculate the associated i. By (7):

$$2^2 < 5 \leq 2^3.$$

Concluding that $k = 2$. Using this k in (6):

$$a_{5,1} = 5 + \sum_{j=1}^{2} 2^j = 11.$$

With which we obtain the first element of this equivalence class. Finally, by (4):

$$i = log_2(\frac{131-5+2}{11-5+2}) + 1 = 5$$

Therefore, 131 is the fifth element of A_5. In relation to the hotel, our result says that we must assign room number 131 to the fifth tourist of contingent number 5. Although in this way what we did was assign a room to a tourist, we can do the reverse procedure as we will show below. If we wanted, for example, to see which piece to assign to the sixth tourist in contingent 3, what we have to do is see which natural number occupies position 6 in A_3. We must calculate the first element of this class and then recursively the other elements through the recursion that characterizes A_3, in this case, double the previous element to take its successor. The general recursion relation behind all A_n is the following:

$$a_{n,i+1} = 2^i a_{n,1} - (n-2)(2^i - 1). \tag{8}$$

Where both n and i are taken from 1 and each $a_{n,1}$ is computed according to (6). This function is the inverse of the one that assigns a pair (n,i) to each x. Let's see that this agrees with the first A_n that we showed at the beginning. If $n = 1$, then

$$a_{1,i+1} = 2^i a_{1,1} + (2^i - 1).$$

What generates the succession $1, 3, 7, 15, ...$ starting from its first element, $a_{1,1} = 1$.
Si $n = 2$,

$$a_{2,i+1} = 2^i a_{2,1}.$$

For $n = 3$,

$$a_{3,i+1} = 2^i a_{3,1} - (2^i - 1).$$

And in this way it can be corroborated for the rest of the numbers.
Returning to the hotel, we were interested in seeing which room to give to the sixth member of contingent 3.Let us first compute $a_{3,1}$. According to the equation (7),

$$2^1 < 3 \leq 2^2$$

implies that $k = 1$. So, by (5):

$$a_{3,1} = 3 + \sum_{j=1}^{1} 2^j = 5.$$

And finally we get (from (8)):

$$a_{3,6} = 2^5(5) - (2^5 - 1) = 129.$$

Therefore, if we were in the framework of the *Hilbert's Hotel* problem, we would award room 129 to the traveler in question. Something essential for our procedure is that the sum of the lengths of the sets C_l is not bounded. Thus, we have at least countable examples, which we can effectively generate, that show this property of countable sets with respect to the cardinality of their union. The equivalence relation behind all these partitions, regardless of the base that generates it, is the following: *two naturals x, y are related if and only if they belong to the same A_n, that is , if they have associated the same n* (4). Having the same n assigned proves reflexive, symmetric, and transitive.

The previous line of reasoning can be generalized for the case of a generic base $b \in \mathbb{N} \setminus \{1\}$.Their corresponding equations are:

$$a_{n,i+1} = b^i a_{n,1} - ((b-1)n - b)(\frac{b^i - 1}{b - 1}) \tag{9}$$

$$a_{n,1} = n + \sum_{j=1}^{k} b^j \tag{10}$$

$$b^k < n \leq b^{k+1} \tag{11}$$

The program that performs the corresponding base 2 partition can be downloaded at the following link. Given a natural x, it assigns the corresponding n and i, as well as showing some elements of the A_n in question, and if enters the pair (n, i), returns the natural that corresponds according to our equations: `https://drive.google.com/file/d/0B-rDGOh8gCl2OE1RdE5BVGhac1E/view`

2.3. **Generalized DNPs.** In this part, we will generalize the recursive DNP generation method. This generalization was not presented by its authors in [1] and can be considered the first contribution of our article. Such a generalization is not necessary to understand the basic procedures related to the semantics that we will present (we only use it in some of the applications at the end of the paper). It could be useful if we wanted, in the near future, to generalize the method presented in the next section. Therefore, any reader will not have comprehension problems if they want to skip this subsection and go to 3.

As we have seen, a particular DNP consists of generating a partition of $\mathbb{N}$ into enumerable classes, each one enumerable. As each of the classes is equinumerable with the starting set, we can associate a new partition to each one. In this way, we will get a new partition associated with the original one. Since this process can be iterated as many times as one wishes, we will have an effective way to accomplish this task. Such a form continues in the framework of recursive processes and allows, among other things, to obtain an alternative (algorithmic) proof of the equinumerability of the sets $\mathbb{N}$ and $\mathbb{N}^n$ for all n nature. It is easy to see that the proposed method for generating the DNPs provides an effective method to prove that $\mathbb{N}$ and $\mathbb{N}^2$ are equipotent. One only has to take into account the bijection that a DNP, in a given base, establishes between each natural number x and the pair (n,i).Where the first element of the pair indicated the class of x and the second, the position within it. That is, to each natural x an A_n is associated, such that $x \in A_n$ in the i-th position. If we apply the same procedure to A_n (and to all the other sets of that partition), we will have associated a triad (n,i,j) with each natural x. That is, to each x we biunivocally associate an $a_{n,i,j}$, which means that we apply a DNP to the class A_n (with the same original base) obtaining a new partition doubly countable, $\{B_m\}_{m\in\mathbb{N}}$, where $x \in B_i$ in the j-th position. Let us show this procedure in a little more detail.

If in a first DNP of $\mathbb{N}$ we obtain the classes $\{A_m\}_{m\in\mathbb{N}}$, then we have that

$$\mathbb{N} = \bigcup_{i=1}^{\infty} A_i$$

, where

$$\begin{aligned} A_1 &= \{a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{1,n}, \ldots\} \\ A_2 &= \{a_{2,1}, a_{2,2}, a_{2,3}, \ldots, a_{2,n}, \ldots\} \\ &\vdots \\ A_n &= \{a_{n,1}, a_{n,2}, a_{n,3}, \ldots, a_{n,n}, \ldots\} \\ &\vdots \end{aligned}$$

If we apply the same effective procedure to each class A_m of the previous ones, we obtain

$$A_m = \bigcup_{j=1}^{\infty} A_{m,j}$$

, with

$$\begin{aligned} A_{m,1} &= \{a_{m,1,1}, a_{m,1,2}, a_{m,1,3}, \ldots, a_{m,1,n} \ldots\} \\ A_{m,2} &= \{a_{m,2,1}, a_{m,2,2}, a_{m,2,3}, \ldots, a_{m,2,n}, \ldots\} \\ &\vdots \\ A_{m,n} &= \{a_{m,n,1}, a_{m,n,2}, a_{m,n,3}, \ldots, a_{m,n,n} \ldots\} \\ &\vdots \end{aligned}$$

Therefore, to a natural x that in the first partition the pair (n,i) was paired, that is, we associated the number $a_{n,i}$, after a second DNP, will have associated the triad (n,m,j), where $i = a_{m,j}$.

$$a_{n,m,j} = a_{n,a_{m,j}}.$$

For example, in the partition associated with the equation 8, we associated the number 31 with the class A_1, in position 5, that is, $a_{1,5}$. In turn, 5 was in class A_3, position 1. Therefore, after a second partition, $a_{1,5}$ is associated with $a_{1,3,1}$.

$$31 \longleftrightarrow a_{1,5} \longleftrightarrow a_{1,3,1}.$$

It can also be confirmed that the number 19 is related to the pair $(5,2)$, that is, with $a_{5,2}$. Since 2 is related to the pair $(2,1)$, then 19 will correspond, in the second application of the DNP, el $a_{5,2,1}$.

$$19 \longleftrightarrow a_{5,2} \longleftrightarrow a_{5,2,1}.$$

Since our assignments through the DNPs are bijective, there is no way that more than one $a_{n,m,j}$ is associated to each $a_{n,k}$ in this way. All these associations are one-to-one, therefore, we have an effective mechanism to prove the equipotence between $\mathbb{N}, \mathbb{N}^2$ and $\mathbb{N}^3$. The procedure can be generalized to the general case as follows:

$$a_{n_1,n_2,\ldots,n_{k-2},n_{k-1},n_k} \longleftrightarrow a_{n_1,n_2,\ldots,n_{k-2},m}$$

with $m = a_{(n_{k-1},n_k)}$.

For example, we know that $a_{2,3,4,1,1}$, which is a coefficient product of performing the same partition 4 times, will have associated $a_{2,3,4,a_{(1,1)}} = a_{2,3,4,1}$, since in the considered DNP, $1 = a_{(1,1)}$.

$$a_{2,3,4,1,1} \longleftrightarrow a_{2,3,4,1} \longleftrightarrow a_{2,3,a_{(4,1)}} \longleftrightarrow a_{2,a_{(3,a_{(4,1)})}}$$

using the program
`https://drive.google.com/file/d/0B-rDGOh8gCl2OE1RdE5BVGhac1E/view`
, that the DNP performs in base 2, we can calculate all the coefficients that appear and finish the assignment. According to this program, $a_{4,1} = 6$. Therefore,

$$a_{2,3,4,1,1} \longleftrightarrow a_{2,3,4,1} \longleftrightarrow a_{2,3,a_{(4,1)}} = a_{2,3,6} \longleftrightarrow a_{2,a_{(3,6)}} = a_{2,129}$$

Of course, all these accounts could be done by hand with what is presented in the formalization part of the DNP. Therefore, we have an algorithmic method that proves the equipotence between $\mathbb{N}$ and $\mathbb{N}^n$ for all $n \in \mathbb{N}$.

3. Application of DNPs to valuations

We are already in a position to relate the DNPs with our semantics. This is the core of our work.

We will consider the DNP, on the desired basis, fixed. We will work with the DNP generated in base 2, but the results naturally extend to any base greater than 1. Once the partition of the Naturals into countable sets is determined, we establish the following relationship:

$$A_1 = \{1, 3, 7, 15, 31, \ldots\} \longleftrightarrow D_p$$

$$A_2 = \{2, 4, 8, 16, 32, \ldots\} \longleftrightarrow N_p$$

$$A_3 = \{5, 9, 17, 33, \ldots\} \longleftrightarrow D_{\neg p}$$

$$A_4 = \{6, 10, 18, 34, \ldots\} \longleftrightarrow N_{\neg p}$$

$$A_5 = \{11, 19, 35, 67, \ldots\} \longleftrightarrow D_{p\vee q}$$

$$A_6 = \{a_{6,1}, a_{6,2}, a_{6,3}, a_{6,4}, \ldots\} \longleftrightarrow N_{p\vee q}$$

$$A_7 = \{a_{7,1}, a_{7,2}, a_{7,3}, a_{7,4}, \ldots\} \longleftrightarrow D_{p\wedge q}$$

$$A_8 = \{a_{8,1}, a_{8,2}, a_{8,3}, a_{8,4}, \ldots\} \longleftrightarrow N_{p\wedge q}$$

$$A_9 = \{a_{9,1}, a_{9,2}, a_{9,3}, a_{9,4}, \ldots\} \longleftrightarrow D_{p\to q}$$

$$A_{10} = \{a_{10,1}, a_{10,2}, a_{10,3}, a_{10,4}, \ldots\} \longleftrightarrow N_{p\to q}$$

Where with D and N we denote the designated and undesignated sets respectively associated with each connective. The first two sets, D_p, N_p, are the sets of named and unnamed values corresponding to atomic statements. If $\odot$ denotes a connective of our language, then $D_{p\odot q}, N_{p\odot q}$ are the corresponding sets associated with *wff* of complexity 1 (for simplicity we do not specify the superscript 1). In general, we will denote them by $D^{\alpha}_{\odot}, N^{\alpha}_{\odot}$, where the superscript α corresponds to the complexity of the formula associated with the set. In this way, for each complexity, we have associated 8 sets A_i of our partition. For example, the 8 associated with complexity 2 formulas are:

$$A_{11} \longleftrightarrow D^2_{\neg}$$

$$A_{12} \longleftrightarrow N^2_{\neg}$$

$$A_{13} \longleftrightarrow D^2_{\vee}$$

$$A_{14} \longleftrightarrow N^2_{\vee}$$

$$A_{15} \longleftrightarrow D^2_{\wedge}$$

$$A_{16} \longleftrightarrow N^2_{\wedge}$$

$$A_{17} \longleftrightarrow D^2_{\rightarrow}$$

$$A_{18} \longleftrightarrow N^2_{\rightarrow}$$

$D^{\alpha}_{\odot}$ must be interpreted as *the set of values designated for the complexity formulas* α *whose principal connective is* $\odot$. Analogously, $N^{\alpha}_{\odot}$ is interpreted. In this way, we are assured of different sets (named and unnamed) for all formulas of different complexity, even though they share their main connective. Since there are as many A_i (disjoint and countable) as there are complexities associated with *fbf*, the process can be continued indefinitely.

> Remark: Our reasoning is independent of whether the designated and undesignated sets of values consist only of $true$ and $false$ or whether there are several different types of these elements in each set. Each of our arrays is countable and could contain different types of named (or unnamed) values. For the moment, we can think that all the elements of each of the sets $D_{\odot}$ will at some point end up in the truth value $true$ (and those of $N_{\odot}$ in $false$).
>
> It can also be noted that we have arbitrarily selected as primitive connectives $\neg, \vee, \wedge \rightarrow$, when we could do with less. We do this to have the greatest possible degree of independence and generality. If we want them to be interdefinable, we will make their respective suitability conditions and truth tables relate appropriately. In our next examples, we will also take double implication as a primitive connective and it will be clear that the number of these we take does not affect the line of our reasoning.

At this point we have two alternatives, which we will explore in parallel: $a)$ analyze properties of these interpretation sets, as if they were interpretation sets for a non-deterministic semantics of Nmatrices. That is, to explore properties of *adequacy* of each set, the relationship that each presents with the valuations, etc. $b)$ use these sets to define evaluations in a functional way, such that each evaluation gives different values to each *wff* of the language. Both problems are strongly interconnected.

Let's start by looking at how to define injective valuations using our sets. We will consider the functions $\mathfrak{v} : \mathscr{F}rm_{\mathscr{L}} \to \mathbb{N}$, such that they assign different values to all atomic propositions (p_n) (countable) on sets assigned to propositional variables (D_p, N_p). That is, if we restrict the domain of $\mathfrak{v}$ to atomic statements, the function $\mathfrak{v} : Atom \to D_p \cup N_p$ must be injective. This is not only possible, but it is easy to prove that there is an uncountable number of valuations with these characteristics. We further ask that such assessments meet the following criteria, which we will call *fitness criteria* for the disjunction, conjunction, and implication interpretation sets (see section 3 of [14]), together with a corresponding criterion for denial.

(1) $\widetilde{\wedge}$:

$$\text{If} \quad a \in D \quad \text{and} \quad b \in D, \quad \text{then} \quad a\widetilde{\wedge}b \subseteq D$$
$$\text{If} \quad a \notin D, \quad \text{then} \quad a\widetilde{\wedge}b \subseteq V \setminus D$$
$$\text{If} \quad b \notin D, \quad \text{then} \quad a\widetilde{\wedge}b \subseteq V \setminus D$$

(2) $\widetilde{\vee}$:

$$\text{If} \quad a \in D, \text{ then} \quad a\widetilde{\vee}b \subseteq D$$
$$\text{If} \quad b \in D, \quad \text{then} \quad a\widetilde{\vee}b \subseteq D$$
$$\text{If} \quad a \notin D \quad \text{y} \quad b \notin D, \quad \text{then} \quad a\widetilde{\vee}b \subseteq V \setminus D$$

(3) $\widetilde{\to}$:

$$\text{If} \quad a \notin D, \quad \text{then} \quad a\widetilde{\to}b \subseteq D$$
$$\text{If} \quad b \in D, \quad \text{then} \quad a\widetilde{\to}b \subseteq D$$
$$\text{If} \quad a \in D \quad \text{and} \quad b \notin D, \quad \text{then} \quad a\widetilde{\to}b \subseteq V \setminus D$$

(4) $\widetilde{\neg}$:

$$\text{If } a \in D, \text{then } \widetilde{\neg}(a) \subseteq V \setminus D$$
$$\text{If } a \notin D, \text{then } \widetilde{\neg}(a) \subseteq D$$

In the section 4 we will analyze these conditions in more detail, but for the moment, and to understand the procedure, we will keep this basic form. In the above conditions, D refers to any of the designated sets for any connective and complexity.

> *Remark*: This property will be mentioned several times throughout our article. Sometimes we will predicate the adequacy of interpretation sets corresponding to a single connective to denote that the corresponding property holds for that connective (and a given complexity).

In all cases, the set D of designated values is considered the countable union of all A_i with odd i. Therefore, $V \setminus D := N$, the set of non-designated values, is the countable union of A_i with i even. This is,

$$D = \bigcup_{i=1}^{\infty} A_{2i-1} \quad ; \quad N = \bigcup_{i=1}^{\infty} A_{2i} \tag{12}$$

This *adequacy* criterion (for $\vee, \wedge$ and $\to$) is proposed in the framework of Nmatrix semantics to ensure that the positive segment of Classical Logic is validated. We will see that our case is similar, since for each connective, and given the input values a, b, we have uniquely assigned a set where the corresponding connective must be interpreted. The main difference with the standard case of non-deterministic semantics (Nmatrix) is that we can propose different *adequacy* criteria for connectives depending on the complexity of each *wff*. In addition, we could provide, if we wanted, an extra criterion to univocally determine the position that each

valuation will assign to each formula within the corresponding set.
If $\odot$ denotes a dyadic connective and $\odot(p_1, p_2)$ is a formula of complexity α, then we impose the following conditions on $\mathfrak{v}$ (for *wff* of higher complexity than 0):

$$\text{If } v(p) = a_{i',j'}, \text{ then } v(\neg p) = a_{i,k} \in A_i \quad ; \quad k = 2^{i'} 3^{j'} \tag{13}$$

$$i = \left\{ \begin{array}{ll} 8\alpha - 5 & \text{If i' is even} \\ 8\alpha - 4 & \text{If i' is odd} \end{array} \right\}$$

In the case of denial, α is the complexity of $\neg p$

$$\text{If } v(p) = a_{i',j'} \quad ; \quad v(q) = a_{k',l'}, \text{ then } v(p \odot q) = a_{i,k} \in A_i \quad ; \quad k = 2^{i'} 3^{j'} 5^{k'} 7^{l'} \tag{14}$$

$$\text{If } \odot \in \{\vee\}, \quad i = \begin{array}{ll} 8\alpha - 3 & \text{If k' o i' is odd} \\ 8\alpha - 2 & \text{If k', i' son pares} \end{array}$$

$$\text{If } \odot \in \{\wedge\}, \quad i = \begin{array}{ll} 8\alpha - 1 & \text{If k', i' they are odd} \\ 8\alpha & \text{If k' o i' is even} \end{array}$$

$$\text{If } \odot \in \{\rightarrow\}, \quad i = \begin{array}{ll} 8\alpha + 1 & \text{If k' is odd o i' is even} \\ 8\alpha + 2 & \text{If k' par e i' is odd} \end{array}$$

In the equation for negation, the subscript i is determined from the subscript i' of $a_{i'.j'}$ corresponding to $v(p)$. If i' is odd, that is, $v(p) = a_{i',j'} \in D$, then i must be even, and consequently $a_{i,k} \in N$. In order to know which is the assigned A_i, of those that belong to N, one must take into account the complexity of the valued formula at that stage. There are 8 sets that are used to interpret formulas of complexity α (greater than 0) with principal connective $\odot$. The *wff* of complexity 0, propositional variables, are assigned the sets A_1 and A_2. Therefore, if a formula has complexity $\alpha > 0$, the 8 assigned interpretation sets are

$$A_{8\alpha-5} \quad , \quad A_{8\alpha-4} \quad , \quad A_{8\alpha-3} \quad , \quad A_{8\alpha-2} \quad , \quad A_{8\alpha-1} \quad , \quad A_{8\alpha} \quad , \quad A_{8\alpha+1} \quad , \quad A_{8\alpha+2} \tag{15}$$

The first two sets correspond to designated and undesignated values for negation, the next two, $A_{8\alpha-3}, A_{8\alpha-2}$, to the respective sets for disjunction, then, $A_{8\alpha-1}, A_{8\alpha}$, are the two sets assigned to the conjunction, and finally there are the two sets that correspond to designated and not designated for the material implication. This way of associating the k-th position within an A_i is only one of the possible alternatives to ensure that two different formulas are never evaluated to the same value. Of course, other criteria can be chosen to ensure the same. We have selected this one for simplicity and because it is enough to show what we want, but none of the reasoning would change if we had another procedure to assign positions. It can be noted that our criteria leaves many elements of each A_i unused by the valuations. For example, since $i > 0$, all k will be even. On the other hand, k can never be a prime greater than 7. That is, all $a_{i,p}$ with p strict prime greater than 7 will not be the image of any valuation (of course, this set is included in the set of odd numbers). Therefore, we have numerable vacant places within each A_i, which could be used if necessary. Finally, we can name that the valuations will use different elements within those that are possible. To name just one example, the element $a_{4,6}$ will only be used by a valuation $\mathfrak{v}$, such that $\mathfrak{v}(p) = a_{1,1}$ for some atomic statement p. It can also be directly verified that these valuations meet the definitions 1.3, 1.4,1.5 of functionality of connectives.

Let's see an application example. Let's calculate the truth value of the following formula of complexity 6, whose main connective is negation.

$$\neg(\neg(p \vee q) \longrightarrow (\neg r \wedge s))$$

From what has been said before, since we have to value a wff of complexity 6, the eight interpretation sets assigned will be $A_{43}, A_{44}, \ldots, A_{50}$. Since its main connective is a negation, we are left with A_{43} and A_{44} (designated and not designated for negation respectively). In order to know which of these sets the valuation of the formula under consideration will belong to, we must know if the value of the formula of complexity 5 that is being negated is designated or not and apply the negation criterion shown in 4.$\widetilde{\neg}$. Using the suitability criteria for the interpretation sets of the connectives given above (1.$\widetilde{\wedge}$, 2.$\widetilde{\vee}$, 3.$\widetilde{\rightarrow}$, 4.$\widetilde{\neg}$), it can be calculated that the set to which the valuation value will belong is A_{44}. To properly develop the method with all its details, we are going to assume the following values even for the propositional variables:

$$\mathfrak{v}(p) = a_{1,1} \in D_p = A_1 \quad ; \quad \mathfrak{v}(q) = a_{2,5} \in N_p = A_2 \quad ; \quad \mathfrak{v}(r) = a_{1,3} \in D_p = A_1 \quad ; \quad \mathfrak{v}(s) = a_{2,2} \in N_p = A_2$$

Let's calculate the value that the valuation assigns to $p \vee q$.As $p \vee q$ is a formula of complexity 1 with principal connective $\vee$, its corresponding set of interpretation will be $D_{p\vee q} = A_5$. Therefore, $\mathfrak{v}(p \vee q) = a_{5,k_1}$, with $k_1 = 2^1 3^1 5^2 7^5$. Where the subscripts of the $a_{i,j}$ and the corresponding powers of the prime factors of k_1 have been intentionally highlighted with the same colors so that their origin can be easily identified. We proceed in the same way to calculate the truth value of the remaining formulas. Let's calculate the value that the valuation assigns to $\neg r$. Since this is a formula of complexity 1 with principal connective $\neg$ and the value of r is designated, then its corresponding interpretation set is $A_4 = N_{\neg p}$. Therefore, $\mathfrak{v}(\neg r) = a_{4,k_2}$, $k_2 = 2^1 3^3$.We are now in a position to see the value of $\mathfrak{v}(\neg r \wedge s)$. We have a formula of complexity 2 whose main connective is a conjunction and we know that its value will not be designated. This means that we will value it in the set A_{16}. That is to say, $\mathfrak{v}(\neg r \wedge s) = a_{16,k_3}$, con $k_3 = 2^4 3^{k_2} 5^2 7^2$. Now let's go back to the formula $p \vee q$ to establish the value that the valuation assigns to its negation. $\mathfrak{v}(\neg(p \vee q))$ is a formula of complexity 2 whose main connective is negation, which means that it will be interpreted in A_{11}. $\mathfrak{v}(\neg(p \vee q)) = a_{11,k_4}$, with $k_4 = 2^5 3^{k_1}$. We still have two steps left, to assign value to the implicative formula $\neg(p \vee q) \rightarrow (\neg r \wedge s)$ in order to then be able to negate it. The implicative formula is of complexity 5, therefore the sets that interpret formulas of this complexity whose main connective is the implication matter of of are A_{41}, A_{42}. Since our implicative formula has an undesignated antecedent, it will have a designated value. That is, the set that corresponds to it will be A_{41}. So, $\mathfrak{v}(\neg(p \vee q) \rightarrow (\neg r \wedge s)) = a_{41,k_5}$, where $k_5 = 2^{11} 3^{k_4} 5^{16} 7^{k_2}$. Finally, $\mathfrak{v}(\neg(\neg(p \vee q) \rightarrow (\neg r \wedge s))) = a_{44,k_6} \in A_{44} = N^6_{\neg}$, with $k_6 = 2^{41} 3^{k_4}$.

In this way we obtain what we want, our valuation assigns within the set A_{44} the element that is in position k_6. This value can be calculated with the algorithms presented in the 2.2.1 section. Regardless of the specific natural number assigned, we know that no other *wff* will be able to correspond to that value through the considered valuation. Since the reasoning shown only depended on the initial values that the valuation assigns to the propositions, if another valuation $\mathfrak{v}'$ assigned the same values, but reversing some order, for example $\mathfrak{v}'(p) = \mathfrak{v}(q)$, $\mathfrak{v}'(q) = \mathfrak{v}(p), \mathfrak{v}'(r) = \mathfrak{v}(r), \mathfrak{v}'(s) = \mathfrak{v}(s)$, then this new valuation would send the statement $\neg(q \vee p) \rightarrow (\neg r \wedge s)$ to the same value found for the original proposition. It should also be noted that the original valuation $\mathfrak{v}$ assigns different values to the two statements considered, since the order of the values assigned to the atomic statements is considered in the calculation. As we well named, the conditions (1.$\widetilde{\wedge}$, 2.$\widetilde{\vee}$, 3.$\widetilde{\rightarrow}$) guarantee the positive fragment of Classical Logic. By adding the condition $(4.\widetilde{\neg})$, we ensure the behavior of classic negation. Therefore, these conditions guarantee that the classical inferences are validated. That is, the valuations presented will satisfy the same theorems as the classical bivalued valuations for CL (see section 4.1).

4. INTERPRETATION SETS FOR DIFFERENT COMPLEXITIES AND SUITABILITY CRITERIA

In this section we will present the general adequacy criteria for each complexity. We will begin by analyzing the case of disjunction. The case of complexity 1 has already been presented, therefore we will apply the same reasoning for complexity 2. If we have a *wff* of complexity 2 whose main connective is a disjunction, two cases can occur: the disjunction connects a propositional variable to the right with a *wff* of complexity 1 to the left or vice versa. In either case our valuations are assigned the same sets of designated and undesignated values. In the deterministic case presented above, the criteria used to select the corresponding k in each case guarantees that they can never occupy the same position within any of the corresponding sets.Therefore, we must take into account that the input values for $\widetilde{\vee}_2(a,b)$ can belong to both $D_p = A_1, N_p = A_2$, propositional case, and to any of $A_3, A_4, ..., A_{10}\,(D_{\neg p}, N_{\neg p}, D_{p\vee q}, N_{p\vee q}, D_{p\wedge q}, N_{p\wedge q}, D_{p\to q}, N_{p\to q})$ if it is a *wff* of complexity 1. If we denote with D^1, N^1, V^1 the sets of designated, undesignated values and truth values corresponding to complexity 1, that is,

$$D^1 = D^1_{\neg}\cup D^1_{\vee}\cup D^1_{\wedge}, D^1_{\to} = A_3\cup A_5\cup A_7\cup A_9 = \bigcup_{l=-1}^{2} A_{7-2l}$$

$$N^1 = N^1_{\neg}\cup N^1_{\vee}\cup N^1_{\wedge}, N^1_{\to} = A_4\cup A_6\cup A_8\cup A_{10} = \bigcup_{l=-1}^{2} A_{8-2l}$$

$$V^1 = D^1\cup N^1 = \bigcup_{l=-2}^{5} A_{8-l}$$

, then we can introduce the *adequacy* criterion for the disjunction associated with *wff* of complexity 2 as follows:

$$\widetilde{\vee}_{2(a,b)} : \left\{ \begin{array}{l} \text{If } a\in D_p \text{ and } b\in V^1, \text{ then } \widetilde{\vee}_2 \subseteq D^2_{\vee} = A_{13} \\ \text{If } a\in N_p \text{ and } b\in N^1, \text{ then } \widetilde{\vee}_2 \subseteq N^2_{\vee} = A_{14} \\ \text{If } a\in N_p \text{ and } b\in D^1, \text{ then } \widetilde{\vee}_2 \subseteq D^2_{\vee} = A_{13} \\ \text{If } a\in D^1 \text{ and } b\in V_p, \text{ then } \widetilde{\vee}_2 \subseteq D^2_{\vee} = A_{13} \\ \text{If } a\in N^1 \text{ and } b\in D_p, \text{ then } \widetilde{\vee}_2 \subseteq D^2_{\vee} = A_{13} \\ \text{If } a\in N^1 \text{ and } b\in N_p, \text{ then } \widetilde{\vee}_2 \subseteq N^2_{\vee} = A_{14} \end{array} \right) \tag{16}$$

We must remember that, because of how our sets were defined, the unions are all disjoint. For all complexity, the following relations hold:

$$D^{\alpha} = D^{\alpha}_{\neg}\cup D^{\alpha}_{\vee}\cup D^{\alpha}_{\wedge}\cup D^{\alpha}_{\to} \quad ; \quad N^{\alpha} = N^{\alpha}_{\neg}\cup N^{\alpha}_{\vee}\cup N^{\alpha}_{\wedge}\cup N^{\alpha}_{\to} \quad ; \quad V^{\alpha} = V^{\alpha}_{\neg}\cup V^{\alpha}_{\vee}\cup V^{\alpha}_{\wedge}\cup V^{\alpha}_{\to} \quad ; \quad V^{\alpha} = D^{\alpha}\cup N$$

In the same way, our DNP produces the following relations even to the interpretation sets of the connectives:

$$\widetilde{\neg} = \bigcup_{\alpha\in\mathbb{N}} \widetilde{\neg}_{\alpha} \quad ; \quad \widetilde{\vee} = \bigcup_{\alpha\in\mathbb{N}} \widetilde{\vee}_{\alpha} \quad ; \quad \widetilde{\wedge} = \bigcup_{\alpha\in\mathbb{N}} \widetilde{\wedge}_{\alpha} \quad ; \quad \widetilde{\to} = \bigcup_{\alpha\in\mathbb{N}} \widetilde{\to}_{\alpha}$$

Therefore, the corresponding *adequacy* criteria in *complexity* α can be expressed as:

$$\widetilde{\vee}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in D^{\beta} \text{ and } b \in V^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq D^{\alpha}_{\vee} = A_{8\alpha-3} \\ \text{If } a \in N^{\beta} \text{ and } b \in D^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq D^{\alpha}_{\vee} = A_{8\alpha-3} \\ \text{If } a \in N^{\beta} \text{ and } b \in N^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq N^{\alpha}_{\vee} = A_{8\alpha-2} \end{array} \qquad (17)$$

$$\widetilde{\wedge}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in N^{\beta} \text{ and } b \in V^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq N^{\alpha}_{\wedge} = A_{8\alpha} \\ \text{If } a \in D^{\beta} \text{ and } b \in N^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq N^{\alpha}_{\wedge} = A_{8\alpha} \\ \text{If } a \in D^{\beta} \text{ and } b \in D^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq D^{\alpha}_{\wedge} = A_{8\alpha-1} \end{array} \qquad (18)$$

$$\widetilde{\rightarrow}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in N^{\beta} \text{ and } b \in V^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq D^{\alpha}_{\rightarrow} = A_{8\alpha+1} \\ \text{If } a \in D^{\beta} \text{ and } b \in D^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq D^{\alpha}_{\rightarrow} = A_{8\alpha+1} \\ \text{If } a \in D^{\beta} \text{ and } b \in N^{\gamma};\, \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq N^{\alpha}_{\rightarrow} = A_{8\alpha+2} \end{array} \qquad (19)$$

$$\widetilde{\neg}_{\alpha(a)}: \begin{array}{l} \text{If } a \in D^{\beta};\, \beta=\alpha-1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq N^{\alpha}_{\neg} = A_{8\alpha-4} \\ \text{If } a \in N^{\beta};\, \beta=\alpha-1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq D^{\alpha}_{\neg} = A_{8\alpha-5} \end{array} \qquad (20)$$

Remark: Unlike the previous form (16), this compact versions are symmetric with respect to α and β. But it is clear that we could formally maintain the asymmetry if necessary.

This partitioning of the interpretation sets for each connective in function of complexity *allows an alternative interpretation* of logical connectives. Normally, we assign an interpretation set for each connective. We now have countably disjoint sets of interpretation for every connective and every given complexity. Therefore, we could think that *we have different connective numerables*, one for each of the previous sets of interpretations. That is, we could think that the disjunction connecting two *wff* of complexities α_1 and α_2 respectively is a different disjunction from the one connecting two formulas of complexities α_3 and α_4, while $\alpha_1+\alpha_2 \neq \alpha_3+\alpha_4$. If equality were given in the previous expression, both connectives would have the same set associated, namely $\widetilde{\vee}_{\alpha_5+1}$, with $\alpha_5 = \alpha_1+\alpha_2 = \alpha_3+\alpha_4$. In addition, the respective conditions for *adequacy (or not)* can be treated independently, so we could have different criteria for each of the sets. This would be, in some sense, similar to having distinct connective numerables in our language.

We could also ask ourselves the following: is the condition $\alpha_1+\alpha_2 \neq \alpha_3+\alpha_4$ inescapable or could we have different connectives also for the case of equality? Because of the way we present our method, we require that the sums are not equal, but it is easy to adapt the method so that cases where the sums agree can be discerned. We could have presented it like this from the beginning. For this, it would be necessary to assign a different set of interpretation to each disjunction (or whatever connective) that connects *wff* of complexity α for each possible decomposition of it as the sum of two complexities. That is, we would need a partition of $\widetilde{\vee}_{\alpha}$ made up of s sets (equivalence classes), where s is the number of ways to add $\alpha-1$. For example, if $\alpha=3$, then $s=3$ (0+2,2+0 and 1+1). The DNP method allows this partition to be carried out in enumerable classes, so we should only apply the method to each A_i set of the original partition again. In this way, we would keep the first s arrays and leave the rest unused (this would be just one of many ways to implement it). That is, we would need a generalized DNP of three subscripts, as we effectively computed in the 2.3 section. With these generalized DNPs, we could not only distinguish sets for complexities α_1, α_2 with the same sum, but we could also have countable and disjoint sets that discriminate the main connective of the main formula and that of all the subformulas that compose it. . That is, a countable and disjoint interpretation set could be assigned that takes into account the entire formation tree of the *wff* in question. In the last section, we will present an example that goes along these lines. We will show how to isolate bad behavior or “anomalies” in our interpretation sets if we want them not to propagate over all the higher complexities. This example will show the use of generalized DNPs and will make it clear how to proceed if one wanted the interpretive sets of connectives to accompany the formula formation chain. Our valuations may change their behavior depending on the complexities involved, since different sets of interpretation for different complexities may have different criteria regarding their *adequacy*. This means that *equivalent replacement* might fail if the replaced formulas had different complexities. Of course, we would recover such a property by collapsing the *adequacy criteria* in the standards, which do not discriminate complexity. Two formulas could be equivalent within the classical framework and yet not be equivalent in the complexity formalism.

4.1. **Functional relationships.** We are in a position to establish some functional relations that can be enlightening. Remember that $Frm_{\mathscr{L}}$ denotes the set of *wff* of a language $\mathscr{L}$, which in our case is the propositional. In future works we will study the feasibility of generalizing the method to incorporate first-order (and even arithmetic) languages. Each of the doubly countable partitions of natural numbers (DNP) generated in section 2.2, is an element of the set $\mathscr{P}_{num}(\mathscr{P}_{num}(\mathbb{N}))$, where

$$\mathscr{P}_{num}(\mathbb{N})) = \{A \subseteq \mathbb{N} : |A| = |\mathbb{N}|\}.$$

That is, the set formed by all the infinite subsets of the natural ones. To establish the interpretation sets shown in the previous section, we have selected a particular DNP, which is in base 2 (for additional details and properties, [1] is recommended). Each of the interpretation sets for the connectives is an infinite subset of the natural ones, that is, an element of $\mathscr{P}_{num}(\mathbb{N})$. We have established an effective method to assign to each formula, based on its main connective and its complexity, certain sets of interpretation (designated and not designated), which are countable subsets of the natural ones:

$$\begin{aligned} v' : Frm_{\mathscr{L}} &\to \mathscr{P}_{num}(\mathbb{N}) \\ wff &\to A_i \end{aligned} \tag{21}$$

For example, in the section 3, we assign the set A_{44} to the formula $\neg(\neg(p \vee q) \longrightarrow (\neg r \wedge s))$. Furthermore, the valuation assigned it a set position within that set, which could not be shared by any other formula (for that given valuation). If we restrict the codomain of v' to the established DNP, that is, to $\{A_i\}_{i\in\mathbb{N}}$, we can make our function surjective.

Let's define the following function, v'', whose domain is the image of v':

$$v'' : \{A_i\}_{i\in\mathbb{N}} \to \{0,1\} \tag{22}$$

$$A_i \to 1 \quad \text{If i is odd} \quad ; \quad A_i \to 0 \quad \text{If i is even}$$

Therefore, we can obtain a new valuation, v, by performing the composition $v'' \circ v'$:

$$v : Frm_{\mathscr{L}} \to \{0,1\} \tag{23}$$

Controlling the *adequacy* criteria imposed on each of the sets of interpretations of the connectives for each complexity, we can make these last functions coincide with the classical valuations. Seen from the point of view of interpretation sets, the above can be thought of as follows: for each formula of complexity α and principal connective $\odot$, we have a function that assigns it (depending on the values of true a,b) any of the interpretation sets $D^{\alpha}_{\odot}, N^{\alpha}_{\odot}$. Each of these sets ends up, in the classical case, at $\{1\},\{0\}$, and if the semantics are not deterministic, at certain $\widetilde{\odot}_{(a,b)} \subseteq D$ or $\widetilde{\odot}_{(a,b)} \subseteq V \setminus D$. In some way, it could be considered that if we stay at the level of the function v', without doing the composition with v'', we

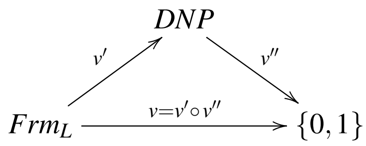

FIGURE 1. Relationship between classical bivalued ratings and complexity ratings.

have a higher degree of precision in our language, since we can handle the evaluations independently for each connective and complexity. We lose this richness when we compose with the function that takes us from the classes of the DNP in question to $\{0,1\}$. Therefore, if we work at the level of A_i we can have an "enriched"semantics (in terms of precision), with the security of converging on Boolean semantics if desired (imposing fitness conditions for all performance sets). At this intermediate level, we would have *a truth table*, which may or may not be deterministic, for each connective and complexity. Numerous truth tables that converge, under the right conditions, to the classical Boolean tables. That is, we could interpret our results as having connective numerals, with their respective associated tables, which, if required, converge to classical logic.

Another equivalent way of understanding complexity ratings is as follows: each rating is a function, with domain $Frm_{\mathscr{L}}$ and codomain $\mathbb{N}$, defined by cases as follows:

$$v(\psi) = \begin{array}{l} a_{i,j} \in D_p \cup N_p \quad \text{If } compl(\psi) = 0 \\ a_{i,j} \in \widetilde{\neg}_{\alpha(v(\phi))} \subseteq D^{\alpha}_{\neg} \cup N^{\alpha}_{\neg} \quad \text{If} \quad \psi = \neg\phi \quad \wedge \quad compl(\psi) = \alpha \\ a_{i,j} \in \widetilde{\vee}_{\alpha(v(\phi_1),v(\phi_2))} \subseteq D^{\alpha}_{\vee} \cup N^{\alpha}_{\vee} \quad \text{If} \quad \psi = \phi_1 \vee \phi_2 \quad \wedge \quad compl(\psi) = \alpha \\ a_{i,j} \in \widetilde{\wedge}_{\alpha(v(\phi_1),v(\phi_2))} \subseteq D^{\alpha}_{\wedge} \cup N^{\alpha}_{\wedge} \quad \text{If} \quad \psi = \phi_1 \wedge \phi_2 \quad \wedge \quad compl(\psi) = \alpha \\ a_{i,j} \in \widetilde{\rightarrow}_{\alpha(v(\phi_1),v(\phi_2))} \subseteq D^{\alpha}_{\rightarrow} \cup N^{\alpha}_{\rightarrow} \quad \text{If} \quad \psi = \phi_1 \rightarrow \phi_2 \quad \wedge \quad compl(\psi) = \alpha \end{array} \tag{24}$$

The sets $D^{\alpha}_{\odot}$ and $N^{\alpha}_{\odot}$ are defined for each complexity and connective according to our algorithm, but in the most general case, where no *adequacy criteria have yet been imposed.* for connectives, we cannot decide which of these disjoint sets the valuation belongs to. In summary, the previous function can be expressed as:

$$v(\psi) = \begin{array}{l} a_{i,j} \in D_p \cup N_p \quad \text{If } compl(\psi) = 0 \\ a_{i,j} \in \widetilde{\neg}_{\alpha(v(\phi))} \subseteq D^{\alpha}_{\neg} \cup N^{\alpha}_{\neg} \quad \text{If} \quad \psi = \neg\phi \quad \wedge \quad compl(\psi) = \alpha \\ a_{i,j} \in \widetilde{\odot}(v(\phi), v(\phi_2)) \subseteq D^{\alpha}_{\odot} \cup N^{\alpha}_{\odot} \quad \text{If} \quad \psi = \phi_1 \odot \phi_2 \quad \wedge \quad compl(\psi) = \alpha \end{array} \tag{25}$$

Remark: The central idea behind these valuations is that, somehow, we perform a DNP in the domain of well-formed formulas, another DNP in the natural numbers, and relate them in a convenient way for our purposes. The DNP of the naturals can be, for example, the one generated in base 2, and that of the domain could be thought to have as its first class the set of propositional variables, the second class formed by all the formulas of complexity one that can be formed. with the propositional variables of the first class and the nth class by all the formulas of complexity n whose subformulas belong to the previous classes.

Comparative example. We will show an example to compare the complexity semantics with the classical two-valued one. Take a formula of complexity 2, such as, for example,

$\Psi = p \vee (\neg q)$ and, for each semantics, a valuation v, such that it designates both atomic propositions, that is $v(p) \in D$ and $v(q) \in D$ (where the set of values designated corresponds to the system we are dealing with).

With classical two-valued semantics, we have: $D = \{1\}$

$$v(p) = v(q) = 1 \quad ; \quad v(\neg q) = 0 \in \widetilde{\neg}_{(1)} \quad ; \quad v(p \vee (\neg q)) = 1 \in \widetilde{\vee}_{(1,0)}$$

If we use complexity ratings:

$$v(p) = a_{1,j} \in D_p = A_1 \quad ; \quad v(q) = a_{1,k} \in D_p = A_1$$

$$v(\neg q) = c \in \widetilde{\neg}^{1}_{(a_{1,k})} \subseteq N_{\neg p} = A_4$$

That is, $c = a_{4,l} \in A_4$, where l indicates the position of the element in this set.

Finally,

$$v(p \vee \neg q) = d \in \widetilde{\vee}^{2}_{(a_{1,j},a_{4,l})} \subseteq D^2_{\vee} = A_{13}$$

This is, $d = a_{13,m}$.

<u>Conclusion</u>:

$$\text{classically} \longrightarrow v(p \vee (\neg q)) = 1 \in \widetilde{\vee}_{(1,0)} = \{1\}$$

$$\text{complexity} \longrightarrow v(p \vee \neg q) = d = a_{13,m} \in \widetilde{\vee}^{2}_{(a,c)} \subseteq D^2_{\vee} = A_{13}$$

What is gained in the case of complexity is that, by having the information given by $v(p \vee \neg q) = a_{13,i}$, we can know the complexity and principal connective of the valued formula. Something that is not within the possibilities of the standard semantic system.

5. Examples of application to multivalued logics

In this section we will show how the DNP method can be adapted to known cases of multivalued logics. We will focus on the following two systems: trivalued system of Łukasiewics and fourvalued FDE. This will make it clear how to adapt the method to more general cases, as we will briefly show in section 5.3 . Strictly speaking, the semantics presented so far were multivalued, since the interpretation sets were infinite, but now we have more classes of countable sets, instead of two, D and N, as in the previous case.

5.1. **Trivalued logic of Łukasiewicz.**

Continuing the line of reasoning presented for the standard case, to atomic propositions, we assign the sets V_p, I_p, F_p, corresponding to A_1, A_2, A_3 respectively. We will follow the canonical interpretation for this system. V will represent true values (v), with I denoting indeterminate values (i) and F denoting false values (f). Which of these values will or will not be designated usually comes hand in hand with the way in which the logical consequence relationship is defined. For example, if it is defined as preserving designated values, then only v is usually taken as designated. We will say more about this shortly.

For any complexity α greater than or equal to 1, we assign the following fifteen sets:

$$A_{15\alpha-11}, A_{15\alpha-10}, A_{15\alpha-9}, A_{15\alpha-8}, A_{15\alpha-7}, A_{15\alpha-6}, A_{15\alpha-5}, A_{15\alpha-4}$$

$$A_{15\alpha-3}, A_{15\alpha-2}, A_{15\alpha-1}, A_{15\alpha}, A_{15\alpha+1}, A_{15\alpha+2}, A_{15\alpha+3}$$

Each connective $\odot$ has associated, for each complexity, three sets: $V^{\alpha}_{\odot}, I^{\alpha}_{\odot}, F^{\alpha}_{\odot}$. And we consider 5 different connectives even for each complexity: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$. We do this, as we have already mentioned above, to maintain the maximum possible generality. If you wish to have only three or four independent ones, you only have to join the corresponding interpretation sets and leave a single *adequacy* criterion (or table).

Remark: Now we will present the criteria of *adequacy* corresponding to the connectives, but it is important to clarify the following point. The expressions below (including those corresponding to the FDE case that we show later) are functions of the sets $V^{\alpha}_{\odot}, I^{\alpha}_{\odot}, F^{\alpha}_{\odot}$ ($V^{\alpha}_{\odot}, b^{\alpha}_{\odot}, n^{\alpha}_{\odot}, F^{\alpha}_{\odot}$ for FDE), rather than based on $D^{\alpha}_{\odot}$ and $N^{\alpha}_{\odot}$. We do this to become independent (or not have to take a position until the end) of the role that these sets have when dealing with the logical consequence. That is, when defining the relation of logical consequence, we could take as designated values only the $V^{\alpha}_{\odot}$, or also include the $I^{\alpha}_{\odot}$. Since we don't want to commit to this yet at this point, we use all the sets presented. Once decided which ones are part of the designated ones, the expressions that we will show can be simplified. Therefore, the conditions that for familiarity and convenience we continue to call *suitability conditions* are halfway between certain non-deterministic truth tables and proper *suitability* conditions. We think that this will not generate problems when it comes to understanding the presentation, since from these conditions the proper expressions of *adequacy* can be obtained if the set D is made explicit.

For each complexity α, $\mathfrak{V}^{\alpha} = V^{\alpha} \cup I^{\alpha} \cup F^{\alpha}$ represents the set of all truth values associated with that complexity.

Conditions of *adequacy*.

$$\widetilde{\vee}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in V^{\beta} \text{ and } b \in \mathfrak{V}^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{15\alpha-8} \\ \text{If } a \in I^{\beta} \text{ and } b \in V^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{15\alpha-8} \\ \text{If } a \in I^{\beta} \text{ and } b \in I^{\gamma} \cup F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq I^{\alpha}_{\vee} = A_{15\alpha-7} \\ \text{If } a \in F^{\beta} \text{ and } b \in V^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{15\alpha-8} \\ \text{If } a \in F^{\beta} \text{ and } b \in I^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq I^{\alpha}_{\vee} = A_{15\alpha-7} \\ \text{If } a \in F^{\beta} \text{ and } b \in F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq F^{\alpha}_{\vee} = A_{15\alpha-6} \end{array} \tag{26}$$

$$\widetilde{\wedge}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in F^{\beta} \text{ and } b \in \mathfrak{V}^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{15\alpha-3} \\ \text{If } a \in I^{\beta} \text{ and } b \in V^{\gamma} \cup I^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq I^{\alpha}_{\wedge} = A_{15\alpha-4} \\ \text{If } a \in I^{\beta} \text{ and } b \in F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{15\alpha-3} \\ \text{If } a \in V^{\beta} \text{ and } b \in V^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq V^{\alpha}_{\wedge} = A_{15\alpha-5} \\ \text{If } a \in V^{\beta} \text{ and } b \in I^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq I^{\alpha}_{\wedge} = A_{15\alpha-4} \\ \text{If } a \in V^{\beta} \text{ and } b \in F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{15\alpha-3} \end{array} \tag{27}$$

$$\widetilde{\rightarrow}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in F^{\beta} \text{ and } b \in \mathfrak{V}^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq V^{\alpha}_{\rightarrow} = A_{15\alpha-2} \\ \text{If } a \in I^{\beta} \text{ and } b \in V^{\gamma} \cup I^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq V^{\alpha}_{\rightarrow} = A_{15\alpha-2} \\ \text{If } a \in I^{\beta} \text{ and } b \in F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq I^{\alpha}_{\rightarrow} = A_{15\alpha-1} \\ \text{If } a \in V^{\beta} \text{ and } b \in V^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq V^{\alpha}_{\rightarrow} = A_{15\alpha-2} \\ \text{If } a \in V^{\beta} \text{ and } b \in I^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq I^{\alpha}_{\rightarrow} = A_{15\alpha-1} \\ \text{If } a \in V^{\beta} \text{ and } b \in F^{\gamma};\ \beta+\gamma=\alpha-1, \text{ then } \widetilde{\rightarrow}_{\alpha} \subseteq F^{\alpha}_{\rightarrow} = A_{15\alpha} \end{array} \tag{28}$$

$$\widetilde{\neg}_{\alpha(a)}: \begin{array}{l} \text{If } a \in V^{\beta};\ \beta=\alpha-1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq F^{\alpha}_{\neg} = A_{15\alpha-9} \\ \text{If } a \in F^{\beta};\ \beta=\alpha-1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq V^{\alpha}_{\neg} = A_{15\alpha-11} \\ \text{If } a \in I^{\beta};\ \beta=\alpha-1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq I^{\alpha}_{\neg} = A_{15\alpha-10} \end{array} \tag{29}$$

Remark: Although we omit it, the corresponding set for double implication can be presented in the same way.

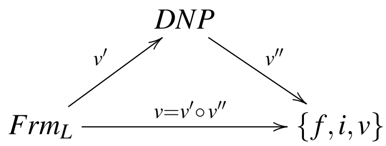

FIGURE 2. Relationship between trivalued ratings of Łukasiewicz and complexity ratings.

In the same way that we did in 21 for the bivalued case, we now pose

$$v' : Frm_{\mathscr{L}} \to \mathscr{P}_{num}(\mathbb{N}) \qquad (30)$$
$$wff \to A_j$$

$$v'' : \{A_j\}_{j\in\mathbb{N}} \to \{f,i,v\} \qquad (31)$$

$$A_j \to f \quad \text{si } j \equiv 0\,(3) \quad ; \quad A_j \to i \quad \text{si } j \equiv 2\,(3) \quad ; \quad A_j \to v \quad \text{si } j \equiv 1\,(3)$$

Therefore, we can obtain a new valuation, v, by performing the composition $v'' \circ v'$:

$$v : Frm_{\mathscr{L}} \to \{f,i,v\} \qquad (32)$$

If we ask that the criteria given by (26), (27), (28), (29), then this semantics will have the same logical consequences as its standard counterpart.

5.2. **FDE.** To adapt our partition to the goals of having a semantic for FDE, we proceed as follows. In FDE "$\neg,\vee,\wedge$" are taken as primitive connectives, but for greater generality, we will take implication and double implication as primitives as well. If we want to get back to classic FDE, we just need to ignore the partition elements associated with implication and double implication, as well as their respective *suitability* conditions. In the same way that if we want to dispense with the distinction by complexities, we simply unite (for each connective) the sets associated with all the complexities. Since the standard semantics for FDE have four truth values, $0,n,b,1$ (interpreted as *false only, neither true nor false, true and false, true only*), then we define

$$\mathfrak{V} = V \cup n \cup \mathbf{b} \cup F = \bigcup_{\alpha} [(V_{\neg}^{\alpha} \cup V_{\vee}^{\alpha} \cup V_{\wedge}^{\alpha} \cup V_{\to}^{\alpha} \cup V_{\leftrightarrow}^{\alpha}) \cup (\mathbf{b}_{\neg}^{\alpha} \cup \mathbf{b}_{\vee}^{\alpha} \cup \mathbf{b}_{\wedge}^{\alpha} \cup \mathbf{b}_{\to}^{\alpha} \cup \mathbf{b}_{\leftrightarrow}^{\alpha})$$
$$\cup (n_{\neg}^{\alpha} \cup n_{\vee}^{\alpha} \cup n_{\wedge}^{\alpha} \cup n_{\to}^{\alpha} \cup n_{\leftrightarrow}^{\alpha}) \cup (F_{\neg}^{\alpha} \cup F_{\vee}^{\alpha} \cup F_{\wedge}^{\alpha} \cup F_{\to}^{\alpha} \cup F_{\leftrightarrow}^{\alpha})]$$

We will take four sets for each connective and complexity. Therefore, associated with the zero complexity we will have the sets $V_p, \mathbf{b}_p, n_p, F_p$, linked respectively with A_1, A_2, A_3, A_4. Then we have, for each (non-zero) complexity α, 20 sets. Four sets for each connective.

$$V_{\neg}^{\alpha}, \mathbf{b}_{\neg}^{\alpha}, n_{\neg}^{\alpha}, F_{\neg}^{\alpha} \quad (A_{20\alpha-15}, A_{20\alpha-14}, A_{20\alpha-13}, A_{20\alpha-12})$$
$$V_{\vee}^{\alpha}, \mathbf{b}_{\vee}^{\alpha}, n_{\vee}^{\alpha}, F_{\vee}^{\alpha} \quad (A_{20\alpha-11}, A_{20\alpha-10}, A_{20\alpha-9}, A_{20\alpha-8})$$
$$V_{\wedge}^{\alpha}, \mathbf{b}_{\wedge}^{\alpha}, n_{\wedge}^{\alpha}, F_{\wedge}^{\alpha} \quad (A_{20\alpha-7}, A_{20\alpha-6}, A_{20\alpha-5}, A_{20\alpha-4})$$
$$V_{\to}^{\alpha}, \mathbf{b}_{\to}^{\alpha}, n_{\to}^{\alpha}, F_{\to}^{\alpha} \quad (A_{20\alpha-3}, A_{20\alpha-2}, A_{20\alpha-1}, A_{20\alpha})$$
$$V_{\leftrightarrow}^{\alpha}, \mathbf{b}_{\leftrightarrow}^{\alpha}, n_{\leftrightarrow}^{\alpha}, F_{\leftrightarrow}^{\alpha} \quad (A_{20\alpha+1}, A_{20\alpha+2}, A_{20\alpha+3}, A_{20\alpha+4})$$

***Adequacy* criteria for FDE.** For reasons of length, we will present only the criteria associated with the sets of interpretation of negation, disjunction, and conjunction. The conditions even for the rest of the connectives are those inherited by their interdefinition (if one wishes to stay within the standard FDE framework). If required, independent criteria can be presented including involvement and double involvement, which adapt to the needs.

$$\widetilde{\neg}_{\alpha(a)}: \begin{array}{l} \text{If } a \in V^{\beta}\,;\, \beta = \alpha - 1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq F^{\alpha}_{\neg} = A_{20\alpha-12} \\ \text{If } a \in \mathbf{b}^{\beta}\,;\, \beta = \alpha - 1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq \mathbf{b}^{\alpha}_{\neg} = A_{20\alpha-14} \\ \text{If } a \in n^{\beta}\,;\, \beta = \alpha - 1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq n^{\alpha}_{\neg} = A_{20\alpha-13} \\ \text{If } a \in F^{\beta}\,;\, \beta = \alpha - 1, \text{ then } \widetilde{\neg}_{\alpha} \subseteq V^{\alpha}_{\neg} = A_{20\alpha-15} \end{array} \tag{33}$$

$$\widetilde{\vee}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in V^{\beta} \text{ and } b \in \mathfrak{V}^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{20\alpha-11} \\ \text{If } a \in \mathbf{b}^{\beta} \text{ and } b \in V^{\gamma} \cup n^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{20\alpha-11} \\ \text{If } a \in \mathbf{b}^{\beta} \text{ and } b \in \mathbf{b}^{\gamma} \cup F^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\vee}_{\alpha} \subseteq \mathbf{b}^{\alpha}_{\vee} = A_{20\alpha-10} \\ \text{If } a \in n^{\beta} \text{ and } b \in V^{\gamma} \cup \mathbf{b}^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\vee}_{\alpha} \subseteq V^{\alpha}_{\vee} = A_{20\alpha-11} \\ \text{If } a \in n^{\beta} \text{ and } b \in n^{\gamma} \cup F^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\vee}_{\alpha} \subseteq n^{\alpha}_{\vee} = A_{20\alpha-9} \\ \text{If } a \in F^{\beta} \text{ and } b \in X^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\vee}_{\alpha} \subseteq X^{\alpha}_{\vee} \end{array} \tag{34}$$

Where X denotes, to simplify the expression, any of the sets $V, \mathbf{b}, n, F$.

$$\widetilde{\wedge}_{\alpha(a,b)}: \begin{array}{l} \text{If } a \in F^{\beta} \text{ and } b \in \mathfrak{V}^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{20\alpha-4} \\ \text{If } a \in n^{\beta} \text{ and } b \in V^{\gamma} \cup n^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq n^{\alpha}_{\wedge} = A_{20\alpha-5} \\ \text{If } a \in n^{\beta} \text{ and } b \in \mathbf{b}^{\gamma} \cup F^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{20\alpha-4} \\ \text{If } a \in \mathbf{b}^{\beta} \text{ and } b \in V^{\gamma} \cup \mathbf{b}^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\wedge}_{\alpha} \subseteq \mathbf{b}^{\alpha}_{\wedge} = A_{20\alpha-6} \\ \text{If } a \in \mathbf{b}^{\beta} \text{ and } b \in n^{\gamma} \cup F^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ ,then } \widetilde{\wedge}_{\alpha} \subseteq F^{\alpha}_{\wedge} = A_{20\alpha-4} \\ \text{If } a \in V^{\beta} \text{ and } b \in X^{\gamma};\, \beta+\gamma = \alpha - 1, \text{ then } \widetilde{\wedge}_{\alpha} \subseteq X^{\alpha}_{\wedge} \end{array} \tag{35}$$

Continuing what was done in the previous cases,

$$v' : Frm_{\mathscr{L}} \rightarrow \mathscr{P}_{num}(\mathbb{N}) \tag{36}$$

$$wff \rightarrow A_j$$

$$v'' : \{A_j\}_{j\in\mathbb{N}} \rightarrow \{f, n, b, v\} \tag{37}$$

$$A_j \rightarrow f \quad \text{si } j \equiv 0\,(4) \quad ; \quad A_j \rightarrow n \quad \text{si } j \equiv 3\,(4) \quad ; \quad A_j \rightarrow b \quad \text{si } j \equiv 2\,(4) \quad ; \quad A_j \rightarrow v \quad \text{si } j \equiv 1\,(4)$$

Therefore, we can obtain a new valuation, v, by performing the composition $v'' \circ v'$ (figure 3):

$$v : Frm_{\mathscr{L}} \rightarrow \{f, n, b, v\} \tag{38}$$

If we ask for the corresponding *adequacy* criteria to be checked, then our semantics will validate the same formulas as the standard case of FDE.

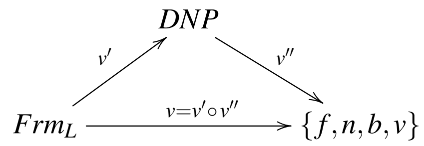

FIGURE 3. Relationship between tetravalued FDE ratings and complexity ratings.

5.3. **General case.** For the case in which there are m truth values and s independent connectives (both finite), we must associate the elements of our partition as follows.

The atomic propositional level will be assigned the sets of truth values (including designated and undesignated)

$$V_1, V_2, ..., V_m \quad \text{(corresponding to } A_1, A_2, ..., A_m\text{)}$$

For complexity α greater than zero, we will have m sets for each of the s connectives, that is, we will have a total of $s.m$ sets

$$V^{\alpha}_{1,\odot_i}, V^{\alpha}_{2,\odot_i}, ..., V^{\alpha}_{m,\odot_i} \quad i \in \{1, ..., s\}$$

If the complexity is 1, the $s.m$ associated sets are:

$$A_{m+1}, A_{m+2}, \ldots, A_{s.m+m}$$

And for generic α:

$$A_{m(s\alpha-s+1)+1}, \ldots, A_{ms\alpha+m}$$

Note that for the particular case $m=4, s=5$, the sets used in the FDE case are reproduced (for each complexity). For example, $m=4, s=5, \alpha=1$, generates the $s.m=20$ sets $A_5, ..., A_{24}$. In the same way, it can be corroborated that it also complies with what was seen in the case of Łukasiewicz. The criteria that the sets must meet must be introduced in each case and taking into account the compatibility with the semantics that one wants (for the case that discriminates complexity)

5.4. **Isolating unwanted behaviors: two examples of modularization.** The objective of our final section is to show, through two examples, a possible effective procedure that can be implemented when it is desired to have interpretation sets with particular behaviors for some complexities, without affecting the other sets. That is, when we want to isolate certain valuation behaviors within some particular regions. We mentioned this possibility before when we asserted that with generalized DNPs it is possible to build sets of interpretations that take into account the entire chain of formation of the *wff*. In order not to overextend this section and to make the example as clear as possible, we will assume that we have only two independent connectives, $\neg, \vee$. The actual procedure shown will make it clear how to generalize for the case of an arbitrary number of connectives (even infinite). It will also allow extracting the general procedure to control behaviors by regions of *complexity* for any connective. The following examples are only intended to show the general idea, not to formalize a method. Although such formalization will be postponed for future work, its plausibility as an algorithmic method to be implemented will be clear.

Example of modularization. Suppose we have a negation with a totally *inadequate* behavior when it comes to formulas of *complexity* 1. Can we make this bad behavior not affect the other formulas of our language? That is, we want the bad behavior of the negation to be isolated for *wff* of *complexity* 1, that is, the values of this non-standard negation have no effect when evaluating formulas such as $\neg(\neg p)$ or $(p \vee \neg q)$. For our first example, let us consider that the interpretation set for formulas of *complexity* 1 with principal connective $\neg$ is such that:

$$\widetilde{\neg}^1_{(a)} \subseteq N^1_{\neg} \quad \forall a \in V_p \tag{39}$$

However, we want its behavior from this complexity to be classical (and not to have "error" carryover for higher complexities). We are going to say that this negation is *anomalous*, or that it behaves *anomalous*, for *complexity* 1. With this set of interpretation, all valuations give undesignated values to the negations of propositions , no matter what truth value they give to the propositional variables. We expose this extreme case to make it clear how to implement the *modularization* procedure, not because we think that this case can have direct application.

It is clear that it is not enough to impose a standard behavior on all sets of larger complexities, that is,

$$\widetilde{\neg}^{\alpha}_{(a)} \subseteq D^{\alpha}_{\neg} \quad \text{si } a \in N^{\alpha-1}_{\neg}; \quad \widetilde{\neg}^{\alpha}_{(a)} \subseteq N^{\alpha}_{\neg} \quad \text{si } a \in D^{\alpha-1}_{\neg} \quad ; \quad \alpha \geq 2 \tag{40}$$

Since in this way, all the formulas of *complexity* 2 with principal connective "$\neg$" will have designated values and we will drag the initial bad behavior. Let's see how to give a possible solution. Let's start by establishing the standard DNP associated with the system. Remember that to simplify the exposition, we take only two independent connectives $(\neg, \vee)$.

$$\begin{aligned}
A_1 &\longleftrightarrow D_p \\
A_2 &\longleftrightarrow N_p \\
A_3 &\longleftrightarrow D^1_{\neg} \\
A_4 &\longleftrightarrow N^1_{\neg} \\
A_5 &\longleftrightarrow D^1_{\vee} \\
A_6 &\longleftrightarrow N^1_{\vee} \\
A_7 &\longleftrightarrow D^2_{\neg} \\
A_8 &\longleftrightarrow N^2_{\neg} \\
A_9 &\longleftrightarrow D^2_{\vee} \\
A_{10} &\longleftrightarrow N^2_{\vee} \\
&\vdots
\end{aligned}$$

Suppose that the only sets that have non-*adequate* behavior, outside of the classical, are those associated with $\widetilde{\neg}^1$. How should we make our valuations isolate this anomalous behavior so that we can continue to hold, for example, $\neg(\neg p) \models p$? (or that $p \vee (\neg q)$ is designated when neither p nor q is designated).

The general procedure consists of acting on the sets from where it is desired to restore a given behavior. In our case, we want to restore the classical behavior for negation starting from *complexity* 2, so we act on $\widetilde{\neg}^2$ and all sets whose input values involve elements of $\widetilde{\neg}^1$. That is, we must also make similar changes to the designated and undesignated sets associated with $\widetilde{\vee}^{\alpha}$, since a *wff* of *complexity* $\alpha \geq 2$ with main connective $\vee$, of the form, for example, $(\phi \vee q) \vee (\neg r)$, it will be committed to the bad behavior of negating statements (q and r are statements and $compl(\phi) = \alpha - 3$).

To concretize, we will begin by calculating the truth value that one of these evaluations gives to the proposition $p \vee (\neg q)$. Then we will give one last example that commits to the double negative.

Let $v(p) = a_{2,1} \in N_p = A_2$ and $v(q) = a_{2,2} \in N_p = A_2$, we want to obtain $v(p \vee \neg q)$. Let us call c the value that our valuation gives to $\neg q$. Then, $v(\neg q) = c \in \widetilde{\neg}^1 \subseteq N^1_\neg$, that is, our valuation will give an undesignated value to the negation of q. If the behavior of the disjunction is considered classical, then the value that the valuation will give to $p \vee \neg q$ will be undesignated. But this will be due, not to the bad behavior of the disjunction, but to the carryover of the undesired value that the valuation gives to $\neg q$. To get around this obstacle, we perform a new DNP on the sets corresponding to the disjunction of *complexity* 2. For this, we can use the procedure shown in the section 2.3. That is, we use *internal degrees of freedom* associated with $\widetilde{\vee}^2$, since $v(p \vee (\neg q)) \in \widetilde{\vee}^2_{(a_{2,1},c)}$. We know that the unwanted input value, c, that will be taken by our array $\widetilde{\vee}^2$ is in $N^1_\neg$, when, in "reality", it should belong to $D^1_\neg$ (if we didn't have the negation anomaly). So let's internally develop $D^2_\vee$ to fix that value internally.

> Remark: we must internally develop this set, and not $N^2_\vee$, because we want to recover some classical behavior from a situation that, in principle, is totally opposite. And in a classical context, we know that, from the values that the valuations gave to the propositional variables, it follows that the truth value of the considered formula must be designated. That is, from complexity 1 we want to return to an adaptation of the connectives of a classical nature. As we have already said, for each complexity one can impose its adequacy criteria. The only new thing now is that we want to have the possibility that the criteria selected for lower complexities do not affect higher ones.

The whole *ruse* of the question lies in recalculating, from the the truth values of the atomic statements (belonging to D_p, N_p), the truth values of the subformulas associated with $\widetilde{\vee}^2$, without having to take anomalous input values (in this case c) corresponding to sets that have the undesired behavior. Performing a new partition on A_9, we obtain countable subsets of the form $A_{9,j}$. We show only the first ones because this is enough for us to make the named changes. We associate to the first two sets of $D^2_\vee$, that is, $A_{9,1}, A_{9,2}$, the sets $D_p(A_1)$ and $N_p(A_2)$ respectively , which is where the desired information unaffected by the complexity denial 1 anomaly is located.

$$A_9 \longleftrightarrow D^2_\vee : \begin{array}{l} {}^2D_p \longleftrightarrow A_{9,1} \equiv A_1 (\text{we associate } a_{9,1,i} \longleftrightarrow a_{1,i}) \\ {}^2N_p \longleftrightarrow A_{9,2} \equiv A_2 (\text{we associate } a_{9,2,i} \longleftrightarrow a_{2,i}) \\ {}^2D^1_\neg \longleftrightarrow A_{9,3} \\ {}^2N^1_\neg \longleftrightarrow A_{9,4} \\ {}^2D^1_\vee \longleftrightarrow A_{9,5} \\ {}^2N^1_\vee \longleftrightarrow A_{9,6} \\ {}^2D^2_\vee \longleftrightarrow A_{9,7} \end{array} \tag{41}$$

We have put left superscripts to distinguish, for the moment, the sets of the original partition from those obtained by this new generalized DNP. It is important to see that $N^1_\neg$ y ${}^2N^1_\neg$ they are different, one is associated with A_4 (DNP in the base 2 with two independent connectives) and the other, $A_{9,4}$. While the anomalous negation of complexity 1 is fully committed to $\widetilde{\neg}^1_{(a)} \subseteq N^1_\neg = A_4$ for every input value a, the same is not true of the calculation, through internal degrees of freedom, that we reproduce using $A_{9,4}$. This is left to the new (and independent) criteria that we impose on the new partition. Since we want to restore a standard behavior, let's impose:

$$D^2_\vee: \begin{array}{ll} \widetilde{\neg}^1_{(a)} \subseteq {}^2D^1_\neg = A_{9,3} & \text{if } a \in N_p \\ \widetilde{\neg}^1_{(a)} \subseteq {}^2N^1_\neg = A_{9,4} & \text{if } a \in D_p \\ \widetilde{\vee}^1_{(a,b)} \subseteq {}^2D^1_\vee = A_{9,5} & \text{if } a \in D_p \text{ or } b \in D_p \\ \widetilde{\vee}^1_{(a,b)} \subseteq {}^2N^1_\vee = A_{9,6} & \text{if } a \in N_p \text{ and } b \in N_p \\ \widetilde{\vee}^2_{(a,b)} \subseteq {}^2D^2_\vee = A_{9,7} & \text{if } a \in {}^2D^\beta \wedge b \in V^\gamma \quad \text{or} \quad b \in {}^2D^\gamma \wedge a \in V^\beta \quad ; \quad \beta+\gamma=1 \end{array} \tag{42}$$

Thus, when we internally evaluate $\neg q$ to get $v(p \vee \neg q)$, we will not use the sets $D^1_\neg$ and $N^1_\neg$ (A_3 and A_4), but ${}^2D^1_\neg$ and ${}^2N^1_\neg$ ($A_{9,3}$ and $A_{9,4}$).

Let's take a closer look at this: how $v(p) = a_{2,1} \in N_p$ y $v(q) = a_{2,2} \in N_p$, then using (42) we have to $v(\neg q) \in \widetilde{\neg}^1_{(a_{2,2})} \subseteq {}^2D^1_\neg = A_{9,3}$, since of $a_{2,2} \in {}^2N_p = A_{9,2} \equiv A_2$. That is, $a_{2,2}$ is $a_{9,2,2}$, according to what we said in the first two lines of (41). Therefore, $v(\neg q)$ is designated, let's assign the value $a_{9,3,j}$ (where j represents the position within the set $A_{9,3}$). Continuing in this way, $v(p \vee \neg q) \in \widetilde{\vee}^2_{(a_{2,1},a_{9,3,j})} \subseteq {}^2D^2_\vee = A_{9,7}$ (by last line of 42). With which we obtain that our proposition acquires a designated value, for example, $a_{9,7,k}$, restoring the given anomaly due to the interpretation set of *complexity* 1 for the negation. It is important to remark, that since all this calculation depends only on the *internal degrees of freedom*, it does not affect the previous valuations. That is, propositional valuations continue to maintain original values and $\neg q$ remains undesignated. The only change will appear from *complexity* 2, restoring a classic behavior, for those interpretation sets that take input values corresponding to ${}^2D^2_\vee$ (which for the purposes of higher complexities, can already be be taken directly as $D^2_\vee$, disregarding the left superscript).

Remark: It is evident that the generalized DNP procedure, as was already highlighted in the section 2.3, along with the actual procedure that grants the *internal degrees of freedom* are entirely algorithmic. Therefore, our recursive procedure can be automatically implemented on a computer. Complexity estimations, including generalized DNPs with *internal degrees of freedom*, represent an effectively implementable procedure with which to construct estimations. The procedure shown developing the *internal degrees* allows compositionality and functionality (within the range allowed by the Nmatrix framework) to be restored to previous steps and not necessarily to the immediately previous one. It is not simply a "to forget what was previously calculated" and set new values to convenience. Effective criteria can be algorithmically implemented complexity by complexity and recalculated, if necessary, from new algorithmic criteria at each step. In our example we set the "compositional" dependency on the values of the atomic statements, skipping the *complexity* level 1. It's as if we could "bypass" certain problem regions and re-establish the dependency with certain previous areas . Taken to the extreme, and despite at all times moving within a non-deterministic framework of Nmatrices, we can make all valuations, regardless of complexity, depend directly on the atomic level without being affected by the intermediate levels. So to speak, this method allows for independent reshuffling and rerolling for each *complexity*. Obviously, when no internal level of freedom is expanded and the *suitability* conditions become independent of complexity, we recover the starting semantics (as we already showed in previous chapters). We can call this characteristic of the interpretation sets for Nmatrix connectives with *internal degrees of freedom dependence on* *initial conditions*. So these initial conditions would recursively determine the behavior of the interpretation sets independently for each *complexity*. All could eventually, if we wished, depend only on the initial conditions, although this dependency may be different for each *complexity*.

Last example. In view of what has just been shown and considering the anomaly in the negation, what would we have to do so that any valuation that does not designate p also does not designate $\neg(\neg p)$? As we have seen, we cannot count on the value that each valuation gives to $\neg p$, since they are all non-designated. What we must do is take the value that each valuation assigns to each propositional variable and, based on this, calculate the set where $\neg(\neg p)$ will be interpreted, which is $\tilde{\neg}^2$. The main difference is that we want this set to be a function of $v(p)$, instead of depending on $v(\neg p)$ (as it would be in the standard way). Or, put equivalently, we want $\tilde{\neg}^2$ to be a function of a new value of $v(\neg p)$, (we could call it $v_{int}(\neg p)$, by internal valuation) that is not committed to the anomaly and which is *only used for the internal calculation of the double negation of p without changing the values established up to now by the valuation*. That is, even if we internally recalculate the truth value of $\neg p$ to obtain an untainted value of $\neg(\neg p)$, we do not change the original anomalous value that the valuation gives to any denial of complexity 1.

They are two equivalent ways of looking at it:

(1) we want $\tilde{\neg}^2_{(v(p))}$, instead of $\tilde{\neg}^2_{(v(\neg p))}$. In other words, to calculate the set where all *wff* of complexity 2 with negation main connective will be valued, we will not take into account the truth value of $\neg p$, but that of p.
(2) we want $\tilde{\neg}^2_{(v_{int}(\neg p))}$, instead of $\tilde{\neg}^2_{(v(\neg p))}$. That is, using a generalized DNP and the internal degrees of freedom associated with $\tilde{\neg}^2$, we will calculate the truth value of $\neg p$ imposing new criteria of *adequacy*, in order to to obtain a value for $\neg(\neg p)$ unaffected by the anomaly of the interpretation set for the negation of complexity 1.

In both points of view we are using the new sets associated with a generalized DNP and imposing independent criteria on them.

As we have already seen in the previous example, we must maintain the information on the truth values that each valuation gives to the propositional variables to calculate the set of interpretation where the formulas in question are going to be valued. Since we want, in principle, to value $\neg(\neg p)$ and $v(\neg(\neg p)) \in \tilde{\neg}^2$, the information must be stored in one of the sets associated with $\tilde{\neg}^2$. We will show how to proceed to $\tilde{\neg}^2$.

The interpretation set for the negation of *complexity* 2 is associated with A_7 and A_8 ($D^2_{\tilde{\neg}}, N^2_{\tilde{\neg}}$). On these two sets we apply the standard procedure to generate a new DNP for each one. In the previous example we performed this procedure only with $D^2_{\tilde{\vee}}(A_9)$, because we had assumed given values for the propositional variables from the outset. But doing it on both sets won't cause any confusion. For this, we apply the previous procedure, developed in the section 2.3. That is, A_7 is separated into countable and disjoint sets $A_{7,1}, A_{7,2}, \ldots, A_{7,n}, \ldots$ and A_8, en $A_{8,1}, A_{8,2}, \ldots, A_{8,n}, \ldots$.

As we already mentioned, the *trick* is to recalculate, from the the truth values of the atomic statements (D_p, N_p), the truth values of the subformulas associated with $\tilde{\neg}^2$, without having to take anomalous input values corresponding to the sets that have the unwanted behavior. For this, in our case, we associate the first two sets of $D^2_{\tilde{\neg}}, N^2_{\tilde{\neg}}$, that is, $A_{7,1}, A_{7,2}$ and $A_{8,1}, A_{8,2}$, the sets $D_p(A_1)$ and $N_p(A_2)$ respectively, which is where the desired information not affected by the complexity denial 1 anomaly is found. From $A_{7,3}$ and $A_{8,3}$ onwards, we proceed in the way that we will now show:

$$A_7 \longleftrightarrow D^2_\neg : \begin{array}{l} {}^2D_p \longleftrightarrow A_{7,1} \equiv A_1 (\text{we associate } a_{7,1,i} \longleftrightarrow a_{1,i}) \\ {}^2N_p \longleftrightarrow A_{7,2} \equiv A_2 (\text{we associate } a_{7,2,i} \longleftrightarrow a_{2,i}) \\ {}^2D^1_\neg \longleftrightarrow A_{7,3} \\ {}^2N^1_\neg \longleftrightarrow A_{7,4} \\ {}^2D^1_\vee \longleftrightarrow A_{7,5} \\ {}^2N^1_\vee \longleftrightarrow A_{7,6} \\ {}^2D^2_\neg \longleftrightarrow A_{7,7} \end{array} \tag{43}$$

The first two sets $A_{7,1}$ y $A_{7,2}$ were replaced by $A_1(D_p)$ y $A_2(N_p)$, which are the ones that keep the original information on the valuations of atomic propositions. In the last line, we arrive at ${}^2D^2_\neg$, which is the final designated set, product of the treatment of these internal degrees of freedom, which will be used in the *adequacies*, without carrying over previous anomalies. This set, ${}^2D^2_\neg$, will be the one we will use, for example, when a connective of *complexity* needs a designated value corresponding to a negation of *complexity* 2. In this way, we were able to calculate the set ${}^2D^2_\neg$ from the original values that the valuation gives to the atomic propositions. It is this set that we will use if we want the truth value of the double negative to coincide with that of the propositional variable. Si $v(p) \in D_p$, then $v(\neg(\neg p)) \in \tilde{\neg}^2_{(v(p))} \subseteq {}^2D^2{}_\neg = A_{7,7}$

We proceed in the same way for $N^2_\neg$:

$$A_8 \longleftrightarrow N^2_\neg : \begin{array}{l} {}^2D_p \longleftrightarrow A_{8,1} \equiv A_1 \\ {}^2N_p \longleftrightarrow A_{8,2} \equiv A_2 \\ {}^2D^1_\neg \longleftrightarrow A_{8,3} \\ {}^2N^1_\neg \longleftrightarrow A_{8,4} \\ {}^2D^1_\vee \longleftrightarrow A_{8,5} \\ {}^2N^1_\vee \longleftrightarrow A_{8,6} \\ {}^2N^2_\neg \longleftrightarrow A_{8,7} \end{array} \tag{44}$$

These *internal degrees of freedom* will allow *fitting* the interpretation sets without dragging on unwanted previous errors. Now it will be seen, as soon as we apply the classic *adequacy* criteria for our example, that these generalized DNPs allow the valuation calculations to be reproduced internally, without taking into account results of previous complexities (except from where we decided to start, in our case D_p and N_p) and without altering them. What we do in these sets (*internal degrees of freedom*) and the successive ones will not affect the valuations of *complexity* 1.

Let's proceed to carry out the *adequacy* of the *internal degrees* for $D^2_\neg$.

$$D^2_\neg : \begin{array}{l} \tilde{\neg}^1_{(a)} \subseteq {}^2D^1_\neg = A_{7,3} \quad \text{If } a \in N_p \\ \tilde{\neg}^1_{(a)} \subseteq {}^2N^1_\neg = A_{7,4} \quad \text{If } a \in D_p \\ \tilde{\vee}^1_{(a,b)} \subseteq {}^2D^1_\vee = A_{7,5} \quad \text{If } a \in D_p \text{ or } b \in D_p \\ \tilde{\vee}^1_{(a,b)} \subseteq {}^2N^1_\vee = A_{7,6} \quad \text{If } a \in N_p \text{ and } b \in N_p \\ \tilde{\neg}^2_{(a)} \subseteq {}^2D^2_\neg = A_{7,7} \quad \text{If } a \in {}^2N^1_\neg \cup {}^2N^1_\vee := {}^2N^1 = A_{7,4} \cup A_{7,6} \end{array} \tag{45}$$

For $N^2_\neg$, we get:

$$N_{\neg}^{2}: \begin{array}{l} \tilde{\neg}^{1}_{(a)} \subseteq {}^{2}D^{1}_{\neg} = A_{8,3} \quad \text{If } a \in N_p \\ \tilde{\neg}^{1}_{(a)} \subseteq {}^{2}N^{1}_{\neg} = A_{8,4} \quad \text{If } a \in D_p \\ \tilde{\vee}^{1}_{(a,b)} \subseteq {}^{2}D^{1}_{\vee} = A_{8,5} \quad \text{If } a \in D_p \text{ or } b \in D_p \\ \tilde{\vee}^{1}_{(a,b)} \subseteq {}^{2}N^{1}_{\vee} = A_{8,6} \quad \text{If } a \in N_p \text{ and } b \in N_p \\ \tilde{\neg}^{2}_{(a)} \subseteq {}^{2}N^{2}_{\neg} = A_{8,7} \quad \text{If } a \in {}^{2}D^{1}_{\neg} \cup {}^{2}D^{1}_{\vee} := {}^{2}D^{1} = A_{8,3} \cup A_{8,5} \end{array} \tag{46}$$

The final sets, ${}^{2}D^{2}_{\neg}, {}^{2}N^{2}_{\neg}$, product of the process of *adequacy* of the *internal degrees of freedom*, are the ones we take to replace the original $D^{2}_{\neg}, N^{2}_{\neg}$. Therefore, we are now in a position to express the *adequacy* condition for $\tilde{\neg}^{2}$, without it being affected by the anomalies of the previous level.

$$\tilde{\neg}^{2}: \begin{array}{l} \tilde{\neg}^{2}_{(a)} \subseteq D^{2}_{\neg}({}^{2}D^{2}_{\neg}) \quad \text{si } a \in N^{1} = N^{1}_{\neg} \cup N^{1}_{\vee} = A_{7,4} \cup A_{7,6} \\ \tilde{\neg}^{2}_{(a)} \subseteq N^{2}_{\neg}({}^{2}N^{2}_{\neg}) \quad \text{si } a \in D^{1} = D^{1}_{\neg} \cup D^{1}_{\vee} = A_{8,3} \cup A_{8,5} \end{array} \tag{47}$$

These last *fits* are a function of the truth value of the previous complexity formula, in our case $v(\neg p)$, but it is equivalent to calculating it from the original value of $v(p)$, because the sets are already adequated in the correct way to correspond to classical behavior. Therefore, we can choose (for the case of negation) to take as input value the truth value of the propositional variable or the value of the previous complexity subformula. This procedure allows us to value the valuations of *complexity* 2 within sets that are not affected by the *anomalies* of the previous level.

Summarizing: thanks to generalized DNPs and *internal degrees of freedom*, each interpretation set of a certain connective of complexity α, you can keep an internal copy of all previous interpretation sets that are needed to recalculate valuations without having to resort to input values that carry errors or unwanted behavior. The method without *internal degrees of freedom* allowed us to provide *suitability* criteria independently for each *complexity*. By incorporating these new *degrees of freedom*, we can further apply these criteria by avoiding misbehavior loopholes that may have arisen in the environment, may have arisen for some reason, or may have been produced intentionally.

It can be seen that with this procedure, by being able to correct the *anomaly* caused by the negation of *complexity* 1, our valuation does not represent a counterexample for the following relation

$$\neg(\neg p) \models p$$

, even though all valuations give undesignated values to $\neg p$.

Take the special case where $v(p_1) = a_{2,j} \in N_p$ and let's see what value this valuation would give us for $v(\neg(\neg p_1))$. As $v(\neg(\neg p_1)) \in \tilde{\neg}^{2}$, if we did not take into account what has just been said, then $v(\neg(\neg p_1)) \in \tilde{\neg}^{2}_{(b)}$, with $b = v(\neg p_1) \in N_p$, since all valuation grants non-designated values to $\neg p_1$. Like, of course, $\tilde{\neg}^{2}(b)$ It has a classic behavior, $\tilde{\neg}^{2}_{(b)} \subseteq D^{2}_{\neg} = A_7$. That is, the valuation designates the double negation of p_1. with which we have to $\neg(\neg p_1) \not\models p_1$.

If, on the other hand, we consider the internal degrees of freedom so as not to depend on the outlier value of the complexity negation 1, we have $v(\neg(\neg p_1)) \in \tilde{\neg}^{2}_{(v(p_1))}$ (instead of $v(\neg(\neg p_1)) \in \tilde{\neg}^{2}_{(v(\neg p_1))}$) and we follow instructions according to (46). This is for (44), we have to $v(p_1) = a_{8,2,j} \in {}^{2}N_p \equiv A_{8,2}$. Therefore, $v(\neg p_1) \in {}^{2}D^{1}_{\neg} = A_{8,3}$. And finally, $v(\neg(\neg p_1)) \in {}^{2}N^{2}_{\neg} = A_{8,7}$, giving an undesignated value to the double negative of p_1.

6. Conclusions

The DNPs constitute a powerful method to disjoint the natives into countable countable sets. The process is algorithmic and there are as many ways to do it as there are real numbers. We show that these partitions can be the basis of a semantic for a propositional language. We have shown that the method allows building a general semantic framework where not only logical systems such as LC, LP, $Ł_3$ and FDE are embedded, but also allows meaning to be given to the formulas depending on their complexity. That is, this general framework admits semantics with sufficient precision to discriminate (and signify) formulas that, having the same logical form, differ in their complexities. The construction method of complexity valuations allows generalizing, in terms of precision, certain semantics, such as the classical two-valued, the trivalued Łukasiewics and the four-valued FDE, among others. The method gives us the guarantee of being able to recover these systems in their standard form if necessary, simply by joining certain disjoint sets. The reasoning presented was fully compatible with the non-deterministic semantics of Nmatrices, so we will be able to present applications in this field in future works. As one of the many consequences of this line of reasoning, we show that it is possible to construct valuations that consistently assign a different truth value to each formula in the language.

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