The Relationship Between Lagrangian and Hamiltonian Mechanics: The Irregular Case*

Clara Bradley

Department of Logic and Philosophy of Science,
University of California, Irvine

Abstract

Lagrangian and Hamiltonian mechanics are widely held to be two distinct but equivalent ways of formulating classical theories. Barrett (2019) makes this intuition precise by showing that under a certain characterisation of their structure, the two theories are categorically equivalent. However, Barrett only shows equivalence between "hyperregular" models of Lagrangian and Hamiltonian mechanics. While hyperregularity characterises a large class of theories, it does not characterise the class of gauge theories. In this paper, I consider whether one can extend Barrett's results to show that Lagrangian and Hamiltonian formulations of gauge theories are equivalent. I argue that there is a precise sense in which one can, and I illustrate that exploring this question highlights several interesting questions about the way that one can construct models of Hamiltonian mechanics from models of Lagrangian mechanics and vice versa, about the role that constraints play, as well as the definition and interpretation of gauge transformations.

* *Draft of August 27, 2024.*

1 Introduction

Lagrangian and Hamiltonian mechanics are widely held to be two distinct but equivalent ways of formulating classical theories. Although there have been recently challenges to this view[1], Barrett (2019) makes precise the sense in which one can maintain that Lagrangian and Hamiltonian mechanics are equivalent: as long as one characterizes the structure of these theories in a certain natural way, one can show that they are theoretically equivalent, where the standard of theoretical equivalence is *categorical equivalence.*

However, Barrett's equivalence result is restricted in an important way: he only shows equivalence between "hyperregular" models of Lagrangian and Hamiltonian mechanics. While hyperregularity characterizes a large class of theories, it does not characterize the class of *gauge theories*: theories that have local, time-dependent symmetries. The question of whether Lagrangian and Hamiltonian mechanics are equivalent in the context of gauge theories is one that has not been discussed directly in the philosophical literature, despite the fact that it bears on other debates that are prominent in the literature. In particular, there has been a recent debate about the correct characterization of the gauge transformations in the Hamiltonian formalism. Several authors have criticized the standard view on the basis that the resulting theory is inequivalent to the Lagrangian formalism.[2] However, one fails to find a clear exposition of which formulations of Lagrangian and Hamiltonian mechanics in the presence of gauge symmetries are equivalent and in what sense.

In this paper, I aim to fill this gap. I demonstrate that the relationship between Lagrangian and Hamiltonian mechanics is made significantly more complicated when the assumption of hyperregularity is dropped, and I argue that the literature has so far failed to establish more than a notion of *dynamical equivalence* in the non-hyperregular context. However, I show that one can extend Barrett's result to prove an equivalence result in the irregular case by constructing hyperregular models of Lagrangian and Hamiltonian

[1]See North (2009) and Curiel (2014).

[2]See in particular Pitts (2014*a*,*b*), Gracia & Pons (1988).

gauge theories through a process known as 'symplectic reduction'. In doing so, I argue that the claims in the literature that the standard approach to gauge transformations renders Hamiltonian mechanics inequivalent to Lagrangain mechanics are false: there is a natural formulation of Lagrangian mechanics in the irregular context that is equivalent to the formulation of Hamiltonian mechanics under the standard definition of gauge transformations.

While ultimately the paper supports the equivalence between Lagrangian and Hamiltonian mechanics in the context of gauge theories, exploring this question will highlight several interesting questions about the way that one can construct models of Lagrangian mechanics from models of Hamiltonian mechanics and vice versa, about the role that constraints play in relating the kinematics and dynamics of a theory, as well as the interpretation of gauge transformations.

In section 2, I spell out the equivalence result in Barrett (2019), paying particular attention to the parts of the result that require the assumption of hyperregularity. In Section 3, I discuss how the situation changes when one considers gauge theories, and present the standard Hamiltonian approach to determining the gauge transformations in terms of a constraint formalism. In Section 4, I consider the arguments in the literature regarding equivalence between Lagrangian and Hamiltonian gauge theories, and I discuss why they fall short of providing an account of *theoretical* equivalence. In Sections 5 and 6, I show that one can reformulate Lagrangian mechanics as a constraint theory in a way that is analogous to formulating a Hamiltonian constraint theory, drawing from the work of Gotay & Nester (1979), and I show that the models of the reformulated Lagrangian gauge theory are related to the models of the Hamiltonian constraint theory in a natural way. In Section 7, I prove an equivalence result that extends the result in Barrett (2019) to the context of gauge theories. Finally, in Section 8 I discuss the upshots of this equivalence result and some possible responses.

2 The Regular Case

The relationship between Lagrangian and Hamiltonian mechanics in the 'regular' case has been widely discussed. On the one hand, North (2009) defends the view that Hamiltonian mechanics has less structure than Lagrangian mechanics. On the other hand, Curiel (2014) agrees that Hamiltonian and Lagrangian mechanics ascribe different structure, but argues that Lagrangian mechanics is a better representation of the structure of classical systems. More recently, Barrett (2019) argues that this debate hinges on how one defines the structure of the two theories: while one can maintain that they are inequivalent by defining the structure of the two theories in certain ways, there is also a natural way of spelling out the structure of the two theories that renders them equivalent under a widely defended account of theoretical equivalence, namely, categorical equivalence.[3]

In light of this debate, let us distinguish three views that one might hold regarding the equivalence between Lagrangian and Hamiltonian mechanics in the 'regular' case:

Lagrangian-first View: Lagrangian mechanics better represents physical systems than Hamiltonian mechanics.

Hamiltonian-first View: Hamiltonian mechanics better represents physical systems than Hamiltonian mechanics.

Equivalence View: Lagrangian and Hamiltonian mechanics are (categorically) equivalent, and so equally well represent physical systems.

Our focus here will be whether the Equivalence View is one that can also be maintained in the irregular case, and so it will be important for our purposes to see how the Equivalence View is defended in the regular case.

Lagrangian mechanics has state space given by the tangent bundle of configuration space, T_*Q, whose points consist of the pair $(q_i, \dot{q}_i)$ encoding the positions and velocities of the particles. The dynamics is given by specifying a Lagrangian function $L(q_i, \dot{q}_i)$,

[3] See Halvorson (2012, 2016), Weatherall (2016*a*,*b*, 2019) for discussion and defense of categorical equivalence as a standard of theoretical equivalence.

with dynamical equations given by the Euler-Lagrange equations, which in coordinate-dependent form are given by: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$.

The fiber derivative of L is called the *Legendre transformation* and it is the map $FL : T_*Q \rightarrow T^*Q$ from the tangent to cotangent bundle that is defined as taking the point $(q_i, \dot{q}_i)$ to $(q_i, \frac{\partial L}{\partial \dot{q}_i})$. We say that L is *regular* iff FL is a local diffeomorphism. When FL is a global diffeomorphism i.e. it is also invertible, we say that the model (T_*Q, L) is *hyperregular.*

Hamiltonian mechanics has as its state space the cotangent bundle of configuration space, T^*Q, whose points consist of the pair (q_i, p_i) encoding the positions and canonical momenta of the particles. The dynamics is given by specifying a Hamiltonian function $H(q_i, p_i)$, with dynamical equations given by Hamilton's equations: $\frac{dq}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial q}$.

The fiber derivative of H is the map $FH : T^*Q \rightarrow T_*Q$ from the cotangent to tangent bundle that is defined as taking the point (q_i, p_i) to $(q_i, \frac{\partial H}{\partial p_i})$. When FH is a (global) diffeomorphism, we say that the model (T^*Q, H) is (hyper)regular.

The cotangent bundle naturally comes equipped with a symplectic (closed, non-degenerate) two-form ω. We can write the equations of motion in terms of this two-form: $\omega(X_H, \cdot) = dH$ where X_H is the vector field associated with the Hamiltonian, which is unique by the non-degeneracy of the symplectic two-form. The integral curves of X_H correspond to solutions.

We can also use this symplectic structure to define a two-form on the tangent bundle, $\Omega = FL^*(\omega)$. Ω is symplectic when FL is a (local or global) diffeomorphism. We can then show that the Euler-Lagrange equations are equivalent to $\Omega(X_E, \cdot) = dE$ where X_E is the vector field associated with the energy function $E = FL(\dot{q}_i)\dot{q}^i - L$. The integral curves of X_E correspond to solutions.

The structure preserving maps of tangent space are given by point$_*$ transformations T_*f, defined as follows: given a diffeomorphism $f : M_1 \rightarrow M_2$, $T_*f : (q, v) \rightarrow (f(q), f_*(v))$. Similarly, the structure preserving maps on cotangent space are given by point* transformations: given a diffeomorphism $f : M_1 \to M_2$, $T^*f : (q,p) \to (f^{-1}(q), f^*(p))$.

Let us restrict ourselves to hyperregular models of Lagrangian and Hamiltonian mechanics. Define the functor F between a hyperregular model of Lagrangian mechanics and a hyperregular model of Hamiltonian mechanics as $F : (T_*Q, L) \to (T^*Q, E \circ FL^{-1})$, $F : T_*f \to T^*(f^{-1})$.

Similarly, define the functor G between a hyperregular model of Hamiltonian mechanics and a hyperregular model of Lagrangian mechanics as $G : (T^*Q, H) \to (T_*Q, (\theta_a(X_H)^a - H) \circ FH^{-1})$, $G : T^*f \to T_*(f^{-1})$ where θ_a is the canonical one-form such that $\omega_{ab} = -d_a\theta_b$. These translation maps preserve empirical content, in the sense that they preserve the base integral curves.[4]

Define the categories **Lag** and **Ham** in the following way:

1. An object in the category **Lag** is a hyperregular model (T_*Q, L). An arrow $(T_*Q_1, L_1) \to (T_*Q_2, L_2)$ is a point$_*$ transformation $T_*f : T_*Q_1 \to T_*Q_2$ that preserves the Lagrangian in the sense that $L_2 \circ T_*f = L_1$.

2. An object in the category **Ham** is a hyperregular model (T^*Q, H). An arrow $(T^*Q_1, H_1) \to (T^*Q_2, H_2)$ is a point* transformation $T^*f : T^*Q_1 \to T^*Q_2$ that preserves the Hamiltonian in the sense that $H_2 \circ T^*f = H_1$.

Then:

> **Theorem** (Barrett (2019)): F : **Lag** $\to$ **Ham** and G : **Ham** $\to$ **Lag** are equivalences that preserve solutions.

The upshot is that as long as one is concerned with *hyperregular* Lagrangian and Hamiltonian models, there is a clear sense in which these theories are equivalent in terms of categorical equivalence. Indeed, the proof of the above theorem relies on hyperregularity in several ways. First, notice that the functors F and G rely on the maps FL^{-1} and FH^{-1}

[4]See Abraham & Marsden (1987, Theorem 3.6.2) for more details.

in order to define a Hamiltonian model in terms of a Lagrangian model and vice versa. These maps are only well-defined functions (globally) if FL and FH are (global) diffeomorphisms. Second, Barrett (2019) proves the above theorem by showing that F and G are inverses in the sense that $GF(T_*Q, L) = (T_*Q, L)$ and $FG(T^*Q, H) = (T^*Q, H)$ (and similarly are inverses on arrows). This relies on the fact that $FL^{-1} = FH$ and $FH^{-1} = FL$, which is only true in the hyperregular context.

Given the importance of hyperregularity in reaching the conclusion that the categories of Lagrangian and Hamiltonian models are equivalent, one might conclude that the class of irregular Lagrangian and Hamiltonian theories cannot be categorically equivalent.[5] However, there are several physically important theories that do not have hyperregular, or even regular, models; most notably, gauge theories are such that the Legendre transformation defines a submanifold of T^*Q. It would be surprising, and significant, if the class of Lagrangian gauge theories and the class of Hamiltonian gauge theories were not equivalent. Therefore, it is worthwhile to consider whether one could set up an equivalence result as strong as categorical equivalence in the context of gauge theories. But to do this, we first need to define the models of the corresponding Lagrangian and Hamiltonian gauge theories. So let us start by considering the way that gauge theories are usually formulated.

3 The Irregular Case

We say that the Lagrangian is *irregular* when the Hessian $W_{ij} = \frac{\partial L}{\partial \dot{q}^i \dot{q}^j}$ is not invertible i.e. when it is singular. A class of irregular Lagrangian theories can be characterized by the fact that the Legendre transformation $FL(T_*Q)$ is a submanifold of T^*Q called the *primary constraint surface* Σ_p, defined by the satisfaction of a collection of (primary) constraints $\varphi(q_i, p_i) = 0$. It is this class of irregular Lagrangian theories that we will take to constitute the gauge theories.

[5] Indeed, in a footnote (16), Barrett (2019) says: "One can, of course, consider the more general case, but I conjecture that there the theories will be inequivalent according to any reasonable standard of equivalence."

Given that the Legendre transformation defines a submanifold of the cotangent space in the context of gauge theories, it seems natural that we should formulate the Hamiltonian theory on this submanifold if we want to relate the two theories. Indeed, if we start with a Hamiltonian theory on T^*Q, then one can specify the theory on the primary constraint surface. First, we can define an induced presymplectic two-form $\tilde{\omega} = i^*\omega$ where $i : \Sigma_p \to T^*Q$ is the inclusion map. The null vector fields of $\tilde{\omega}$ are the vector fields corresponding to the *primary first-class constraints*, which geometrically correspond to the primary constraints whose vector field is tangent to the constraint surface (while the *second-class* constraints are those constraint whose vector field is not tangent to the constraint surface).

Using this presymplectic two-form, the equations of motion on this submanifold can be written as $\tilde{\omega}(X_H, \cdot) = dH$ where H is the Hamiltonian on T^*Q restricted to the constraint surface (this is sometimes called the Hamilton-Dirac equation). Notice that since $\tilde{\omega}$ is degenerate, the solutions to this equation of motion are not unique; we can think of this fact as related to the gauge nature of the theory.

This provides a well-defined theory on the primary constraint surface. However, as Dirac (1964) and others noticed, there are inconsistencies that might arise with this theory. In particular, it may not be that the primary constraints hold at all points along a solution, which corresponds to the fact that the vector fields X_H that define the solutions to this equation may not be tangent to the constraint surface. In order for the solutions to be tangent to the constraint surface, it must be that $\tilde{\omega}(X_H, Z) = dH(Z) = 0$ for vector fields Z associated with the primary constraints. This may define a further collection of constraints called *secondary constraints*, and we can think of these additional constraints and leading to the specification of a further submanifold.

Continuing this process of requiring that the solutions to the equations of motion are tangent to the constraint surface terminates in a *final constraint surface*, $(\Sigma_f, \tilde{\omega}_f, H|_{\Sigma_f})$, defined by the satisfaction of the full collection of $M + S$ constraints, where the null vector fields of $\tilde{\omega}_f$ are those M vector fields associated with the M first-class constraints, and S is the number of second-class constraints. The integral curves of the null vector fields are called the *gauge orbits*. They are M-dimensional surfaces on the constraint surface spanned by the null vectors. In this way, on the final constraint surface, the gauge transformations are given by transformations along the integral curves of the null vector fields associated with first-class constraints.

The equations of motion $\tilde{\omega}(X_H, \cdot) = dH$ only defines X_H up to arbitrary combinations of null vectors when $\tilde{\omega}$ is presymplectic. So following standard usage, let us define the 'Total Hamiltonian' as the equivalence class of Hamiltonians defined up to arbitrary combinations of *primary* first-class constraints i.e. the equivalence class of Hamiltonians on the primary constraint surface. Similarly, we define the 'Extended Hamiltonian' as the equivalence class of Hamiltonians defined up to arbitrary combinations of *primary and secondary* first-class constraints i.e. the equivalence class of Hamiltonians on the final constraint surface.

Going forward, we will use the term 'Total Hamiltonian formalism' to refer to the formulation of irregular Hamiltonian mechanics on the primary constraint surface and 'Extended Hamiltonian formalism' to refer to the formulation of irregular Hamiltonian mechanics on the final constraint surface.[6]

4 Inequivalence Argument

In the previous section, we showed that a Hamiltonian gauge theory is naturally formulated on the final constraint surface with the Extended Hamiltonian as the equivalence class of Hamiltonians. However, we also pointed out that if we start with a Lagrangian theory, the Legendre transformation defines the primary constraint surface, which corresponds to the Total Hamiltonian being the right equivalence class of Hamiltonians (see Figure 1). This fact has led some authors to conclude that Extended Hamiltonian formalism is inequivalent to the Lagrangian formalism, and that this is reason to think that the Extended Hamiltonian formalism is mistaken.

[6] For further motivation, see Bradley (2024*a*,*b*).

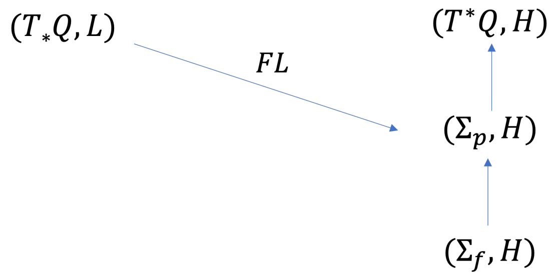

Figure 1: The irregular case.

For example, Gracia & Pons (1988) state:

> "No "extended hamiltonian" is needed, since it would introduce new solutions of the equations of motion that would break the equivalence between Lagrangian and Hamiltonian formalisms".

Similarly, Pitts (2014*b*) argues:

> "The extended Hamiltonian breaks Hamiltonian-Lagrangian equivalence. Requiring Hamiltonian-Lagrangian equivalence fixes the supposed ambiguity permitting the extended Hamiltonian".

Such claims have been used to argue that the right definition of a gauge transformation in the Hamiltonian formalism is not given by a transformation relating solutions to the Extended Hamiltonian, but rather it is a transformation relating solutions to the Total Hamiltonian. And one can show that the transformations relating solutions to the Total Hamiltonian are not given by arbitrary combinations of first-class constraints but rather by a *particular* combination of first-class constraints, contrary to the standard definition.[7] Therefore, the claim that the Lagrangian formalism is equivalent only to the Total Hamiltonian formalism has significant implications not only for how one formulates Hamiltonian gauge theories but also for the characterization of the gauge transformations themselves.

[7]For more discussion on this debate, see Pitts (2014*b*), Pons (2005), Pooley & Wallace (2022).

However, to evaluate these claims, we ought to understand the sense of (in)equivalence that is at stake. This hasn't been discussed in detail in the literature; indeed, what one finds are references to certain results that are taken to show that the solutions to the Euler-Lagrange equations are equivalent to the solutions to the Hamilton-Dirac equations on the primary constraint surface. One particular result that is widely cited is found in Batlle et al. (1986), so let us spell out this result and consider the notion of equivalence that it supports.

Theorem (Batlle et al. (1986)): If $(q_i(t), \dot{q}_i(t))$ satisfies the Euler-Lagrange equations, then $FL(q_i(t), \dot{q}_i(t))$ satisfies the Hamilton-Dirac equations on the primary constraint surface. Similarly, if $(q_i(t), p_i(t))$ satisfies the Hamilton-Dirac equations on the primary constraint surface, then $FL^{-1}(q_i(t), p_i(t))$ satisfies the Euler-Lagrange equations, where $FL^{-1}(q_i(t), p_i(t))$ is constructed via:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + v_a(q_i, \dot{q}_i)\frac{\partial \phi_a}{\partial p_i}$$

$$-\frac{\partial L}{\partial q^i} = \frac{\partial H}{\partial q^i} + v_a(q_i, \dot{q}_i)\frac{\partial \phi_a}{\partial q^i}$$

where ϕ_a are the primary constraints and $v_a(q_i, \dot{q}_i)$ is arbitrary.[8]

The theorem shows that the solutions to the Euler-Lagrange equations map to the solutions to the Hamilton-Dirac equations on the primary constraint surface and vice versa. But notice that the inverse Legendre transformation maps one point on the primary constraint surface to multiple points on the tangent space since it is defined in terms of arbitrary functions v_a. It therefore maps one solution on the primary constraint surface to multiple solutions on tangent space. If these solutions are not considered equivalent from the perspective of the Lagrangian formalism, then this result cannot

[8]See Batlle et al. (1986) for proof.

establish that a Lagrangian gauge theory defined on tangent space and its corresponding Hamiltonian theory defined on the primary constraint surface have equivalent solutions.

Moreover, even if we do interpret these points/solutions as equivalent, it seems that the most that this theorem can tell us is that there is a *dynamical* equivalence between Lagrangian mechanics and Hamiltonian mechanics on the primary constraint surface. One cannot use Barrett's result to establish categorical equivalence since we do not have a way of translating the models and symmetries of one theory to those of the other. In particular, it was important for Barrett's result that $FL^{-1} = FH$, which followed from these maps being global diffeomorphisms. The maps between tangent space and the primary constraint surface do not satisfy this property. Therefore, the results in Batlle et al. (1986) are not sufficient to infer theoretical equivalence between Lagrangian gauge theories and Hamiltonian gauge theories defined on the primary constraint surface.

However, this theorem does provide the tools to infer that there is a dynamical, and therefore theoretical, *inequivalence* between Lagrangian gauge theories and the Extended Hamiltonian formalism: what the theorem shows is that the equivalence class of solutions to the Euler-Lagrange equations on tangent space are in one-to-one corresponds to the equivalence class of solutions to Hamilton's equations on the primary constraint surface. That is, once we take into account the *symmetries* of the equations of motion, then the two formalisms agree about which solutions are distinct from one another. On the other hand, the symmetries of Hamilton's equations on the final constraint surface, are wider than symmetries of Hamilton's equations on the primary constraint surface (there are distinct solutions of the Total Hamiltonian formalism that are equivalent in the Extended Hamiltonian formalism). Therefore, the Lagrangian formalism and the Extended Hamiltonian formalism are inequivalent because their equivalence classes of solutions are different.

Indeed, it is this dynamical inequivalence that seems to be the core of the arguments that the Extended Hamiltonian gets the gauge transformations wrong, from the perspective of the Lagrangian formalism: there is a mismatch of the symmetries of the equations of motion. However, there are some lingering puzzles.

First, there is a sense in which the Total Hamiltonian formalism is *empirically equivalent* to the Extended Hamiltonian formalism: if we take secondary constraints to be a physical requirement in the Total Hamiltonian formalism, then the solutions one gets when one takes the solutions to the Total Hamiltonian and restricts to the final constraint surface are just the solutions to the Extended Hamiltonian on the final constraint surface.[9] Therefore, the fact that the equivalence classes of solutions are different doesn't seem to allow for the inference that the Extended Hamiltonian formalism is wrong without some further reason to think that the Lagrangian equivalence class of solutions is the right one. Another way to put this worry is that without an account of theoretical equivalence, one cannot fully evaluate the claim that the Total Hamiltonian formalism is the right formulation from the perspective of the Lagrangian formalism.

Second, given that we have motivated two formulations of Hamiltonian mechanics in the presence of gauge symmetry – the Total Hamiltonian formalism and the Extended Hamiltonian formalism – it is natural to ask whether in the context of gauge theories, one could also motivate a new formulation of *Lagrangian* mechanics whose equivalence class of solutions matches the Extended Hamiltonian formalism. If we could, then this would suggest that the dynamical inequivalence that we find between Lagrangian mechanics and the Extended Hamiltonian formalism is an accident of the way we set up the Lagrangian formalism in the first place.

These puzzles lead to the following questions: First, is there some empirically equivalent formulation of Lagrangian mechanics that is dynamically equivalent to the Extended Hamiltonian formalism? Second, can one provide a stronger account of theoretical equivalence between formulations of Lagrangian and Hamiltonian gauge theories?

In what follows, I will argue that the answer to both questions is yes: we can both reformulate Lagrangian mechanics in the presence of gauge symmetry such that the resulting theory is dynamically equivalent to the Extended Hamiltonian formalism, and

[9]For a more detailed argument of this kind in the context of Electromagnetism, see Pooley & Wallace (2022).

one can set up a categorical equivalence result that renders these formulations equivalent. This will refute the claim that from the perspective of (equivalence with) the Lagrangian formalism, the Total Hamiltonian formalism is motivated over the Extended Hamiltonian formalism.

More carefully, I will first demonstrate, drawing from Gotay & Nester (1979), that one can formulate Lagrangian gauge theories on a constraint submanifold of tangent space, and that the relationship between the Lagrangian constraint surface and the Hamiltonian final constraint surface is the same as the relationship between tangent space and the Hamiltonian primary constraint surface. I will use this to show that the equivalence class of solutions of the reformulated Lagrangian theory match the equivalence class of solutions of the Extended Hamiltonian formalism.[10] Next, I argue that there is a way to redefine the models of these theories using a process known as *reduction* such that one can set up a categorical equivalence result between classes of models of the reduced theories. This will demonstrate that there is a sense in which Lagrangian and Hamiltonian gauge theories are theoretically equivalent, but not in a way that supports the view that the Extended Hamiltonian formalism is wrong; to the contrary, it demonstrates that there is a natural formulation of Lagrangian mechanics that is theoretically equivalent to the Extended Hamiltonian formalism.

5 Lagrangian Constraint Formalism

To see how we can think of constraints in the Lagrangian formalism, let us start by writing the Euler-Lagrange equations as:

$$W_{ij}\ddot{q}^j + K_i = 0 \tag{1}$$

[10] In Gryb & Thébault (2023, ch.8) it is argued that the symmetries of the Extended Hamiltonian can be motivated from the Lagrangian perspective through careful consideration of Noether's Second Theorem. I take this to be complementary to the argument presented here. The reason for using the formalism in Gotay & Nester (1979) is that it directly allows us to compare the geometric structure of the two theories. However, it would be interesting to explore the extent to which the results here agree with the analysis in Gryb & Thébault (2023).

where $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is the Hessian and $K_i = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}$. The singular case is characterized by the vanishing of the determinant of W_{ij}. Let us say that the rank of W_{ij} is $n - m_1$ so that W_{ij} has m_1 null vectors, φ_μ, such that $W_{ij}\varphi^j_\mu = 0$. We call these "gauge identities" because they hold at all points in T_*Q.

Contracting the equations of motion with the null vectors, we get:

$$\chi^{(1)}_\mu = K_i \varphi^i_\mu = 0 \tag{2}$$

We call these the first m_1 "Lagrangian constraints". We now require for consistency that these constraints are preserved under time evolution i.e. $\frac{d}{dt}\chi^{(1)}_\mu = 0$. This gives rise to new Lagrangian constraints $\chi^{(2)}_{\mu'}$. We can continue this process until we are left with all of the Lagrangian constraints. As in the Hamiltonian case, there are certain constraints whose time evolution allows one to determine some of the undetermined accelerations; as we will see, these constraints correspond to the second-class constraints on the Hamiltonian side.

It will be helpful to consider the picture more geometrically.[11] We can define, as in the regular case, the Lagrangian state space to be endowed with a two form $\Omega = FL^*\omega$ that is given in coordinate form by $\Omega = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^j$. When the Hessian $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is non-invertible, Ω is degenerate and so it is a pre-symplectic two-form. We call this the *irregular* case.

The geometric equations of motion can be written as before as:

$$\Omega(X_E, \cdot) = dE \tag{3}$$

Because Ω is not symplectic in the irregular case, there will not be a unique solution to the equations of motion; indeed there may not be any solution at some points. However, the null vector fields of Ω allow us to define a submanifold where one can solve the equations at every point, in the following way. The null vector fields Z of Ω are such

[11] For details, see Gotay & Nester (1979).

that $\Omega(Z, \cdot) = 0$. So, in order for the equations of motion to hold, and be tangent to T_*Q, we must have that $dE(Z) = 0$. This motivates restricting to the submanifold P_1 defined by $dE(Z) = 0$ for null vector fields Z. We can therefore think of $dE(Z)$ as *constraints*.

Next, we require that the solutions to the equations of motion everywhere lie tangent to P_1 i.e. that the constraints hold at all points along a solution. But this is just to require that $dE(Y) = 0$ where Y is a null vector field of Ω restricted to P_1, which we can write as Ω_1. So we should restrict to a submanifold where in addition $dE(Y) = 0$. Therefore, we can think of $dE(Y)$ as further constraints.

Reiterating this process, we find a constraint surface P_k for K constraints where the solutions of the equations of motion $\Omega_k(X_E, \cdot) = dE$ are tangent to the constraint surface.[12] The null vector fields of Ω_k correspond to the null vector fields of Ω and the vector fields associated with the constraints. Therefore, we can think of this formalism as providing a way on the Lagrangian side to associate constraints with gauge transformations: the vector fields associated with the constraints generate (a subset of) the gauge transformations, understood as transformations along the integral curves of the null vector fields.

However, there are some constraints $K_i \varphi^i_\mu = 0$ that are not accounted for by this geometric procedure. These are the constraints that do not correspond to null vector fields of the (induced) presymplectic two-forms. As Gotay & Nester (1980) show, these constraints are determined by requiring that the equation of motion is second-order, which corresponds to requiring that a solution to the equation of motion, written in coordinate-dependent form as $X = \alpha^i \frac{\partial}{\partial q^i} + \beta^i \frac{\partial}{\partial \dot{q}^i}$, is such that $\alpha^i = \dot{q}^i$ (this follows from the two-form written in coordinate form above). If constraints of this kind arise, we can find their time derivative and thereby determine potentially new constraints. So take the final constraint surface to be given by $(P_f, \Omega_f, L|_{P_f})$ where P_f is the sub-manifold defined by the satisfaction of $K + J$ constraints where J is the number of constraints arising from the second-order condition.

[12]Here, the energy function E should be thought of as the energy function on T_*Q restricted to the points of the constraint surface P_k.

6 Relationship between Final Constraint Surfaces

We have seen that we can construct submanifolds of the tangent bundle in a similar way to the construction of submanifolds in the Hamiltonian formalism through constraints, and that we can write the equations of motion intrinsically on these submanifolds. So the natural question is whether the theory defined on the final constraint submanifold on the Lagrangian side is equivalent to the theory defined on the final Hamiltonian constraint manifold. To present an equivalence result of this kind, we will start by using the results in Gotay & Nester (1979) to show that the relationship between the models on the final constraint manifolds is the same as the relationship between the original Lagrangian model and the model on the primary constraint surface.[13]

We will restrict ourselves, following Gotay & Nester (1979), to *almost regular* Lagrangian models. An almost regular Lagrangian model is associated with two assumptions. First, FL is a submersion onto its image i.e. its differential is surjective. Second, the fibers $FL^{-1}(FL(q, \dot{q}))$ are connected submanifolds of T_*Q. These two assumptions guarantee that $FL^*H = E$ defines a single-valued Hamiltonian, since they imply that the energy function E is constant along the fibers $FL^{-1}(FL(q, \dot{q}))$. We can think of the almost regular Lagrangian models as characterizing the Lagrangian gauge theories: they are the models of Lagrangian mechanics for which there is a well-defined corresponding Hamiltonian theory on the primary constraint surface.

We also assume that we have no ineffective constraints,[14] which means that there is a clear separation between first-class and second-class constraints i.e. a first-class constraint does not become second-class when considering its time derivative and vice versa. To start, we will assume that we just have first-class constraints on the Hamiltonian side and constraints that correspond to null vector fields on the Lagrangian side.

Let us first consider the relationship between T_*Q and the corresponding primary

[13] Although the results in this section can be found in Gotay & Nester (1979), they do not discuss in detail the kind of equivalence that these results imply, nor do they draw the implications that we do here for the debate about the Total vs. Extended Hamiltonian.

[14] An ineffective constraint is one whose gradient vanishes weakly. For discussion, see Gotay & Nester (1984).

Hamiltonian surface Σ_p. Take i_p to be the inclusion map $i_p : \Sigma_p \to T^*Q$. Then we can define the Legendre transformation between T_*Q and Σ_p as $i_p \circ FL_p = FL$ where $FL : T_*Q \to T^*Q$ is the Legendre transformation. Since FL is assumed to be a submersion onto its image and its image is precisely Σ_p, FL_p is also a submersion.

> **Proposition 1:** If Z is a null vector field on T_*Q, then $FL_{p*}(Z)$ is well-defined and is a null vector field on Σ_p. Similarly, if Y is a null vector field on Σ_p, then $FL_p^*(Y)$ is a null vector field on T_*Q.[15]

Proposition 1 tells us for every null vector field on tangent space, there is a corresponding null vector field on the primary Hamiltonian constraint surface and vice versa. Notice that this does not mean that there is a one to one correspondence between null vector fields. In fact, the relationship between null vector fields is many to one from the Lagrangian side to the Hamiltonian side, with the difference in dimension of null vector fields being given by the dimension of $Ker(FL_{p*})$ (the kernel of FL_{p*} i.e. the null vector fields of the pushforward of the Legendre transformation). The dimension of $Ker(FL_{p*})$ is equal to the number of primary first-class constraints. This is as expected, since we know that FL_p is a surjective submersion.

It turns out that the same relationship holds between the final constraint surfaces P_f and Σ_f. Define the induced Legendre transformation between these spaces as follows. Define $i_L : P_f \to T_*Q$ to be the inclusion map from the final Lagrangian constraint surface to the tangent space and $i_H : \Sigma_f \to T^*Q$ to be the inclusion map from the final Hamiltonian constraint surface to the cotangent space. Then $FL_f : P_f \to \Sigma_f$ is given implicitly by $i_H \circ FL_f = FL \circ i_L$.

> **Proposition 2:** If Z is a null vector field on P_f, then $FL_{f*}(Z)$ is well-defined and is a null vector field on Σ_f. Similarly, if Y is a null vector field on Σ_f, then $FL_f^*(Y)$ is a null vector field on P_f.[16]

[15] See Appendix for proof.
[16] See Appendix for proof.

Proposition 2 tells us that the relationship between null vector fields on the final constraint surfaces is such that the number of null vector fields on P_f is equal to the number of null vector fields on Σ_f plus the dimension of $Ker(FL_{f*})$. One can also show that $Ker(FL_{f*}) = Ker(FL_{p*})$, and so $Ker(FL_{f*})$ has dimension equal to the number of primary first-class constraints.

We can also show that the solutions to the equations of motion are related in a similar way:

> **Proposition 3:** Take a solution X_E to the equations of motion $\Omega_f(X_E, \cdot) = dE$. Then $FL_{f*}(X_E)$ is well-defined and satisfies $\tilde{\omega}_f(FL_{f*}(X_E), \cdot) = dH$. Similarly, if X_H satisfies $\tilde{\omega}_f(X_H, \cdot) = dH$, then $FL_f^*(X_H)$ satisfies $\Omega_f(FL_f^*(X_H), \cdot) = dE$.[17]

Proposition 3 implies that every solution to the Lagrangian equations of motion on the final constraint surface corresponds to a solution to the Hamiltonian equations of motion on the final constraint surface and vice versa. Moreover, there is not a one-to-one correspondence of solutions in the same way that there is not a one-to-one correspondence of null vector fields. To see this, notice that if X_L satisfies the Lagrangian equations of motion, then $X_L + X_N$ satisfies the equations of motion where $X_N \in Ker(FL_{f*})$. But since $X_N \in Ker(FL_{f*})$, $FL_{f*}(X_L + X_N) = FL_{f*}(X_L)$. This means that $FL_f^*(X_H)$ does not define a unique vector field on P_f.

This shows that the relationship between the Lagrangian and Hamiltonian theories defined on the final constraint surfaces is the same as the relationship between the theory defined on T_*Q and the theory defined on the primary constraint surface: we can map solutions to solutions, but only up to symmetries on the Lagrangian side, where the symmetries are generated by null vector fields. Therefore, we can say that the theories formulated on the final constraint surfaces are dynamically equivalent, in the sense that they agree on the equivalence class of solutions. This provides a a partial response to the claim that the Extended Hamiltonian formalism is inequivalent to the Lagrangian

[17] See Appendix for proof.

formalism: there is in fact an alternative formulation of Lagrangian gauge theories that is dyanmically equivalent to the Extended Hamiltonian formalism in the same way that the original formulation of Lagrangian gauge theories is dynamically equivalent to the Total Hamiltonian formalism (see Figure 2).

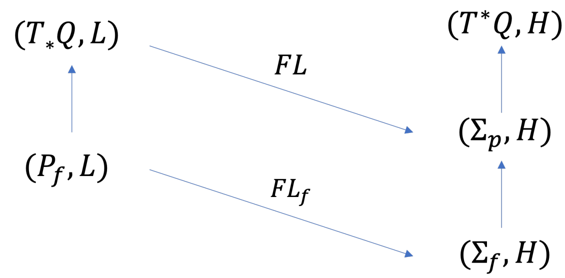

Figure 2: Relationship between final constraint surfaces.

However, we do not yet have a way to provide a (stronger) theoretical equivalence result.[18] In particular, we need a way of characterizing the claim that X_L is equivalent to $X_L + X_N$, or more generally, we need a way of characterizing the structure of the theories that includes the transformations generated by the null vector fields. Moreover, although we have defined a map from the final Lagrangian constraint surface to the final Hamiltonian constraint surface, and we can use this to pull-back structures from the Hamiltonian model to the Lagrangian model, we do not yet have the tools to set up a categorical equivalence result analogous to the result in Barrett (2019), since it is not the case that the induced Legendre transformation is related to the fiber derivative of the Hamiltonian on the constraint surface in the right way.

Before turning to how we might set up a categorical equivalence result, let us consider how the situation changes when we also have second-class constraints on the Hamiltonian side. Since we assumed that there are no ineffective constraints, this means that we only need to consider the case where we have primary second-class constraints, since the time

[18] Gotay & Nester (1979) suggest that Propositions 2 through 4 do suffice for equivalence between almost regular Lagrangian models and the corresponding Hamiltonian models. However, they spell out equivalence in terms of Proposition 3, i.e. as dynamical equivalence, which we are taking to be weaker than theoretical equivalence.

derivative of these constraints will generate any additional second-class constraints.

We have shown that we can relate the first-class constraints to null vector fields on the Lagrangian side. But since second-class constraints do not correspond to null vector fields, we cannot relate them to a Lagrangian constraint in the same way. However, it turns out that for every (distinct) primary second-class Hamiltonian constraint, there is a corresponding (distinct) Lagrangian constraint whose associated vector field is not null. In particular, the additional Lagrangian constraints are the pullback under the (induced) Legendre transformation of the time derivative of a second-class Hamiltonian constraint.[19] Generalizing, the final Lagrangian constraint surface will be reduced in dimension by the number of second-class constraints on the Hamiltonian side.

7 Reduction and Equivalence

Although we now have a picture under which both the Lagrangian formalism and the Hamiltonian formalism can be written intrinsically on constraint manifolds that are systemically related, we do not yet have a theoretical equivalence result. Recall that the barrier is that we do not have a way to define a translation from Lagrangian to Hamiltonian models and vice versa via the relationship between FL, FL^{-1}, FH and FH^{-1} since the final constraint submanifolds are not of the same dimension.

However, there is an indication that we should be able to set up an equivalence result: while the dimensions of the final constraint surfaces are different, the difference seems to be due to *arbitrariness* in the Lagrangian formalism coming from the null vector fields in the kernel of FL_*. Indeed, if we take null vector fields to generate symmetries, then there is an argument that once we have accounted for all of the symmetries, the two formalisms are in complete agreement. One way of thinking about 'accounting for the symmetries' is to consider whether there is a way to characterize the theories in terms of the equivalence class of states along the integral curves of the null vector fields. In fact, there is a well-known construction for specifying a Hamiltonian theory in terms of the

[19] For details, see Batlle et al. (1986), Pons (1988).

equivalence class of states called *reduction*: the process of reduction defines a manifold that "quotients out" the gauge transformations. This is not a construction that one often finds discussed for a Lagrangian theory.[20] However, we have shown that we can think of a Lagrangian gauge theory in an analogous way to the Hamiltonian formalism as defined on a pre-symplectic manifold. This suggests that we should be able to equally construct a reduced space for the final Lagrangian constraint surface. The question then becomes: are the *reduced* versions of Lagrangian and Hamiltonian gauge theories categorically equivalent?

The reason that reduction will help us to set up a categorical equivalence result is that one can show that reduction induces a symplectic two-form on the reduced space. Recall that being symplectic means that the Lagrangian/Hamiltonian models are *regular*: the two-form is non-degenerate and so we can, at least locally, define the inverse of the fiber derivatives. Therefore, if we can show that the Legendre transformation of a reduced Lagrangian model gives rise to a reduced Hamiltonian model and vice versa, then this suggests that we can set up an equivalence result in an exactly analogous way to Barrett (2019), if we restrict to the hyperregular reduced models.

In order to show that this is indeed possible, we will show that the the dimensions of the reduced spaces related by FL_f are the same, that the structures defined on this space can be inherited from the final constraint surface in a natural way, and that the image of the Legendre transformation of the reduced Lagrangian space is precisely the corresponding reduced Hamiltonian space. These will provide the tools to prove categorical equivalence between classes of models of the reduced theories.

Consider first a presymplectic Hamiltonian manifold $(\Sigma, \tilde{\omega}, H)$ that is foliated by the gauge orbits at each point. We can define a smooth, differentiable manifold $\bar{\Sigma}$ by taking the quotient of Σ by the kernel of $\tilde{\omega}$ i.e. the null vector fields of $\tilde{\omega}$. Recall that the integral curves of the null vector fields define the gauge orbits, and so the points of the quotient manifold are just the equivalence class of points along the gauge orbits. This is

[20] An exception is Pons et al. (1999).

well-defined since the gauge orbits foliate the constraint surface in such a way that one can define a transverse manifold that meets each leaf of the foliation in at most one point i.e. through each point there is only one gauge orbit.[21] Recall that on the final constraint surface, the dimension of the gauge orbits is the number of first-class constraints M and the dimension of Σ_f is $2N - M - S$ where N is the dimension of configuration space and S is the number of second-class constraints. So the quotient manifold of the final Hamiltonian constraint surface $\bar{\Sigma}$ has dimension $2N - 2M - S$.

Define an open, surjective projection map $\pi : \Sigma_f \to \bar{\Sigma}$ such that we define the reduced two-form $\bar{\omega}$ via $\tilde{\omega}_f = \pi^*(\bar{\omega})$, which acts according to $\bar{\omega}(\bar{X}, \bar{Y}) = \tilde{\omega}_f(X, Y)$ where $X = \pi^*(\bar{X})$. One can show that $\bar{\omega}$ is well-defined and is symplectic.[22] We can also define a reduced Hamiltonian $\bar{H}$ as the value of H on the equivalence class of points along the gauge orbits i.e. $H = \pi^*(\bar{H})$. This is well-defined because H is constant along the gauge orbits on the final constraint surface (since the solutions to the equations of motion are tangent to the final constraint surface). We can therefore write the equations of motion on the reduced space in terms of the reduced Hamiltonian $\bar{H}$, and the solutions are just the projection of the solutions to the equations of motion on Σ_f to $\bar{\Sigma}$: they are just the solutions defined for the gauge-invariant quantities.

To summarize, there is a well-defined Hamiltonian theory on the reduced space of the final Hamiltonian constraint surface in terms of a symplectic two-form and a reduced Hamiltonian function. However, this only required that we had a presymplectic manifold with a foliation induced by the null vector fields of the associated two-form and that the Hamiltonian function was constant along the gauge orbits. Given that the same is true for the Lagrangian final constraint surface, we can do the same reduction procedure on the Lagrangian side to produce a reduced Lagrangian space. This will have dimension $2N - 2K - J$ where K is the number of Lagrangian constraints associated with null vector fields and J is the number of additional Lagrangian constraints. As in the Hamiltonian

[21] See Souriau (1997) §5 and §9 for details.

[22] It is well-defined since the value of $\tilde{\omega}_f$ doesn't depend on which point along the gauge orbit one considers. It is closed since $\tilde{\omega}_f$ is closed and π is a surjective submersion, and it is non-degenerate since $Ker(\bar{\omega}) = Ker(\tilde{\omega}_f)/Ker(\tilde{\omega}_f) = 0$.

case, the equations of motions are well-defined because the energy function E is constant along gauge orbits on final constraint surface, and so the reduced Lagrangian function $\bar{L}$ will be well-defined as well.

Let us now turn to the relationship between the models of the reduced theory. First, let us consider the relationship between the dimensions of the reduced spaces corresponding to models on the final constraint surfaces P_f, Σ_f that are related via FL_f. Recall that the dimension of the Lagrangian final constraint surface P_f is equal to the dimension of the Hamiltonian final constraint surface Σ_f plus the number of primary first-class constraints. But recall also that the dimension of the kernel of Ω_f is equal to the number of first-class constraints plus the number of primary first-class constraints. Therefore, the dimension of the reduced Lagrangian space $\bar{P}$ is equal to the dimension of the Hamiltonian constraint surface Σ_f minus the number of first-class constraints. But this is just the dimension of the reduced Hamiltonian space, $\bar{\Sigma}$. Therefore, the dimensions of the reduced Lagrangian final constraint surface and the reduced Hamiltonian final constraint surface are equal.

Now define an induced transformation $\bar{F} : \bar{P} \to \bar{\Sigma}$ that satisfies $\pi_H \circ FL_f = \bar{F} \circ \pi_L$ where $\pi_H : \Sigma_f \to \bar{\Sigma}$ and $\pi_L : P_f \to \bar{P}$ are the projection maps. This provides a way to map from the reduced Lagrangian space to the corresponding reduced Hamiltonian space. Moreover, notice that since $\bar{L}$ is regular (since the induced two-form is symplectic), the Legendre transformation on $\bar{P}$ will be a local diffeomorphism. And since $\bar{P}$ and $\bar{\Sigma}$ have the same dimension, the induced transformation $\bar{F}$ is precisely the Legendre transformation on $\bar{P}$, $F\bar{L}$. That is, $\bar{F} : \bar{P} \to \bar{\Sigma}$ is the Legendre transform on T_*Q, FL, projected to the reduced space. Similarly, since $\bar{H}$ is regular, the fiber derivative of $\bar{H}$, $F\bar{H}$, will be a local diffeomorphism and it will map $\bar{\Sigma}$ to $\bar{P}$. Using the reduced Legendre transformation, one can also show that the reduced symplectic two-forms are related via $F\bar{L}^*(\bar{\omega}) = \bar{\Omega}$ and the reduced Hamiltonian and energy function are related via $F\bar{L}^*(\bar{H}) = \bar{E}$.[23]

[23] To see this, notice that $\pi_L^*(\bar{FL}^*\bar{\omega}) = FL_f^*(\pi_H^*\bar{\omega}) = FL_f^*\tilde{\omega}_f = \Omega_f$. Since π_L is a surjective submersion, this implies that $F\bar{L}^*(\bar{\omega}) = \bar{\Omega}$. The second follows by similar reasoning.

Finally, since (P_f, L_f) is, by assumption, an almost regular system, $(\bar{P}, \bar{L})$ will also be almost regular. This implies that $F\bar{L}$ is injective.[24] Moreover, the image of $F\bar{L}$ is $\bar{\Sigma}$ by construction so $F\bar{L}$ is surjective. But this means that $F\bar{L}$ is a global diffeomorphism, and so $(\bar{P}, \bar{L})$ is in fact a *hyperregular* system. Therefore, we can define the inverse $F\bar{L}^{-1} : \bar{\Sigma} \to \bar{P}$. This allows us to define $\bar{H} = \bar{E} \circ F\bar{L}^{-1}$.

Therefore, for an almost regular Lagrangian model defined on the final constraint surface we can construct a reduced model such that this model is hyperregular and its Legendre transformation is precisely the (hyperregular) reduced model of the corresponding Hamiltonian final constraint surface. This implies that as long as we are concerned with almost regular Lagrangian models and their corresponding Hamiltonian models, the reduced formulations of these theories bear exactly the same relationship as hyperregular models of Lagrangian and Hamiltonian mechanics (see Figure 3).

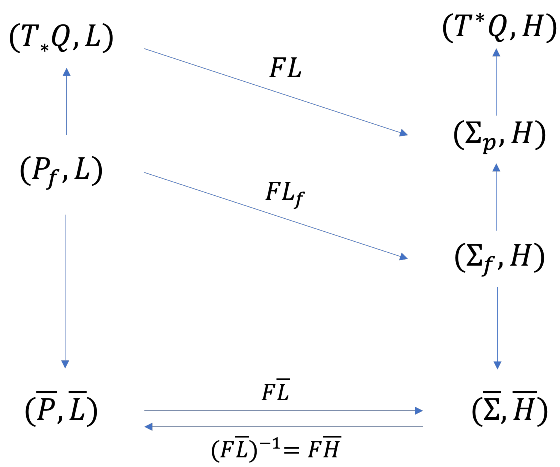

Figure 3: Relationship between reduced spaces.

We are now at the point where we can set up an equivalence result. Recall that in order to do so, we need to define the models and symmetries of the associated theories. In the hyperregular case given by Barrett (2019), the symmetries were the point-

[24] The reason is that for an almost regular system the image of the Legendre transformation is the leaf space of the foliation generated by the kernel of the pushforward of the Legendre transformaiton. When a system is regular, this kernel is zero, and so it must be injective.

transformations that preserved the Lagrangian/Hamiltonian. However, in order for the point-transformations to be well-defined for the reduced theories, we need that the reduced state space has the form of a (co)tangent bundle. This is not guaranteed by the above; at least, it will depend upon the particular nature of the constraints and the gauge transformations.[25] On the other hand, we do have that the reduced spaces are symplectic manifolds. Therefore, it seems that the natural notion of symmetry is rather given by *symplectomorphisms:* diffeomorphisms that preserve the symplectic two-form on the reduced space (and preserve the Lagrangian/Hamiltonian).

So let us define the category **LagR** as having objects $(\bar{P}, \bar{\Omega}, \bar{L})$ and take the arrows between objects $(\bar{P}_1, \bar{\Omega}_1, \bar{L}_1)$ and $(\bar{P}_2, \bar{\Omega}_2, \bar{L}_2)$ to be given by symplectomorphisms i.e. diffeomorphisms $f : \bar{P}_1 \to \bar{P}_2$ such that $f^*(\bar{\Omega}_2) = \bar{\Omega}_1$, that preserve the Lagrangian in the sense that $f^*\bar{L}_2 = \bar{L}_1$.

Similarly, define the category **HamR** as having objects $(\bar{\Sigma}, \bar{\omega}, \bar{H})$ and take the arrows between objects $(\bar{\Sigma}_1, \bar{\omega}_1, \bar{H}_1)$ and $(\bar{\Sigma}_2, \bar{\omega}_2, \bar{H}_2)$ to be given by symplectomorphisms $g : \bar{\Sigma}_1 \to \bar{\Sigma}_2$ such that $g^*(\bar{\omega}_2) = \bar{\omega}_1$, that preserve the Hamiltonian in the sense that $g^*\bar{H}_2 = \bar{H}_1$.

Define the functor J that takes the object $(\bar{P}, \bar{\Omega}, \bar{L})$ to $(\bar{\Sigma}, \bar{\Omega} \circ F\bar{L}^{-1}, \bar{E} \circ F\bar{L}^{-1})$ and that takes the arrow $f : \bar{P}_1 \to \bar{P}_2$ to $F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1}$. Similarly, define the functor K that takes models $(\bar{\Sigma}, \bar{\omega}, \bar{H})$ to $(\bar{P}, \bar{\omega} \circ F\bar{H}^{-1}, (\bar{\theta}_a(X_{\bar{H}})^a - \bar{H}) \circ F\bar{H}^{-1})$ where $\bar{\theta}$ is the reduced one form, and arrows $g : \bar{\Sigma}_1 \to \bar{\Sigma}_2$ to $F\bar{H}_2 \circ g \circ F\bar{H}_1^{-1}$.

Proposition 4: J : **LagR** $\to$ **HamR** and K : **HamR** $\to$ **LagR** are equivalences that preserve solutions.[26]

[25] Moreover, even if one could think of the reduced state space as having the structure of (co)tangent space, it isn't clear that one would want the symmetries to be given by point-transformations. As Barrett (2015) shows, there are point$_*$-transformations that don't preserve an arbitrary symplectic two-form on T_*Q. One might conclude from this that point$_*$-transformations are not the relevant symmetries to consider for the reduced Lagrangian models, since the symplectic two-form is an integral part of the construction of these reduced models.

[26] See Appendix for proof.

8 Upshots

Proposition 4 tells us that there is a formulation of irregular Lagrangian mechanics that it is theoretically equivalent to a formulation of irregular Hamiltonian mechanics. More precisely, it tells us that the the categories of hyperregular reduced models of the final constraint surfaces are equivalent. This is significant for several reasons.

First, we discussed in Section 4 the view that the correct Hamiltonian formulation is the Total Hamiltonian formalism on the basis that it is equivalent to the Lagrangian formalism in the context of gauge theories. But our arguments have suggested that in fact the Extended Hamiltonian formalism can be motivated in a similar, and even a stronger, way: there are reasons to move to the final Lagrangian constraint surface from the perspective of the Lagrangian formalism, and not only can the models formulated on the Lagrangian final constraint surface be said to be dynamically equivalent to models of the Extended Hamiltonian formalism, one can also give a stronger, theoretical equivalence result between the reduced versions of such models.

In order to deny that such results provide support for the Extended Hamiltonian formalism, one would have to maintain that there is something mistaken about the Lagrangian constraint formalism that we presented. One avenue might be to argue that we shouldn't think of Lagrangian constraints as imposing a restriction on the state space of Lagrangian mechanics: they should be thought of as *dynamical* constraints and not *kinematical* constraints, and therefore they should place a restriction only on the dynamically possible models and not the kinematically possible models. On this view, the correct formulation of the kinematically possible models is given by the usual tangent bundle formulation and the formulation on the Hamiltonian primary constraint surface. In further support for this view, one might point to the fact that the categorical equivalence result that we presented goes through for this characterization of irregular Lagrangian and Hamiltonian models: it corresponds to the special case where there are no Lagrangian/secondary constraints.[27]

[27] Although, such an equivalence result is complicated by the fact that the solutions to the equations

Although this response highlights interesting questions about the role of kinematics vs. dynamics in evaluating constraints, I think that there are good reasons to think that this distinction is not significant. For one, the dynamical solutions that we get are the same whether we define the equations of motion intrinsically on the final constraint surface or we consider the equation of motion on the tangent bundle and then impose the constraints. Therefore, there isn't any clear empirical difference between these formulations. Second, there is a natural sense in which the formulations on the final constraint surfaces have less structure: there are more symmetries of the theories formulated on the final constraint surfaces than on the tangent bundle/primary constraint surface since there are more null vector fields.[28] And so, if one is motivated by parsimony considerations, it is natural to think that the final constraint surface is the right intrinsic formulation of the theory.

Second, showing that there is an equivalence between Lagrangian and Hamiltonian gauge theories suggests that it is wrong to view one formulation as more fundamental than the other since they have the same underlying structure. This is interesting because the usual way of setting up the Hamiltonian formalism in the presence of constraints is by starting with a Lagrangian formulation and using it to define the primary constraints and Total Hamiltonian, which suggests that the Lagrangian formulation is more fundamental. On the other hand, the equivalence result suggests that one can instead start with a Hamiltonian theory with constraints, reduce the final constraint surface, and use this to define the corresponding Lagrangian theory.

Moreover, it is often assumed that in order to find the gauge-invariant degrees of freedom, one ought to use the Hamiltonian formulation. For example, Earman (2002) says: "Is there then some non-question begging and systematic way to identify gauge freedom and to characterize the observables? The answer is yes, but specifying the details involves a switch from the Lagrangian to the constrained Hamiltonian formalism."

of motion are not tangent to the constraint surface in the case where there are Lagrangian/secondary constraints that are not represented in the structure of the state space, and so the reduced equations of motion are not well-defined. See Pons et al. (1999) for further discussion.

[28] This is spelt out for the Hamiltonian case in Bradley (2024*b*).

The reason is that the constrained Hamiltonian formulation clearly draws out the connection between constraints and gauge-symmetry: the gauge transformations are those transformations generated by arbitrary combinations of first-class constraints, and we can define the observables as just those quantities whose Poisson bracket with the first-class constraints is zero. On the usual way of expressing the Lagrangian formulation, we find the symmetries by using Noether's second theorem, which doesn't directly connect the idea of constraints and observables. But the geometric formulation shows that if the focus is on the null vector fields of the associated two-form, then the Lagrangian formulation draws out the gauge transformations in the exact same way.

However, there are several subtleties with the equivalence result given by Proposition 4. For one, we restricted to a subset of the Lagrangian models, the 'almost regular' ones, and then considered the corresponding Hamiltonian models defined via the Legendre transformation. While we were able to show that the almost regular Lagrangian models have hyperregular reduced models, and therefore that the Hamiltonian models defined from these models also have hyperregular reduced models, we did not show that this exhausts the class of hyperregular reduced models. It would therefore be interesting to consider whether there are hyperregular reduced models that cannot be thought of as coming from a 'gauge theory' in the sense of being an almost regular Lagrangian model or its corresponding Hamiltonian model. Moreover, 'almost regularity' referred to the Lagrangian model, but there doesn't seem to be a clear Hamiltonian analogue: the fiber derivative of the Hamiltonian on the primary/final constraint surface does not construct an almost regular Lagrangian model. Therefore, it seems that we need some alternative way to characterize the relevant class of gauge theories in Hamiltonian terms.[29] These subtleties suggest that there is more work to be done in motivating the physical reasonableness of restricting to hyperregular reduced models to show equivalence between irregular models of Lagrangian and Hamiltonian mechanics.

[29] For example, is it the case that any regular Hamiltonian theory with the addition of constraints give rise to a constraint surface model that is the Legendre transformation of some almost regular Lagrangian model?

Another subtlety of the equivalence result is that symplectic reduction can lead to counter-intuitive conclusions, which has led several authors to argue that once should not reduce gauge theories (at least in certain contexts). The most notable example of this is the *Problem of Time*: when one reduces theories that are time reparameterization invariant, one ends up with a theory with no meaningful notion of evolution. If one finds these arguments convincing, then one might think that the equivalence result given by Proposition 4 is irrelevant; what matters is not whether the reduced theories are equivalent, but whether the unreduced theories are.

I take this to be an important limitation of the arguments presented here. However, one response is to point out that all one has done by moving to the theory formulated on the reduced space is to equivocate between states/solutions that are symmetry-related in the theory formulated on the final constraint surface. Therefore, if we interpret symmetry-related states/solutions as equivalent, then arguably the theories defined on the final constraint surface and on the reduced space have the same (symmetry-invariant) content. This suggests that even if one doesn't have a categorical equivalence result directly between classes of models on the final constraint surface, one can infer that they are equivalent from the fact that the reduced theories are equivalent.[30]

There is an interesting connection here to another strand of literature: the difference between 'reduction' and 'sophistication' (Dewar (2019)). A sophisticated version of a theory is, broadly, one where the all the transformations that we take to be symmetries are isomorphisms of the models of the theory. Dewar (2019) conjectures that the sophisticated and the reduced versions of a theory are categorically equivalent. Here, we have defined and compared the reduced versions of Lagrangian and Hamiltonian gauge theories. But what does the corresponding *sophisticated* versions of the theories look like? Arguably, the theories formulated on the final constraint surfaces are 'sophisticated', in the sense that the symmetries – the gauge transformations – are isomorphisms of the

[30]Indeed, I think one *could* spell out a categorical equivalence result directly between classes of models on the final constraints surface. However, setting up such a functor is less clear than it is for the reduced theories, which is why this was not the approach taken in this paper.

models of the final constraint surface.[31] Whether it is correct to characterize the theories formulated on the final constraint surfaces as the sophisticated version of the theory is a question that I hope to consider in future work.

9 Conclusion

To conclude, I have argued that there is a sense in which Lagrangian and Hamiltonian gauge theories are equivalent by showing that one can formulate these theories geometrically on a presymplectic constraint manifold such that the hyperregular class of reduced models of these constraint models are categorically equivalent and agree dynamically. This provides an extension to the result in Barrett (2019) that hyperregular Lagrangian and Hamiltonian theories are categorically equivalent. Moreover, this extension sheds light on philosophical debates regarding the definition and interpretation of gauge transformations. In particular, in showing that one could motivate a formulation of Lagrangian gauge theories that is equivalent to Extended Hamiltonian formalism, we thereby demonstrated that the Extended Hamiltonian can be motivated from the perspective of the Lagrangian formalism, contrary to claims found in the literature.

However, this equivalence result relied on several important assumptions that are not relevant in the case considered by Barrett (2019). First, it depended on how we understand the role of constraints in the construction of the models of the theories. Second, it depended upon an interpretation of the null vector fields of a presymplectic two-form as generating the (gauge) symmetries of the theory. Finally, it depended upon reduction, and restricting to the class of hyperregular reduced models, being justified. Inasmuch as all of these assumptions are disputable, there remain interesting questions regarding the relationship between Lagrangian and Hamiltonian gauge theories.

Moreover, while categorical equivalence suggests that we can move back and forth interchangeably between Lagrangian and Hamiltonian gauge theories, there were several subtleties regarding the way that we defined the categories of models of these theories

[31] This is proved in Bradley (2024*b*).

that suggest possible avenues for maintaining that one framework is a more natural expression of gauge theories than the other. For example, the (pre)symplectic structure of Lagrangian mechanics was motivated by thinking about the Hamiltonian (pre)symplectic structure, and the class of Hamiltonian models for which categorical equivalence held were defined in terms of the Lagrangian models that they were related to. Whether one should think that being (pre)symplectic is faithful to the structure of Lagrangian mechanics, and whether one can motivate the class of Lagrangian gauge theories in terms of Hamiltonian quantities, are open questions that would further deepen the understanding of the relationship between Lagrangian and Hamiltonian gauge theories.

10 Appendix

10.1 Proposition 1

First, notice that because FL_p is a submersion, we can not in general push forward vector fields on the Lagrangian side. However, null vector fields are such that they define the same vector at points along the integral curves of $Ker(FL_{p*})$. That is, if $FL_p(x) = FL_p(y)$ for points $x, y \in T_*Q$ then $Z(x) = Z(y)$ where Z is a null vector field on T_*Q. The reason is that $\Omega(Z, Y) = 0$ for $Y \in Ker(FL_{p*})$, which means that the flow of Z along the intergral curves of Y is 0 i.e. Z is constant along the intergral curves of $Ker(FL_{p*})$. Therefore, the pushforward of null vector fields on T_*Q under FL_p is well-defined. Indeed, for any vector field tangent to T_*Q, we can define its pushforward in this way, since $Ker(FL_{p*}) \subset Ker(\Omega)$.[32]

Suppose that Z is a null vector field on T_*Q. This means that

$$0 = \Omega(Z, \cdot) = FL_p^* \tilde{\omega}_p(Z, \cdot) \tag{4}$$

since $\Omega = FL^*(\omega)$. Now we can define Z as $FL_p^*(FL_{p*}(Z))$ since the push-forward of Z is well-defined. So:

[32]This can be seen through the definition of Ω.

$$0 = FL_p^*(\tilde{\omega}_p(FL_{p*}(Z), \cdot)) \tag{5}$$

Since $FL_p^*(\tilde{\omega}_p(FL_{p*}(Z), \cdot))$ is defined by $\tilde{\omega}_p(FL_{p*}(Z), \cdot) \circ FL_p$ and FL_p is a submersion, the above implies that $\tilde{\omega}_p(FL_{p*}(Z), \cdot) = 0$. But this just means that if Z is a null vector field of Ω, then its pushforward along FL_p is a null vector of $\tilde{\omega}_p$.

Second, we want to show that if X is a null vector field of $\tilde{\omega}_p$, then its pullback along FL_p is a null vector field. We can see this as follows: If $\tilde{\omega}_p(X, \cdot) = 0$ then $\tilde{\omega}_p(X, \cdot) \circ FL_p = 0$ i.e. $FL_p^*(\tilde{\omega}_p(X, \cdot)) = 0$. But since $FL_p^*(\tilde{\omega}_p) = \Omega$, this implies that $\Omega(FL_p^*(X), \cdot) = 0$ i.e. the pullback of X is a null vector field.

This shows that the pushforward of every null vector field on T_*Q is a null vector field on Σ_p and every null vector field on Σ_p can be pulled back to a null vector field on T_*Q. Notice that this is not the same as saying that the spaces have an equal number of null vector fields. Indeed, they do not: what the above shows is that the number of null vector fields on T_*Q is equal to the number of null vector fields on Σ_p+ the dimension of $Ker(FL_{p*})$, since FL_p is a submersion.

10.2 Proposition 2

In order to use the same proof that was used for Proposition 1, we need that FL_f is a submersion. To see why this is the case, notice that Proposition 1 implies that if $dE(Z)$ is a Lagrangian constraint where Z is a null vector field on T_*Q, then $dH(FL_{p*}(Z))$ is a Hamiltonian constraint. Similarly, if $dH(X)$ is a Hamiltonian constraint where X is a null vector field on Σ_p, then $dE(FL_p^*(X))$ is a Lagrangian constraint. Moreover, $dE(Y)$ for $Y \in Ker(FL_{p*})$ is automatically zero, since by assumption of almost regularity E is constant along the fibers $FL^{-1}(FL(q, \dot{q}))$. This means that there will be a one to one correspondence between first generation Lagrangian constraints of this kind and the first generation of secondary Hamiltonian (first-class) constraints. Reiterating, the same will be true of all further constraint submanifolds, and so since each constraint reduces the dimension by one, the relationship between P_f and Σ_f will be the same relationship as between T_*Q and Σ_p: the induced Legendre transformation FL_f will be a submersion, where $Ker(FL_{f*}) = Ker(FL_{p*})$ has dimension equal to the number of primary first-class constraints.

Therefore, we can use the same proof as the proof for Proposition 1 to show that every null vector field on P_f can be pushed forward to a null vector field on Σ_f, and every null vector field on Σ_f can be pulled back to a null vector field on P_f, where the number of null vectors on P_f is equal to the number of first-class constraints + the dimension of $Ker(FL_{f*})$.

10.3 Proposition 3

Take a solution X_L to the equations of motion $\Omega_f(X_L, \cdot) = dE$. This is equivalent to $FL_f^* \tilde{\omega}_f(X_L, \cdot) = d(FL_f^*(H))$. Since X_L is tangent to the constraint surface, we can write this as $FL_f^*(\tilde{\omega}_f(FL_{f*}(X_L), \cdot)) = FL_f^*(dH)$ since the pushforward is well-defined. Since FL_f is a submersion, this means that $FL_{f*}(X_L)$ satisfies $\tilde{\omega}_f(FL_{f*}(X_L), \cdot) = dH$ which is the equation of motion on the final Hamiltonian constraint surface. The other direction is similar: if X_H satisfies the Hamiltonian equations of motion, then its pullback satisfies the Lagrangian equations of motion.

10.4 Proposition 4

To show that J is a functor, we need to to show that J takes objects of **LagR** to objects of **HamR** and arrows to arrows. The first is trivial. To show the second, take an arrow f between objects $(\bar{P}_1, \bar{\Omega}_1, \bar{L}_1)$ and $(\bar{P}_2, \bar{\Omega}_2, \bar{L}_2)$. Since f is a symplectomorphism, $f^*\bar{\Omega}_2 = \bar{\Omega}_1$. Since $\bar{\Omega} = F\bar{L}^*\bar{\omega}$ by construction, this means that $f^*(F\bar{L_2}^*\bar{\omega}_2) = F\bar{L_1}^*\bar{\omega}_1$. We want to show that $F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1}$ is an arrow in **HamR**. That is, we want to show that $(F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1})^*\bar{\omega}_2 = \bar{\omega}_1$ and $(F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1})^*(\bar{E}_2 \circ F\bar{L_2}^{-1}) = \bar{E}_1 \circ F\bar{L}_1^{-1}$. The first follows from the fact that $f^*(F\bar{L_2}^*\bar{\omega}_2) = F\bar{L_1}^*\bar{\omega}_1$. The second follows from the fact that $f^*\bar{E}_2 = \bar{E}_1$ since $f^*\bar{L}_2 = \bar{L}_1$. Similar reasoning can be used to show that K is a functor.

Since $F\bar{L}$ and $F\bar{H}$ are global diffeomorphisms, one can define the inverse $F\bar{L}^{-1} = F\bar{H}$ and $F\bar{H}^{-1} = F\bar{L}$. This implies that the functors J and K are inverses on objects and similarly on arrows. That J and K preserve solutions follows from the fact that FL_f preserves solutions (Proposition 3), and that the solutions that are equivocated through reduction are just the gauge-related solutions.

References

Abraham, R. & Marsden, J. E. (1987), *Foundations of Mechanics, Second Edition*, Addison-Wesley Publishing Company, Inc.

Barrett, T. W. (2015), 'Spacetime structure', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **51**, 37–43.

Barrett, T. W. (2019), 'Equivalent and inequivalent formulations of classical mechanics', *The British Journal for the Philosophy of Science* **70**(4), 1167–1199.

Batlle, C., Gomis, J., Pons, J. M. & Roman-Roy, N. (1986), 'Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems', *Journal of Mathematical Physics* **27**(12), 2953–2962.

Bradley, C. (2024*a*), 'Do first-class constraints generate gauge transformations? a geometric resolution'. Unpublished Manuscript.

Bradley, C. (2024*b*), 'The extended hamiltonian is not trivial'. Unpublished Manuscript.

Curiel, E. (2014), 'Classical mechanics is lagrangian; it is not hamiltonian', *The British Journal for the Philosophy of Science* **65**(2), 269–321.

Dewar, N. (2019), 'Sophistication about symmetries', *The British Journal for the Philosophy of Science* **70**(2), 485–521.

Dirac, P. A. M. (1964), *Lectures on Quantum Mechanics*, Dover.

Earman, J. (2002), Tracking down gauge: An ode to the constrained hamiltonian formalism, *in* K. Brading & E. Castellani, eds, 'Symmetries in Physics: Philosophical Reflections', Cambridge University Press, pp. 140–62.

Gotay, M. J. & Nester, J. M. (1979), Presymplectic lagrangian systems. i: the constraint algorithm and the equivalence theorem, *in* 'Annales de l'institut Henri Poincaré. Section A, Physique Théorique', Vol. 30, pp. 129–142.

Gotay, M. J. & Nester, J. M. (1980), Presymplectic lagrangian systems. ii: the second-order equation problem, *in* 'Annales de l'institut Henri Poincaré. Section A, Physique Théorique', Vol. 32, pp. 1–13.

Gotay, M. J. & Nester, J. M. (1984), 'Apartheid in the dirac theory of constraints', *Journal of Physics A: Mathematical and General* **17**(15), 3063.

Gracia, X. & Pons, J. (1988), 'Gauge generators, dirac's conjecture, and degrees of freedom for constrained systems', *Annals of Physics* **187**(2), 355–368.

Gryb, S. & Thébault, K. (2023), *Time Regained: Volume 1: Symmetry and Evolution in Classical Mechanics*, Oxford University Press.

Halvorson, H. (2012), 'What scientific theories could not be', *Philosophy of Science* **79**(2), 183–206.

Halvorson, H. (2016), Scientific theories, *in* P. Humphreys, ed., 'The Oxford Handbook of Philosophy of Science', Oxford University Press.

North, J. (2009), 'The "structure" of physics: A case study', *The Journal of Philosophy* **106**(2), 57–88.

Pitts, J. B. (2014*a*), 'Change in hamiltonian general relativity from the lack of a time-like killing vector field', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **47**, 68–89.

Pitts, J. B. (2014*b*), 'A first class constraint generates not a gauge transformation, but a bad physical change: The case of electromagnetism', *Annals of Physics* **351**, 382–406.

Pons, J. M. (1988), 'New relations between hamiltonian and lagrangian constraints', *Journal of Physics A: Mathematical and General* **21**(12), 2705.

Pons, J. M. (2005), 'On dirac's incomplete analysis of gauge transformations', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **36**(3), 491–518.

Pons, J., Salisbury, D. & Shepley, L. (1999), 'Reduced phase space: quotienting procedure for gauge theories', *Journal of Physics A: Mathematical and General* **32**(2), 419.

Pooley, O. & Wallace, D. (2022), 'First-class constraints generate gauge transformations in electromagnetism (reply to pitts)', *arXiv preprint arXiv:2210.09063* .

Souriau, J.-M. (1997), *Structure of dynamical systems: a symplectic view of physics*, Vol. 149, Springer Science & Business Media.

Weatherall, J. O. (2016*a*), 'Are newtonian gravitation and geometrized newtonian gravitation theoretically equivalent?', *Erkenntnis* **81**, 1073–1091.

Weatherall, J. O. (2016*b*), 'Understanding gauge', *Philosophy of Science* **83**(5), 1039–1049.

Weatherall, J. O. (2019), 'Part 2: Theoretical equivalence in physics', *Philosophy Compass* **14**(5), e12591.