Computable Qualitative Probability

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Theories of qualitative probability provide a justification for the use of numerical probabilities to represent an agent's degrees of belief. If a qualitative probability relation satisfies a set of well-known axioms then there is a probability measure that is compatible with that relation. In the particular case of subjective probability this means that we have sufficient conditions for representing an agent as having probabilistic beliefs. But the classical results are not constructive; there is in no general method for calculating the compatible measure from the qualitative relation. To address this problem this paper introduces the theory of *computable qualitative probability.* I show that there is an algorithm that computes a probability measure from a qualitative relation in highly general circumstances. Moreover I show that given a natural computability requirement on the qualitative relation the resulting probability measure is also computable. Since computable probability is a growing interest in Bayesian epistemology this result provides a valuable interpretation of that notion.

1 Introduction

Bayesians assert that an agent's beliefs should be representable by a probability measure. Critics object that while one might certainly feel more or less strongly about whether some events will occur—say, whether it will rain tomorrow—it is a rather large leap to say that these feelings have all the structure of a probability measure. I am not sure which infinitely precise real number best represents my confidence that it will rain tomorrow. If I can't put a number on it, how can I ensure my beliefs are truly coherent, the way a Bayesian says they ought to be? Or suppose I am a behavioral scientist—how can I test whether human agents really are approximately Bayesian?

Clearly we need a system of *measurement* for belief. And we have such a system: qualitative probability. This is a tradition with roots in the work of Ramsey ([1]), Koopman ([2]), de Finetti ([3]), and Good ([4]), but extends into modern day research. On this account, we assume that we have access to some set of *probability judgments* of the form "B is at least as likely as A". We can write this judgment as the formula "$A \preceq B$". We then take the relation $\preceq$ itself as the object of study. Using the relation we prove a *representation theorem*, a result of the form: given some axioms on $\preceq$ there corresponds a (sometimes unique) probability measure P such that $A \preceq B \iff P(A) \leq P(B)$. Many results have been proved in this vein ([5], [6]; see [7], [8] for an introduction). The result is a common set of agreed-upon axioms; any relation satisfying these axioms is called a *qualitative probability.*

Qualitative probability seems to resolve our primary concerns. First, some authors such as Koopman ([2]) and Good ([4]) take qualitative probabilities to be more fundamental than numerical probabilities. On this view, numerical probabilities serve as a *measurement scale* for qualitative belief, much in the way that meters are a measurement scale for distances. A representation theorem as sketched above justifies the introduction of numerical probabilities as an appropriate measurement scale. So it is no problem that I cannot put a precise number to my beliefs. If I can produce a set of qualitative judgments that satisfy the axioms, then my beliefs can be represented probabilistically. Second, it is empirically verifiable whether an agent's judgments really conform to the axioms of qualitative probability. One need only check whether an agent's judgments satisfy the axioms of qualitative probability.

Qualitative probability results do not settle all our issues. The classical theorems tell us that, assuming $\preceq$ adheres to a (small) set of axioms, there is a corresponding probability measure. These results do not tell us how to determine *which* probability measure represents an agent's beliefs. Suppose you are given an agent's qualitative probability relation $\preceq$ and an event A. Can you always determine $P(A)$? This question is not resolved by the classical results. What is required is an *algorithm* that takes a qualitative probability as input and outputs a probability measure. This suggests that we should investigate *computable* qualitative probability structures.

A related (though independent) motivation for this study comes from a recent interest in computable Bayesian agents in the philosophical literature. For example Zaffora Blando ([9]) studies the Blackwell-Dubins "merging of opinions" theorem for agents whose beliefs are given by a computable probability measure. Belot ([10], [11]) has studied notions of "chance laws" for agents with computable beliefs. And Huttegger, Walsh, and Zaffora Blando ([12]) have recently proved a wide range of martingale "convergence to the truth" results for computable Bayesian agents. There is an open question of interpretation: what does it mean to say that an agent's beliefs are computable? When can we truly say that an agent's beliefs are computable? Is there a precise method for delineating agents with computable beliefs from those without?

This paper answers both motivations: (i) how to determine an agent's beliefs from their qualitative judgments, and (ii) how to interpret the "computability" aspect of computable Bayesian agents. In particular I develop the theory of computable qualitative probability. I introduce a natural computability assumption on qualitative probability structures. The main result (Theorem 2) shows that given this assumption (together with the classical axioms of qualitative probability) there exists a unique *computable* probability measure that represents a computable qualitative probability structure (answering motivation (ii)), and moreover one can compute this probability measure from the qualitative probability (answering motivation (i)).

In §2 I review the classical results from the theory of qualitative probability. My proof strategy for the main result is a computable variant of the classical proof strategy. §3 introduces the definition of computable qualitative probability. In §4 I show that one can compute finite partitions of equally probable sets which themselves have important computability properties. Finally, §5 uses these partition results to prove the main theorem. I conclude with a discussion of future directions for this work.

2 Classical Results

We begin with a discussion of the classical results in qualitative probability.

Definition 1. Let $\mathscr{A}$ be a σ-algebra of subsets of some set Ω. Let $\preceq$ be a relation on $\mathscr{A}$. Assume (i) $\preceq$ is a total preorder with initial element $\emptyset$ and terminal element Ω; and (ii) if $B_1 \cap B_2 = \emptyset$ and if $A_1 \preceq B_1$ and $A_2 \preceq B_2$ then $A_1 \cup A_2 \preceq B_1 \cup B_2$. Moreover if in one of the first two inequalities $\preceq$ is replaced with $\prec$, then the last one holds with $\preceq$ replaced with $\prec$. Then we call $\preceq$ a *qualitative probability*. If both $A \preceq B$ and $B \preceq A$ then we write $A \sim B$.

Definition 2. Let $\preceq$ be a qualitative probability on a σ-algebra $\mathscr{A}$. We say that $\preceq$ is *monotonely continuous* if given a monotone increasing sequence $A_n \uparrow A$ and an event B such that for all n, $A_n \preceq B$, then $A \preceq B$.

Monotone continuity was introduced by [6] to ensure that the resulting probability measure is countably additive. To simplify exposition we will only be interested in countably additive measures, so we follow Villegas in assuming monotone continuity in the results that follow.

Definition 3. If $\mathscr{A}$ is a σ-algebra and $\preceq$ is a monotonely continuous qualitative probability, then we call $(\mathscr{A}, \preceq)$ a *qualitative probability σ-algebra*.

The end goal of qualitative probability is to show that there is a probability measure that is *compatible* with the relation, that is, the measure preserves the ordering.

Definition 4. Let $(\mathscr{A}, \preceq)$ be a qualitative probability σ-algebra. Let P be a probability measure on $\mathscr{A}$. P is *compatible with* $\preceq$ if for all $A, B \in \mathscr{A}$,

$$P(A) \leq P(B) \iff A \preceq B.$$

The rest of this section outlines the classical strategy for proving the existence of a compatible probability measure.

First, as with probability measures, monotone continuity implies nice convergence properties for qualitative probabilities. The following is due to Villegas ([6], Lemma 3.2); it says, roughly, that if in a sequence of events each event has at most half the probability of the previous event, then this sequence converges to a null event.

Lemma 1. *If $(\mathscr{A}, \preceq)$ is a qualitative probability σ-algebra, and $\{A_n\}_{n\in\omega}$ is a monotone decreasing sequence of events, such that $A_{n+1} \preceq A_n \setminus A_{n+1}$, then* $\lim_{n\to\infty} A_n \sim \emptyset$.

All previous work on qualitative probability assumes some form of *atomlessness*. Analogous to atomless probability measures, a qualitative probability is atomless if every set of positive probability has a subset of positive probability.

Definition 5. Let $(\mathscr{A}, \preceq)$ be a qualitative probability algebra. We say that $(\mathscr{A}, \preceq)$ is *atomless* if for every $A \in \mathscr{A}$ such that $A \succ \emptyset$ there is $B \subset A$ such that $B \succ \emptyset$.

Atomlessness ensures that the resulting probability measure is unique; while a qualitative probability need not be atomless, we will in general only consider the atomless case.

The classical proof strategy, e.g. followed by [6],[7],[5], and others, is roughly as follows. One begins by showing that the qualitative probability can partition the space in a nice way. In particular one shows that for any event A in the σ-algebra $\mathscr{A}$ there is a partition of A into two equally likely events B, C. This equipartition process can be applied recursively to Ω to produce a binary tree structure. We can then assign any given point $x \in \Omega$ a binary code σ_x, the code for the "branch" of events containing x. In effect we map our qualitative probability σ-algebra $(\mathscr{A}, \preceq)$ to a qualitative version of Cantor space with Lebesgue measure. From there we map Cantor space to the unit interval using the standard bijection between reals $r \in [0,1]$ and their binary expansions. Thus we have a qualitative version of the unit interval with Lebesgue measure. We call the map from $\mathscr{A}$ to the unit interval with (qualitative) Lebesgue measure a *uniform random variable*:

Definition 6. A function $X : \Omega \to [0,1]$ is a *uniform random variable* if whenever $I, J \subseteq [0,1]$ are intervals with $|I| \leq |J|$, $X^{-1}(I) \preceq X^{-1}(J)$;

These steps are formalized in the following proposition.

Proposition 1 ([6], Theorem 3.5)**.** *In a qualitative probability σ-algebra $\mathscr{A}$ the following are equivalent:*

1. *$(\mathscr{A}, \preceq)$ is atomless;*
2. *every event can be partitioned into two equally probable events;*
3. *there is a uniform random variable.*

We then use the uniform random variable to define our probability measure. The important details are provided in the following sketch of the proof due to Villegas.

Proposition 2 ([6], Theorem 5.3)**.** *If a qualitative probability σ-algebra $(\mathscr{A}, \preceq)$ is atomless then there is a unique compatible probability measure, and it is countably additive.*

Proof. We give the important details. By Proposition 1, since $\mathscr{A}$ is atomless, there is a uniform random variable X. Given a number $x \in [0,1]$ consider the event $F(x) := X^{-1}[0,x]$. Clearly $F(0) = \emptyset$, $F(1) = \Omega$ and $F(x)$ is a monotone function of x. It is also continuous in the sense that if $x_n \to x$ then $\lim_{n\to\infty} F(x_n) = F(x)$. By a continuity argument it can be shown that given an event A there is a unique $a \in [0,1]$ such that $A \sim X^{-1}[0,a]$. We then define $P(A) = a$, i.e., $A \sim X^{-1}[0, P(A)]$. It is quick to show that P so defined is additive.

To show that it is unique, since X is a uniform random variable note that the events

$$\{\omega \mid (i-1)/n \leq X(\omega) \leq i/n\}$$

for $i = 1, \dots, n$ constitute a uniform partition of Ω. For any rational a and any compatible probability measure Q,

$$Q\{\omega \mid 0 \leq X(\omega) \leq a\} = a = P\{\omega \mid 0 \leq X(\omega) \leq a\}.$$

Finally, P is countably additive because it is compatible with $(\mathscr{A}, \preceq)$ and it is monotonely continuous. □

In what follows we will be concerned with *countably generated* σ-algebras, i.e., σ-algebras $\mathscr{A}$ for which there is a countable algebra $\mathscr{R}$ such that $\mathscr{A} = \sigma(\mathscr{R})$. Countably generated σ-algebras are perhaps the most common setting for probability theory (e.g. [13], [14]), so this is not a restrictive assumption. Moreover they have nice properties, as the following two results demonstrate.

Lemma 2. *Let $(\mathscr{A}, \preceq)$ be an atomless qualitative probability σ-algebra. Let P be the induced probability measure on $\mathscr{A}$. Let $A \in \mathscr{A}$. Then for any dyadic rational $q \in [0,1]$ there is an event $B \subseteq A$ such that $P(B) = qP(A)$.*

Proof. Suppose $q = n/2m$. By Proposition 1.2, there is a partition of A into $2m$-many equally likely events $\{A_n\}$, so for any $n \leq 2m$, $P(A_n) = P(A)/2m$; then set $B := \bigcup_{i=1}^{n} A_n$. □

The following is a basic result in measure theory (see for example [15], Theorem 1.3.11). We present a standard proof in order to highlight a particular case that will be important in what follows.

Proposition 3. *Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let $\mathscr{R}$ be an algebra such that $\sigma(\mathscr{R}) = \mathscr{A}$. For any $\epsilon > 0$ and $A \in \mathscr{A}$ there is a set $B \in \mathscr{R}$ such that $\mu(A \Delta B) < \epsilon$.*

In particular, suppose $A \in \mathscr{A}$, $C \in \mathscr{R}$, $A \subseteq C$, and $\epsilon > 0$. Then there is a subset $B \subset C$ such that $B \in \mathscr{R}$ and $\mu(A \Delta B) < \epsilon$.

Proof. Let $\mathscr{G}$ be the closure of $\mathscr{R}$ under countable unions. Since P is montonely continuous, for any $U := \bigcup_{n=1}^{\infty} U_n$ with $U_n \in \mathscr{R}$ for all n there is $N > 0$ such that $P(U) - P(\bigcup_{n=1}^{N} U_n) < \epsilon$, so the result holds for $\mathscr{G}$.

For arbitrary $F \in \mathscr{A}$, note that

$$P(F) = \inf\{P(G) \mid G \in \mathscr{G} \wedge G \supseteq A\},$$

in which case we can approximate F via some sequence $\{G_n\}$ of elements of $\mathscr{G}$, and each G_n can itself be approximated by a sequence $\{U^n_m\}$ of members of $\mathscr{R}$.

For the second statement, relativize the above proof as follows. Approximate A via sets $\{G_n \cap C\}$, which are themselves approximated by sets $\{U^n_m \cap C\}$, each of which is a member of $\mathscr{R}$. □

3 Computable Qualitative Probability σ-Algebras

In this section we define a *computable* qualitative probability σ-algebra, our primary object of study. We start with the standard notion of a computable algebra.[1]

Definition 7. Let $\mathscr{R}$ be a countable Boolean algebra, and fix some enumeration $\{R_n\}$ of elements of $\mathscr{R}$. We say that $\mathscr{R}$ is *computable* if, given $R_n, R_m \in \mathscr{R}$, the Boolean operations

$$\begin{aligned} R_n \cap R_m &= R_l \\ R_n \cup R_m &= R_j \\ \neg R_n &= R_k \end{aligned}$$

[1] We use standard definitions from computable analysis; see [16].

are uniformly computable in n, m, i.e., there are total computable functions f, g, h such that

$$f(n,m) = l$$
$$g(n,m) = j$$
$$h(n) = k.$$

In general the σ-algebra $\mathscr{A}$ of interest is uncountable, so this definition will not suffice for our purposes. Instead we will work with countably-generated σ-algebras, and assume that they are generated by a computable algebra.

Definition 8. A *computable qualitative probability σ-algebra* is a tuple $(\mathscr{A}, \mathscr{R}, \preceq)$ where

(1) $(\mathscr{A}, \preceq)$ is an qualitative probability σ-algebra;
(2) $\mathscr{R} = \{R_i\}_{i\in\omega}$ is a computable countable Boolean algebra such that $\sigma(\mathscr{R}) = \mathscr{A}$;
(3) for all $i \in \omega$,

$$(\leftarrow, R_i) = \{R_j \in \mathscr{R} \mid R_j \prec R_i\}$$
$$(R_i, \rightarrow) = \{R_j \in \mathscr{R} \mid R_i \prec R_j\}$$

are c.e. sets uniformly in i.[2]

As in the classical case, the goal is to show that for any computable qualitative probability σ-algebra there is a unique compatible probability measure—but in this case, we wish to show that this probability measure is itself computable. To this end we must define a notion of computability for probability measures.

Definition 9. A real number $r \in \mathbb{R}$ is *computable* if there is a uniformly computable sequence of rationals $\{q_n\}_{n\in\omega}$ such that for all n, $|r - q_n| \leq 2^{-n}$. We call the sequence $\{q_n\}_{n\in\omega}$ a *fast Cauchy sequence*. Let $(\mathscr{A}, \mathscr{R}, \preceq)$ be a computable qualitative probability σ-algebra, and suppose P is a compatible probability measure on $\mathscr{A}$. Then P is a *computable* probability measure if for all $R \in \mathscr{R}$, $P(R)$ is a computable real uniformly in R.[3]

The algebra $\mathscr{R}$ consists of the "simple" sets with nice computability properties. In particular the unique compatible probability measure maps these sets to computable real numbers; this fact is reflected in axiom 3, which parallels the fact that strict inequality ($<$) is a c.e. relation on computable reals. We can also define a kind of set that is not itself a member of $\mathscr{R}$ but is sufficiently like $\mathscr{R}$ to be useful for constructing partitions. We call these sets "almost-computable", and it will turn out that these sets are also assigned computable measure.

Definition 10. Let

$$U := \bigcup_{i\in I} R_i$$

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[2]This definition extends that of [17], which defines a "computable σ-algebra´´ as a structure $(\mathscr{A}, \mathscr{R})$ such that $\mathscr{A}$ and $\mathscr{R}$ satisfy (2).

[3]One can show that this definition is equivalent to the standard definition of "computable probability measure´´ as given, for example, by [18], [19].

where $R_i \in \mathscr{R}$ for all i, and suppose its complement $V = \Omega \setminus U$ is of the form

$$V := \bigcup_{j \in J} R_j$$

where again $R_j \in \mathscr{R}$ for all j. If $I, J \subseteq \omega$ are c.e. sets then we say that U, V are *almost-computable sets.*

Lemma 3. *Almost-computable events are closed under computable unions.*

Proof. Immediate from the fact that c.e. sets are closed under computable unions. □

The following lemma says, qualitatively, that the compatible probability measure assigns computable reals to almost-computable sets. We formulate this qualitatively by noting that strictly inequality ($<$) is a c.e. relation on computable reals.

Lemma 4. *Let U be an almost-computable event with complement V. Then the sets*

$$\{R_k \mid R_k \prec U\}$$

and

$$\{R_k \mid U \prec R_k\}$$

are c.e.

Proof. Let $U = \bigcup_{i \in I} R_i$ and $V = \bigcup_{j \in J} R_j$. If $R_k \prec U$ then there is some finite subset $I_0 \subseteq I$ such that

$$R_k \prec \bigcup_{i \in I_0} R_i \prec U$$

and the first relation is a c.e. condition since $\preceq$ is a computable qualitative probability. For the other case, $U \prec R_k$ if and only if $\Omega \setminus R_k \prec \Omega \setminus U = V$. Then there is some finite subset $J_0 \subseteq J$ such that

$$R_k \prec \bigcup_{j \in J_0} R_j \prec V$$

and the first relation is a c.e. condition. □

In the next section I prove a series of lemmas showing that we can compute equipartitions of almost-computable sets, thus providing an effective version of the standard proof strategy. The following is an obvious but important fact about almost-computable events: it simply says that they can be partitioned at all.

Lemma 5. *If U is almost-computable and $\emptyset \prec U$ then one can compute $B, C \in \mathscr{R}$ such that $\{B, C\}$ is a partition of U, $\emptyset \prec B, C$, and $B \prec C$.*

Proof. This follows immediately from the fact that U is a union of $\mathscr{R}$-elements and $(\mathscr{R}, \preceq)$ is atomless. □

4 Computing Partitions

In this section I show (Lemma 11) that almost-computable sets can be computably partitioned into two equally likely almost-computable sets. To show this we need to do two things: first, I show that the relevant sets always *exist*, and second, I show that we can *effectively find them*. In many cases existence is an immediate consequence of classical versions of these results.

The following lemma is an existence result. It says, roughly, that for any positive probability simple event there exists a subset of that event whose probability is bounded by two other simple events.

Lemma 6. *Let* $(\mathscr{A}, \mathscr{R}, \preceq)$ *be a computable qualitative probability* σ*-algebra. Let* $A, B, C \in \mathscr{R}$ *with* $A \prec B \preceq C$*. Then there is an event* $D \in \mathscr{R}$ *such that* $D \subset C$ *and* $A \prec D \prec B$*.*

Proof. Theorem 2 implies that there is a unique probability measure $P : \mathscr{A} \to [0, 1]$. Thus $P(A) < P(B) \leq P(C)$. Let $r = P(B) - P(A)$. Let $\delta > 0$ and fix a dyadic rational q such that

$$\left|\left(P(A) + \frac{r}{2}\right) - qP(C)\right| = \left|\left(P(B) - \frac{r}{2}\right) - qP(C)\right| < \delta. \tag{1}$$

In words, q is less than δ away from the midpoint $P(A) + \frac{r}{2} = P(B) - \frac{r}{2}$ of the interval $(P(A), P(B))$. Then by Lemma 2 there is an event $D' \subseteq C$ such that $P(D') = qP(C)$. Let $\epsilon = \frac{r}{2} - \delta$. By Proposition 3 there is an event $D \in \mathscr{R}$ such that $D \subset C$ and $P(D \Delta D') < \epsilon$, which implies that

$$P(D) \in \left(qP(C) - \left[\frac{r}{2} - \delta\right], qP(C) + \left[\frac{r}{2} - \delta\right]\right). \tag{2}$$

By (1) we have

$$\left(P(A) + \frac{r}{2}\right) - \delta < qP(C) < \left(P(B) - \frac{r}{2}\right) + \delta. \tag{3}$$

Combining (2) with (3) we have

$$\begin{aligned} P(D) &\in \left(qP(C) - \left[\frac{r}{2} - \delta\right], qP(C) + \left[\frac{r}{2} - \delta\right]\right) \\ &\subseteq \left(\left[P(A) + \frac{r}{2} - \delta\right] - \left[\frac{r}{2} - \delta\right], \left[P(B) - \frac{r}{2} + \delta\right] + \left[\frac{r}{2} - \delta\right]\right) \\ &= (P(A), P(B)), \end{aligned}$$

that is, $A \prec D \prec B$. □

The next lemma says that if we are given a partition of some event and each element of the partition has probability bounded away from zero, then that partition is finite. Finiteness is, of course, an extremely useful property if we wish to *compute* partitions.

Lemma 7. *Let $(\mathscr{A}, \mathscr{R}, \preceq)$ be a computable qualitative probability σ-algebra. Let $\{A_n\}$ be a partition of an event $A \succ \emptyset$. If for all n, $A_n \succ B \succ \emptyset$, then $\{A_n\}$ is finite.*

Proof. Again by Theorem 2 let P denote the unique probability measure on $\mathscr{A}$. Then for all n we have $P(A_n) > P(B) > 0$, and so there must be only finitely many A_n. $\square$

From here I follow the proof strategy due to Fishburn, specifically §14.2. In the following results (Lemmata 8, 9, 10, and 11) we assume that $(\mathscr{A}, \mathscr{R}, \preceq)$ is an atomless computable qualitative probability σ-algebra. Each of these results has a classical analogue proved by [7], so we need not worry about existence.

Lemma 8. *Suppose that $A, B \in \mathscr{R}$ and $A \prec B$. Then one can compute a finite partition $\{C_n\}$ of Ω such that $\forall i \leq n, C_i \in \mathscr{R}$ and $A \cup C_i \prec B$.*

Proof. For any $C_i \in \mathscr{R}$ the condition

$$A \cup C_i \prec B \tag{4}$$

is c.e. If we were working with a probability measure we could rewrite (4) in terms of $P(C_i)$ as $P(C_i) < P(B) - P(A)$. But working qualitatively we cannot do this, since we do not assume $A \subseteq B$ (in which case it could be that $B \setminus A' = \emptyset$). To fix this, note that by assumption $A \prec B \preceq B$, so by Lemma 6 there is a subset $A' \subseteq B$, $A' \in \mathscr{R}$ such that

$$A \prec A' \prec B. \tag{5}$$

To find the elements of the partition we use the following algorithm:

1. Begin enumerating $C_i \in \mathscr{R}$ that satisfy condition (4). Let C_1 be the first such set enumerated.
2. Check if $A \cup (\Omega \setminus \bigcup_{i=1}^{n} C_i) \prec B$, where n is the largest integer such that C_n has been defined.
   (a) If yes, define $C_{n+1} := A \cup (\Omega \setminus \bigcup_{i=1}^{n} C_i)$. Halt.
   (b) Else compute $C_{n+1} \in \mathscr{R}$ such that
      (i) $C_{n+1} \subseteq (\Omega \setminus \bigcup_{i=1}^{n} C_i)$ (a computable condition since $\mathscr{R}$ is a computable algebra);
      (ii) $C_1 \prec C_{n+1} \prec B \setminus A'$.
3. Return to 2.

First we show that Step 2.(b) is well-defined. By the argument preceding the algorithm, if the initial condition of Step 2 has not been satisfied then $B \setminus A' \prec (\Omega \setminus \bigcup_{i=1}^{n} C_i)$. But then, by Lemma 6, there must exist $C_{n+1} \subseteq (\Omega \setminus \bigcup_{i=1}^{n} C_i)$ such that

$$C_1 \prec C_{n+1} \prec B \setminus A', \tag{6}$$

which implies

$$A \cup C_{n+1} \prec A' \cup C_{n+1} \prec B. \tag{7}$$

Importantly, condition (6) is a c.e. condition uniformly in C_{n+1}, and condition (7) is the desired property for elements of the partition.

Finally we show that the resulting set $\{C_n\}$ is a finite partition of Ω. By (i) the C_n are disjoint. By (ii) and Lemma 7 the set $\{C_n\}$ is finite. Therefore Step 2.(a) will be satisfied at some finite stage and the algorithm halts. □

Lemma 9. *If $A, B, C \in \mathscr{R}$ are pairwise disjoint with $A \preceq B$, $B \prec A \cup C$, then one can compute $D \subseteq C$ such that $\emptyset \prec D$ and $B \cup D \prec A \cup (C \setminus D)$.*

Proof. Clearly $\emptyset \prec C$. It follows from Lemma 8 that there one can compute $D_1 \in \mathscr{R}$, $D_1 \subseteq C$ such that $\emptyset \prec D_1$ and $B \cup D_1 \prec A \cup C$. By Lemma 3 D_1 can be computably partitioned into $D, D' \in \mathscr{R}$ with $\emptyset \prec D \prec D'$. Therefore

$$B \cup D \cup D' \prec A \cup (C \setminus D) \cup D,$$

which implies $B \cup D' \prec A \cup (C \setminus D)$. Thus since $D \prec D'$, $B \cup D \prec B \cup D'$, so $B \cup D \prec A \cup (C \setminus D)$. □

Lemma 10. *If $A, B \in \mathscr{R}$ are such that $\emptyset \prec A, B$ and $A \cap B = \emptyset$ then B can be computably partitioned into C and D such that $C \preceq D \preceq A \cup C$.*

Proof. By Lemma 8 compute a finite partition $\{G_n\}$ such that $\forall i \leq n, \emptyset \prec G_i \prec A$. It follows classically from Lemma 10 that there is m such that $\bigcup_1^m G_i \preceq \bigcup_{m+1}^n G_i \preceq \bigcup_1^{m+1} G_i$ (see Fishburn 14.C7, p. 196). We want to show that this m can be effectively found.

Since $\bigcup_1^m G_i \prec \bigcup_1^{m+1} G_i$, there are three possibilities:

1. $\bigcup_1^m G_i \prec \bigcup_{m+1}^n G_i \prec \bigcup_1^{m+1} G_i$;
2. $\bigcup_1^m G_i \sim \bigcup_{m+1}^n G_i \prec \bigcup_1^{m+1} G_i$;
3. $\bigcup_1^m G_i \prec \bigcup_{m+1}^n G_i \sim \bigcup_1^{m+1} G_i$.

To compute m we begin enumerating the following four c.e. sets:

$$\Gamma_0 = \{m \leq n \mid \bigcup_1^m G_i \prec \bigcup_{m+1}^n G_i\},$$

$$\Gamma_1 = \{m \leq n \mid \bigcup_1^m G_i \succ \bigcup_{m+1}^n G_i\},$$

$$\Delta_0 = \{m \leq n \mid \bigcup_{m+1}^n G_i \prec \bigcup_1^{m+1} G_i\},$$

$$\Delta_1 = \{m \leq n \mid \bigcup_{m+1}^n G_i \succ \bigcup_1^{m+1} G_i\}.$$

If (1) holds then m is eventually enumerated into Γ_0 and Δ_0, and the process halts.

If (2) holds then m is eventually enumerated into Δ_0. And, for $\ell < m$, $\bigcup_1^\ell G_i \prec \bigcup_{\ell+1}^n G_i$, and for $L > m$, $\bigcup_1^L G_i \succ \bigcup_{L+1}^n G_i$. Therefore m is the unique number that is *not* enumerated into either Γ_0 or Γ_1; since n is finite, m can be determined by elimination after finitely many steps.

Similarly if (3) holds then m is eventually enumerated into Γ_0 and is the unique number that is not enumerated into either Δ_0 or Δ_1, and so can be determined after finitely many steps.

Therefore we can compute $m \leq n$ satisfying $\bigcup_1^m G_i \preceq \bigcup_{m+1}^n G_i \preceq \bigcup_1^{m+1} G_i$. Thus $C := \bigcup_1^m G_i \in \mathscr{R}$ and $D := \bigcup_{m+1}^n G_i \in \mathscr{R}$ can be computed and satisfy $C \preceq D \preceq G_{m+1} \cup C$; since $G_{m+1} \prec A$ by construction, we have $C \preceq D \prec A \cup C$. □

A nice consequence of the above argument is: if there is some union of partition elements that is "half the measure" (equally as likely as its complement), then we can effectively find this union. More precisely, let $A \in \mathscr{R}$, $\emptyset \prec A$, and let $\{B_n\}$ be a finite computable partition of A. If there is some set $I \subseteq \{0, 1, \ldots, n\}$ such that $\bigcup_{i \in I} B_i \sim \bigcup_{i \notin I} B_i$ then I is computable. Put this way it's quite obvious, since there are only finitely many such I to check and each of them is finite, so we can simply use a brute-force search.

Lemma 11. *If A is almost-computable and $\emptyset \prec A$ then A can be computably partitioned into almost-computable sets B, C with $B \sim C$.*

Proof. It follows from Lemma 3 that A can be computably partitioned into $B_1, C_1, D_1 \in \mathscr{R}$ such that $B_1 \preceq C_1 \cup D_1$ and $C_1 \preceq B_1 \cup D_1$. If $B_1 \sim C_1 \cup D_1$ or $C_1 \sim B_1 \cup D_1$ then this can be determined effectively, and we're done.

So instead assume that $B_1 \prec C_1 \cup D_1$ and $C_1 \prec B_1 \cup D_1$. Thus $\emptyset \prec D_1$. Assume without loss of generality that $B_1 \preceq C_1$. Then Lemma 9 implies that we can compute $C^2 \in \mathscr{R}$ such that $C^2 \subseteq D_1, \emptyset \prec C^2$, and $C_1 \cup C^2 \preceq B_1 \cup (D_1 \setminus C^2)$ (see Figure 1). Then $\emptyset \prec D_1 \setminus C^2$, so by Lemma 10, $D_1 \setminus C^2$ can be computably partitioned into $B^2, D_2 \in \mathscr{R}$ with

$$B^2 \preceq D_2 \preceq C^2 \cup B^2.$$

Since by assumption $B_1 \preceq C_1$ we have

$$B_1 \cup B^2 \preceq C_1 \cup D_2 \prec C_1 \cup D_2 \cup C^2.$$

We then let $B_2 := B_1 \cup B^2$ and $C_2 := C_1 \cup C^2$. Repeating this process we can compute a sequence of three-part partitions $\{B_n, C_n, D_n\}$ such that for all n,

1. $B_n, C_n, D_n \in \mathscr{R}$;
2. $B_n \prec C_n \cup D_n$ and $C_n \prec B_n \cup D_n$;
3. $B_n \subseteq B_{n+1}, C_n \subseteq C_{n+1}, D_{n+1} \subseteq D_n$;
4. $D_{n+1} \preceq D_n \setminus D_{n+1}$.

By (4) and Lemma 1 we have $D_n \downarrow \emptyset$.

Now let $B := \bigcup_{n=1}^\infty B_n$, $C := (\bigcup_{n=1}^\infty C_n) \cup (\bigcap_{n=1}^\infty D_n)$. $\{B, C\}$ is clearly a partition, and B, C are almost-computable. Below we reiterate Fishburn's argument that $B \sim C$, but note that the effective content is finished—we have already shown that B and C can be computed.

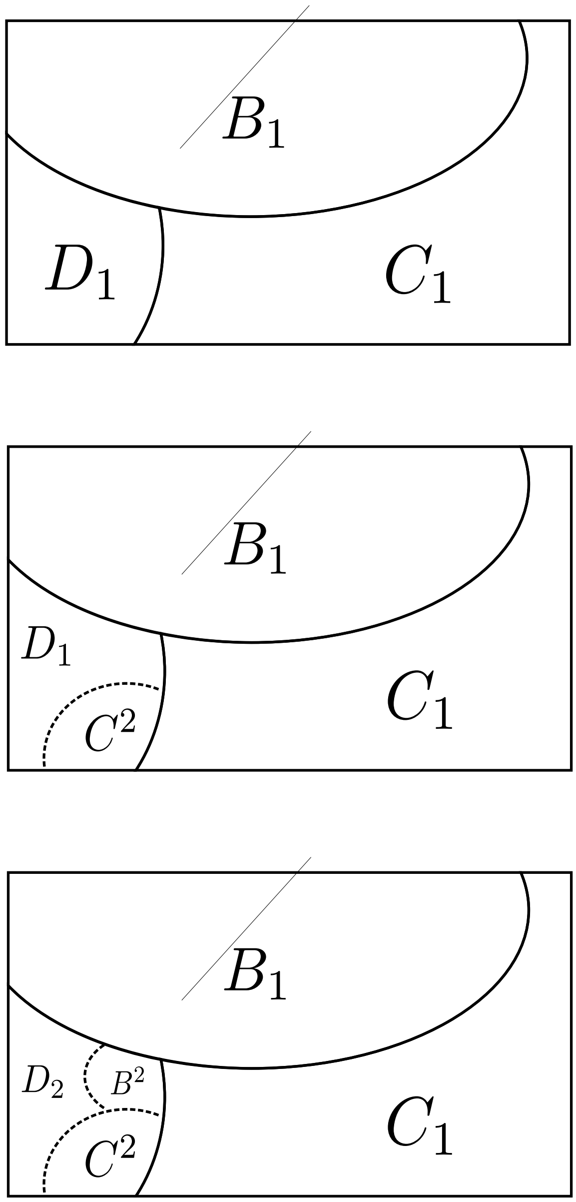

Fig. 1 A diagram of the partition procedure described in Lemma 11

Suppose toward a contradiction that $B \prec C$. Then $B \prec \bigcup_{n=1}^{\infty} C_n$ since $D_n \downarrow \emptyset$, so by Lemma 9 we can compute $G \subseteq \bigcup_{n=1}^{\infty} C_n$ with $\emptyset \prec G$ and such that, for sufficiently large N,

$$B \cup G \prec \left(\bigcup_{n=1}^{N} C_n \right) \setminus G. \tag{8}$$

Since $B \cap G = \emptyset$ and for all $n, B_n \preceq B$, we have

$$B_n \cup G \preceq B \cup G. \tag{9}$$

For sufficiently large N (not necessarily the same N as before), $D_N \prec G$, so

$$B_N \cup D_N \prec B_N \cup G. \tag{10}$$

Furthermore,

$$\begin{aligned} \bigcup_{n=1}^{\infty} C_n &= \bigcup_{n=1}^{\infty} (C_n \setminus G) \cup G \\ &= \left(\bigcup_{n=1}^{\infty} C_n \setminus D_n \right) \cup \left(\left(\bigcup_{n=1}^{\infty} C_n \right) \cap D_n \right) \end{aligned}$$

But for sufficiently large N we have $D_N \cap \left(\bigcup_{n=1}^{N} C_n \right) \prec G$, and hence for sufficiently large N we have

$$\left(\bigcup_{n=1}^{N} C_n \setminus G \right) \preceq \left(\bigcup_{n=1}^{N} C_n \right) \setminus D_n. \tag{11}$$

Finally it is clear that

$$\left(\bigcup_{n=1}^{N} C_n \right) \setminus D_N \prec C_N. \tag{12}$$

We therefore have, for sufficiently large N,

$$\begin{aligned} B_N \cup D_N &\prec B_N \cup G && \text{by (10)} \\ &\prec B \cup G && \text{by (9)} \\ &\prec \left(\bigcup_{n=1}^{N} C_n \right) \setminus G && \text{by (8)} \\ &\preceq \left(\bigcup_{n=1}^{N} C_n \right) \setminus D_n && \text{by (11)} \\ &\prec C_N && \text{by (12)} \end{aligned}$$

which contradicts the construction of B_N, C_N, D_N (see 2 above). Therefore $B \not\prec C$. A parallel argument establishes that $C \not\prec B$. □

5 Constructing the Measure

In this section I prove the main result: every atomless computable qualitative probability σ-algebra has a unique *computable* probability measure that is compatible with it. To do so we need to introduce a few more technical devices.

Definition 11. A function $X : \Omega \to [0,1]$ is a *computable uniform random variable* if:

1. whenever $I, J \subseteq [0,1]$ are intervals with $|I| \leq |J|$, $X^{-1}(I) \preceq X^{-1}(J)$;
2. for all $q \in \mathbb{Q}$, the sets $X^{-1}[0,q), X^{-1}(q,1]$ are almost-computable uniformly in q.

Theorem 1. *If $(\mathscr{A}, \mathscr{R}, \preceq)$ is an atomless computable qualitative probability σ-algebra then there is a computable uniform random variable $X : \mathscr{A} \to [0,1]$.*

Proof. The argument closely follows the classical result due to Villegas (Theorem 5). Define a sequence of binary random variables $X_n : \mathscr{A} \to \{0,1\}$ recursively as follows. By Lemma 11 computably partition Ω into almost-computable sets $[1],[0] \in \mathscr{A}$ such that $[1] \sim [0]$. Let $X_1(x) = 1$ iff $x \in [1]$. Again by Lemma 11 we can computably partition $[1],[0]$ into equally likely events $[11],[10] \subseteq [1]$ and $[01],[00] \subseteq [0]$. Let $X_2(x) = 1$ iff $x \in [11] \cup [01]$. Continuing in the obvious manner we define an infinite sequence of equipartitions of Ω, where the n^{th} element of the sequence has cardinality 2^n, and a corresponding sequence X_n of uniform binary random variables.

In particular for each $x \in \Omega$ there is an infinite binary sequence σ_x such that for all n, $x \in [\sigma_x \restriction n]$. Thus the sequence $\{X_n\}_{n\in\omega}$ can be used to define a uniform random variable X in a natural way (see [6] for more details). We now show that X must be computable. Let $q \in [0,1]$ be rational. Then q has a computable binary expansion $0.x_1x_2\ldots = \sigma_q$. We write $s \sqsubset t$ to mean that s is a prefix of t. Then we have

$$X^{-1}[0,q) = \bigcup \left\{[t0] \mid t \in 2^{<\omega} \wedge t1 \sqsubset \sigma_q\right\},$$

Similarly,

$$X^{-1}(q,1] = \bigcup \left\{[t1] \mid t \in 2^{<\omega} t0 \sqsubset \sigma_q\right\}.$$

(See Figure 2.) Since σ is computable both $X^{-1}[0,q)$ and $X^{-1}(q,1]$ are computable unions of almost-computable events and hence are almost-computable uniformly in q. □

We can now prove our main result, Theorem 2. It says that for every atomless computable qualitative probability σ-algebra there is a unique computable probability measure compatible with it.

Theorem 2. *If $(\mathscr{A}, \mathscr{R}, \preceq)$ is an atomless computable qualitative probability σ-algebra then one can compute a unique computable probability measure $P : \mathscr{A} \to [0,1]$ that is compatible with $\preceq$.*

Proof. By Theorem 1 there exists a computable uniform random variable X on $\mathscr{A}$. As in the classical proof we define $P(A)$ for any $A \in \mathscr{A}$ as the unique real number such that

$$A \sim X^{-1}[0, P(A)].$$

The existence, uniqueness, and countable additivity of P, and the fact that P is compatible with $\preceq$, follow from the fact that $(\mathscr{A}, \mathscr{R}, \preceq)$ is atomless and monotone, via [6] Theorem 4.3. It remains to verify that P is computable.

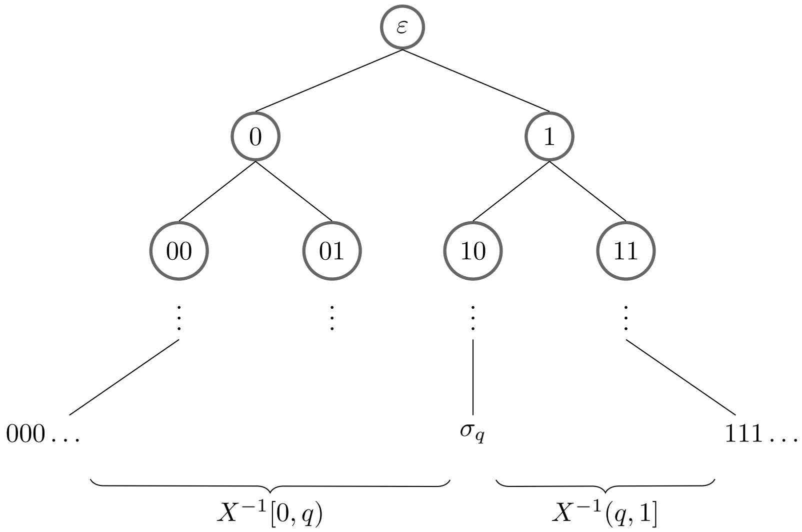

Fig. 2 $X^{-1}[0, q)$ and $X^{-1}(q, 1]$.

Let $V = X^{-1}[0, q)$, an almost-computable event. By definition of P, $P(V) = q$. Thus by Lemma 4 the sets

$$\{R \in \mathscr{R} \mid P(R) < q\}$$
$$\{R \in \mathscr{R} \mid q < P(R)\}$$

are c.e., or in other words, the conditions "$P(R) < q''$ and "$q < P(R)''$ are c.e conditions uniformly in R. Now fix $R \in \mathscr{R}$. It follows that the condition

$$p < P(R) < q \tag{13}$$

for some $p, q \in \mathbb{Q}$ is c.e. It follows that the set of $R \in \mathscr{R}$ that satisfy condition (13) and for which, given $m \in \omega$, $(q - p) \leq 2^{-(m-1)}$, is c.e., since this latter condition is computable. In other words the set

$$\Delta_R = \{(p, q, m) \in \mathbb{Q}^2 \times \omega \mid p < P(R) < q \; \& \; (q - p) \leq 2^{-(m-1)}\}$$

is a c.e. set. We then define a fast Cauchy sequence approximating $P(R)$ as follows. Begin enumerating Δ_R. Define the total computable function $f(n) := \langle p_n, q_n, n \rangle$, the (code for the) first tuple enumerated into Δ_R whose final coordinate is n. For each n define $r_n := (q_n - p_n)/2$. Since by definition of Δ_R we know that $P(R) \in (p_n, q_n)$ and $(q_n - p_n) \leq 2^{-(n-1)}$, we know that $|P(R) - r_n| \leq 2^{-n}$ for all n. Thus $\{r_n\}_{n \in \omega}$ is a fast Cauchy sequence of rationals whose limit is $P(R)$. In other words, for all $R_i \in \mathscr{R}$ the probability $P(R_i)$ defined by the computable uniform random variable X is a computable real uniformly in i. □

Note that in the statement of Theorem 2 we say that we can *compute* the representing measure P, not merely that it exists. Indeed given any event $R_i \in \mathscr{R}$ we can compute $P(R)$ from the relation $\preceq$ as follows. For each n compute a $2n$-fold equipartition $\{B_n\}$ of Ω into almost-computable sets. If for some $k \leq 2n$ we have

$$R_i \sim \bigcup_{j=1}^{m} B_j$$

then, as in Lemma 10, this can be decided. Otherwise there is a unique ℓ such that

$$\bigcup_{j=1}^{\ell} B_j \prec R_i \prec \bigcup_{j=1}^{\ell+1} B_j$$

and so we know that

$$\left| P(R_i) - \frac{\ell}{2n} \right| \leq 2^{-n},$$

and thus we compute a fast Cauchy sequence with limit $P(R_i)$. Since $\mathscr{R}$ generates $\mathscr{A}$ the values $P(R_i)$ determine P.

6 Conclusion

We have seen that given a natural computability condition on a qualitative probability σ-algebra there corresponds a unique computable probability measure. In this way we have a representation of degrees of belief for computable agents. Theorem 2 therefore tells us (i) when an agent is properly represented as having computable beliefs, and (ii) how to compute those beliefs given the agent's qualitative probability structure.

This work suggests further avenues for research. The most natural extension is to derive a computable version of Savage's representation theorem. Savage's theorem is considered the central result in the literature on representation theorems for decision theory, since it derives both utility and probability from a common preference structure. One might hope to show that given a single (hopefully natural) computability condition on the preference relation one can derive both a computable probability and computable utility function; this would provide a unified foundation for studying computable agents in decision theory.

It also suggests a study of much more bounded agents than those representable by a universal Turing machine. For example, work in cognitive science ([20]) and bounded rationality ([21]) often studies much less sophisticated agents representable via finite state automata or time-bounded Turing machines. Analogous representation theorems for these architectures are, to my knowledge, currently lacking.

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