What Is a Reference Frame in General Relativity?

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Abstract

In General Relativity, reference frames must be distinguished from coordinates. The former represent physical systems interacting with the gravitational system, aside from possible approximations, while the latter are mathematical artefacts. We propose a novel three-fold distinction between Idealised Reference Frames, Dynamical Reference Frames and Real Reference Frames. This paper not only clarifies the physical significance of reference frames, but also sheds light on the similarities between idealised reference frames and coordinates. It also analyses the salience of reference frames to define local gauge-invariant observables and to propose a physical interpretation to diffeomorphism gauge symmetry.

The best papers are the unpublished ones (not yet).

Anonymous

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1 Introduction

In the 'post-Einsteinian' physical and philosophical literature, due to some ambiguity which can be traced back to Einstein himself, it has been customary to conflate the terms 'reference frame' and 'coordinate system', which have been used somewhat interchangeably, or at least have not been always clearly distinguished. This problem has been highlighted in its philosophical-historical components in Norton (1989) and Norton (1993).[1]

This paper analyses what a reference frame is in General Relativity (GR), expanding on existing definitions in the literature. This analysis helps to further clarify the distinction between reference frames and coordinate systems. We will show that such a distinction plays a role in the physical interpretation of diffeomorphism gauge freedom, as well as the definition of local, gauge-invariant observables.[2]

The use of coordinates, or even manifold points, to spatiotemporally localise quantities in GR poses two main, closely interconnected problems:

(P1) We cannot define local gauge(diffeomorphism)-invariant observables

(P2) We have an interpretation of diffeomorphism gauge symmetries as mathematical redundancies or 'descriptive fluff' (Earman (2004)).

We will show that both problems are naturally solved when neither coordinates, nor manifold points, are used to localise physical quantities, but (spatiotemporal) reference frames.[3] As we will specify later (footnote 51, section 3.3), the mention of the use of manifold spacetime points is to emphasise that we do not intend to limit our proposal to the case where a coordinate representation of the theory is chosen. In this regard, we identify the gauge group of GR with the group of *active*

[1] According to Norton's analysis, 'Einstein's coordinates' had physical meaning, in that they were purely *mathematical* structures in $\mathbb{R}^4$, called the 'Einstein's manifold' which are *instantiated* by *physical* spacetimes. Modern practice has, however, confused 'Einstein's coordinates' with 'coordinate charts', here called coordinate systems, which are merely labellings of geometric structures defined in a smooth, differentiable manifold $\mathscr{M}$, and have no physical *instantiation*. See also Gomes (2023a). Here we call '*instantiation*' the representational relationship between a model and the physical possibility that is is supposed to model.

[2] Having this distinction clear also has a pedagogical role: it should be avoided to confuse two terms, even if one uses them consistently.

[3] We consider **(P2)** to be a problem, as gauge symmetries permeate throughout all of known physics, and it is reasonable to desire a physical interpretation for the ubiquity of gauge.

diffeomorphisms.[4]

Gauge transformations lead to redundant descriptions of physical states. This redundancy (usually understood as mathematical redundancy) poses challenges to identify physically meaningful quantities. In the context of Hamiltonian theories this freedom or redundancy appears in the form of certain kinds of constraints that the variables have to satisfy, called first class constraints. Dirac (2001) argued that only quantities that remain unaffected under gauge transformations are defined as observables, ensuring their physical relevance. Thus, he defined observables of a theory with first-class constraints as quantities which commuted with all of the constraints. In fact, in the Hamiltonian formalism, the action of constraints on quantities via the Poisson bracket generates infinitesimal gauge transformations, so commutation implies gauge-invariance. This is the formal definition; in practice explicit observables may be hard to find. This is especially the case in vacuum General Relativity (GR). Given a three-dimensional foliation of spacetime, the first-class constraints of the theory are equivalent to spatial diffeomorphisms along the leaves of the foliation and to diffeomorphisms whose generators act in the normal directions to the leaves ('refoliations'). For spacetimes which satisfy the Einstein equations, there is a neat correspondence between these '3+1-dimensional' Hamiltonian symmetries and the four-dimensional spacetime diffeomorphisms of spacetime (Lee and Wald (1990)). Since geometrical objects depend on the points of the manifold, and the GR gauge group is thought to be the four-dimensional diffeomorphism group which ''moves points around'' (Isham, 1993, p.170), objects that are represented locally are not gauge-invariant. This problem was addressed by Rovelli (2002b), in which he argued for a distinction between two notions of 'observables' in GR. *Partial observables* can be observed, in the sense of measured, even if not gauge-invariant, while *complete observables* are constructed by relating gauge-dependent partial observables in a gauge-invariant manner and characterise the predictions of the theory, coinciding with the notion of Dirac observables. Although theories can only predict

[4] We do not have space to adequately introduce this topic. For a good introduction to the distinction between active and passive diffeomorphisms see Rovelli and Gaul (2000). For a critique of this clear-cut distinction, see (Weatherall, 2018, p.14). The idea that diffeomorphisms should straightforwardly be regarded as gauge symmetries for GR is not unanimously shared. For example, Belot (2017) highlights a mismatch between (isomorphisms induced by) diffeomorphisms and gauge-symmetries (understood as transformation *not* generating new possibilities) in the case of asymptotically flat spacetimes at spatial infinity in vacuum. In particular, given the diffeomorphism $d \in Diff(\mathscr{M})$, since the two isomorphic models (see below for the meaning of symbols) $\langle \mathscr{M}, g_{ab} \rangle$ and $\langle \mathscr{M}, [d^* g]_{ab} \rangle$ have to be flat on *the same* asymptotic region, we need to use a notion of *identity* of the base set $\mathscr{M}$. Thus, the diffeomorphisms that preserve flatness, *but are not the identity in the asymptotic region* are taken to relate different physical possibilities, so they are not gauge-symmetries. Only asymptotically trivial diffeomorphisms are. See also Ashtekar et al. (1991).

gauge-invariant Dirac observables, *i.e.* complete ones, gauge-dependent partial observables are crucial in physical theories, as they describe physical observations.[5] The construction of complete observables, via composition of partial ones, implements the idea that the physical content of GR lies in the relations between dynamic quantities represented by partial observables. Thus, the idea is that locality is to be understood relationally between fields, rather than with respect to some background unobservable spatiotemporal structure (Westman and Sonego (2009)). As we shall see, the problem of constructing gauge invariant local quantities in GR is closely related to the relevance of recognising a physical meaning to reference frame.

We introduce three classes of reference frames in GR, when considered as (a set of variables representing, or *instantiated by*) a material system.[6] The benefit of this constraint is due to the fact that GR is deparametrisable only for some specific material models, for which it is possible to construct local gauge-invariant Dirac observables.[7] Thus, there is the hope that the deparametrisation techniques used for these matter toy models can provide hints to better understand the case of GR coupled with realistic standard matter.[8]

We call Idealised Reference Frames (**IRFs**) those reference frames that represent systems in which both the dynamical equations and the stress-energy contribution to the Einstein Field Equations (EFEs) are neglected. The second class is that of Dynamical Reference Frames (**DRFs**), whose stress-energy content is neglected, but the frame satisfies a specific dynamical equation. As we will show, these are the reference frames that can be associated with what are usually referred to as 'test particles'. Finally, we name Real Reference Frames (**RRFs**) those ones whose stress-

[5]For further discussion on the concept of 'observable' in GR, see Bergmann (1961), Gryb and Thébault (2016), Pitts (2022).

[6]For simplicity, throughout the paper we will often alternate between saying that a reference frame *is* a physical system and that reference frame is a set of variables in a mathematical model *representing* a physical system. Although the difference is conceptually relevant and we think the latter is the more correct way of saying, this distinction does not invalidate our work in any way. We thank Erik Curiel (private correspondence) for this suggestion.

[7]We will directly give some examples of what deparametrising means in Section 3. Naively, deparametrising a theory means choosing a set of material degrees of freedom $\{\phi\}$ that can be used (at least locally) as a spatiotemporal reference frame. This means being able to invert (at least locally) these degrees of freedom when expressed in any coordinate system (see fn. 40). In the strict sense of the term, however, deparametrisation is often defined as a global procedure. More technically, it means to rewrite the constraints in the form $C = \pi + h$, where π are the conjugate momenta of ϕ and where h, called the 'physical Hamiltonian', does not depend in any way on (ϕ, π). See Thiemann (2006); Tambornino (2012).

[8]However, using material reference frames is not the only option. Early attempts to use purely gravitational degrees of freedom as a reference frame in order to write local gauge-invariant Dirac observables were proposed, *e.g.*, in Komar (1958). Incidentally, here the author refers to the degrees of freedom constituting the reference frame as 'intrinsic coordinates'.

energy contribution to the EFEs is taken into account, as well as their dynamics. Although **RRFs** are systems of great interest, as they are physically more realistic in principle, in the remainder of the paper we deal exclusively with **IRFs** and **DRFs**. The proposed classification offers a possible reason why the notions of reference frame and coordinate system have not been carefully distinguished, with some notable exceptions already mentioned. The confusion stems from the practical, but not conceptual, equivalence that exists between **IRFs** and coordinate systems.[9]

Two other distinct notions are often confused, namely, 'idealisation' and 'approximation'. In Norton (2012)'s view, an idealisation is a novel, (typically) fictitious, system that replaces the target system under study and that is simpler to analyse. Approximations are inexact descriptions of the target system. Basically, the crucial difference lies in whether one introduces a novel system (in the case of idealisation) or not (in the case of approximation). Here, we understand reference frames to be structures which we identify in a model of GR. The most general models of the theory — labeled *kinematically possible models* (KPMs) (so as to distinguish them from those models that satisfy equations of motions, which are labeled *dynamically possible models* (DPMs) — are given by tuples $\langle \mathscr{M}, g_{ab}, \phi \rangle$. Here, $\mathscr{M}$ is a smooth manifold, g_{ab} is a Lorentzian metric, and ϕ represents the material degrees of freedom to be possibly used as spatiotemporal reference frames and *instantiated* by a target system.[10] **IRFs** (as well as **DRFs**) are supposed to be derived from successive approximations to such model structures modelling the target system playing the role of the reference frame. On the other hand, coordinates are idealisations, in particular, they are mathematical objects in their own right, without a target physical referent to model, since they are not *instantiated variables* (see Sections 2 and 4). Thus, they are not structures within our physical possible models.[11]

This paper clarifies the nature of an important and ubiquitous concept in physics: that of reference frame. In fact, whenever we set up an experiment or formalise the behaviour of a physical

[9] We mean that at least in light of problems **(P1)** and **(P2)** above, the use of **IRFs** or coordinates is indistinguishable. This does not mean that being aware of which one is being used is irrelevant (see below).

[10] Here, we are using the abstract index notation (see Penrose and Rindler (1987)) to stress that it is a geometrical object independent from the choice of a coordinate representation.

[11] This distinction between idealisations and approximations is not the only one possible. A more recent and different perspective on this matter is found in Frigg (2022). Here, an idealisation does not constitute a novel system but rather is the name given to the model-target system relation and '*must have* a physical interpretation' (ivi p.318). An approximation, on the other hand, 'operates *solely* at the mathematical level' and 'involves no reference to a model' (ivi p. 318). It goes beyond the scope of this work, but it is interesting to assess whether the distinction between reference frames and coordinate systems fits also into such a framework.

system, we *implicitly* or explicitly use a reference frame. In Anderson's words:

> All measurements are comparisons between different physical systems. (Anderson, 1967, p.128)

But see also Rovelli (1991b):

> Any measurement in physics is performed in a given reference system ,

or (Landau and Lifshitz, 1987, p.1):

> For the description of processes taking place in nature, one must have a *system of reference.*

The *implicit* (or omitted) use of reference frames, which will be addressed in Section 4, is relevant in light of the fact that it makes little sense to define a quantity that we measure experimentally using a set of *uninstantiated* coordinates and it would be more accurate to describe phenomenology in terms of reference frames.[12]

The main question of the present work is what a reference frame in GR can be. One reason why it is important to reach this end is that researchers in contemporary physics and philosophy of physics are interested in quantum reference frames (see Rovelli (1991a), Giacomini et al. (2019)). In fact, all physical systems are, to our knowledge, ultimately quantum. Our work can help to understand what kind of quantum reference frames are adopted when considering gravitational situations (Giacomini (2021)). We argue that, before we can really have a discussion on quantum reference frame in such framework, we should know properly what reference frames are in classical GR.

As a final note of caution, which it will be useful to keep in mind when we come to Section 4, we would like to emphasise that our work is not intended to delegitimise the valuable and often ingenious approximation procedure that follows *any* modelling in physics (Elgin (2017)). Our

[12] Similarly, by presenting a theory in the coordinate-free language of differential geometry, the quantities cannot be defined in a diff-invariant way, since they covary under the action of the group of diffeomorphisms. Thus, they cannot describe the behaviour of physical, gauge-invariant, systems. The only alternative would be to present a theory in terms of equivalence classes, such as $[g_{ab}]$. However, this formulation is not expressible by a single model as explicit functions of spacetime points and it is at least questionable to interpret the behaviour of physical systems in these terms.

work is meant to be a kind of '*memento*' that sometimes approximating 'without remembering anymore' can cause problems in theory (see **(P1)** and **(P2)** above).

The paper is structured as follows.

In Section 2, we revise the role of reference frames and coordinate systems in gravitational physics, as well as the main definitions of a reference frame adopted in the literature.

Section 3 contains the detailed classification of reference frames in GR, supported by some concrete examples.

In Section 4, we provide our proposal on the origin of the lack of care in using the notions of reference frame and coordinate system in GR.

In section 5 we analyse the differences between **DRFs** and coordinates. We also highlight the usefulness of reference frames to address the problem of local gauge-invariant observables and to have a physical interpretation of diffeomorphism gauge symmetries in GR.

2 Reference Frames vs. Coordinate Systems

This section is not intended to be a comprehensive review of all possible reference frame definitions in all spatiotemporal theories. The intention is to provide a context in which to place our proposal within GR, which extends the existing literature on the subject. We also briefly introduce the differences between coordinate systems and reference frames in gravitational and non-gravitational physics.

Let us now summarise some main definitions of reference frame, from which all others that may be encountered can be derived.

In (Rovelli, 1991b, p.303) a reference frame is defined as a set of variables representing a material system, for example a discrete set of physical bodies or a matter field, that can be used to define spatiotemporal localisation in a relational sense. This definition will ground our classification in Section 3. From Rovelli (1991b) also comes the suggestion that in GR reference frames can be considered 'dynamically uncoupled' (to the dynamical system of interest, i.e. the metric field, that we want to write in terms of the chosen reference frame), only if they are approximated. By 'dynamical coupling' we mean that: given a matter field ϕ which is dynamically coupled to the metric g_{ab}, then it is the case that the dynamical possible solutions of the matter field are affected by those of the gravitational field (*and viceversa*). In a more formal way, $\langle \mathscr{M}, g_{ab}, \phi \rangle$ is a DPM *iff* $\langle \mathscr{M}, [d^*g]_{ab}, d^*\phi \rangle$ is $\forall d \in Diff(\mathscr{M})$. That is, the couples $(g_{ab}, d^*\phi)$, or $([d^*g]_{ab}, \phi)$ are not dynamically allowed.[13] Our proposal will deepen and extend Rovelli's seminal and partial analysis on reference frames in GR.

On the other hand, in the work of Norton and Earman (see Earman and Friedman (1973); Earman (1974); Earman and Glymour (1978)), as is now common in the vast majority of the physical and philosophical literature of GR, a reference frame is defined by a smooth, timelike *4-velocity* field U^a tangent to the worldlines of a material system to which an equivalence class of coordinates is locally adapted (see e.g the recent (Bradley, 2021, p.1042),(Jacobs, 2024, p.4)). It is straightforward that to *fully* consider the physical 'referentiality' of such a reference frame, the 4-velocity field, representing *e.g.* massive particles' worldlines, should take into account the coupled dynamics between the particle system and the gravitational degrees of freedom. Since this is the leading and most widely used definition, we will discuss it in a separate section (Section 3.2.3).

Closely linked to this characterisation of a reference frame in terms of 4-vectors, is that of defining a reference frame in GR in terms of *tetrads*, also called '*orthonormal frames*' ((Wald, 1984, ch.3.4); Duerr (2021)). The tetrads $e_a^{(A)}$ are four smooth 4-vector fields, satisfying the orthonormality condition $e_a^{(A)} e_b^{(B)} g^{ab} = \eta^{AB}$, where $\eta^{AB} = diag(-1,1,1,1)$. The indices A, B are also called the '*internal Minkowski indices*' and label the four vectors spanning the tangent space $T_p\mathscr{M}$ and forming an orthonormal basis; the latin index a is the usual index of the coordinate-free notation. Geometrically, a tetrad is a map from the tangent space $T_p\mathscr{M}$ to Minkowski spacetime $\mathbb{M}^4$.[14] In terms of the tetrads, it is possible to rewrite *locally* the components of a given geometric object, say the metric $T_p\mathscr{M} \times T_p\mathscr{M} \ni g_{ab} = e_a^{(A)} e_b^{(B)} \eta_{AB}$; or a vector field $T_p^*\mathscr{M} \ni U_a = u_{(A)} e_a^{(A)}$, $(u_{(0)}, \cdots, u_{(3)}) \in \mathbb{R}^4$. So, tetrads are a particular basis used to provide a simplified local description of geometric objects in a small neighbourhood of each point of the manifold, as if we were in Minkowski spacetime. In the 'tetrad frame', we can associate four independent scalar values $X^{(A)}$ (instead of four coordinates x^μ) with each point $p \in \mathscr{M}$. In a nutshell, each tetrad is a 4-vector rep-

[13] A similar concept is also present in (Wallace, 2022, p.242), in terms of 'dynamical isolation'.

[14] The tangent space $T_p(\mathscr{M})$ at each point p of a Lorentzian manifold is already a Minkowski vector space. Thus, describing tetrads as a map from $T_p(\mathscr{M})$ to the Minkowski space $\mathbb{M}^4$ might seem redundant or confusing. To clarify, the tetrads do not map $T_p(\mathscr{M})$ to another separate Minkowski space. Instead, the tetrads serve to establish a local orthonormal basis in $T_p(\mathscr{M})$ itself.

resenting a *direction* in spacetime. The temporal tetrad $e^a_{(0)}$ represents the (local) temporal direction (often associated with the 4-velocity of the *comoving* observer). The spatial tetrads $e^a_{(1)}, e^a_{(2)}, e^a_{(3)}$ represent the (local) spatial directions. The set of these orthonormal spacetime directions (four orthonormal axes at each point) is the so-called 'tetrad frame'.

In other recent works, the conventional, but rather informal way to distinguish a reference frame from a coordinate system is to point out that only a reference frame has a link to an observer's state of motion (DiSalle (2020)). Moreover, (Pooley, 2022, Sec.4.3) defines a reference frame as a set of standards (such as a standard of rest and a standard of simultaneity in Newton theory) relative to which a body's motion can be quantified. This paper shows that these definitions do not fully exhaust the characterisation of reference frames in GR.

The literature dealing with quantum reference frame agrees that reference frames are associated with physical material systems, which are ultimately quantum. Relative to these objects, we determine the properties of the physical systems we wish to study, such as spatiotemporal localisation Castro-Ruiz et al. (2020). In this context, not considering reference frames as physical degrees of freedom, but as mere coordinates, has obvious consequences as one misses their quantum nature. Nonetheless, we frequently find in the quantum reference frames literature an assumption of non-backreacting, and non-dynamical, material reference frames (de la Hamette et al. (2023); Kabel et al. (2024)).

Some ambiguity still appears on the classica level, *e.g.* in Read (2020), in which the two terms are not clearly distinguished. In fact, at p. 215 a 'non-tensorial object' — that is thought of as a 'coordinate-dependent object' (*ivi*, p. 217) — is defined as a (reference) frame-dependent object. Also in Lehmkuhl (2014), following Einstein's idiom which emerges in many quotations (*ivi*, p. 321), the terms reference frame and coordinate system are used interchangeably, without much concern.[15]

[15]These cases are perfect examples of what was said in the introduction: the distinction between coordinates and reference frames does not always cause internal problems for the coherence of a work, but it is good that this distinction is recognised, expressed and maintained in the literature.

2.1 GR vs. Pre-GR Physics

It is worth resuming the meaning of reference frames and coordinates in GR (understood as the best representative of gravitational physics), as opposed to non-gravitational theories, which we will refer to as 'pre-GR theories'.

Let us briefly recall our coarse-grained definition of:

Reference Frame: a set of dynamically coupled and *instantiated* physical degrees of freedom[16]

Coordinate System: a set of non-dynamical and *uninstantiated* mathematical labellings

Pre-GR Our claim is that the need to separate the two concepts is not an urge in pre-GR physics, since a reference frame *can* be identified '*directly*', that is *without any approximation procedure* involving the degrees of freedom of the reference frame, to an '*instantiated* coordinate system'.

Following (Henneaux and Teitelboim, 1994, p.27), an *instantiated* coordinate system can be thought of as being 'brought in from the outside'. Here we interpret 'outside' to mean 'dynamically uncoupled' to the relevant dynamical system under study (for the resolution of the dynamical problem of interest, this is equivalent to considering the system non-dynamical).

Agreed: even in pre-GR there remains the important conceptual distinction between reference frames and coordinates: being (frames) or not being (coordinates) instantiated by real, physical objects. Actually, the fact that reference frames *can* be identified as (instantiated) coordinates does not imply that they *have to* be. For example, in general-covariant formulations of pre-GR physics, coordinates retain their role as mere uninstantiated parameters without physical meaning, as in GR (on that point, see Giovanelli (2021)'s distinction between formal vs. operational understandings of coordinates).[17] This shows that the distinction between coordinates and reference frames also applies in pre-GR.

Let's now see a pre-GR example of reference frame understood as an *instantiated* coordinate system. Within Maxwell's theory in Minkowskian spacetime, the Maxwell field is understood as a subsystem of the Universe that does not affect the global inertial reference frame that can be defined as a rigid scaffolding on the fixed background structure, defined by the Minkowski

[16]Recall, again, that the dynamical coupling is here understood *with respect to the dynamical system of interest* that we want to write in terms of the chosen reference frame. In GR, this is the metric field.

[17]For a further analysis on this point see also Pooley (2017); Westman and Sonego (2009).

metric $\eta_{\mu\nu} = (-1,1,1,1)$. For instance, we can define locations in spacetime by means of non electrically charged rods and clocks, which constitute the reference frame. This point is elucidated in the following passage in (Einstein, 1905, p.38):

> The theory to be developed—like every other electrodynamics—is based upon the kinematics of rigid bodies, since the assertions of any such theory concern relations between rigid bodies (systems of coordinates), clocks, and electromagnetic processes.

This passage can be interpreted to mean that the special relativistic Maxwellian theory is concerned with the relations between electromagnetic processes and material bodies that are dynamically uncoupled to (outside) the electromagnetic system under study, and that Einstein calls 'systems of coordinates'. Thus, the point is that in pre-GR a reference frame can be 'definitionally' dynamically uncoupled, but instantiated and can be identified with a less strict notion of coordinate system: an '*instantiated* coordinate systems': a sort of halfway between reference frames and coordinates.[18][19] Consequently, the definitions of reference frame and coordinate system in pre-GR theories need to be amended as follows:

Reference Frame: a set of dynamically coupled and *instantiated* physical degrees of freedom[20]

Reference Frame/Coordinate System (interchangeable): a set of 'definitionally' dynamically uncoupled and *instantiated* labellings (without any approximation being necessary)

[18] As we will see in the next section, this kind of reference frames are also present in GR: they are **IRFs** (agreed: **IRFs** are treated as non-dynamical degrees of freedom.). The relevant difference with pre-GR case is that in GR it is mandatory to implement some approximations. There can't be 'definitionally' dynamically uncoupled frames.

[19] These variables are those that, through the parameterisation procedure, are parameterised and consequently their dynamics are made explicit. Then, through the de-parametrisation procedure, they are used as reference frames. See section 3.2.1 for a toy model.

[20] For example, imagine we choose four complex scalar fields as reference frame for an electrodynamical system. There will be a coupling between the electromagnetic field *(the dynamical quantity of interest)* and the four charged scalar fields used as reference frame. Strictly speaking, they are the sources of the electromagnetic field and are affected by it. Also, think of the standard $U(1)$ Lagrangian density

$$\mathscr{L} = [(\partial_a \phi \partial^a \phi^* - m^2 \phi \phi^*) - \frac{1}{4} F^{ab} F_{ab}] + e^2 [A_a A^a \phi \phi^* + \frac{1}{e} A^a J_a],$$

where we have the free field terms of the complex scalar field and the Maxwell field and the coupling terms; e is the electric charge; m is the mass of the scalar field; $J^a = ie[-\phi \partial^a \phi^* + \phi \partial^a \phi]$ is the conserved current. More naively, just think of *electrically charged* rods and clocks in the case proposed by Einstein and quoted above of an electrodynamic theory in Minkowski.

Coordinate System: a set of non-dynamical and *uninstantiated* mathematical labellings (typical in the case of general covariant formulations).

Note the difference between being 'dynamically uncoupled' and being 'non-dynamical'. In the former case, the instantiated object has its own dynamics that is, however, not relevant to the dynamical problem of interest. In the latter case, on the other hand, since there is no instantiation, no dynamics can be defined for the variables that constitute the coordinate system.[21] It is evident that non-dynamical implies dynamically uncoupled. It is also the case that being dynamically uncoupled is *as if* one were non-dynamical, as far as the dynamics of interest of the problem is concerned.

GR In contrast, GR has no available 'outside'. Namely, we are not allowed to consider reference frames as dynamically uncoupled, *barring some approximation procedure*. This follows from the fact that no existing, real physical system is gravitationally neutral. Consequently, there is no way to disregard the (mutual) interaction between the gravitational field and the degrees of freedom that characterise the reference frame, unless dynamical approximations are adopted.[22] But it is precisely here that the lines become blurry. As we showed above in the Maxwellian case, in non-gravitational, pre-GR physics there is *no need* for approximations: it is always possible to identify a reference frame as a dynamically uncoupled and *instantiated* set of parameters and to make it coincide with the notion of a coordinate system for all practical use.[23] In contrast, in GR the concept of a coordinate system, even if is widely used throughout the general relativistic practice, can *only* be considered as a mathematical artefact without a physical referent (that is, without an *instantiation*).[24] If this were not the case, it would be a set of gravitationally charged degrees

[21] At most, gauge conditions can be defined for a coordinate system.

[22] Taking the same example from footnote 20, think of GR coupled with a Maxwell field and a charged scalar field. In this case, the Lagrangian density is:

$$\mathscr{L} = \sqrt{-g}(g^{ab}D_a\phi D_b\phi^* - m^2\phi\phi^*) - \frac{1}{4}g^{ab}g^{cd}F_{ac}F_{bd} - R,$$

where $D_a\phi = \nabla_a\phi + ieA_a\phi$ is the covariant gauge derivative; ∇_a is the curved connection and R the Ricci scalar constructed from g_{ab}. We notice that we loose the free field terms, because gravity *sticks* through the connection.

[23] The reference frame might be considered dynamically uncoupled due to the irrelevance of interaction effects compared to the experimental precision. Nevertheless, in this context, we refer to the property of being dynamically uncoupled independent of experimental errors and constraints.

[24] Following Kuchar (1990), this fact is closely linked to the fact that GR does differ from pre-GR physics in lacking a non-general covariant formulation.

of freedom and thus not dynamically uncoupled. In the following section, we will make precise our claim that the only way to make a reference frame dynamically uncoupled is to implement dynamical approximations. In conclusion, it is clear that the notions of 'reference frame' and 'coordinate system' cannot coincide in GR, neither from a conceptual point of view (which is also strictly true in pre-GR), but also from the point of view of theoretical practice.

2.1.1 The Newtonian case

We now face the elephant in the room. Under the label of pre-GR physics there is also Newtonian physics. It is clear, therefore, that 'pre-GR' and 'non-gravitational' are not and cannot be used as synonyms, as we have done so far. Also Newtonian gravity has no available 'outside'. Gravity is a universal interaction. This may lead one to think that the correct distinction is between gravitational and non-gravitational physics, as far as the possibility of identifying reference frames with coordinates is concerned, with no room for misunderstanding given by considering the broader category of pre-GR physics in the discussion.

To shed light on the matter, it is useful for us to follow (Pooley, 2022, sec. 8.10)'s analysis on the meaning of coordinates. Pooley identifies two possible interpretations of coordinates in pre-GR theories:

ESR (Einstein–Stachel–Rovelli) interpretation: [25] coordinates are *anchored* to material reference frames, such as physical synchronised clocks and rods that establish time and distance intervals. This anchoring ensures that coordinates are *instantiated* by material objects, 'outside' of the system under study. However, in the Newtonian case, this type of interpretation does not ensure that to define a reference frame interchangeably as an (*instantiated*) and dynamically uncoupled coordinate system, no approximations are required. In this regard, it is eloquent to quote (Rovelli, 2004, pp.61-62) at length:

> For Newton, the coordinates $\vec{x}$ that enter his main equation
>
> $$\vec{F} = m\frac{d^2\vec{x}(t)}{dt^2} \tag{2.152}$$

[25] This approach is the one we adopt throughout in this paper.

> are the coordinates of absolute space. However, since we cannot directly observe space, the only way we can coordinatize space points is by using physical objects. The coordinates $\vec{x}$ [...] are therefore defined as distances from a chosen system O of objects, which we call a ''reference frame'' [...] Notice also that for this construction to work it is important that the objects O forming the reference frame are not affected by the motion of the object A. *There shouldn't be any dynamical interaction between A and O.* (Our emphasis).

Of course, for there to be no dynamical interaction, due to the universality of gravitation, an approximation procedure is implied. Thus, according to **ESR**, the correct demarcation for the different meaning of coordinates would be between gravitational and non-gravitational physics. Only in the latter, coordinates *can* be understood as reference frames (and viceversa), as they *can be* 'definitionally' dynamically uncoupled.

ATF (Anderson–Trautman–Friedman) interpretation: Special coordinates are not *instantiated* by external material objects, but through gauge-fixing conditions that encode physically meaningful spacetime structures. In the Newtonian case, KPMs being of the kind: $\langle \mathbb{R}^4, t_a, h^{ab}, \Gamma^a_{bc} \rangle$ such gauge-fixing consists of imposing the connection $\Gamma^\mu_{\nu\rho} = 0$, the temporal metric $t_\mu = (1,0,0,0)$ and the spatial metric $h^{\mu\nu} = diag(0,1,1,1)$. Thus, it means they are *instantiated* by background spacetime structure, and not by material objects.[26]. This position is based on a substantivalist view of spacetime in which coordinates *encode* spatiotemporal physical magnitudes that exist even without material physical bodies disclosing them. In the **ATF** view, coordinates must still be anchored to the real world, but this anchoring does not depend on material reference frames. This allows an interpretation of reference frames that is independent of specific material objects. Also, this approach allows an understanding of physical properties that is *intrinsically* (or *internally*) linked to the spatiotemporal structure of the theory, rather than to external objects.

Then, we have two possibilities as to whether Newtonian physics can be subumed into the broader category of 'pre-GR physics', or not. But first, let us recall Newton (1687)'s definitions of absolute space and time:

[26]For an analysis about the spacetime-matter distinction see Martens and Lehmkuhl (2020)

Absolute Space: Absolute space, in its own nature, without regard to anything external, remains always similar and immovable

Absolute Time: Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external

Case 1: Absolute space and time are part of ontology In this case, **ATF** allows us to consider 'GR vs pre-GR' as the relevant distinction regarding the meaning of coordinates. The 'definitionally' *instantiated* and dynamically uncoupled coordinates are absolute space and absolute time. So, we can understand reference frames as *instantiated* and dynamically uncoupled coordinates, exactly as in Maxwellian theory in Minkowski spacetime, where electrically neutral rods and clocks are used. Notice that the category of *instantiation* is valid since absolute space and time are considered to be part of the ontology. However, this is not immediate, considering that Newton (1999), in the *General Scholium*, also refers to absolute space as a '*sensorium Deï*'. This quote shows how Newton conceived absolute space not as a physical entity, but as a manifestation of the divine presence, a '*sensorium Dei*' (God's sensorium), through which God perceives the universe. And this brings us to the second case.

Case 2: Absolute space and time are not part of ontology In this case, they are not *instantiated* parameters, since the category of *instantiation* is no longer applicable. Space and time revert to being mere *uninstantiated* and non-dynamical mathematical labellings, just as in GR. Since this approach discards substantivalism about spacetime, since Newton's absolute space-time is no longer a physical substance (on the critique of spacetime as belonging to the category of 'substance', see also Brown and Lehmkuhl (2013)), the rationale behind the **ATF** approach loses its validity. The only way to define a spatiotemporal reference frame (which is a necessarily *instantiated* set of variables) is through material objects. However, in order to have a set of variables that can play the role of absolute space and time, it is necessary, as **ESR** suggests (see above), to apply approximation procedures to such material objects in order to 'screen out' the universality of gravitational interaction, thus having a set of dynamically uncoupled variables. It is therefore not possible to understand the term 'reference frame' and 'coordinate system' interchangeably, because there are no *instantiated* and dynamically uncoupled coordinates, without the application of some approximation procedure. Therefore, the different meaning of reference frames and coordinates relies on the distinction between gravitational physics (GR and Newtonian gravitation, where reference frames can be understood as *instantiated* and dynamically uncoupled coordinates *only by means of an approximation procedure*; while coordinates are definitionally non-dynamical and *uninstantiated*) and non-gravitational physics (where coordinates can be definitionally dynamically uncoupled and *instantiated* and coincide with a reference frame, without any approximation procedure).

3 IRF, DRF, RRF

We define a spatiotemporal reference frame, at the most basic level, as a set of variables *instantiated* by a material system. In particular, we need at least four scalar independent degrees of freedom representing experimentally accessible quantites.[27] Since most of the fundamental physical fields (fermionic or bosonic) are not scalars, the required four scalar quantites could also represent 'phenomenological' properties of matter (e.g. in FLRW comsology the entropy of the cosmological fluid can be used as a reference clock (Schutz (1970, 1971); Cianfrani et al. (2009); Campolongo and Montani (2020))); or be constructed out of fundamental fields.[28]

We now introduce a novel three-fold distinction of possible reference frames, based on the degree of dynamical approximation applied to (model of) the target material system, which makes its dynamics more or less intertwined with that of the gravitational field. Although the paper adheres to the rationale presented in Rovelli (1991b) — that adopting approximations to define a reference frame blurs its physical significance, while considering its stress-energy presence and its dynamics brings its full physical significance back into focus — our proposal provides an independent contribution to this topic. In particular, our paper engages with more philosophical literature than Rovelli's original proposal and complements his work with additional theoretical tools and objec-

[27] The stated necessary condition for spatiotemporal reference frames as giving four independent scalars is coherent with the proposal to use a timelike four-vector field as a reference frame. It is sufficient to considers the scalars be the components of the timelike four-vector.

[28] This 'level zero' definition of a reference frame is still too broad. Conditions must be imposed on the set of four parameters representing space and time. See, *e.g.*, footnote 36.

tives. To give a concrete example, in Rovelli (1991b) the author overlooks the class of **DRFs**, which are dealt with extensively here. This fact is curious since, as we shall see, a particularly relevant physical example of **DRF** is precisely GPS coordinates, introduced in Rovelli (2002a). Moreover, we place special emphasis on **IRFs** as possible reference frames. In fact, we argue that they are crucial for understanding the role of coordinates in GR (Section 4).

Also, this work analyses the standard definition of reference frame provided in the literature in the light of a new perspective and adopts a methodology that can provide a clear conceptual map useful for discerning between possible reference frames in GR. The main motivations of our proposal, which will be discussed below, concern the physical interpretation of diffeomorphism gauge freedom, as well as the need to define local gauge-invariant observables (Section 3.2 and Section 5); the interpretation of vacuum GR; the fictitious role of coordinates in GR and the exposition of a new perspective on why the two concepts are sometimes used interchangeably without much care (Section 4).

3.1 Idealised Reference Frame

In the case of an Idealised Reference Frame (**IRF**), *any* dynamical interaction of the material system represented by the reference frame is ignored. In particular, two approximations are adopted:

(a) In the EFEs, the stress-energy tensor contribution of the material reference frame is neglected

(b) The set of equations that determine the dynamics of the material reference frame is neglected

Step (b) renders indeterministic the dynamics of the metric field, even if written in terms of the matter degrees of freedom. We argue that the class of **IRFs** are similar to what Rovelli (2004, p.62) calls 'undetermined physical coordinates'. The reason for this designation is clearly expressed by the author:

> We obtain a system of equation for the gravitational field and other matter, expressed in terms of coordinates X^μ that are interpreted as the spacetime location of reference objects whose dynamics we *have chosen* to ignore. This set of equation is underdetermined: same initial conditions can evolve into different solutions. However, the interpretation of such underdetermination is simply that we have chosen to neglect part of the equations of motion.

However, we refrain from adopting this nomenclature as it may cause unnecessary confusion between the terms reference frames and coordinates, and because we believe it is more appropriate in this context to speak of indeterminism of the dynamics than underdetermination, which is usually a term related to a 'supernumerary' of possible choices, not necessarily linked to dynamical considerations. Basically, when we use **IRFs**, similarly to the use of coordinates in GR, the system appears to be not deterministic. To be precise, we have not a real indeterminism. In the case of coordinates, it is merely the result of an unexpressed gauge freedom in the dynamics, that allows the same initial data to evolve into different solutions. Different solutions with the same initial conditions represent two gauge-related configurations.[29] To explore the origin of indeterminism in the case of **IRFs**, suppose we have a metric field g_{ab} satisfying the EFEs, and four scalar fields $\{\phi^{(A)}\}$ representing the **IRF**. We assume that the independent values of $\{\phi^{(A)}\}$ define a (local) diffeomorphism $U \subseteq \mathscr{M} \to \mathbb{R}^4$. Given any doublet $(g_{ab}, \phi^{(A)})$, the metric can be parametrised by the four values of the scalar fields, used as **IRF**. Thus, we can write the so-called 'relational observable' $g_{AB}(\phi) := g_{ab} \circ (\phi^{(A)})^{-1}$. The reason behind the apparent indeterminism when using **IRFs** is that both $(g_{ab}, \phi^{(A)})$ and $([d^*g]_{ab}, \phi^{(A)})$ are legitimate models for the dynamics, for all diffeomorphisms $d \in Diff(\mathscr{M})$. This is due to the fact that **IRF** is dynamically uncoupled from the metric field. Thus, we have still a redundancy in the frame representation of the metric. Hence, the apparent indeterminism. The matter degrees of freedom participate in the definition of the metric, whose evolution is determined up to four arbitrary functions because of approximation (b).[30] The difference between **IRFs** and coordinates is very subtle and will be discussed in Section 4. In brief, an **IRF** can be thought of as an *instantiated coordinate system*. We believe that the origin of the confusion between the concepts of reference frame and coordinate system stems

[29] As for the connection between the use of coordinates in GR and indeterminism, we refer the reader to the well-known problem of the hole argument Earman and Norton (1987), Weatherall (2018), Pooley and Read (2021).

[30] We point out that the logical possibility that *only* approximation (b) is adopted leads to the same conclusion about indeterminism. In this case, the material reference frame contributes to the curvature of spacetime and thus to the particular solution of the EFEs, but its dynamic equations are not considered. The fact that this is an approximation is immediately apparent here. From (differential) Bianchi identities follows the condition $\nabla^a G_{ab} = 0$. Therefore, from the validity of the EFEs, one immediately has $\nabla^a T_{ab} = 0$. The Euler-Lagrange equations for the matter fields in GR are essentially equivalent to the imposition of $\nabla^a T_{ab} = 0$. This is why the Einstein equations are said to contain the dynamical equations of matter. Therefore, having a non-null T_{ab} and thus being able to implement $\nabla^a T_{ab} = 0$ and yet not considering the equations of motion of matter is evidently an approximation. This case is also present in pre-GR physics, where we can consider objects that are sources of the field, but are unaffected by it. For example in electromagnetism, the Gauss constraint $div(E) = \rho$ involves the charge density ρ, which is treated as a source, unaffected by the electric field E. There is also a close analogy with the Poisson equation $\nabla\phi = \rho_m$ in Newtonian gravity, where the mass density ρ_m sources the potential. Of course, even in pre-GR these are approximations.

from the pragmatic equivalence of formulating the general-relativistic dynamics in terms of **IRFs** or coordinates, as far as **(P1)** and **(P2)** are concerned.

3.2 Dynamical Reference Frame

If we assume only the first of the above approximation, namely (a), we get a Dynamical Reference Frame (**DRF**). Consequently, we now have the possibility of using the dynamical equations of matter and obtain a deterministic dynamical system, since we can interpret the equations of motion of the material reference frame as the set of gauge-fixing conditions that we would apply if we used the parameters constituting the reference frame as coordinates. This fact supports the position expressed in Rovelli (2014) (see also (Gomes, 2023b, Sec. 2.3)), according to which the existence of gauge freedom is not a redundancy of the formalism, rather it suggests the (overlooked) relational nature of physical degrees of freedom.[31] This is also consistent with the definition given in (Henneaux and Teitelboim, 1994, p.3) of a gauge theory as a theory

> [...] in which the dynamical variables are specified with respect to a 'reference frame'.

In the following, we will give three examples of a **DRF**.

3.2.1 Warm-up: parametrisation and de-parametrisation

Before doing so, to give a simple toy example of what it means to write our dynamics using a **DRF**, we propose a parallel to the case of a parametrised Newtonian system in one spatial dimension, described by canonical variables $[q(t), p(t)]$. This is by no means intended to introduce a proper example of what a **DRF** is, but at best a valuable analogy.

Through the *parametrisation procedure*[32] we extend the configuration space $\mathscr{C} = \{q(t)\} \to \mathscr{C}_{\text{ext}} = \{q(\tau), t(\tau)\}$ and 'unfreeze' the time coordinate t (which corresponds to some 'external' clock), which can now be treated as a dynamical variable on the same footing of the q variable.

[31] In this paper, we choose to take Rovelli's position. In this context one should not *reduce* the state space of the theory by quotienting under the action of symmetries, since quotienting is throwing away physical information encoding how systems interact. The topic is broad. For the sake of completeness, we refer the reader to some replies on various fronts to Rovelli's 'relational proposal' on 'why gauge'. See, *e.g.*, Teh (2015) and Weatherall (2016).

[32] See (Henneaux et al., 1990, ch.4), or Tambornino (2012).

Both depend on an arbitrary parameter τ. The extended action of the parametrised system reads as

$$S_{\text{ext}} = \int d\tau \left[p_t \frac{dt}{d\tau} + p \frac{dq}{d\tau} - N(\tau) \left(p_t + \frac{p^2}{2m} \right) \right], \tag{1}$$

while the Hamilton equations are

$$\begin{cases} \dfrac{dt}{d\tau} = N(\tau), & \dfrac{dp_t}{d\tau} = 0 \\ \dfrac{dq}{d\tau} = N(\tau)\dfrac{p}{m}, & \dfrac{dp}{d\tau} = 0 \end{cases} \tag{2}$$

The extended system is subject to the reparametrisation symmetry $\tau \to \tau'(\tau)$ and different choices of the Lagrange multiplier $N(\tau)$, also known in GR as 'lapse function', amount to considering the gauge dynamics in different parametrisations. This is the analogue of the diffeomorphism symmetry in GR.

We can *partially* gauge fix the system, through the gauge choice $N = 1$, which amounts to having a parametrisation in which $t(\tau)$ grows linearly. However, the dynamics still has some redundancy, since $(t(\tau), q(\tau))$ are not reparametrisation-invariants. The dynamics is expressed in terms of the (introduced-by-hand) arbitrary parameter τ, therefore the evolution is physically meaningless.

A well-known approach to constructing local, *gauge-invariant* quantities is to impose the canonical gauge condition $t(\tau) \equiv t_0$, which completely eliminates any residual gauge redundancy.[33] Geometrically, this condition defines a slice that cuts all the so-called gauge orbits on the constraint surface — generated by the first class constraint $\mathrm{C} := p_t + \dfrac{p^2}{2m}$ — once and only once. That way, we can write observables which are understood as *unique* gauge-invariant quantities. In particular, an observable can be defined as the '*coincidence*' of q with t, that is the value of q when t reads the value t_0.[34] Explicitly: $q(\tau)|_{t(\tau)=t_0} := q(\tau) + p/m\,[t_0 - t(\tau)]$. Note that this is the definition of an evolving constant of motion (see Belot and Earman (2001)).

[33] Here we use the term in the most basic sense: ''there is more than one set of values of the canonical variables representing a given physical state'' (Henneaux et al., 1990, p.17).

[34] This is the analogous of the notion of *spacetime coincidence* in GR. See Giovanelli (2021) for a recent review on this topic.

An equivalent approach is to pick the variable t as the 'temporal reference frame' (also referred to as the 'relational clock') by inverting the relation $t(\tau) = \tau \leftrightarrow \tau(t) = t$. Note that this can be done since we are able to solve Hamilton equations for the considered system. Furthermore, in this case, the function $t(\tau)$ is globally invertible, but this is not always the case. By inserting the quantity $\tau(t)$ within the gauge-dependent quantity $q(\tau)$ we obtain a gauge-invariant relational observable $q(t)$, defined for any given value of t. This consists in *de-parametrising* the system.[35] Consequently, we recover the formalism of the original 'unparametrised' case in which t represented a mere non-dynamical coordinate. However, in such a case the physical interpretation of the time t as a dynamical variable is now revealed and, in this sense, it represents a good analogy of (the temporal component of) a **DRF**. In fact, now $q(t)$ describes the gauge-invariant, relational evolution of q with respect to the dynamical variable t. Furthermore, the dynamical theory written in relational terms becomes deterministic and without any gauge redundancy.

In a nutshell, what we wanted to show in this warm-up section is that the de-parametrisation procedure allows some dynamical variables to be used as a reference frame, getting the use of coordinates (parameters) out of the way. This becomes valuable when dealing with GR, which 'naturally' comes in a parameterised form. In fact,

> The already [parametrised] system "per excellence" is the gravitational field in general relativity. (Henneaux and Teitelboim, 1994, p.102)

3.2.2 Four Klein-Gordon Scalar Fields

The first proper example of a **DRF** is the so-called *test fluid* reference frame. In short, the test fluid is affected by the metric field (it is acted upon), but the metric field is not affected by the test fluid (it does not act): thus, the back-reaction on gravity (namely, its stress-energy tensor on the *r.h.s.* of the EFEs) is approximated to be negligible. As a toy model for a test fluid, we consider a set of four real, massless, free, Klein-Gordon scalar fields in a curved spacetime (which is solution of some EFEs where the stress-energy tensor relative to the scalar fields is neglected). Each scalar

[35] Apart from technical details and setting aside the obvious difference in theoretical framweork, this is the rationale according to which complete observables are constructed (see Rovelli (2002b), Dittrich (2006), Tambornino (2012)).

field $\phi^{(A)}, A = 1,2,3,4$ has its own equations of motion (in abstract index notation)

$$\Box_g \phi^{(A)} \equiv \nabla^a \nabla_a \phi^{(A)} = 0 \tag{3}$$

and the system of the four scalar fields can be used as a (generally local) reference frame (a clock and three rods) with respect to which local observables can be defined.[36] More clearly, to describe the dynamics of the scalar fields we first need to know the metric g_{ab} (deriving from some EFEs, e.g. $R_{ab} = 0$), in order to define the compatible connection ∇_a. In that sense, the metric acts on the test fluid, but it is not affected by it.[37]

This example clarifies what we have said about the correspondence between using a **DRF** and a gauge-fixed formulation of the theory written in coordinates. In fact, the four scalar fields satisfy the same equation that is written when De Donder gauge-fixing is imposed on coordinates. Hence, we have a straightforward example of a gauge-fixing condition, understood 'relationally' as a set of dynamical equations.[38]

When we use the set of Klein-Gordon fields $\phi^{(A)}$ as the reference frame, we recover a complete set of local gauge-invariant observables $g_{AB}(\phi) := g_{ab} \circ (\phi^{(A)})^{-1}$. No gauge redundancy appears, since when $(g_{ab}, \phi^{(A)})$ is a solution, then $([d^*g]_{ab}, \phi^{(A)})$ is not, for a generic $d \in Diff(\mathscr{M})$. Hence, given some initial data, we have a *unique* representation for $g_{AB}(\phi)$. This is due to the fact that **DRF** is *dynamically coupled* to the metric field.[39] With the notation $g_{AB}(\phi)$, it is emphasised that the 'physical metric' is not localised on points of the spacetime manifold $\mathscr{M}$, but on a somewhat four-dimensional 'physical' manifold $\mathscr{T}$, which is the space of ordered four-tuples of fields' real values.

[36]Arguably, the scalar field selected to play the role of the timelike variable (say $\phi^{(1)}$) needs to satisfy some properties such as a homogeneity condition $\nabla^i \nabla_i \phi^{(1)}(x^\mu) = 0$, where $i = 1,2,3$ are spatial indices in some coordinates $\{x^\mu\}$. We could also assume a 'monotonicity condition' connected with some assumptions on its potential (when it is considered).

[37]Please, be careful: this does not mean that the metric is *given* in the sense that it is an *absolute field*: i.e. the same (up to isomorphism) in every DPM, or a *fixed field*: i.e. the same in every KPM (Anderson (1967), James Read (2023)). It only means that we do not consider back-reaction. We must not have a *given* metric in the above meanings, because that would reduce the discussion to a Klein-Gordon theory in a curved background. Here, we deal instead with GR which is a background independent dynamical theory of the gravitational field. The case examined here is introduced in (Pooley, 2022, sec.8.7) under the name **GR2**. More generally, one can also have a dynamic metric that takes into account the stress-energy tensor of other material fields, but not that of the reference frame.

[38]As stated in Gomes (2023b): 'Though De Donder gauge is still not complete—it requires initial conditions on the metric and its time derivative (cf. (Landsman, 2021, p.161))—it suffices to render evolution deterministic'.

[39]However, due to the approximation procedure, the dynamical coupling between scalar fields and gravity is *not symmetrical* here. That is, differently from what was said in Section 2, the dynamically possible solutions of the scalar fields do not influence those of the metric, but depend on them.

Thus, reference fields $\phi^{(A)}$ can be understood as diffeomorphisms $\phi^{(A)} : g_{ab} \in \mathscr{M} \to g_{AB} \in \mathscr{T}$ and the metric can be parametrised by the (four) values of the scalar fields, used as reference frames.[40] [41] Taking away any reference to points of $\mathscr{M}$, one obtains a well-defined notion of local gauge-invariant observables in GR in terms of 'Einsteinian coincidences' (Einstein (1916)). Physical objects do not localise relative to the manifold, but relative to one another. This constitutes what is referred to as *relational localisation* (see Rovelli (2024), Goeller et al. (2022)). Basically, as we have shown, one must express the spatiotemporal localisation of observables through matter fields, which play the role of reference frames. Thus, we designate all the spatiotemporal locations by the values of four scalar fields. As also argued in Gomes (2023a), this way of understanding localisation in terms of physical field values is very similar to Einstein's original understanding of coordinates (see footnote 1 in the Introduction).

3.2.3 DRFs in the Orthodox View

In accordance with the previously mentioned literature (see Section 2), a reference frame can also be represented by a timelike *4-velocity* field U^a tangent to a congruence of worldlines of a system of test particles, or a test matter fluid. We choose to review this particular case, as we believe it warrants a closer examination and it is particularly significant from an historical and philosophical perspective.

The 'orthodox' point of view—that is how we call the view introduced in the philosophical

[40] In some coordinates $\{x^\rho\}$, we have explicitly $g^{\alpha\beta}[\phi^\gamma] = (\partial_\mu \phi^\alpha)(\partial_\nu \phi^\beta) g^{\mu\nu}(x^\rho)$. With a slight abuse of notation we have defined $\phi^\gamma(x^\rho) := \{\phi^0(x), \phi^i(x)\}, i = (1,2,3)$. See (Tambornino, 2012, p.11), or (Westman and Sonego, 2009, fn.17). The initial letters $(\alpha, \beta, ...)$ of the Greek alphabet refer to reference frame's indices. The final letters $(\mu, \nu, ...)$ refer to coordinates' indices. Let's notice that it is difficult to think of a realistic situation in which a reference frame would cover the entire manifold. In fact, four Klein-Gordon scalars won't generally form a bijection. For example, they could end up having the same values everywhere on $\mathbb{R}^4$, thus representing only one (physical) point. Finally, in order to indicate viable reference frames, each ϕ should be *at least locally* invertible, i.e. in some open set $U \subset \mathscr{M}$ and for a given chart, $\det(\partial \phi^\gamma / \partial x^\mu) \neq 0$.

[41] The four degrees of freedom necessary to define a spatiotemporal reference frame can be understood as the four components of $U^a := (\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \phi^{(3)})$. This establishes a connection with the previous proposal in Section 2 to use a 4-vector field as a reference frame. Notice also that in the case of (orthonormal) tetrads, which we recall are four smooth 4-vector fields, we can give a physical *instantiation* to tetrads if we write them directly in terms of four scalar fields, that is: $e^\alpha_\mu(x^\rho) := \partial \phi^\alpha / \partial x^\mu$ (see (Westman and Sonego, 2009, p.16)). The initial letters $(\alpha, \beta, \dots)$ of the Greek alphabet here refer to *tetradic indices* when coordinates are explicited. This relationship suggests that the scalar fields can be interpreted as an 'internal parameterisation' associated with each point in spacetime. Thus, for example, the metric tensor in the (orthonormal) tetrad frame can be written as $\eta^{\alpha\beta}[\phi^\gamma] = (\partial_\mu \phi^\alpha)(\partial_\nu \phi^\beta) g^{\mu\nu}(x^\rho)$. In the case of *non-orthonormal* tetrads, the tetradic metric $g^{\alpha\beta}[\phi^\gamma]$ is not necessarily Minkowskian. Furthermore, assuming the invertibility of the fields $\{\phi^\alpha\}$, the tetrads are non-degenerate (i.e. $det(e^\alpha_\mu) \neq 0$) (see the analogy with footnote 40 above).

literature by Earman and Norton[42] —recognises as a viable characterisation of a reference frame the expression of matter's *state of motion*, *i.e.* the assignment of a 4-velocity vector field U^a tangent to the worldlines of a material system, satisfying some dynamical equation and to which locally adapt a coordinate system (x^0, x^1, x^2, x^3).

We quote at length the definitions of a reference frame provided by Earman and Norton:

> In this context a reference frame is defined by a smooth, timelike vector field V. [...]Alternatively, a frame can, at least locally, be construed as an equivalence class of coordinate systems. The coordinate system $\{x^i\}$, $i = 1,2,3,4$, is said to be adapted to the frame F if the trajectories of the vector field which defines F have the form $x^a = \text{const}, a = 1,2,3$. If $\{x^i\}$ is adapted to F, then so is $\{x'^i\}$ where $x'^a = x'^a(x^b)$, $x'^4 = x'^4(x^b, x^4)$; such a transformation is called an internal coordinate transformation. F may be identified with a maximal class of internally related class of coordinate systems. (Earman, 1974, p.270)

> [...] it is now customary to represent the intuitive notion of a physical frame of reference as a congruence of time-like curves. Each curve represents the world line of a reference point of the frame. [...] A coordinate system $\{x^i\}$ $(i = 1,2,3,4)$ is said to be 'adapted' to a given frame of reference just in case the curves of constant x^1, x^2 and x^3 are the curves of the frame. These three coordinates are 'spatial' coordinates and the x^4 coordinate a 'time' coordinate. (Norton, 1985, p.209)
>
> Thus a frame of reference is introduced in standard practice as a congruence of timelike curves defined on the manifold (with metric). The frame, if smooth, assigns a velocity, its tangent vector, to every event in the manifold. (Norton, 1989, p.1242)

Since we defined (at the most basic level) a **DRF** as a material system that satisfies equations of motion and whose dynamics depends on, but not affects, that of the gravitational field, we can firmly assert that the orthodox view considers **DRFs** as possible reference frames. (However, there is no mention of reference frames corresponding to **IRFs**, or **RRFs**).

[42] It is also the most supported in the physical literature (see Wald (1984), Malament (2012)).

A comparison between reference frame 'à la Earman-Norton' and **DRFs** would be useful both for a better understanding of different possible types of **DRFs**, but above all for providing a delimited conceptual context to reference frames as they are usually employed in the literature.

To this purpose, we would like to comment briefly that there are possible differences, to be analysed, between a **DRF** (as we have introduced it) and a reference frame defined in the orthodox sense, as encoded by a 4-velocity U^a. However, we will not fully engage in this project on this occasion, but we will only give a few hints. In the light of footnote 40, the first noticeable difference is that for us a reference frame can be understood as a 4-vector in the sense that its components are *directly* the components of the 4-vector. In contrast, when it is said that a 4-velocity constitutes a reference frame, the components (one temporal and three spatial) of the reference frame are *not* directly the components of the 4-velocity *itself*.[43] In fact, basically using a 4-velocity $U^a = dx^a/d\tau$ (with τ being the proper time of the *comoving* observer) as a reference frame means employing *comoving* coordinates (where $U^a = (1/\sqrt{-g_{00}}, 0, 0))$ a with respect to which to define the components of a geometric object of interest. In particular, the components of a generic 4-vector can be organised as follows: its time component is that *along* the direction of the 4-velocity itself, which gives the direction along which the proper time of each worldline of the fluid increases; its spatial components are defined as those *orthogonal* to the direction of the 4-velocity. Therefore, we can see that a formalism in terms of coordinates is maintained, that is, we are using what Earman (1974) and Norton (1985) call the 'adapted' coordinate system (i.e. the coordinate system that a *comoving* observer with the matter fluid uses to define space and time).[44] To give an example of the substantial difference between the two approaches we have contrasted, note that (Rovelli, 1991b, p.309) also uses a set of particles or a fluid of matter moving along timelike geodesics as a reference frame, but does not use the 4-velocity as a reference frame. Instead, he uses *four scalar*

[43] The same can be said in the case of the tetrad frame, since the four tetrads are used to define four spatiotemporal *directions*. Thus each tetrad, being a 4-vector, does not *itself* constitute a standard of time or space, as is the case when we define a 4-vector as consisting of four scalar fields.

[44] Again, the same applies in the case of the tetrad frame: *when we drop the abstract formalism* in order to have a spatiotemporally explicit characterisation, we retain the formalism in coordinate terms to express the components of any geometric object in that reference frame.

degrees of freedom associated with such a material system.[45] So, in short, there is a difference between using a 4-velocity vector as a reference frame and using a 4-vector consisting of four scalar components.

Along these lines, we mention that, contrary to our definition of a **DRF**, the orthodox characterisation of a reference frame as '*a maximal class of adapted coordinate systems*' can lead to come conceptual confusion between the set of adapted coordinates on the reference frame and the reference frame itself. This is also stated in (Earman and Glymour, 1978, p.254):

> Of course, a reference frame can be represented by a maximal class of adapted coordinate systems. [...] *But such a coordinate representation can easily lead to a blurring of the crucial distinctions* [between reference frames and coordinate systems] *mentioned above*. (Our italics).

3.2.4 GPS Frame

To conclude, we argue that a realistic example of **DRF** is given by the set of the so-called *GPS coordinates*, introduced in Rovelli (2002a). The idea is to consider the system formed by GR coupled with four test bodies, referred to as 'satellites', which are deemed point particles following timelike geodesics of a given metric g_{ab}, and meeting at some (starting) point O.[46] Each particle is associated with its own proper time ϕ. Accordingly, we can uniquely associate four numbers $\phi^{(A)}, A = 1,2,3,4$ to each spacetime point P. These four numbers represent the four physical variables that constitute the **DRF**. Geometrically they constitute the proper timelike distances between the four intersection points with the past lightcone of P and the starting point O. Such quantites are broadcasted by the satellites and received in P. See Fig. 1. This example in particular highlights the usefulness of our classification as a clear and easy-to-use conceptual framework for categorising

[45] Another well-known example in this direction is the case of the so-called 'Brown and Kuchař dust'. The dust of pressure-less and freely-falling particles is described by eight scalar fields, four of which (T, Z^i) represent the spatiotemporal degrees of freedom to be used as reference frame, thus each point of $\mathscr{M}$ is labelled by the set (T, Z^i). In particular, the Z^i are constant along the geodesics of the dust and the T measures the proper time parametrising the geodesics of the dust. We can foliate the spacetime through spacelike $T = constant$ hypersurfaces, and label each point in these hypersurfaces through the Z^i degrees of freedom. This emphasises that such a dust fluid is a special case of a *global* reference frame in GR cpupled with matter field (unless there are singularities or boundaries of spacetime).

[46] Regarding the meaning to be attributed to '*given*', see footnote 37.

reference frames introduced in the literature.[47]

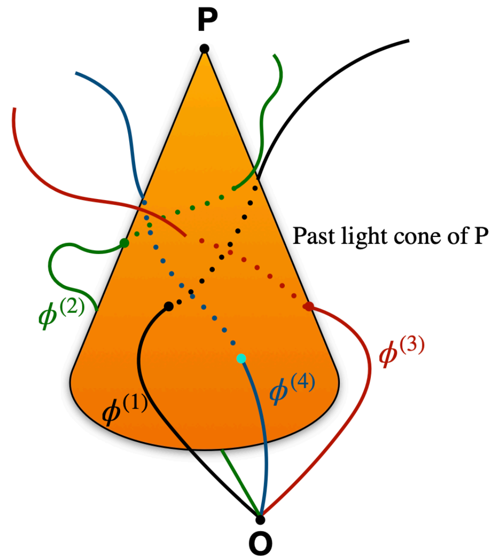

Figure 1: Construction of the set of GPS reference frame $\phi^{(A)}, I = 1,2,3,4$.

3.3 Real Reference Frame

When we take into account both the dynamics of the chosen material reference frame and its stress-energy tensor, we get a Real Reference Frame (**RRF**). Examples of **RRFs** include pressureless dust fields Brown and Kuchař (1995) and massless scalar fields Rovelli and Smolin (1994). We point out that **RFFs** are the *least* mathematically convenient and *most* phenomenologically relevant reference frames. Although we will not use it in the following discussion, it is possible to propose a sub-classification of **RRFs**. In brief, we define a $\mathbf{RRF}_{\text{dep}}$ as a **RRF** that permits the theory to be deparametrised, so that complete observables can be *analytically* described (see Tambornino (2012) for a review). As a matter of fact, in some special cases approximations can be made to the Hamiltonian of the material field used as the **RRF**, thereby implementing a deparametrisation procedure.[48]

[47] Think also of Fletcher (2013)'s *light clocks*, which are nothing but a type of temporal **DRF**. In fact, also in this case, the 'temporal reference' is given by the measurement of proper time ''experienced by a point particle along a timelike curve with the length of that curve as determined by the metric'' (*ivi*, p.1369).

[48] Therefore, even in the case of **RRFs** we have room to make some approximations in the sense of Norton (2012). Actually, even in the case of **DRFs**, we may or may not obtain a (global) deparametrisation of the theory, depending if the field used as a reference frame entirely covers the manifold or not.

In the remainder of the paper we will primarily focus on **IRFs** and **DRFs**, leaving the study of **RRFs** for future work.[49]

Summing Up The relevance of introducing our classification can be summarised as follows:[50]

- **Conceptual Framework**: it provides a clear conceptual framework for discussing issues in GR related to reference frames as material systems coupled to gravity. In particular, it helps contextualising the issues presented in the Introduction and clarifying the implications of using different types of reference frames:

  (P1) We cannot define local gauge-invariant observables when we use **IRFs**, or coordinates

  (P2) The gauge freedom of GR is interpreted as a mere mathematical redundancy if we use **IRFs**, or coordinates.

  However, using reference frames and by relaxing some of the approximations, thus considering **DRFs**, or **RRFs**, both problems **(P1)** and **(P2)** find a natural resolution (see Section 5).[51] Agreed: our proposal is not intended to be a new proposal to solve these problems. Rather, it provides a new theoretical structure in which to frame such problems.

- **Effective Communication**: it provides a valuable semantic clarification, thus enabling clear communication on the use of reference frames in all general-relativistic sectors

- **Enhanced Understanding**: it enhances understanding on the role of material reference frames in GR, confirming their practical significance to resolve the identified issues. In ad-

[49] In our opinion, **RRFs** play a major role in quantum gravity phenomenology. For example, whenever a quantum material reference frame is in superposition, no matter how small the mass, the spacetime in which it lives must split into two separate spacetimes (this effect is called the Bose-Marletto-Vedral effect (Bose et al. (2017); Marletto and Vedral (2017))). However, as also expressed in Adlam et al. (2022): ''for the small masses we deal with in current quantum experiments the difference between the spacetimes is experimentally insignificant and thus it is typically assumed that we can completely discount any effects of gravita- tional back-reaction''.

[50] We point out that the fourth logical possibility, that only approximation (b) is adopted, is not admissible in GR. In contrast, in pre-GR physics, it is entirely admissible to have objects that are sources of the field, but are unaffected by it. For example in electromagnetism, the Gauss constraint involves the charge density, which is treated as a source, unaffected by E. There is also a close analogy with the Poisson equation in Newtonian gravity and its modern descendant Newton-Cartan gravity, where the mass density sources the potential but does not self-gravate. Of course, even in pre-GR these are approximations.

[51] We recall that our approach is not restricted to a coordinate formulation of a theory. Gauge freedom under active diffeomorphisms appears as a mathematical redundancy because one adopts physical spatiotemporal localisation in terms of manifold points. Once one adopts relational localisation, by replacing the manifold points with the four scalar components of a reference field, one obtains gauge-invariant observables and it becomes clear why gauge existed.

dition, it also helps to understand the relationship between coordinates and reference frames in GR (see next Section 4).

We conclude by considering another interesting implication in considering **IRFs** and **DRFs** as possible classes of reference frames. If we disregard also any stress-energy contribution from other material sources, the solutions of the EFEs will be vacuum solutions. In support of (Rovelli, 1991b, p.304), we can say that vacuum GR can be seen as an approximate theory for an *observable* metric, in which we make use of **IRFs**, or **DRFs**. In other words we are suggesting, without any pretense of making a claim, that *exact* vacuum solutions may not exist in nature, but only *approximated* matter solutions that behave like vacuum solutions could be permitted. Further discussion is required in this regard.

4 IRFs vs. Coordinates: What Is the Source of the Confusion Between Reference Frames and Coordinate Systems?

> [I]t is not often that experiments are done under the stars. Rather they are done in a room. Although it is physically reasonable that the walls have no effect, it is true that the original problem is set up as an idealization.
>
> Richard Feynman.[52]

A notable empirical success of GR is the detection of gravitational waves by the LIGO project (Abbott et al. (2016)). The gravitational contribution of the reference frame used to localise the detection of gravitational waves on Earth is completely disregarded. Even theoretically, the components of the metric are calculated within a particular coordinate gauge, namely the so-called Transverse-Traceless gauge (TT gauge). Therefore, in practical uses of GR a reference frame is often intended as a coordinate system.

However, acknowledging that we are *implicitly* using reference frames can help us understand the physical reasons for presence of gauge freedom and why we need to use some gauge-fixing condition. Namely, as already stressed in section 3.2, the presence of diffeomorphisms as gauge

[52] Feynman and Hibbs (1965). This choice of section opening is also found in Wallace (2022).

redundancy indicates the tacit assumption of an approximation procedure that overlooks the dynamics of a physical system we are using as a reference frame. Indeed, in the case of gravitational waves, the TT-gauge conditions can be interpreted relationally as a set of dynamical equations satisfied by the reference frame used to make the spatiotemporal measurements. Thus, there is a correspondence between a gauge-fixing procedure and a choice of a particular reference frame.

The puzzle, then, is why such approximations work so well and, analogously, why the idealisation of the target system — which we select to play the role of the reference frame — to a mere coordinate chart works so well that the difference between the two concepts can be overlooked in theoretical and experimental practice. This issue is clearly expressed by Thiemann (2006, p.2) within the cosmological sector:[53]

> Why is it that the FRW equations describe the physical time evolution which is actually observed for instance through red shift experiments, of physical, that is observable, quantities such as the scale parameter? The puzzle here is that these observed quantities are mathematically described by functions on the phase space which *do not Poisson commute with the constraints*! Hence they are not gauge invariant and therefore should not be observable in obvious contradiction to reality.

Simply put, in theoretical and experimental practice reference frames are unwittingly approximated to **IRFs**, and this leads to their being understood as coordinate systems.[54] A sequence of misunderstandings. In both cases, there is gauge-redundancy and no local gauge-invariant observables can be defined. However, this leads to the situation where all general-relativistic physics incorrectly interprets the dynamical equations of systems as physical evolution equations 'rather than what they really are, namely gauge transformation equations' (*ivi*, p.3), as they are written in

[53]Here Thiemann uses the term 'observable' in the sense of Dirac observables.

[54]Local coordinate systems are usually employed to compute solutions of EFEs. A straightforward example is the use of Schwarzschild coordinates (t,r,θ,ϕ). Of course, the Schwarzschild geometry can be expressed in a range of different choices of coordinates. In the 'textbooks interpretation', Schwarzschild coordinates represent spatiotemporal points on that Manifold. Instead, from our point of view Schwarzschild coordinates should represent some physical degrees of freedom that allow the localisation to be defined in a diffeomorphism-invariant manner, rather than representing (spatiotemporal values of) points on a Manifold. However, they are idealised (in the sense of Norton (2012)) to coordinates, without *any* reference to a physical system *instantiating* them.

coordinates (or in dynamically uncoupled reference frames, like **IRFs**).[55] The analysis of such a puzzle deserves a separate discussion, which will not be carried out here.[56] Let us just say that our classification serves us well. In fact, only by acknowledging that we are using **IRFs**, and not coordinates, we can lighten the degree of approximation on the reference frame, for example including its dynamical equations. In this way, the dynamics becomes a physical, relational dynamics of local gauge-invariant observables written in terms of **DRFs** and not a (coordinate) gauge-variant description of reality. Predictions can be made *only* by going through the procedure for constructing relational local observables, which are deterministic.

According to us, the underlying source of the confusion between coordinate systems and reference frames is that reference frames are approximated to such an extent that they play the role of **IRFs**. However, once these approximations are made it becomes impossible to realise that approximated physical systems in the sense of **IRFs**, rather than coordinate systems, are being used. The relevant point is that *in practice* there is no difference between a coordinate system and an **IRF**.[57] Both come in the form of a set of non-dynamical variables that are used to define a spatiotemporal localisation of our relevant quantities. However, the difference between **IRFs** and coordinate systems is conceptually relevant. An **IRF** is *instantiated* by a physical systems that would, by nature, interact with all other degrees of freedom in the theory, but to which we apply *a posteriori* some dynamical approximations. On the other hand, a coordinate system is an idealisation: it is a set of *uninstantiated* mathematical variables that definitionally have no dynamics whatsoever (this is an *exact* property of the idealised novel system). In a nutshell: coordinates are mathematical, *uninstantiated* idealisations; **IRFs** are *instantiated* approximations of some structure within our theory's model, which represents a physical, real, material target system. Hence, the 'confusion' between coordinates and reference frames can be traced back to that between idealisations and

[55] In support of Thiemann (2006), let's notice that the scale factor $a(t)$ is not gauge-invariant because in FLRW model, via homogeneity and isotropy requirements, one fixes the lapse function $N = 1$ and the shift vector $N^i = 0$, but that choice is not sufficient to fix the gauge *completely*. It is only equivalent to a choice of a certain functional form of the gauge time parameter (cosmic time t). Recall the warm-up exercise given in Section 3.2.1.

[56] Another evident problem is that coordinates are mathematical, *uninstantiated* artefacts. What sense does it make to define quantities that we *measure* experimentally as dependent on a set of mathematical labels which are not part of any possible preferred ontology?

[57] Again, we mean with regard to problems **(P1)** and **(P2)** presented in the Introduction, that have to do with the dynamics of a theory. From a technical point of view, the distinction between **IRFs** (*instantiated*) and coordinates (*uninstantiated*) is rooted in the fact that, contrary to coordinates, **IRFs**, as a class of fields that extends over the spacetime manifold, are assumed to covary with reshufflings of points under an active diffeomorphism.

approximations, and ‘the difference matters’ (Norton (2012)).[58]

5 DRFs vs. Coordinates

The differences between **DRFs** and coordinates are clear and can be summarised as follows:

- The gravitational dynamics is deterministic when using a **DRF** and not deterministic (in the sense of the presence of gauge-freedom) when using coordinates
- We can define local gauge-invariant observables in terms of a **DRF**. No local gauge-invariant observables are defined when we use coordinates
- The variables constituting a **DRF** are partial observables describing our phenomenology, while coordinates are not.

Let us give a practical example of such differences. Let $\{x^0, x^i\}$ be a set of coordinates and $\{T, Z^i\}$ a set of four scalar degrees of freedom, describing some matter fluid defined by some dynamical equations and whose backreaction on gravity is neglected. We consider the ADM space+time analysis of such model of GR.[59] In total we have $6\times\infty^3$ degrees of freedom of the 3-metric $h_{ij}(x^0, x^i)$ written in coordinates plus $4\times\infty^3$ physical degrees of freedom describing the material system. By removing $4\times\infty^3$ gauge degrees of freedom of the metric, thorugh a gauge-fixing, $6\times\infty^3$ physical degrees of freedom remain. When we use the matter field as a reference frame, the relational 3-metric $h_{\alpha\beta}(T, Z^i)$ has naturally $6\times\infty^3$ physical degrees of freedom.[60] Thus, the same deterministic dynamical theory, when written in relational terms (*i.e.* using a **DRF**) is well-defined without any gauge condition to be fixed. Furthermore, the (relationally) local quantity

[58] We think that this sort of ‘confusion’ is also rooted in the lack of care Maudlin (2018) highlights regarding the clarification of the ontology of theories by modern philosophers and theoretical physicists. (Gomes and Butterfield, 2024, p.2) also agree, stating: ‘‘No special care is taken to specify: which parts represent ontology, ‘what there is’ (and within that: what is basic or fundamental, and what derived or composite); and which parts represent ‘how it behaves’ (which (Maudlin, 2018, p.4) calls ‘nomology’: in particular, dynamics); and which parts represent nothing physical, but instead mathematics (which, though unphysical, can of course be invaluable for calculation)’’. We also agree with (Gomes and Butterfield, 2024, fn.1) that this necessary clarification on the ontology of theories is not intended to support the claim of logical positivists that the physical theories to be considered should present a once-and-for-all division of facts and conventions (see e.g. Putnam (1975)).

[59] The ADM formalism Arnowitt et al. (1960) is a Hamiltonian formulation of GR. The canonical variables of this formalism are the 3-metric tensor h_{ij} and its conjugate momentum p^{ij}.

[60] Here α and β are (spatial) frame’s indices running from 1 to 3.

$h_{\alpha\beta}(T, Z^i)$ is a local gauge-invariant observable. In fact, the diffeomorphism group act both on the 3-metric and the scalar fields, in such a way to leave $h_{\alpha\beta}(T, Z^i)$ invariant.

This example shows that the use of **DRFs** solves both problems **(P1)** and **(P2)**.

(P1-solved) As also demonstrated with practical examples in section 3.2, we are able to write local gauge-invariant observables. The 'price' to pay is to accept a notion of relational locality between fields. The force of this notion lies in the possibility of constructing gauge-invariant local observables and defining a physical, deterministic dynamics for such quantities

(P2-solved) As far as the interpretation of diffeomorphism gauge freedom in GR, using **DRFs**, we can interpret gauge-fixing conditions as dynamical equations of some physical system chosen as the reference frame. The presence of gauge freedom, in such a view, suggests that we are ignoring the dynamics of some physical system that we are using as a reference frame.[61] In other words, as stated in Rovelli (2014):

> Gauge invariance is not just mathematical redundancy; it is an indication of the relational character of fundamental observables in physics. [...] Gauge is ubiquitous. It is not unphysical redundancy of our mathematics. It reveals the relational structure of our world.
>
> [...] The choice of a particular gauge can be realized *physically* via coupling: with a material reference system in general relativity.

6 Conclusion

The presented work introduced three distinct classes of reference frames in GR, according to their increasing physical relevance in the gravitational dynamics. Indeed, we considered 'idealised' (**IRF**) those reference frames whose physical nature does not enter in any way into the dynamical picture, as 'dynamical' (**DRF**) those one which are associated with a specific set of dynamical

[61] Following Gomes et al. (2022), another answer to **(P2)** is that ''gauge symmetry provides a path to building appropriate dynamical theories—and that this rationale invokes the two theorems of Emmy Noether (1918)''. This approach is an extension of the, well-known, answer amongst practising physicists known as the *gauge argument* of Weyl (1929), which posits that local gauge invariance necessitates the introduction of gauge fields to properly describe fundamental interactions.

equations, as ‘real’ (**RRF**) those whose stress-energy tensor also contributes to the EFEs. In light of the identified problems related to defining local, gauge-invariant observables and interpreting diffeomorphism gauge symmetry not as a mere mathematical redundancy, our novel three-fold classification proved to be a valuable tool, providing a theoretical framework in which to effectively contextualise the aforementioned challenges. In particular, we analysed the role of **DRFs** in this debate. This work complements and extends existing literature on the subject, as it includes past definitions of a reference frame, thus enhancing the coherence and validity of the proposed framework.

The paper also sheds light on the necessity of maintaining conceptual clarity to avoid significant errors in interpretation. Reference frames can be approximated as **IRFs** and inaccurately conflated with coordinate systems, since an **IRF** behaves *as if* it were a coordinate system, as far as the objective of constructing local gauge-invariant, determinstic observables is concerned. On a conceptual level it is a serious mistake to confuse the two notions. In GR, coordinates are mathematical idealisations, constituted by *uninstantiated* variables. **IRFs** are approximations of some structure within our theory’s model, *instantiated* by a target physical, material system.

Our proposal on reference frames could have implications both for the increasingly studied notion of quantum reference frame and for future discussions on the nature of vacuum solutions of EFEs. In particular, it remains to be clarified if vacuum solutions can be reconsidered in terms of approximated matter solutions where the stress-energy tensor is neglected. This is not the only option. Certainly, it is possible to opt for a definition of a non-material reference frame in vacuum GR, for example in terms of purely gravitational degrees of freedom. Likewise, it remains to be clarified why and to what extent the use of reference frames as mere coordinates works so well that the difference between the two concepts can be overlooked, as far the experimental practice is concerned. This observation prompts us to consider why in the cosmological sector equations written in coordinates still yield predictive results, although theoretically, physical predictions should solely arise from equations expressed in terms of reference frames whose dynamics is taken into consideration.

The proposed framework serves as a valuable guide for researchers, offering new perspectives, confirming some well-established research paths and opening avenues for exploration within the field. Overall, the aim of this work is to enrich the study of a sector of GR, by providing a clear and coherent approach to understanding the role of material reference frames. In conclusion, our systematic classification of reference frames may have significant implications for the foundations of Einsteinian theory of gravity.

References

Abbott, B. P. et al. (2016, Feb). Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett. 116*, 061102.

Adlam, E., N. Linnemann, and J. Read (2022). Constructive axiomatics in spacetime physics part iii: A constructive axiomatic approach to quantum spacetime.

Anderson, J. L. (1967, May). *Principles of relativity physics*. San Diego, CA: Academic Press.

Arnowitt, R. et al. (1960, March). Canonical variables for general relativity. *Physical Review 117*(6), 1595–1602.

Ashtekar, A., L. Bombelli, and O. Reula (1991). The covariant phase space of asymptotically flat gravitational fields. In M. Francaviglia (Ed.), *Mechanics, Analysis and Geometry: 200 Years After Lagrange*, North-Holland Delta Series, pp. 417–450. Amsterdam: Elsevier.

Belot, G. (2017, April). Fifty million elvis fans can't be wrong. *Noûs 52*(4), 946–981.

Belot, G. and J. Earman (2001). *Pre-Socratic quantum gravity*, pp. 213–255. Cambridge University Press.

Bergmann, P. G. (1961, October). Observables in general relativity. *Reviews of Modern Physics 33*(4), 510–514.

Bose, S., A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. Kim, and G. Milburn (2017, December). Spin entanglement witness for quantum gravity. *Physical Review Letters 119*(24).

Bradley, C. (2021, December). The non-equivalence of einstein and lorentz. *The British Journal for the Philosophy of Science 72*(4), 1039–1059.

Brown, H. R. and D. Lehmkuhl (2013). Einstein, the reality of space, and the action-reaction principle.

Brown, J. D. and K. V. Kuchař (1995, May). Dust as a standard of space and time in canonical quantum gravity. *Physical Review D 51*(10), 5600–5629.

Campolongo, A. and G. Montani (2020, October). Specific entropy as a clock for the evolutionary quantization of the isotropic universe. *The European Physical Journal C 80*(10).

Castro-Ruiz, E. et al. (2020, May). Quantum clocks and the temporal localisability of events in the presence of gravitating quantum systems. *Nature Communications 11*(1).

Cianfrani, F., G. Montani, and S. Zonetti (2009, 4). Fluid Entropy as Time-Variable in Canonical Quantum Gravity. In *3rd Stueckelberg Workshop on Relativistic Field Theories*.

de la Hamette, A.-C., V. Kabel, E. Castro-Ruiz, and C. Brukner (2023, August). Quantum reference frames for an indefinite metric. *Communications Physics 6*(1).

Dirac, P. (2001). *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, monograph series. Dover Publications.

DiSalle, R. (2020). Space and Time: Inertial Frames. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Winter 2020 ed.). Metaphysics Research Lab, Stanford University.

Dittrich, B. (2006, October). Partial and complete observables for canonical general relativity. *Classical and Quantum Gravity 23*(22), 6155–6184.

Duerr, P. M. (2021, August). Theory (in-)equivalence and conventionalism in f(r) gravity. *Studies in History and Philosophy of Science 88*, 10–29.

Earman, J. (1974, June). Covariance, invariance, and the equivalence of frames. *Foundations of Physics 4*(2), 267–289.

Earman, J. (2004). Laws, symmetry, and symmetry breaking: Invariance, conservation principles, and objectivity. *Philosophy of Science 71*(5), 1227–1241.

Earman, J. and M. Friedman (1973, September). The meaning and status of newton's law of inertia and the nature of gravitational forces. *Philosophy of Science 40*(3), 329–359.

Earman, J. and C. Glymour (1978, December). Lost in the tensors: Einstein's struggles with covariance principles 1912–1916. *Studies in History and Philosophy of Science Part A 9*(4), 251–278.

Earman, J. and J. Norton (1987). What price spacetime substantivalism? the hole story. *British Journal for the Philosophy of Science 38*(4), 515–525.

Einstein, A. (1905). Zur elektrodynamik bewegter körper. *Annalen der Physik 322*(10), 891–921.

Einstein, A. (1916). Hamilton's Principle and the General Theory of Relativity. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1916*, 1111–1116.

Elgin, C. Z. (2017, September). *True Enough*. The MIT Press. London, England: MIT Press.

Feynman, R. P. and A. R. Hibbs (1965, January). *Quantum mechanics and path integrals*. New York, NY: McGraw-Hill.

Fletcher, S. C. (2013, October). Light clocks and the clock hypothesis. *Foundations of Physics 43*(11), 1369–1383.

Friedman, M. (1983). *Foundations of space-time theories*. Princeton Legacy Library. Princeton, NJ: Princeton University Press.

Frigg, R. (2022). *Models and Theories: A Philosophical Inquiry*. Philosophy and Science Series. Routledge.

Giacomini, F. (2021, July). Spacetime quantum reference frames and superpositions of proper times. *Quantum 5*, 508.

Giacomini, F. et al. (2019, January). Quantum mechanics and the covariance of physical laws in quantum reference frames. *Nature Communications 10*(1).

Giovanelli, M. (2021, May). Nothing but coincidences: the point-coincidence and einstein's struggle with the meaning of coordinates in physics. *European Journal for Philosophy of Science 11*(2).

Goeller, C. et al. (2022). Diffeomorphism-invariant observables and dynamical frames in gravity: reconciling bulk locality with general covariance.

Gomes, H. (2023a). Same-diff? conceptual similarities between gauge transformations and diffeomorphisms. *forthcoming*.

Gomes, H. (2023b). Understanding the symmetries of physics. *forthcoming*.

Gomes, H. and J. Butterfield (2024). How to choose a gauge? the case of hamiltonian electromagnetism. *Erkenntnis 89*(4), 1581–1615.

Gomes, H., B. W. Roberts, and J. Butterfield (2022). *The Gauge Argument: A Noether Reason*, pp. 354–376. Cambridge University Press.

Gryb, S. and K. P. Y. Thébault (2016, October). Regarding the 'hole argument' and the 'problem of time'. *Philosophy of Science 83*(4), 563–584.

Henneaux, M. et al. (1990). Gauge invariance and degree of freedom count. *Nuclear Physics B 332*(1), 169–188.

Henneaux, M. and C. Teitelboim (1994, August). *Quantization of gauge systems*. Princeton, NJ: Princeton University Press.

Isham, C. J. (1993). Canonical quantum gravity and the problem of time. *NATO Sci. Ser. C 409*, 157–287.

Jacobs, C. (2024). How (not) to define inertial frames. *Australasian Journal of Philosophy*.

James Read (2023, November). *Background independence in classical and quantum gravity*. London, England: Oxford University Press.

Kabel, V., A.-C. de la Hamette, L. Apadula, C. Cepollaro, H. Gomes, J. Butterfield, and C. Brukner (2024). Identification is pointless: Quantum reference frames, localisation of events, and the quantum hole argument.

Komar, A. (1958, Aug). Construction of a complete set of independent observables in the general theory of relativity. *Phys. Rev. 111*, 1182–1187.

Kuchar, K. V. (1990, 7). Canonical quantization of generally covariant systems.

Landau, L. D. and E. M. Lifshitz (1987, January). *The classical theory of fields* (4 ed.). Oxford, England: Butterworth-Heinemann.

Landsman, K. (2021). *Foundations of General Relativity: From Einstein to Black Holes*. Radboud University Press.

Lee, J. and R. M. Wald (1990, March). Local symmetries and constraints. *Journal of Mathematical Physics 31*(3), 725–743.

Lehmkuhl, D. (2014). Why einstein did not believe that general relativity geometrizes gravity. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 46*, 316–326.

Malament, D. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago Lectures in Physics. University of Chicago Press.

Marletto, C. and V. Vedral (2017, December). Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Physical Review Letters 119*(24).

Martens, N. C. and D. Lehmkuhl (2020). Dark matter = modified gravity? scrutinising the spacetime–matter distinction through the modified gravity/ dark matter lens. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 72*, 237–250.

Maudlin, T. (2018, June). Ontological clarity via canonical presentation: Electromagnetism and the aharonov–bohm effect. *Entropy 20*(6), 465.

Newton, I. (1687). *Philosophiæ Naturalis Principia Mathematica*. London: Royal Society. Scholium to the Definitions.

Newton, I. (1999). *The Principia: Mathematical Principles of Natural Philosophy* (3rd ed.). Berkeley, CA: University of California Press. General Scholium.

Noether, E. (1918). Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1918*, 235–257.

Norton, J. (1985, September). What was einstein's principle of equivalence? *Studies in History and Philosophy of Science Part A 16*(3), 203–246.

Norton, J. (1989, October). Coordinates and covariance: Einstein's view of space-time and the modern view. *Foundations of Physics 19*(10), 1215–1263.

Norton, J. D. (1993). General covariance and the foundations of general relativity: Eight decades of dispute. *Reports of Progress in Physics 56*, 791–861.

Norton, J. D. (2012, April). Approximation and idealization: Why the difference matters. *Philosophy of Science 79*(2), 207–232.

Penrose, R. and W. Rindler (1987, February). *Cambridge monographs on mathematical physics spinors and space-time: Two-spinor calculus and relativistic fields volume 1*. Cambridge, England: Cambridge University Press.

Pitts, J. B. (2022, October). Peter bergmann on observables in hamiltonian general relativity: A historical-critical investigation. *Studies in History and Philosophy of Science 95*, 1–27.

Pooley, O. (2017). Background independence, diffeomorphism invariance and the meaning of coordinates. In *Towards a Theory of Spacetime Theories*, pp. 105–143. Springer New York.

Pooley, O. (2022). *The Reality of Spacetime*. University of Oxford.

Pooley, O. and J. A. M. Read (2021). On the mathematics and metaphysics of the hole argument. *The British Journal for the Philosophy of Science 0*(ja), null.

Putnam, H. (1975). The analytic and synthetic. In H. Putnam (Ed.), *Mind, Language and Reality: Philosophical Papers*, pp. 33–69. Cambridge University Press.

Read, J. (2020, March). Functional gravitational energy. *The British Journal for the Philosophy of Science 71*(1), 205–232.

Rovelli, C. (1991a, February). Quantum reference systems. *Classical and Quantum Gravity 8*(2), 317–331.

Rovelli, C. (1991b, February). What is observable in classical and quantum gravity? *Classical and Quantum Gravity 8*(2), 297–316.

Rovelli, C. (2002a, Jan). Gps observables in general relativity. *Phys. Rev. D 65*, 044017.

Rovelli, C. (2002b, jun). Partial observables. *Physical Review D 65*(12).

Rovelli, C. (2004). *Quantum Gravity*. Cambridge University Press.

Rovelli, C. (2014, January). Why gauge? *Foundations of Physics 44*(1), 91–104.

Rovelli, C. (2024). Princeton seminars on physics and philosophy.

Rovelli, C. and M. Gaul (2000). Loop quantum gravity and the meaning of diffeomorphism invariance. In *Lecture Notes in Physics*, pp. 277–324. Springer Berlin Heidelberg.

Rovelli, C. and L. Smolin (1994, January). The physical hamiltonian in nonperturbative quantum gravity. *Physical Review Letters 72*(4), 446–449.

Schutz, B. F. (1970, December). Perfect fluids in general relativity: Velocity potentials and a variational principle. *Physical Review D 2*(12), 2762–2773.

Schutz, B. F. (1971, December). Hamiltonian theory of a relativistic perfect fluid. *Physical Review D 4*(12), 3559–3566.

Stachel, J. (1993). The meaning of general covariance. In J. Earman (Ed.), *Philosophical Problems of the Internal and External World*, pp. 129–60. University of Pittsburgh Press.

Tambornino, J. (2012, March). Relational observables in gravity: a review. *Symmetry, Integrability and Geometry: Methods and Applications*.

Teh, N. J. (2015). A note on rovelli?s ?why gauge?? *European Journal for Philosophy of Science 5*(3), 339–348.

Thiemann, T. (2006). Solving the problem of time in general relativity and cosmology with phantoms and k – essence.

Trautman, A. (1967). Comparison of newtonian and relativistic theories of space time.

Wald, R. M. (1984, June). *General Relativity*. Chicago, IL: University of Chicago Press.

Wallace, D. (2022, April). Isolated systems and their symmetries, part i: General framework and particle-mechanics examples. *Studies in History and Philosophy of Science 92*, 239–248.

Weatherall, J. O. (2016). Understanding gauge. *Philosophy of Science 83*(5), 1039–1049.

Weatherall, J. O. (2018). Regarding the hole argument. *British Journal for the Philosophy of Science 69*(2), 329–350.

Westman, H. and S. Sonego (2009). Coordinates, observables and symmetry in relativity. *Annals Phys. 324*, 1585–1611.

Weyl, H. (1929, April). Gravitation and the electron. *Proceedings of the National Academy of Sciences 15*(4), 323–334.