DOES IDENTITY HAVE SENSE?

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Abstract. In this paper we present novel conceptions of identity arising in and motivated by a recently emerged branch of mathematical logic, namely, Homotopy Type theory (HoTT). We consider an established 2013 version of HoTT as well as its more recent generalised version called Directed HoTT or Directed Type theory (DTT), which at the time of writing remains a work in progress. In HoTT, and in particular in DTT, identity is not just a relation but a mathematical structure which admits for an interpretation in terms of Homotopy theory (directed Homotopy theory in the case of DTT), which in its turn is supported by common intuitions concerning identity of material objects through time, change and locomotion. The DDT-based conception of identity presented in the paper is non-symmetric: here identity is "directed" or has a "sense". We compare the HoTT-based conceptions of identity with standard theories of identity based on the Classical Predicate calculus, and show how the HoTT-based identity helps to treat traditional logical and philosophical problems related to identity and time. In the concluding part of the paper we explore some ontological implications of the HoTT-based identity and show how HoTT and DTT can serve for designing formal process ontologies. The paper is self-contained and comprises expositions and informal explanations of all relevant philosophical, logical and mathematical contents.

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1. Introduction

The concept of identity is a recurrent topic in the history of philosophy that dates back at least to Aristotle. It remains popular in today's philosophical logic and analytic metaphysics where it is usually treated with standard logical tools such as Classical First- and Second-Order Logic and their modal extensions. These formal means help philosophers to develop a common language, establish a commonly agreed standard of logical rigour, and avoid the hermeneutical jungles of natural languages used in informal philosophical discussions. But these advantages of using the formal approach in philosophy come with a price. Since the concept of identity is itself either logical or at least logically-related, no system of logic can be quite neutral with respect to problems concerning identity. Classical First-Order Logic is not an exception. This is why using this standard formal tool may severely delimit a philosophical discussion about logical matters and prohibit certain interesting conceptual possibilities.

Fortunately the dilemma between using only natural languages and using standard logical tools is not exhaustive. In this paper we discuss a family of novel identity concepts recently emerged and formalised in mathematics, namely in a version of Type theory called Homotopy Type theory (HoTT for short) and its offspring called Directed Type theory (DTT). It will be shown that in DTT the identity concept can be construed in a non-symmetric way: the fact that X is identical to Y, generally, does not imply that Y is identical to X. Thus question Whether identity has sense? in the title should be understood both as a general philosophical query about the meaning of identity and as a more specific question concerning the notion of *directed identity* featuring in DTT.

Thus the principal aim this paper is to apply HoTT to some traditional metaphysical problems of identity and time and see whether its claims about identity are philosophically relevant. We argue in what follows in favour of a positive answer and show how HoTT helps to solve (or dissolve) these problems. But our argument does not reduce to a demonstration of power of a new formal technique. Following David Corfield [**?**] we conceive here of HoTT and HoTT-based accounts of identity as an object of philosophical reflexion, analysis and critique. An interesting outcome of our analyse is that HoTT supports the view according to which the concept of identity is tightly related to space and time. Such a view of identity dates back Aristotle and Leibniz but since the rise of formal approaches in philosophy in the 20th century it became rather unpopular. HoTT demonstrates the need to reconsider this issue once again.

Corfield describes HoTT as a "new logic" that today has the same general significance for philosophy as the new logic of Frege and Russell used to have, when it emerged and produced a revolution in philosophical thought more than a century ago [**?**, Ch. 1]. Just like Frege's novel ideas in logic essentially rely on his contemporary mathematics Corfield's "new logic" is intended as a deep revision of the received philosophical logic in view of HoTT and related developments in today's mathematics and Computer Science. Since today HoTT is primarily a mathematical subject a lot of work remains to be done in order to communicate these new mathematical ideas and approaches to philosophers and justify the claim of their general philosophical significance. Corfield's works make a significant progress in this direction. Our present work is conceived as a part of the same general project focused on the concept of identity, which is central in HoTT non only from a philosophical but also from a technical viewpoint. We also make a special emphasis on ontological implications of HoTT (including DTT) and show its relevance for process ontology. Since HoTT and its mathematical ingredients including Higher Category theory so far remain largely unknown to philosophers, a large part of the present work has an expository character and aims at communicating basic concepts and principles of HoTT to the general philosophical readership. References to mathematical literature found in this paper can help the interested reader to study HoTT and related mathematical subjects more systematically[1]

The rest of the paper is as follows. In Section **??** we recall of some well-known problems concerning identity and time in their usual setting. In Section **??** we recall of standard theories of identity based on the Classical First- and Second-Order Logic. In Section **??** we introduce elements of generic HoTT and show how this theory sheds new light on the problematic issues of identity described in Section **??**. In Section **??** we treat DTT in a similar manner. In the concluding Section **??** we discuss the relevance of DTT in the process ontology.

2. Identity and Time

The aim of this Section is to point to some popular logical and metaphysical questions about identity and time.

2.1. **Frege on Morning Star and Evening Star.** In his classical work [**?**] Gottlob Frege asks: How it is possible that sentence

(1) Morning Star is Morning Star

is trivially true but sentence

[1] Having said that we cast doubt on the idea according to which philosophy cannot avoid making some form of logic into an orthodoxy, then heroically fighting this orthodoxy using novel mathematical approaches, and finally completing the cycle by establishing a new logical orthodoxy for another century. Even if, as a matter of historical fact, such cycles repeatedly occur in some philosophical schools and establishments we aim here at a more historically informed type of philosophical work that may give us some sense of continuity between the past and the anticipated future of philosophical l. Thus we do not urge here for a new revolution in the philosophical logic but urge for the need of continuing critical philosophical reflexion on the current developments in mathematics as well as on its history.

(2) Morning Star is Evening Star

is not self-obvious and needs an astronomical justification? This is in spite of the fact that "Morning Star" and "Evening Star" are just two different names of the same planet Venus. What makes the difference between (**??**) and (**??**) from a logical point of view? A similar question can be asked about arithmetical identities like

(3) $$12 = 12$$

and

(4) $$7 + 5 = 12$$

A possible way of tackling these questions, which has been around among Frege's contemporaries [**?**, p. 8], is to diversify the identity concept. It may be argued, for example, that in (**??**) the equality sign = should be understood not as the strict logical identity but as a weaker form of equivalence, so that (**??**) says that the left and the right parts of the formula are "equal in magnitude" without being literally the same. In the 20th century this approach was further developed by Peter Geach who argued that the notion of strict "absolute" identity has no content, and the identity relation is always "relative", so the two parts of (**??**) can be identical *as magnitudes* but be different as symbolic expressions and in many other respects [**?**].

Frege's solution is different. He firmly believes that "identity is a relation given to us in such a specific form that it is inconceivable that various kinds of it should occur " [**?**, p. 254](quoted by [**?**, p. 3]), and tries to find a way to account for the identity sentences like (**??**)-(**??**) with a single "absolute" and "strict" identity concept conceived as a relation "in which each thing stands to itself but to no other thing" [**?**, p. 1]. For this end Frege distinguishes between the *sense* and the *reference* of a given linguistic expression. Names "Morning Star" and "Evening Star" refer to the same thing, namely planet Venus, but the senses of these two expressions are different: the former name describes Venus as a "star" appearing in mornings and the latter name — as a "star" appearing in evenings. Grasping the sense of each expression does not give any clue to the fact that that both expressions have the same reference.

Frege's idea is that identity sign "=" in judgements having the form $A = B$ (including the cases of $A = A$ and $B = B$) refers to the identity relation between the *references* of names (expressions) "A" and "B" but not between their senses (if any). Under this proviso one is in a position to claim that both (**??**) and (**??**) are *true* because planet Venus is a shared reference of names "Morning Star" and "Evening Star", and this planet is identical to itself (like any other thing); the case of (**??**) and (**??**) is similar. Thus the distinction between sense and reference helps Frege to avoid a diversification of the identity concept in this case. The truth of a (true) proposition Frege qualifies as its reference. Thus according to Frege there is no difference between (**??**) and (**??**) as far as their references (and the references of the involved expressions "Morning Star" and "Evening Star") are concerned.

But the corresponding senses of sentences (**??**) and (**??**) are different. (**??**) expresses the thought that the thing called "Morning Star" is identical to itself. (When I say that a sentence "expresses a thought" I mean, after Frege, that the thought constitutes the *sense* of this sentence). Since every thing is identical to itself and not identical to any other thing this thought is trivial: it is just an instance of a general metaphysical principle [2] .Sentence (**??**), in its turn, expresses the thought that names "Morning Star" and "Evening Star" have the same reference. This latter thought is not trivial and it cannot be justified simply by grasping or analysing the senses of these names but needs an independent astronomical

[2]This reasoning does not apply to expressions of the form $A = A$ in cases when name "A" has no reference, in particular, stands for a fictional character. This interesting problem is out of the scope of the present paper.

evidence. This explains the obvious difference between (**??**) and (**??**) without diversifying the identity concept.

A further detail of Frege's theory of sense and reference concerns propositions (aka thoughts), including those of the form $A = B$, which are not asserted (i.e., do not constitute self-standing judgements) but *mentioned* in a so-called indirect speech. As an example consider sentence

(5) Frege believes that Morning Star is Evening Star

that can be expressed using our earlier introduced notation as "Frege believes that (**??**)". According to Frege, when (**??**) is used in this way its sense takes place of its reference. So (**??**) expresses the thought that Frege has a particular epistemic attitude to another thought; the semantic analysis of (**??**) distinguishes between the reference of name "Frege" (who is an epistemic agent) and that of expression (**??**) making part of (**??**), which in this position is the same as its sense [3].

Frege does not explicitly thematise the issue of time in his [**?**] but this theme is obviously present in his *Venus* example: for saying that the Morning Star and the Evening Star are one and the same planet one needs to identity these objects trough the time span between the two moments when they are observed.

2.2. **Endurance and Perdurance.** Endurance and Perdurance are two competing accounts of how spatio-temporal objects that persist over time eventually changing some of

[3]Frege compares senses of linguistic expressions in this respect with "objective images" provided by telescopes, which can be used and shared by many people. Perhaps a photograph would serve as a better example. Frege distinguishes such sharable images from individual subjective sensual impressions, which he calls "ideas" [**?**, p. 39-40]. Unlike individual ideas senses of linguistic expressions can be shared among users of the same language. Notice that on this account an ontological counterpart of the sense of a given linguistic expression is an entity of a different type than the ontological counterpart of its reference. For example, planet Venus and its photograph are entities of different types.

their properties [**?**]. The *perdurantists* argue that such objects along with their spatial parts also have *temporal parts* like Venus-Yesterday and Venus-Today. On this account, what is usually referred as planet Venus is the mereological sum of its temporal parts. The sum of temporal parts is thought here on equal footing with the sum of spatial parts like the sum of parts of the body of a person. Under this view the Morning Star and the Evening star are not literally one and the same object but rather two different temporal parts of one and the same object usually called Venus. Remark that the perdurantism implies that each spatio-temporal object is ultimately a *process* that extends both in time and in space. A basic ontological unit under this view is an *event* rather than an object; events sum into objects when their some satisfy some vaguely defined conditions of temporal and spatial continuity. Such ontological view is called *four-dimensionalism* meaning that on the top of three spatial dimensions objects have one more temporal dimension. The above basic version of perdurantism and four-dimensionalism is motivated by the school-level Newtonian mechanics but some more advanced version of the same view use Relativity theory and other theories of today's physics that have bearing on space, time and spacetime.

The *endurantists* argue that objects preserve their identity through time and change by being *wholly* rather than only partially present at each moment of their existence. On this account Morning star and Evening star are not different temporal parts of the same process but rather different names and different descriptions of the same enduring object that exists at different times and is capable to change, that is, may have different properties at different times. Such ontology is called *three-dimensional* because its basic elements are spatial particles moving in a three-dimensional space and preserving their identities during these motions. A challenge for an endurantist is to explain how it is possible for an enduring object a to have a certain property $P(a)$ at time t_1 and not have the same property at different time t_2. For these assumptions arguably lead to contradiction $P(a), \neg P(a)$. A possible defence strategy here is to argue that property P is relational and needs to be formalised not as a monadic predicate but as a binary predicate (relation) of the form $P(x,t)$ where variable t stands for a moment of time; then one may argue that object a verifies both $P(a, t_1)$ and $\neg P(a, t_2)$ and that the two conditions are mutually consistent.

2.3. **Theseus Ship.** The *Theseus Ship* is an old logical puzzle reported by Plutarch, who was writing in the beginning of the 2nd century A.D. According to Plutarch, Athenians made the old wooden ship of their hero Theseus into a long-preserved monument by continuously replacing its decaying parts by newly made copies. So the Ship of Theseus, says Plutarch, "became an illustration to philosophers of the doctrine of growth and change, as some argued that it remained the same, and others, that it did not remain the same" [**?**, p. 33]. In 1655 Th. Hobbes introduced a further element into the puzzle by suggesting that the original details of the ship could be collected and at certain point (when the last original detail of the ship is replaced) once again reassembled [**?**, 2.11]. Thus instead of one original ship A there emerge two different ships: ship B obtained via the continuous replacement of all details in A, and ship C reassembled from the original details of A. The challenge is to answer which of B, C, if any, is the same as A. Notice that hypotheses (i) $A = B$ and (ii) $A = C$ are supported by different ideas of how identity is preserved through time: while (i) is based on the idea that an enduring object needs to preserve continually its spatial form but not necessarily preserve its material constitution, (ii) is based on the idea that the invariance of identity of that kind of objects requires the invariance of its material constitution while the preservation of its form can be not contuous.

A perdurantist can easily solve — or rather dissolve — the *Theseus Ship* puzzle by describing the processes of building up B and C in terms of elementary point-like events $P(x, y, z, t)$ of the concerned region of space-time, where (x, y, z, t) are their spatio-temporal coordinates, using the modern mathematical construal of physical space-time as a set of atomic events with the structure of four-dimensional Euclidean space or Riemannian manifold. For this end the perdurantist has, first of all, to specify set of atomic events E that make part of the story: it includes the worldline of the original ship A, the process of preparing the replacing details, the continuous replacement of original details by new ones that eventually transforms A into B, and, finally, the re-assembly of the original details that produces C. Then the perdurantist may use some geometrical properties of subsets of E for specifying appropriate subsets $T_i \subset E$ where $i \in \{A, B, C\}$ that qualify as enduring objects in the usual sense of the word. In particular, it can be argued that in order to qualify as enduring object T needs to be path-connected (that is, to consist of a single "chunk", possibly with some holes). But even if this project is not quite successful the perdurantist is already in a position to claim that since the notion of identity is fixed for atomic events and sets thereof [4] , the *Theseus Ship* no longer challenges this basic notion but only makes explicit the problematic character of the concept of enduring spatial object. To emphasise this point they can refer to cases of the "vague identity" of individual clouds on the sky, waves in a sea, etc. It turns out, according to this argument, that the *Theseus Ship* reflects the problematic character of the common notion of objecthood rather than that of identity.

A problem with this straightforward perdurantist solution is that it says nearly nothing about common intuitions, ideas and talks related to identity of moving enduring objects like ships — except saying that these intuitions, ideas, and talks are unreliable. Perdurantism suggests instead a radical revision of the commonsensical view in favour of a counter-intuitive ontology that explains away such objects in terms of the event-based ontology. Such a revision would be readily accepted if it could provide an effective way to improve upon common talks and reasoning involving identity through time. But the above theoretical description of the idea hardly helps by itself to achieve this goal. The *Theseus Ship* puzzle involves only three putative objects, which are denoted above as A, B, C. But sets T_A, T_B, T_C are infinite (since they represent events extended continuously in space-time), and their identity conditions are fixed in terms of those of their elements (as usual in Set theory). Unless the basic perdurantist setting is provided with some additional means of identification for objects like A, B, C (such as geometric means pointed to above) this setting remains ineffective. But when such means are introduced one may wonder whether they need indeed a support of the set-based four-dimensional perdurantis ontology.

[4]How this can be done we discuss in the next Section

Endurantism unlike perdurantism squares well with usual commonsensical intuitions about enduring objects and doesn't imply a radical revision of the related ontological views. But it remains problematic and it doesn't provide any immediate solution of the *Theseus Ship* puzzle. Many tentative solutions of this puzzle are found in the literature and their systematic reviewing is out of the scope of the present paper. In **??** we provide a new solution of the puzzle using basic ideas of Directed Type theory.

3. Standard theories of identity

By "standard" theories of identity we mean theories based on Classical First- and Second-Order Logic, which are used in formal theories like ZFC and widely discussed in today's philosophical logic and analytic metaphysics. In this framework the identity relation can be defined in several different ways, which under some natural conditions that will be discussed shortly are all provably equivalent. The most economic set of axiom for the standard identity relation is the following:

- **Refl**: relation $=$ is reflexive, in symbols $x = x$;
- **InId**: relation $=$ satisfies the principle of *Indiscernibility of Identicals* :

  $$(x = y) \to (F(x) \to F(y))$$

  that says, in words, that if things x, y are identical then anything truly said of x is also true of y [5].

Here **InId** is a *scheme* of first-order axioms; otherwise it can be formulated as a universal formula of the Second-Order Classical logic. In both cases the scope of the domain of predicates that are allowed in the right-hand part of **InId** is crucial as we shall now see.

[5] In case the language contains function symbols f, in addition to **InId**, a similar principle needs to be postulated for functions:

$$(x = y) \to (f(x) = f(y))$$

3.1. **Indiscernibility of Identicals and Identity of Indiscernibles.** Substituting in **InId** for $F(x)$ a scheme of special form $\lambda z.P(z) \to P(x)$ (in the λ-notation) we have $(x = y) \to ((P(x) \to P(x)) \to (P(y) \to (x))$ and then, by the modus ponens and a replacement of the predicate variable, $(x = y) \to (F(y) \to F(x))$. Thus we have proved a stronger and more convenient version of *Indiscernibility of Identicals*, which we denote **InIdS**:

$$(x = y) \to (F(x) \leftrightarrow F(y))$$

By substituting in the consequent of **InIdS** for $F(x)$ predicate $ID_x(z)$ defined as $\lambda z.(z = x)$ we obtain $x = x \leftrightarrow x = y$. So we proved the converse of **InIdS**

$$(x = y) \to (F(y) \leftrightarrow (F(x))$$

known as the principle *Identity of Indiscernibles* or **IdIn**. It can be now shown that the identity relation $=$

(1) is an *equivalence* (i.e., a reflexive, symmetric and transitive relation);

(2) is the *smallest* equivalence (that is, for any equivalence relation $\equiv$, $(x = y) \to (x \equiv y)$;

(3) conversely: an equivalence relation with the property of being smallest satisfies **InIdS**.

If the domain of available predicates for **InId** is limited to the effect of excluding predicates like ID_x then there is a room for doubting whether **Refl** and **InId** alone are sufficient for specifying the identity relation in the intended sense of term. In particular, one may seriously doubt whether a non-symmetric relation may possibly qualify as identity. In **??** we shall defend this option. A similar limitation of the domain of available predicates can be considered for **IdIn** independently of **InId**. Then **IdIn** also becomes controversial since it is not immediately clear whether or not it is sound to think of two or more numerically distinct items sharing all their properties save the property of being mutually identical.

Since this latter controversy plays no special role in what follows I shall not discuss it here.

3.2. **Reflexion.** **Refl** says that any thing x is identical to itself. What exactly this tautology tells us about the identity concept? How being identical to itself is different from simply being itself and from just being something? Those are deep and thorny metaphysical questions which have been a subject of philosophical reflexion throughout the history of philosophy. I shall attempt here to shed some light on these questions by focusing on a syntactic aspect of **Refl** and bringing to the fore the usual linguistic convention according to which repeated occurrences (tokens) of the same symbol-type are understood as repeated references to the same thing. For further references let us denote this convention **C**.

Formula $x = x$ comprises two occurrences of the same symbol-type 'x', which are here before our eyes. How exactly we qualify two or more written or printed occurrences of symbol 'x' as being of the same type is an interesting question, which, however, is out of the scope of my present analysis. For my purposes it is sufficient to notice that this syntactic typing device is working and has a strong intuitive appeal. Every user of a written alphabetic language uses this tool whether on not they understand the mechanism of its working[6].

Let $\{x\}_1$ and $\{x\}_2$ be two different tokens (occurrences) of the same symbol-type $[x]$. Then the fact that $\{x\}_1$ and $\{x\}_2$ are of the same type can be expressed as

$$\{x\}_1 \doteq \{x\}_2 \tag{6}$$

where $\doteq$ is (not identity but) a *sui generis* relation that holds between items of the same type. Now by replacing each of $\{x\}_1$ and $\{x\}_2$ by the type that each of them represents, namely by $[x]$, we can *transport* (**??**) to the level of types by writing

[6]Spoken languages and non-alphabetic written languages similarly involve patterning and the type/token distinctions

$$[x] = [x] \tag{7}$$

where symbol '=' is read as identity. Thus (**??**) makes it clear how anything — in this case type $[x]$ — can hold a certain relation to itself. It does this via its representatives like $\{x\}_1$ and $\{x\}_2$, that is, via the possibility to use the same symbol-type repeatedly. Using convention **C** one can replace in (**??**) syntactic type $[x]$ by its semantic value and thus get back **Refl** in its usual form ($x = x$).

Notice the core of this machinery is purely syntactic; semantic considerations enter into the play only at the last step when symbol-types are given a meaning. A key feature of this machinery is the organisation of the flow of written or spoken items into (locally) stable patterns or types, which admit for (locally) stable meaning. Albeit pointing to this syntactic typing mechanism certainly does not justifies the idea that the identity concept is essentially linguistic, it nevertheless demonstrates that the linguistic aspect of this concept is essential for its analysis. Patterning of written and spoken items and their organisation into stable types provides a syntactic model of how the identity concept works beyond languages.

3.3. **Identity in mathematical theories.** The identity relation defined as above (via **R** and **InId** or equivalently) can make part of any first- or higher-order formal theory like ZFC as a primitive relation. Then it can be conceived of either as a part of the logical apparatus of a given formal theory or as a non-logical relation on equal footing with other non-logical primitive relations (like the relation of membership $\in$ in ZFC). Whether or not the identity relation is logical is an interesting philosophical question which prompts reflections on the scope of logical concepts and criteria of logicality [**?**]. It should be born in mind, however, that such concerns do not have any direct effect on how formal theories using the standard identity relation are built and used.

Another option available in formal theories is to define the identity relation in terms of non-logical terms specific for a given theory. In such cases the identity relation does no longer qualify as primitive but its name serves as a shortcut to some complex relation that does not involve identities. In this sense the identity relation as such becomes dispensable. For example, in ZFC the identity relation can be defined as the *member-congruence*: sets x, y are called member-congruent when they are members of the same sets, in symbols

$$x \in z \leftrightarrow y \in z$$

. Since the member congruence is an equivalence relation satisfying **InIdS** for atomic formulas of the form $x \in y$, in ZFC it is co-extensional with the standard identity relation taken as primitive. A similar formal trick can be made in any first-oder theory with finitely many primitive non-logical symbols [**?**, p. 25-26]. Conceptually such a move is hardly illuminating because it hides the fact that the identity relation — whether one qualifies it as logical or not — applies across a large variety of theories and does not depend on specific features of particular theories.

Apart of general philosophical worries about identity which do not have a direct impact of how the identity concept is formally used, the identity relation in standard formal theories could be seen as unproblematic [7]. But recall that formal mathematical theories are typically built as formalised versions of certain *informal* theories, which continue to be used in their original non-formalised form. In particular, ZFC is a formalised version of Cantor's "naive" Set theory, which today is still studied and used by a much larger mathematical and scientific community than the tiny community of logicians and philosophers using and developing ZFC and its heirs. Formalisation of a given informal theory T is a tricky procedure that hardly admits for a formal description itself. But an usual assumption is that a formal counterpart T_F of T interprets all basic concepts and forms of informal reasoning supported by T improving on their rigour, exactness, and probably some other epistemic qualities. Thus it is assumed that T_F may disqualify and rule out certain elements

[7]Among such general worries about the identity relation is the *Julius Caesar Problem* first formulated by Frege [**?**], see [**?**, ch.9] for a recent discussion.

and fragments of T as erroneous like in the case of Russell paradox formulated in the naive Set theory: the axioms of ZFC prevent the derivation of this and other known set-theoretic paradoxes in this formal theory. At the same time it is commonly assumed that the revision of T implied by its formalisation preserves all the basic traits of T in a new form; otherwise there would be no reason to qualify T_F as a formal version of T. In other words, the formalisation of mathematical and scientific theories by logical means aims at their improvement (at least in some important aspects) but not a sheer replacement by something wholly new and different. So it makes sense to ask whether or not a given method of logical formalisation is *adequate* to those (informal by logician's standard) mathematical and scientific theories, which are supposed to be formalised by this method. In other words, the question is whether the revision of given informal theory T implied by its logical formalisation preserves all its epistemically significant and valuable content. As we shall now see there are strong reasons to doubt that the standard formal theory of identity outlined above adequately formalises the concept of identity (equality) as it is commonly used in today's mathematics.

The problematic and often ambiguous character of conventional mathematical talk about "the same number" or "equal numbers" has been stressed already by Aristotle in his *Metaphysics*. Aristotle distinguishes here between mathematical and "ideal" numbers assuming that only the latter are subjects to strict identity conditions while the former allow for unlimited "copying" that produces *equal* but not identical items. In Aristotle's view, relaxing the strict identity down to mathematical equality, which allows a given entity to exist in form of multiple equal "copies", is necessary for performing usual arithmetical operations like operation $2+2$ that involves two copies of the same ideal prototype $\mathbf{2}$. On this ground Aristotle argues that unlike usual mathematical numbers their ideal prototypes are "inaddible" (*Metaphysics*, Book 13, 1081a).

Surprisingly on not, similar difficulties arise in the modern set-based mathematics of the 20th century where the identity concept is supposed to be fixed once and for all via the underlying formal Set theory, typically ZFC, equipped with the standard identity predicate.

In this foundational framework any mathematical object whatsoever is represented with a set. Notice that in ZFC the question of whether or not two given sets x, y are equal always has a definite (yes-no) answer. But in the "normal" informal mathematics questions like: Whether number π is equal to the exponential function $\lambda x.e^x$? are discharged as ill-formed rather than answered in negative. This observation alone suggests that using the standard identity relation borrowed from ZFC across all mathematical disciplines is hardly adequate, and that the underlying logic of mathematics involves some form of *typing*.

Further, there are difficulties related to the core notion of *mathematical structure* that arise in the set-based mathematics [**?**]. The idea to identify a mathematical structure like an algebraic group with a particular set and treating this set with the standard identity relation making part of the underlying Set theory conflicts with a strong mathematical intuition according to which *isomorphic* groups and other isomorphic structures in relevant contexts are equal or essentially same [**?**]. Formalising the wanted notion of equality using ZFC with its standard identity predicate turns out to be impossible. This problem cannot be easily solved by using Frege's notion of abstraction that justifies a substitution of some equivalence relation (say, the relation of isomorphism in case of groups) for identity, or, to put it into modern mathematical terms, by constructing a *quotient* by the given equivalence. For a working mathematician typically thinks of a "particular" group (say, *the* infinite cyclic group), not as a quotient but rather as an object that exists in multiple isomorphic copies — very much in the same spirit in which Aristotle thinks about natural numbers in his *Metaphysics*. Mathematicians say in such cases that a given structure (for example, a group) is defined *up to isomorphism*. But what does this mean precisely from a logical point of view?[8].

A possible answer amounts to interpreting "reasoning up to isomorphism" as a reasoning where "isomorphism replaces identity" and has just the same formal properties. Since the relation of being isomorphic $\sim$ is obviously reflexive it is sufficient to postulate that it also

[8]See [**?**] and [**?**] for further discussion

verifies **InId** (or **InIdS**, see **??**). Given this new intended interpretation this principle is called *equivalence principle* for set-based structures [**?**] or **EPS** for short:

$$(x \sim y) \to (F(x) \leftrightarrow F(y))$$

EPS says, informally, that if two such structures $x, y$are isomorphic then they have the same properties. But **EPS** is not compatible with the standard ZFC-based foundations of mathematics. Notice that the very relation of being isomorphic $\sim$ is not uniformly defined for all types set-based structures: two groups are isomorphic when there is a *group isomorphism* between these groups, two topological spaces are isomorphic when there is a *homeomorphism* (i.e., an invertible continuous transformation) between these spaces, etc.: each particular *type* of structure comes with its specific notion of isomorphism. Further observe that **EPS** is plausible only when the range of predicate variable F is restricted to a certain class of relevant structural properties that depend on given isomorphic structures x, y: group-theoretic properties for groups, topological properties for topological spaces, etc. But ZFC does not tell us which properties are relevant and "structural" in a given context, and which are not. These observations suggest that **EPS** can be implemented only in a typed formal system but not in an untyped theory like ZFC. In **??** we shall see how **EPS** is implemented in HoTT. For simple explicit counterexamples to **EPS** in the set-based mathematics see [**?**] and [**?**].

3.4. **The standard theories of identity and space-time.** The problematic issues of identity related to time, which have been outlined in Section **??**, were systematically analysed and discussed in recent decades using the standard logical apparatus described earlier in the present Section. This research belongs to a broader area of research that aims at formalisation of reasoning about spatial, temporal and spatio-temporal objects using the First-Order logic and its various model extensions. So we've got today a big lot of useful systems of temporal and spatial logics construed in this way [**?**], [**?**]. In such logical systems the core First-Order logic and the standard identity relation are usually seen as neutral instruments that help to formally express further principles concerning reasoning in space and time by adding appropriate non-logical predicates, axioms and modal operators.

There are, however, some reasons to doubt the philosophical neutrality of this general approach. In order to see this let us recall Aristotle's Law of Non-Contradiction stated by this philosopher in *Metaphysics*(Book 4, 1005b19-20), in the following words:

> It is impossible for the same thing to belong and not to belong *at the same time* to the same thing and in the same respect.

where "belongs to" should be read as belonging of a property to its holder; the italic is mine. In his *Critique of Pure Reason* Kant famously criticises the above Aristotle's wording by arguing that the reference to time is out of place in such a general logical principle (A 152-153); see [**?**] for a more recent critical discussion. This argument may appear self-obvious but in fact it depends on Kant's view of space and time as *forms of intuition* [**?**], which do not reduce to the general logical concepts where, according to Kant, the concept of identity and the related Law of Non-Contradiction belong. The domain of spatio-temporal reasoning, which in Kant's view covers all of mathematics and mathematically-laden science, according to this view, is a subject of a special logical discipline of *transcendental* logic that Kant sharply distinguishes from the *general* logic that regulates reasoning of all sorts including the speculative reasoning not constrained by mathematics and science (A 154).

The thesis according to which spatio-temporal issues are irrelevant in the basic logic becomes problematic when one conceives of space and time in Leibniz' line as *sui generis* relational structures. Indeed, if space and time are nothing but ways to identify, distinguish and relate their elements, then in can be argued that the relevant notions of identity and difference themselves conceptually depend on basic spatial and temporal structures, i.e. are intrinsically spatio-temporal [?] [9]. In what follows we shall see how this line of thought is supported with HoTT and DTT.

The fact that we refer today to Kant as a critic of Aristotle's alleged confusion of fundamental logical principles with spatio-temporal issues, is somewhat ironic because in the late 19th century and in the early 20th century Kant's views were in their turn criticised by Frege, Russell and other proponents of the emerging Analytic philosophy on similar grounds. Frege rejects Kant's view according to which arithmetic essentially involves the intuition of time, and attempts to reconstruct this mathematical discipline as a part of what Kant could qualify as a version of his "general" logic (albeit technically Frege's logic and Kant's general logic are not the same)[?]. Russell extends a similar logicist view onto geometry and the whole of his contemporary mathematics [?]. Albeit today's mainstream view on mathematics in the Analytic philosophy does not endorse its reduction to logic it still does endorse the thesis that spatial and temporal intuitions have no role in the logical foundations of mathematical theories. At least in this respect the standard theory of identity is not philosophically neutral.

[9] It is not wholly clear whether on Leibniz' view certain things can be identified and distinguished without using their spatio-temporal representations. On the one hand, Leibniz states that time and space are modes of representation of *monads* (aka substances) which themselves do not exist in time and space and, by his word, "have no windows". Monads are identified and distinguished by Leibniz via their properties. This suggests that Leibniz applies here a concept of identity unrelated to space and time. But on the other hand, he elaborates on how monads represent each other and states that such a mutual representation of monads is their essential characteristic. This latter remark makes it plausible that talking about monads, about their identity and their differences, Leibniz himself relies on spatio-temporal representations of monads. I shall not go further in the interpretation of Leibniz' writings but only remark that unlike Kant Leibniz makes no sharp distinction between different kinds of logic and doesn't reserve a special place and function for the all-purpose "absolute" concept of identity.

4. Homotopy Type Theory

In this Section we introduce relevant elements of the Homotopy Type theory (HoTT) which is a formalised mathematical theory that can be applied as a general formal framework for mathematical reasoning and prospectively also for reasoning beyond the limits of the pure mathematics. In this and the next Sections we focus on HoTT-related conceptions of identity and apply these conceptions to traditional metaphysical problems of identity through time outlined above in **??**. We acknowledge the fact that the foundational claims of HoTT are debatable and conceive of our present work as an attempt to justify these claims by showing how the HoTT-related identity squares with the concept of identity as it has been construed in the earlier philosophical tradition. We show that even if the HoTT-based account of identity drastically differs from the standard theory outlined in Section **??** it squares well both with common intuitions about the identity through time and with some pre-Fregean analyses of identity including Leibniz's.

While Frege's conception of logic is atemporal and aspatial, HoTT supports a conception of logic as a part of geometry [**?**] that in many cases admits for a straightforward interpretation in intuitive spatio-temporal terms (see **??**, **??** and **??**). This concerns, in particular, the HoTT-based conceptions of identity explained in what follows. This feature of HoTT (including DTT) allows us to apply the relevant HoTT-based conceptions of identity in the problematic cases of identity through time and change *directly* avoiding usual tedious logical and ontological reconstructions, which make these cases to appear so problematic. In **??** we treat in this way Frege's *Venus* example, in **??** we treat the *Ship of Theseus* puzzle, and finally in **??** we come back to the metaphysical issues of endurance and perdurance.

HoTT comprises three basic ingredients: (i) a base Type theory and (ii) its interpretation in terms of Homotopy theory, (iii) its formal category-theoretic semantics. The version of HoTT fragments of which are presented below is the *Book* HoTT so called after [**?**] where the reader will find a systematic presentation of the relevant mathematical material. In this core version of HoTT (i) is a constructive Type theory with dependent types called MLTT after the name of his inventor Per Martin-Löf [?] (see **??**), (ii) is the standard Homotopy theory, and (iii) is groupoid semantics (see **??**). In DTT presented in the next Section **??** (i) is a novel type-theoretic syntax which does not yet exist in a stable form (see **??**), (ii) is *directed* Homotopy theory (see **??**) and (iii) is general (higher) category theory (see **??**).

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4.1. **MLTT.** MLTT is a Gentzen-style typed calculus that comprises no axiom and consists solely of a number or formal rules that allows to produce new formulas from given formulas. Its distinctive feature is the presence of *dependent* types, which are types indexed by terms of their *base* types. Originally MLTT has been conceived by Per Martin-Löf as a formal framework for constructive reasoning in mathematics, which is apt for computational implementations. The first published version of MLTT dates back to 1972 [?]. The homotopical interpretation of MLTT was discovered 34 years later **??**.

Basic formulas in MLTT are called (and normally interpreted as) *judgements*, which should not be confused with *propositions*. Semantically a judgement can be defined as a proposition supported with a proof. But according to Martin-Löf, this order of ideas should be reversed: the concepts of proposition and proof arise via an analysis of the fundamental concept of judgement [?], [?].

MLTT comprises the following four *forms* of judgements:

(i) $A : TYPE$ (A is a type)

(ii) $A \equiv_{TYPE} B$ (types A, B are equal)

(iii) $a : A$ (term a is of type A)

(iv) $a \equiv_A a'$ (terms a, a' of type A are equal).

Sign $\equiv$ that occur in (ii) and (iv) refers to the sort of identity that is called in MLTT *judgemental* or *definitional*. The idea is that such identities do not require and do not admit for a proof. They can be thought of either as terminological conventions like "The Morning Star is Venus" or as axioms justified outside the MLTT. Notice that the definitional equality applies only to terms of the same type (in case (ii) this is the "type of types" denoted $TYPE$); terms of different types cannot be identified.

Martin-Löf offers four alternative semantic interpretations of judgements of form (iii):

(a) a is an element of set A

(b) a is a proof (witness, evidence) of proposition A

(c) a is a method of fulfilling (realising) the intention (expectation) A

(d) a is a method of solving the problem (doing the task) A

and argues that in spite of apparent differences all of them ultimately reduce to the same [10]. As we shall shortly see the homotopical interpretation of MLTT suggests a significant revision of this interpretation and helps to save the common idea that propositions and sets are two different notions.

In addition to the definitional identity MLTT comprises the notion of *propositional* identity (denoted by sign $=$), which equally applies only to terms of the same type. Unlike (iv), $x =_A y$ is not a judgement but a *proposition*, which is, formally, a type. According to the proposed reading (b) judgement $p : x =_A y$ is interpreted as "p proves that terms x, y of type A are equal". When a proposition has a proof it qualifies as constructively true; otherwise it is constructively false.

Basic rules of MLTT concerning the propositional identity are the following:

[10] "If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [...] apart. Instead we simply identify them, that is, treat them as one and the same notion." [**?**, p. 13]

(8)
$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x, y : A}{\Gamma \vdash x =_A y : TYPE}$$

Rule (**??**) is a *formation* rule: it says that whenever in context (i.e., a list of assumption) Γ type A is constructible, its terms x, y (if any) give rise to identity (type) $x =_A y$ (which can be empty).

(9)
$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x : A}{\Gamma \vdash refl(x) : x =_A x}$$

Rule (**??**) is an *introduction* rule. It says that any well-formed identity type of the form $x =_A x$ is inhabited by special term $refl(x)$ called *reflection* of x (and in this sense introduces this term). This rule can be seen as a new form of the reflexivity principle **R** discussed above in **??**. Our above remarks concerning the syntactic character of the reflexivity principle applies in this new formal framework. Like in the standard formal setting in MLTT the syntactic repetition of tokens of the same type plays a fundamental role in sharping the identity concept. But notice that (**??**) does not rule out a possibility that identity type $x =_A x$ has other terms (proofs) than $refl(x)$

(10)
$$\frac{\Gamma \vdash A : TYPE \quad x : A, y : A, p : x =_A y \vdash C(x, y, p) : TYPE \quad \Gamma, x : A \vdash t(x) : C(x, x, refl(x))}{\Gamma, x : A, y : A, p : x =_A y \vdash J^t(x, y, p) : C(x, y, p)}$$

Rule (**??**), which is often referred in the literature as the J-rule, is an *elimination* rule. It shows how $refl(x)$ is "used" and eliminated in the sense that it is no longer present in the conclusion [11]. This rule is a type-theoretic version of **InId** from **??**. It says that given equal terms x, y of type A and "predicate" C, which is satisfied by x, x, and reflexion

[11] For a full exposition of the J-rule see [**?**]

$refl(x)$, the same predicate is satisfied also by x, y and the proof p of their identity. Thus the J-rule can be informally interpreted as a form of *Indiscernibility of Identicals* principle (**InId**): the substitution of term x by identical term y is a substitution *salva veritate*, i.e., substitution that preserves truthness of predicates. Notice, however, that "predicates" $C(x, x, refl(x))$ and $C(x, y, p)$ are not literally the same as in the First-Order logic: these are different types that belong the same family of dependent types. Thus **InId** in MLTT is "relaxed" along with the identity concept itself. We shall shortly see in **??** how the homotopical interpretation of MLTT provides for an intuitive geometrical extension of the above interpretation of the J-rule (**??**) in terms of **InId**.

It remains to discuss rules that relate the definitional and the propositional kinds of identity. The following rule, which is indispensable in MLTT, reflects a given judgemental identity to a propositional one:

$$\frac{\Gamma \vdash x \equiv_A y}{\Gamma \vdash \delta_{x,y} : x =_A y} \tag{11}$$

In other words, by (**??**), the judgemental identity of given terms entails their propositional identity. By the end 1980s all known models of MLTT satisfied also the converse formal property called the *reflexion rule*:

$$\frac{\Gamma \vdash p : x =_A y}{\Gamma \vdash x \equiv_A y} \tag{12}$$

which implies that an identity type can have at most one inhabitant or, in other words, that a proof of identity, if exists, is unique (the uniqueness of identity proofs or **UIP** for short. It was conjectured that either (**??**) or **UIP** could be a consequence of other principles of MLTT. This conjecture appeared then attractive because it could significantly simplify MLTT by trivialising the *higher identity types*, which are built as follows. Given an identity type $x =_A y$ and its two terms p, q. By (**??**) these data allow for building a new identity type of the form $p =_{x=_A y} q$, and this ladder can be continued indefinitely. The emergence of this complex multi-layered structure in the original version of MLTT appeared to be a purely syntactic feature without any reasonable semantic interpretation. **UIP** trivialises this structure in the sense that it implies that p, q as above are nothing but different names of the same unique proof, and similarly for all higher identity types. But in 1993 Thomas Streicher built a model of MLTT that doesn't have this property [**?**] and thus refuted the UIP conjecture. As we will shortly see, in the 2000s the structure of higher identity types in MLTT became the key to the homotopical interpretation of this theory known as HoTT.

4.2. **Homotopy theory and Higher Category theory.** The concept of homotopy in its implicit form was first conceived back in the 18th century in works of Lagrange, further developed in the 19th century in works of Jordan and Klein and Poincaré; the term itself made its first public appearance in 1907 encyclopaedic article by Dehn and Heegaard, see [**?**] for references and further historical details. Here we introduce only a few basic concepts of this theory. For a standard introduction to Homotopy theory, see [**?**].

A central concept of Homotopy theory is that of *path*. A path should not be confused with a geometrical *curve*. A path can be described as a *parametrised* curve with a fixed point of "start" X and a fixed point of "finish" Y (which can be the same point). Intuitively a path represents a continuous *motion* from point X to point Y but not just a spatial location where this motion occurs. Formally a path p can be defined as a continuous map from fixed directed "unit time interval" $T^{\uparrow}$ to some space S, in symbols

$$p : T^{\uparrow} \to S \tag{13}$$

where $p(0) = X$ and $p(1) = Y$ are as above; for making this picture more concrete chose $T^{\uparrow}$ to be real interval $[0, 1] \subset \mathbb{R}$. Obviously the same curve or "trajectory" with endpoints X, Y

in S may geometrically represent many different paths because there are many different ways of moving continuously from given point X to given point Y along the same trajectory: one may first go fast, then slow down or do it the other way round, etc. Thus the relevant concept of path unlike that of curve is essentially *spatio-temporal* but not only spatial.

Whether or not there exists a path between points X, Y of given space S depends on a topological property of S called *path-connectedness*. Space S is called *connected* when, speaking informally, it consists of a single "chunk". Space S is called *path-connected* if any two points of the space are connected by a path. Path-connectedness implies connectedness but not conversely.

Notice that in this standard setting given path p there always exist the *inverse* path p^{-1}, which is formally obtained by the condition

$$p^{-1}(t) = p(1-t)$$

for every moment $t \in [0,1]$; to rewrite this condition in a more abstract way is possible but not quite straightforward as we shall shortly see. If p is thought of as a movie showing the motion of test point P from X to Y then p^{-1} is the same movie played backward.

The concept of (path-) *homotopy* is a generalisation of that of path: informally homotopy can be described as a "path between paths", and defined as a continuous map of the form

$$h : T^{\uparrow} \times T^{\uparrow} \to S \tag{14}$$

that transforms a given path into another path with the same endpoints. If $T^{\uparrow} = [0,1]$ then $[0,1] \times [0,1] = [0,1]^2$ is the real square. Let p, q are paths that share their endpoints X, Y. Then

$$h(t,0) = p(t)$$

$$h(t,1) = q(t)$$

,

$$h(0, s) = p(0) = q(0) = X$$

,

$$h(1, s) = p(1) = q(1) = Y$$

for all pairs of moments $< t, s > \in [0, 1]^2$. A homotopy can be pictured as a two-dimensional *surface* (cell) delimited by a pair of curves but it needs to be once again borne in mind that like a representation of path by a curve such a representation is not faithful (for similar reason).

Like paths homotopies are invertible. Whether or not there exists homotopy h of the form (**??**) between given paths p, q sharing their endpoints X, Y once again depends on topological properties of space S. If it does then paths p, q are called *homotopical*. Figure **??** shows two paths with endpoints OS which are not homotopical because of the hypothetical *wormhole* in the spacetime that prevents a continuous transformation of each of these two paths into the other. (This has the effect of *gravitational lensing* also shown at this picture.)

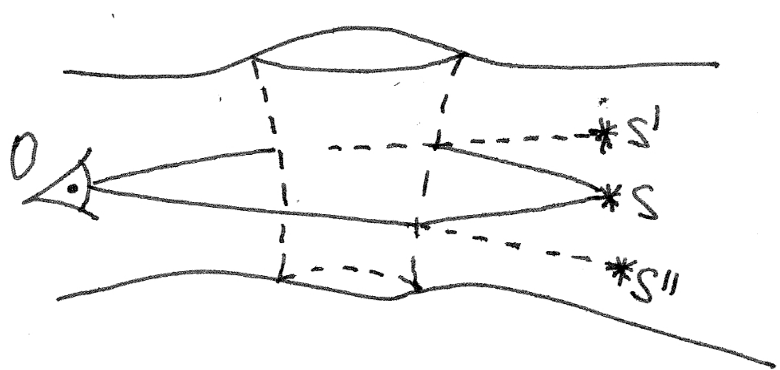

FIGURE 1. Non-homotopic paths around a wormhole

The structure of path-homotopies provides more information about the topology of the underlying space than the structure of paths alone. This ladder can be continued by considering 2-homotopies and n-homotopies for all natural n:

$$h : T^{\uparrow n} \to S \tag{15}$$

For $n > 1$ such maps are called *higher* (path)-homotopies.

A slightly more general concept of homotopy is given by applying the same construction to continuous maps between spaces rather than only to paths. Let f, g be two parallel continuous maps from space S to space S'

$$S \overset{f}{\underset{g}{\rightrightarrows}} S'$$

By a homotopy from f to g we understand continuous map h

$$h : S \times [0,1] \to S' \tag{16}$$

such that for any point $X \in S$, $h(X,0) = f$ and $h(X,1) = f$. If such h exists we call maps f, g homotopical. Now consider two maps i, j going into the opposite directions:

$$S \underset{j}{\overset{i}{\rightleftarrows}} S'$$

If the composition maps $j \circ i$ (j after i) is homotopical to the identity map of S (1_S) that sends each point of that space to itself and $i \circ j$ (i after j) is homotopical to the identity map $1_{S'}$ spaces S, S' are called *homotopy equivalent.* Spaces, which are homotopy equivalent to a point are called *contractible.* For example, the Euclidean space $\mathbb{R}^n$ of any dimension is contractible to a point. Homotopy equivalent spaces are indistinguishable by their homotopy-related properties. In what follows we talk about homotopy equivalent spaces as the same spaces.

We conclude this short presentation of Homotopy theory with a remark concerning algebraic and category-theoretic methods of studying homotopy, which are central in this field.

When the final point of path p coincides with the starting point of path q (let's call this condition *matching*) the two path can be composed into a composition path

$$r = q \circ p \tag{17}$$

In particular, the matching condition is satisfied when p, q are *loops*, i.e., paths that start and end at the same point. Notice that the composition of paths is not defined canonically (i.e., uniquely) because going through p and q takes two units of time but not one, while r should take just one unit of time like any other path. It is a natural solution to stipulate that the test point P goes through p exactly one half of the time unit and goes through q during the other half, so it goes through $q \circ p$ during the unit time interval as required. But this solution is not unique and not canonical: any other continuous mapping of the form

$$s : T^{\uparrow} + T^{\uparrow} \to T^{\uparrow} \tag{18}$$

does the same job. It is easy to see that however the scaling map (**??**) is chosen all the resulted composition paths r, r' are homotopic. This fact suggests the idea to solve the above problem by defining the path composition $\circ$ *up to homotopy*, i.e., replace in (**??**) paths p, q, r by their homotopy classes (i.e. equivalence classes by the relation of being homotopical) $[p], [q], [r]$. In case the paths are loops sharing their base point X_0 this gives rise to the concept of *fundamental group* $\pi_1(S, X_0)$ of the underlying space S first introduced by Poincaré. It is a group because paths are invertible and their composition satisfies group axioms. In case S is path-connected, fundamental group $\pi_1(S, X_0)$ does not depend on the choice of the base point X_0 and thus characterises the given space S itself.

Instead of choosing a base point and considering loops one can take a more general approach and consider *all* paths in a given topological space along with all their available compositions. In this case not all paths are composable because the matching condition is no longer satisfied universally. Thus we get the concept of the *fundamental groupoid* of given space. A groupoid is a group-like algebraic structure where the relevant algebraic operation is partial, i.e., defined for some pairs of its elements (in our case paths) but not for some other pairs. The example of the fundamental groupoid of a given topological space helps to see how the groupoid structure can be geometrically motivated. But from the algebraic and the computational points of view the notion of groupoid unlike that of group appears ineffective. We shall shortly see in **??**, however, how MLTT, via HoTT, makes it computationally effective.

Neither the fundamental group nor the fundamental groupoid of a space reflect properties of individual homotopies like h in (**??**). Such individual path homotopies also can be composed. But the exact definition of composition is more involved in this case. It turns out that in order to realise this idea one needs to apply to path homotopies two different composition operations as shown at the following diagrams:

<u>horizontal composition:</u>

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\beta"] & C \arrow[r, rightsquigarrow] & A \arrow[r, bend left, "k"] \arrow[r, bend right, "l"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

where $k = h \circ f$ and $l = i \circ g$ are compositions of paths and $\gamma = \beta * \alpha$ is the horizontal composition of path homotopies.

<u>vertical composition:</u>

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{description, name=G}] \arrow[r, bend right=60, "h"'{name=H}] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$$

where $\gamma = \beta \star \alpha$ is the vertical composition of homotopies.

The two operations need to be coordinated via appropriate *coherence conditions*, which we skip in this informal presentation. A two-layered algebraic structure that accounts of the composition of paths in a topological space along with the composition of their homotopies is called a *2-groupoid*. The fundamental *2-groupoid* of a space reflects more information about this space than its fundamental "flat" groupoid described above. While in the flat fundamental groupoid the composition of paths is defined up to homotopy, in its 2-dimensional extension all operations on paths and their homotopies are defined up the homotopy of the next level, i.e., up to the 2-homotopy. Continuing to rise the dimension on gets further notions of n-groupoid (where $n = 2, 3, \dots$) and ∞-groupoid.

These higher structures are quite complex and can be hardly faithfully presented with the traditional algebraic syntax. Here the language of *Category theory*, CT for short, including its "higher" extension turns out to be helpful. Usual flat groupoids, 2-groupoids and higher groupoids are *categories* (correspondingly, 2-categories and higher categories) in which all arrows (called in CT *morphisms*) are invertible. The reader who is not familiar with the language of CT can get a relevant intuition by thinking about categories and higher categories along the above pattern of fundamental groupoids and higher groupoids of topological spaces but without assuming that paths and path homotopies of all levels are invertible[12]. As we shall shortly see in **??** the concept of *fundamental category* of a (directed) space plays in DTT the same role as the notion of fundamental groupoid plays in the Book HoTT.

[12]For an elementary introduction to Category theory see [?], for Higher CT from a homotopical perspective see [?]

4.3. **From MLTT to HoTT.** The basic principles of the homotopical interpretation of MLTT concerning identity types are these:

- Types and their terms are interpreted, correspondingly, as spaces and their points; equivalently, types are interpreted as higher-dimensional groupoids. Being so interpreted, types are referred to as *homotopy types*.
- Identity types of the form $X =_S Y$ are interpreted as spaces of paths between points X, Y of space S;
- Judgement of the form $p : X =_S Y$ is justified by exhibiting path p between points X, Y of space S; in other words, two points connected by a path are equal;
- Identity judgement of the second level, i.e., of form $\alpha : p =_{X=_S Y} q$ is justified by exhibiting a homotopy between paths p, q; in other words, homotopical paths are equal;
- higher identity judgements are justified similarly with higher homotopies: the equality of two n-homotopies is justified by exhibiting a $(n+1)$-homotopy between them.

This interpretation was discovered independently by Michael Warren in his Ph.D. thesis supervised by Steven Awodey [**?**] and Vladimir Voevodsky [**?**] in the mid-2000s [**?**, p. 4]. The interpretation extends to the whole of MLTT; in particular, the J-rule (**??**) under the homotopical interpretation amounts to the well-known *lifting property* of homotopy, which is a fundamental principle of abstract Homotopy theory. In order to see how HoTT interprets higher identity types in MLTT consider the following definition:

<u>Definition:</u> We say that space S is of h-level $n+1$ if for all its pairs of points X, Y path spaces $x =_S y$ are of h-level n.

By setting the h-level of point (= contractible space) equal to (-2) one obtains the following stratification of spaces (homotopy types) :

- h-level (-2): single point pt;

- h-level (-1): the empty space $\emptyset$ and the point pt: a type of this level can be either empty or have a single point;
- h-level 0: h-sets aka discrete point spaces: contains distinguishable points;
- h-level 1: flat path groupoids: contain distinguishable points and distinguishable paths between points but no distinguishable homotopies (= non-contractibe surfaces);
- h-level 2: 2-groupoids : contain distinguishable points, paths and surfaces but have no non-contractible volumes;
-
-
- h-level n: n-groupoids
- ...
- h-level ω: ω-groupoids

Notice that h-levels are not equivalence classes of spaces. The homotopical hierarchy is cumulative in the "upside down" way in the sense that all types of h-level n also qualify as types of level m for all $m > n$. For example point pt qualifies as truth-value, as a singleton set, as one-object groupoid, etc.; a set qualifies as a "discrete" groupoid, 2-groupoid, etc. We say that a space (type) is n-space (n-type), when it is of level n but not of level $n - 1$.

The homotopical hierarchy suggests a modification of the intended pluralistic semantic of MLTT mentioned above. Instead of interpreting types as propositions or sets interchangeably we shall interpret (-1)-types as propositions and 0-types as sets. Beware that these sets should not be straightforwardly identified with ZFC-sets. n-types with $n > 0$ are higher-order *structures*.

What the hierarchy of types tells us about identity? First of all, it tells us that the identity type

(19) $$x =_S y$$

generally, is not just a relation. Consider a general identity judgement

(20) $$p : x =_S y$$

If S is a *proposition* (-1-type) then it is either empty (the case of false proposition) or all its terms are equal (the case of true proposition). In that case judgement (**??**) is trivial.

If S is a *set* (0-type) then its two elements x, y taken at random are either the same or not the same. This sounds familiar: in this case the identity concepts behaves as a relation: $x =_S y$ is a proposition, which is true if there exists p such that $p : x =_S y$, and which is false otherwise.

But at the upper levels of the homotopical hierarchy of types things look less familiar and more interesting. If S is a flat groupoid (1-type) there can be different proofs of (**??**). Thus further judgements of the form

(21) $$\alpha : p =_{x =_S y} q$$

generally, are not trivial. In that case terms x, y are not simply either identical or different but they also can be identical “in a number different ways”. In this case the identity type $x =_S y$ may have distinguishable terms $p, q, r, \ldots$ which form a *set*. Recall that the level of the identity type of terms of given k-type is $k - 1$. Thus the limit up to which the identity concept is enriched along the above lines in case the identified terms are of a k-type.

Thus an identity type in HoTT is not necessarily a proposition, which can be either true or false, but a richer homotopical structure (homotopy space), which in lower dimensions is interpretable in intuitive spatio-temporal terms explained in **??** above.

Rendering J-rule (**??**) in terms of the *Indiscernibility of Identicals* principle (**InId**) (as in **??** above) remains compatible with HoTT but in the homotopical environment this interpretation reflects only how this rule is applied at the propositional level. In the general case the same rule is explained and visualised via the geometrical notion of *transport* (along a given path) illustrated with Figure **??**. Here p is a path that proves that given terms x, y of base type A are identical. Now given a family of structures (not necessarily a single property!) $C : A \rightarrow TYPE$ dependent on terms of A (along with the canonical reflection term $t_{refl_x} : C(x) \rightarrow C(x)$) the rule produces a function $t_p : C(x) \rightarrow C(y)$ that "transports" the structure-type C along path p preserving all its structural features .

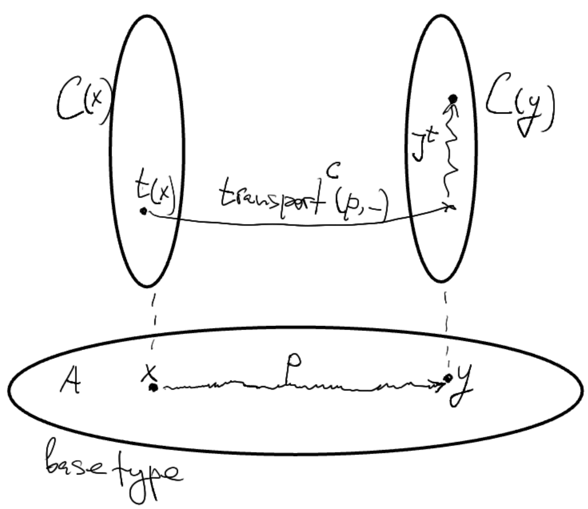

Figure 2. Transport of a structure along a given path in HoTT

[13].

[13]For a more systematic explanation of HoTT in intuitive spatio-temporal terms see [**?**]

Last but not least, it is appropriate to mention here the *Axiom of Univalence* (AU), which solves in HoTT the problem of identity of set-based structures discussed in **??**. The rules of MLTT/HoTT allow one to construct a canonical map of the form

$$e\,(A =_{TYPE} B) \to (A \simeq B) \tag{22}$$

which, to put it informally, witnesses the fact that identity is a special case of equivalence. The *Univalence Axiom* states that this map e has an inverse and thus is itself an equivalence. In other words, AU says that the type

$$(A =_{TYPE} B) \simeq (A \simeq B) \tag{23}$$

is inhabited. If types A, B are propositions then (**??**) reduces to

$$(A = B) \leftrightarrow (A \leftrightarrow B) \tag{24}$$

where $\leftrightarrow$ stands for the familiar logical equivalence. This principle says, informally, that the equality of propositions amounts to their equivalence. If A, B are sets then (**??**) (AU) says that the type of identities $A = B$ (which is a proposition) is logically equivalent to the type of isomorphisms $A \simeq B$. So AU allows one to prove the equivalence principle for set-level structures (**EPS**) that we have formulated in **??**, and thus makes the old Structuralist dream true: developing mathematics on Univalent Foundations one treats isomorphic set-level structures as equal without compromising the logical rigour. Notice that this does not lead to a collapse of identity into equivalence: in UF the two notions remain distinct. It is important to stress however that **EPS** does not automatically extend in UF to mathematical structures of higher homotopy levels. In particular, no analogue of **EPS** holds for all general categories and their equivalences (albeit a version of this principle holds for a special class of *univalent* categories where the category of sets, category of groups and some other familiar categories belong [**?**]).

4.4. ***Venus* homotopically.** The identity concept developed in HoTT is well-motivated within the pure mathematics by Homotopy theory and other areas of today's mathematics where the concept of homotopy appears in some form. Arguably, this extends to the whole of mathematics, or at least this idea is behind the project of building new *Univalent* foundations of mathematics. Since MLTT has been designed as a formal system allowing for a direct computer implementation (and its fragments have indeed been implemented in a number of proof-assistants including Coq) HoTT is also relevant in Computer Science. The relationship between the standard and the HoTT theories of identity is also well understood. By and large, the two theories are equivalent at the propositional level but HoTT further extends the identity concept onto higher homotopy levels. What remains unclear, however, is whether the higher-level homotopical extension of the identity concept has any relevance to the traditional philosophical discussions about identity, which usually draw on common linguistic examples.

In order to give an answer to this question let us consider Frege's *Venus* example (see **??** above) from a homotopical point of view. For simplicity let us assume that "Morning Star" (MS) and "Evening Star" (ES) are names that refer to objects, which are observed only once rather than repeatedly; now the task is to establish that the two objects are in fact the same. The application of HoTT in this case turns out to be surprisingly straightforward: the identity MS=ES is evidenced by a *continuous path* (the trajectory of the moving planet) t that connects the two objects:

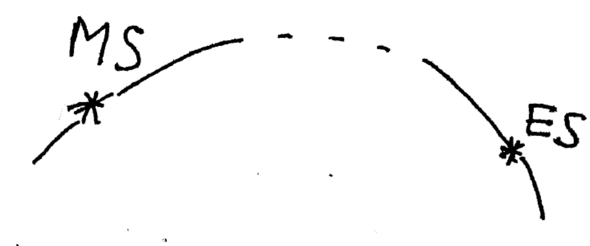

FIGURE 3. the Morning Star is the Evening Star

Unlike MS and ES trajectory is not directly observed but rather theoretically constructed. Nevertheless it existence can be verified empirically (by observing a predicted position of the planet following the trajectory), and there is good reason to think about it realistically as we normally do with other physical concepts from "electrons" to "gravitational waves". It can be also easily visualised as at the above figure. In order to comply with the formalities of HoTT it remains only to assume that MS and ES belong to the same underlying *space* S; then the rules of MLTT/HoTT guarantee the existence of space (type) MS$=_S$ES, which turns out to be inhabited by witness t. A more detailed reconstruction of the same example by means of HoTT is found in our [**?**].

Let us see what the above analysis provides. Admittedly, the Classical kinematic scheme, which we apply to Frege's example, does not perform interesting homotopical features. Whether the physical space is thought of after Aristotle as a finite 3-dimensional ball bounded with a sphere or in the (pre-relativistic) modern way as an infinite 3-space it is contractible, i.e., homotopy equivalent to a point. But since this universal space S is not "empty" but filled with moving mutually non-penetrable bodies (which is roughly a Democritean view) it contracts not to a point but to a *set* (in the sense of HoTT). Thus this scheme rules out non-homotopic paths (trajectories) connecting momentary positions of one and the same moving body: notice that in the above example (Fig.**??**) path t is necessarily unique or at least unique up to homotopy[14]. This shows that the available resources of HoTT are used in this example in a very limited way, and one may question, once again, if the application of this powerful theory to *Venus* example is not an overkill.

In fact even such a limited use of HoTT provides important epistemic gains. First of all, it concerns the very idea of logical analysis of identity through time. The standard approach is based on the idea that the identity concept is logical and independent of space, time and motion. In order to explain what is going on in common talks about "the same person" or "the same place" referred to at different times the standard analysis aims at separating

[14]This applies to the case when MS and ES are thought of as recurrent patterns rather than singular events

spatio-temporal relations from the identity relation and then to combine them together in an appropriate way. In particular, a *perdurantist* analysis (see **??**) may proceed as follows:

(1) Venus is ontologically reconstructed as a wordline (submanifold) V of 4-dimensional space-time S;

(2) Both S and V are thought of as sets of their *points* understood as atomic dimensionless events;

(3) The identity of any particular *point* of V is taken to be unproblematic;

(4) MS and ES are ontologically reconstructed as different non-overlapping *parts* of V (here some formal mereology can be helpful);

(5) The common pre-theoretical view according to which MS and ES are the same is explained away by saying that what is really meant is the facts that MS and ES are parts of the same whole.

According to this analysis what makes the example problematic is the fact that the common talk gets confused by the complicated spatio-temporal mereology, which this analysis puts to the fore and sorts them out. As soon as the analysis reaches the item (3) where the identity relation works on atomic events the difficulty is resolved. The identity relation as such is assumed here to be wholly independent from the spatio-temporal nature of its relata.

HoTT suggests a very different way of thinking about Frege's example. As we have seen in HoTT identity features as a spatio-temporal notion to begin with. It involves a spatio-temporal intuition via Homotopy theory. Surely, it is always possible to strip such intuitions from the foundations of Homotopy theory by developing this theory on set-theoretic foundations (along with the rest of mathematics). But HoTT demonstrates that this is not the only way to attain a formal rigour. HoTT attains the formal rigour without leaving basic spatio-temporal intuitions behind the Homotopy theory. These intuitions are instrumental in the above application of HoTT to Frege's example. Even if we cannot expect that all conceptual resources available in HoTT allow for the same straightforward translation into naive ways of everyday reasoning and common talks about space and time, the fact that the fundamental homotopical concept of *path* does allow for such a translation alone justifies the claim that the concept of identity in HoTT is essentially spatio-temporal. It extends by far the common spatio-temporal intuitions without breaking them down. Thus one who takes the identity concept construed in HoTT epistemologically seriously is in a position to argue that the standard logical analysis of identity through time is misguided: instead of separating the logical concept of identity from spatio-temporal issues one needs to reveal the spatio-temporal character of this concept itself! This is a major change of perspective.

Another interesting consequence of the above HoTT-based analysis of Frege's example is that the underlying space S and the path space $\mathrm{MS}=_S\mathrm{ES}$ are not the same. Correspondingly, MS/ES/Venus and its trajectory t live in different spaces and belong to different types. This is in contrast with item (1) of the above perdurantist reconstruction where Venus is identified with its own worldline (4D ontology). The above HoTT-based analysis of Frege's example supports the *endurantist* view according to which Venus in the morning and Venus in the evening is literally the same planet.

5. Directed Identity

In this Section we briefly present a version of HoTT called *directed* HoTT or DTT (Directed Type Theory), and then discuss a concept of identity supported by this theory. Unlike the standard Book HoTT, DTT is a work in progress, which by the time of writing does not yet exist in a stable form. A recent overview of the current work in DTT is found in the introductory part of [**?**]. Below we present main ideas behind DTT and discuss some elements of just one version its proposed syntax. A comparison of competing approaches in DTT is out of the scope of the present work.

5.1. **Directed Spaces and Directed Sums.** Recall that in the standard Homotopy theory all paths and homotopies of all orders are formally invertible. This corresponds to the usual geometrical intuition according to which a curve connecting two points can be always traced in both directions. This geometrical intuition does not, however, always squares with our everyday experience that tells us that many moves and processes go just in one direction but not in the opposite direction. It does not square either with what our best scientific theories tell us about the physical space, time and spacetime: think about a black hole where a material particle can fall but cannot move out[15]. The *directed* Homotopy theory is a generalisation of the standard Homotopy theory where the invertibility assumption about paths and homotopies is lifted. The theory of directed topological spaces (that includes the directed Homotopy theory) emerged in 1990s. For a comprehensive introduction to this theory see [**?**].

Before we further explore the concept of directed space consider the following elementary example, which show that non-invertible phenomena are more familiar in mathematics than one usually thinks. Consider once again Kant's famous arithmetical example:

$$7 + 5 = 12 \tag{25}$$

The equality sign "=" suggests thinking of (**??**) in a symmetric way: we have here an arithmetical expression on the left, another expression on the right; both expression refer to numbers, and the equality sign says us that the numbers are equal (or one and the same). This is not, however, how an arithmetical *operation* works nor what it is. Operation denoted by sign "+" takes two numbers and outputs another number. To simplify the example we

[15]We are using the black hole example without entering the ongoing discussion on whether or not all physical processes are time-reversible at the fundamental level. It is more appropriate for our present purpose to think about application of mathematics in science in terms of modelling without rising questions about fundamental principles of physics and other sciences. Falling of a particle into a blackhole is a irreversible process at least at some level of its description, whether fundamental or not.

take here the numbers to be *natural* (i.e., non-negative integer) numbers that form set $\mathbb{N}$. Then $+$ can be presented as a map that sends a pair of numbers into a numbers:

$$\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N}$$

.

Map $+$ is *non-invertible* in the usual sense that there exist no map $+^{-1}$ of the form

$$\mathbb{N} \xrightarrow{+^{-1}} \mathbb{N} \times \mathbb{N}$$

such that

- $+^{-1} \circ + = 1_{\mathbb{N}\times\mathbb{N}}$ and
- $+ \circ +^{-1} = 1_{\mathbb{N}}$

where $\circ$ denotes composition of maps and 1_X denote the *identity map* (aka identity function) of given set X that send a given element of that set into itself. Map $+^{-1}$ cannot exist because function $+$ is not injective and hence non invertible. To put it informally, number 12 cannot "remember" the summands of which it is a sum. These summands could be 10 and 2 rather than 7 and 5. It is an essential assumption used in the above argument that $+$ is an operation applicable to all pairs of natural numbers but not an operation specifically designed for a given pair of numbers.

Thus if one wants to stress the fact that represents an arithmetical operation a better notation is

$$7 + 5 \rightarrow 12 \tag{26}$$

that makes its non-invertible character more explicit.

Similar remarks can be made about the popular notation for functions like $y = f(x)$, which should be written rather as $y \leftarrow f(x)$, and in many other cases where the equality sign is similarly (mis)used. These elementary examples demonstrate both the abundance of non-invertible phenomena in mathematics and the power of the traditional conceptual optics that forces us to "symmetrise" such phenomena and let them to pass unnoticed. "Directed sums" mentioned in the title of this subsection are, of course, nothing but the usual arithmetical sums looked at in a particular way.

Let us now return to topology and recall from **??** the notions of fundamental group and fundamental groupoid of a topological space. Every element of a group has an inverse; groupoid is a category where all morphisms are invertible. Lifting the condition of invertibility of paths and homotopies we come to notions of fundamental *monoid* and fundamental category, which in the directed Homotopy theory generalise the notions of fundamental groups and fundamental groupoids. While the concept of fundamental monoid turns out to be of little use [**?**, p. 9], that of fundamental category (including its "higher" version) is the principal algebraic tool of directed Homotopy theory.

DTT is a type theory interpreted in terms of *directed* Homotopy theory in a way similar to which MLTT is interpreted in terms of the standard "reversible" Homotopy theory in the Book HoTT. While the Book HoTT types are interpreted as spaces along with their fundamental (higher) groupoids, in DTT types are interpreted as *directed spaces* along with their fundamental (higher) categories. The fact that general categories in DDT replace groupoids of the Book HoTT makes DDT particularly attractive from a foundational point of view. The idea of developing new foundations of mathematics as a replacement of the received set-theoretic foundations was around at least since mid-1960s. It amounted to replacing the language of sets by the *language of categories* [**?**, ch. 5]. In the mid-2000s Voevodsky proposed new foundations of mathematics that he dubbed *Univalent* on the basis of (then brand-new) HoTT [**?**]. Univalent Foundations (UF) can be described as a partial realisation of the aforementioned project aiming at category-theoretic foundations where general categories (including higher categories) are limited to (higher) groupoids.

Albeit the general category theory can be formalised in UF along with other mathematical theories, general categories, unlike groupoids, do not feature in UF as elementary building blocks. As a consequence in UF categories cannot be treated as easily as groupoids[16] . This is why a replacement of the original UF with hypothetical foundations of mathematics based on DTT where general categories take the place of groupoids appears as a move in the right direction.

5.2. **Syntax of Directed Type theory.** It is far from being obvious how one can modify the syntax of MLTT in order to obtain a version of DTT. At the semantic level one aims here at the replacement of identity types and their terms of form $p : x =_A y$ (as in MLTT) by *hom*-types and terms of form $t : x \to_A y$, i.e., the type of morphisms aka maps (generally, non-invertible) between an ordered pair of terms of the same base type A [17] The idea behind this terminology is, of course, to interpret types in DTT as general categories rather than groupoids as in the standard HoTT. It is not immediately obvious that hom-types of DTT can be understood as generalised *identity* types without losing basic ideas and linguistic intuitions about identity but in what follow we are going to provide some support for this approach. It worths mentioning that this approach is philosophically controversial and is not universally shared by the workers in DTT.

The formation rule (**??**) does not apply in DTT in its original form because one needs to find means to specify at the syntactic level that map t has term x as its domain (source) and terms y as its codomain (target). Below we present a tentative solution of this problem proposed by Paige R. North in [**?**]. First, she introduces two modal operators on types

[16]Thanks to J-rule in UF any function $f : A \to B$ is automatically a functor of groupoids that interpret types A, B. But this is not the case for categories and functors, which are defined in UF "by hand" and require meticulous proofs of correctness of their definitions.

[17]The term "hom-type" derives from the term "hom-set" used in the Category theory. Hom-set $Hom_C(x, y)$ is the set of all morphisms with domain x and codomain y in a given category C. Categories where morphisms form sets are called *locally small*. The prefix "hom-" derives from term "homomorphism" which in this general context is interchangeable with "morphism".

called *polarities* (or *variances* by some other authors). So we get the following formation rules:

$$\frac{\Gamma \vdash A : TYPE}{\Gamma \vdash A^{op} : TYPE} \tag{27}$$

Given type A rule (**??**) forms new type A^{op}. The intended meaning of this operator derives from the idea that A is a category and A^{op} is the *dual* category of A, that is, a category obtained from A via the formal inversion of all its morphisms.

$$\frac{\Gamma \vdash A : TYPE}{\Gamma \vdash A^{core} : TYPE} \tag{28}$$

The intended meaning of A^{core} formed with (**??**) is the maximal groupoid, which is a subcategory of category A; in other words the *core* modality selects from category A all those morphisms which are invertible.

The following two rules connect A^{op} and A^{core} to A:

$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x^c : A^{core}}{\Gamma \vdash x : A} \tag{29}$$

$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x^c : A^{core}}{\Gamma \vdash x^{op} : A^{op}} \tag{30}$$

The formation rule for hom-types is now this:

$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x^{op} : A^{op} \quad \Gamma \vdash y : A}{\Gamma \vdash x \to_A y : TYPE} \tag{31}$$

Notice that in (**??**) two "equal" terms x^{op}, y do not feature in a symmetric way: their different roles in type $x \to_A y$ are specified with the polarisation of base tape A [18].

The corresponding introduction rule is

(32)
$$\frac{\Gamma \vdash A : TYPE \quad \Gamma \vdash x^c : A^{core}}{\Gamma \vdash refl_x : x^{op} \to_A x}$$

The reflexivity term $refl_x$ introduced with (**??**) behaves like the standard identity morphism in general Category theory [19]; the above remarks concerning the Reflexivity Principle (**??**) and its type-theoretic rendering (**??**) remain relevant also in the case of DTT. Notice also that (**??**) uses invertible term $x^c : A^{core}$ in its premises. Notwithstanding the fact that this term belongs to modalized type A^{core} it is taken here as a basic term of "null variance" while types A^{op} and the original type A are thought of types with a further structure that determines the contravariant behavior of x^{op} and the covariant behaviour of x in the dependent hom-type $x^{op} \to_A x$. So syntactically one starts here with a bare type A and then add modalities (via (**??**), (**??**), (**??**), (**??**)) but semantically one thinks of

[18] Here is a further piece of explanation of rule (**??**) that assumes some knowledge of the basic Category theory. A *hom-functor* of the form $\mathbb{C} \xrightarrow{Hom(A,\_)} \mathbb{SET}$ from a given locally small category $\mathbb{C}$ to the category of sets, where A is a fixed object of $\mathbb{C}$, is *covariant* which means that it preserves the directions of morphisms. A hom-functor of the form $\mathbb{C} \xrightarrow{Hom(\_,B)} \mathbb{SET}$, where B is also object of $\mathbb{C}$, is *contravariant* which means that it inverts the direction of all morphisms. The same property can be expressed by saying that the Hom-(bi)-functor $\mathbb{C} \xrightarrow{Hom(\_,\_)} \mathbb{SET}$ is contravariant by the first variable and covariant by the second variable (here both variables are denoted as the empty spaces _). Dependent hom-types in (**??**) model this behaviour of the hom-sets and hom-functors: judgement $x : A^{op}$ expresses the contravariance of the first variable. This remark explains why polarity operators in DTT are also called variances.

[19] It is appropriate to mention here that in the general algebra one can define various "structures without identities", i.e., without algebraic units like units in groups and monoids or identity morphisms in categories. Such are so-called *magmas* (sets with binary operations without further assumptions), semi-groups (associative magmas), semi-categories aka non-unital categories. We leave an analysis of such structures for a future study.

modalized type A^{core} as a basic type to which one adds via (**??**) a further structure that determines its variance. This feature of North's syntactic approach can be described by saying that she builds directed paths on the top of the classical invertible ones. Ideally, one wants a syntax that does not involve such a conceptual twist but supports a version of DTT that does not use specific principles of constructions of the standard "reversible" HoTT in its foundations but nevertheless has the standard HoTT as its special case.

The elimination rule (J-rule) (**??**) of Book HoTT splits in DTT into two separate rules:

Right hom-elimination:

$$\frac{\begin{array}{c}\Gamma \vdash A : TYPE \quad \Gamma, x^c : A^{core} \vdash \Theta(x^c) : TYPE \\ \Gamma, x^c : A^{core}, y : A, t : x^{op} \to_A y, \theta : \Theta(x^c) \vdash C(t, \theta) : TYPE \\ \Gamma, x^c : A^{core}, \theta : \Theta(x^c) \vdash \overrightarrow{f(x)} : C(refl_x, \theta)\end{array}}{\Gamma, x^c : A^{core}, y : A, t : x^{op} \to_A y, \theta : \Theta(x^c) \vdash J_R^t(f, t, \theta) : C(t, \theta)} \tag{33}$$

Left hom-elimination

$$\frac{\begin{array}{c}\Gamma \vdash A : TYPE \quad \Gamma, y^c : A^{core} \vdash \Theta(y^c) : TYPE \\ \Gamma, x^{op} : A^{op}, y^c : A^{core}, t : x^{op} \to_A y, \theta : \Theta(y^c) \vdash C(t, \theta) : TYPE \\ \Gamma, y^c : A^{core}, \theta : \Theta(y^c) \vdash \overleftarrow{f(y)} : C(refl_y, \theta)\end{array}}{\Gamma, y^c : A^{core}, x^{op} : A^{op}, t : x^{op} \to_A y, \theta : \Theta(y^c) \vdash J_L^t(f, t, \theta) : C(t, \theta)} \tag{34}$$

The two rules provide for two kinds of transport of structure (in particular, a property) C along a given directed path $t : x^{op} \to y$: the *forward* transport (along the direction of t) and the *backward* transport in the opposite direction (see again Fig.**??**). The transported structure C is introduced here via an intermediate structure Θ dependent on a core type (with null variance) as in the Book HoTT[20]. Rule (**??**) describes how structure C is transported from starting point x to further point y along directed path t when observed from x; rule (**??**) describes the same move when observed from y back in time. Intuitively these descriptions can be thought of, correspondingly, as prediction and retrodiction. It is essential that the above syntactic rules can be iterated and applied at all homotopical levels. In this way this syntax admits interpretation in $(\infty, 1)$-categories (albeit not general (∞, ∞)-categories) which means that it forces all morphisms of upper levels to be invertible.

To conclude this short presentation of DTT let us mention that the Axiom of Univalence (or univalence property) of the standard HoTT has a natural generalisation in DTT: while AU says (at the propositional level) that every identity path between types is (up to equivalence) a type equivalence (i.e., an invertible function) the generalised directed version of AU says that every *directed* path between types is a (general) *function.* For more details on the directed AU see [**?**].

5.3. **Are Directed Paths Identities?** The analogy between DTT and the standard HoTT suggests considering the type of directed paths (aka morphisms) of the form $x \to_A y$ (let us now ignore the technical issue of polarity) as the type of (directed) *identities* of terms x, y of the same base type A. Thus we get here an identity relation which is reflexive (because of (**??**)), transitive (since two directed paths following one another are composable) and enjoys a form of **InId** (because of the two-directional transport of structures provided by (**??**)-(**??**)) but which is <u>not</u> symmetric. In other words, we get a notion of identity "with a sense". Like in the case of Book HoTT **InId** should be understood in a sense that allow enduring objects to change their properties (as well as higher-order structures that those

[20] Referring to this feature of North's syntax for DTT Altenkirch and Neumann describe North's version of DTT "shallowly polarised" meaning that a "deep" polarisation needs to be applied not only to types but also to contexts [**?**]. Examples of *deeply* polirized versions of DTT are found in [**?**] and in the aforementioned work by Altenkirch and Neumann.

objects support). In this way DTT circumvents the traditional *paradox of change* without violating **InId** or making it trivial.

The DDT-based notion of identity just outlined provides a straightforward solution — or more precisely a *dissolution* — of the *Theseus Ship* puzzle: the coexisting descendant ships B, C share common ancestor A and are identical to this ancestor without being identical to each other:

$$\begin{tikzcd} & B \\ A \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & C \end{tikzcd}$$

This same shape (pattern) of identities through time, which represents a case of *fission* applies in the case of duplicating amoebas, splitting social groups and institutions and in many other similar cases. The diagram below represents the shape of *fusion* of two different entities into one and the same:

$$\begin{tikzcd} A \arrow[dr, "u'"] & \\ & C \\ B \arrow[ur, "v'"'] & \end{tikzcd}$$

Examples of fusion are equally abound in the nuclear physics, once again in biology (cell fusion) in the social and economical life (fusion of companies and institutions), and in many other areas.

The loop shape (other than the reflexion loop) below represents a discrete change of an entity (k-the discrete step of the change corresponds to application of $f^k \circ refl_A$).

$$
\begin{tikzcd}
A \arrow[loop right, "f"]
\end{tikzcd}
$$

For representation of *continuous* change by means of CT see [**?**].

These are only very few very basic patters of identity through change which are straightforwardly mathematically modelled with the (flat) Category theory (CT) as far as one attempts to think about general morphisms (and not only about the *identity morphisms* aka reflexion terms) as representing identities. Higher CT allows for much more.

To get a glimpse of how higher dimensions enter into the picture consider the following example.

Mary and John get married (denote the couple C) and later divorce at certain point. Later they remarry each other again (denote the newly formed couple C'). Are C and C' one and the same couple? Taking the liberal approach to identity explored in this paper we answer in positive and observe, in addition, that C develops into C' in two different ways m and j, which can be thought of as Mary's and John's personal trajectories (or stories) that both involve a divorce and a new marriage (to the same person). At that point it makes perfect sense to wonder how the two stories relate to and interact with each other, how m translates into j and vice versa. Such second-order translations α constitute a new dimension of the complicated history of the couple (see the below diagram).

$$
\begin{tikzcd}
C \arrow[r, bend left, "Mary"] \arrow[r, bend right, "John"'] \arrow[r, phantom, "\Downarrow\alpha"] & C'
\end{tikzcd}
$$

Notwithstanding the above suggestive examples a critical reader may object that taking *all* morphisms in a given category for identities is an absurd idea. Consider the category $\mathbb{SET}$ of sets and functions. Given two arbitrary sets M, N there always exists at least one function between them (at least in one direction). According to our proposal this fact implies that all sets are the same (but not all of them in a symmetric way). Isn't this a sheer absurd?

In order to reply to this objection let us first remark that in the case of Book HoTT one encounters a similar situation. Before the rise of HoTT nobody would call two points of a topological space identical simply because they are connected by a continuous path. This idea comes from MLTT where the identity types have been originally introduced quite independently from the Homotopy theory, and from the homotopic interpretation of this theory that is HoTT. DTT and the proposed interpretation of hom-types as generalised identity types pushes the same idea further forward.

As for the example of $\mathbb{SET}$ observe the familiar way of thinking about sets and their identities is easily recovered without giving up our proposal via changing the category: consider discrete category $\mathbb{D}(\mathbb{SET})$ that has the same objects as $\mathbb{SET}$ but only identity morphisms (in the standard sense of the term) as its morphisms. Notice further the *forgetful functor* $U : \mathbb{SET} \to \mathbb{D}(\mathbb{SET})$ which has non-trivial properties and sheds more light on the concept of set. When CT is used as a foundation (which is the case as far as mathematics is supposed to be formalised with DTT and its default semantics in higher categories) one does not think of categories in terms of their objects and morphisms as if those elements were somehow given in advance. Instead, one turns the tables and reasons in terms of abstract categories specifying via such reasoning what are their objects and their morphisms and thus what these categories are more exactly. As an example of a similar approach we point to the Elementary Theory of Category of Sets (ETCS) proposed by Lawvere back in 1964 [**?**]. But unlike ETCS, which relies upon an external logical framework, namely the Classical First-Order logic, the desired DTT-based foundations of mathematics can rely only on its own internal logical resources like UF.

We are not going to make here an attempt to build Set theory on the basis of DTT but suggest that thinking about all functions as identity-preserving transformations is appropriate and helpful for achieving that goal[21]. ETCS makes it manifest that the standard theory of identity that makes part of ZFC does not meet the needs of Set theory. The principle of equivalence for set-level structures supported by Book HoTT and the Univalent Foundations provide a novel way of thinking about identity that better squares with the existing mathematical practice of working with such structures. But making Category theory into a foundation of mathematics requires a deeper revision of the identity concept than the usual Structuralist reasoning achieves and suggests [**?**]. The anticipated foundations of mathematics based on DTT and Higher CT require still a deeper revision of the standard theory of identity.

A critical reader may also notice that the Book HoTT and DTT both use the concept of *reflexion term*, which interprets as trivial loop in HoTT and as identity morphism (in the standard sense of the term) in DTT, and argue that *only* reflexion terms represent "true" identities while the non-trivial paths (directed or not) represent some other relations and calling them identities is a conceptual mistake. To rebuke this objection it is sufficient to stress once again that the reflexivity of identity is basically a linguistic convention according to which the same symbol has always the same reference when it is used repeatedly. To limit the scope of identity concept to that case is not acceptable from an epistemological point of view because an identification of items called by different names often presents an important piece of knowledge as in Frege's *Venus* example. As other examples given in this paper suggest there is no reason either to limit the identity-preserving transformations to invertible transformations.

[21]The mathematical reader is advised to think of the standard von Neumann's universe of sets V as generated by the empty set $\emptyset$, so every set of this universe results from a transformation of this initial object, and in this sense remains identical to it. Thinking about $\mathbb{SET}$ in a similar manner involves thinking about every function as an identity-preserving transformation. Notice that $\mathbb{SET}$ comprises not only the initial object $\emptyset$ but also the terminal objet, namely singleton $\{*\}$

6. Instead of Conclusion: Identity through Time and the Process Ontology

Ontology is a discipline that treats general features of what there is (and of what there is not). One thing that can be done for treating such a general subject is to build a formal framework that relates basic logical concepts and principles to things and events occurring in the world around us — without wholly losing sight of our cognitive mechanisms that allow us, humans, to conceive of these things and reason about them. This is what the *formal ontology* is doing. Albeit the term "formal ontology" has been coined relatively recently the project itself dates back at least to Aristotle. It turns out that formal ontology is not only philosophically challenging but also practically useful discipline because it helps to apply logic in Computer Science and information technologies. The rise of digital technologies of Knowledge Representation (KR) and Knowledge Engineering (KE) made formal ontologies pertinent in these fields. The title of ontology in its current usage applies not only to a traditional philosophical discipline but also to a software that serves for building and implementing various KR and KE systems used in science, technology, medicine, education, management and in many other areas.

Formal ontologies are typically designed using the Classical First-Order logic and its modal extensions. The existing computer applications of constructive type theories and HoTT, on the other hand, belong to a different kind of software, namely to proof-assistances which are used for a computational representation of reasoning, primarily mathematical proofs, that allows for its automated formal verification. Albeit proof-assistants are rarely viewed as KR tools as a matter of fact they are already used for storing a mathematical knowledge in a formalised form. Since any knowledge (but not only mathematical knowledge) involves some form of reasoning about known objects one may expect that proof assistances and formal ontologies will eventually integrate, and methods of proof-verification which are presently used mostly in the pure mathematics will be applied in other areas of KR as well. To enhance such an integration we provide below a brief analysis of obvious *ontological* implications of HoTT (including DTT as a special case), which enriches its original epistemic semantic inherited from MLTT.

As any other type theory HoTT admits for a two-layer ontology that comprises *points* (terms) and *spaces* (types). HoTT comprises dependent types, i.e., types that depend on terms of base types. Types that depend on the same base type form families which are also types. Thus one gets here an ontology, which is very rich but which remains treatable and observable thanks to its topological character. The topological character of this ontology allows for thinking about its entities in spatio-temporal terms as it has been already demonstrated above. This concerns, in particular, the layer of this ontology that can be described as *propositional* or *logical.* It is a characteristic feature of this ontology that propositions qualify here as entities of a particular sort among other stipulated entities. It makes a sharp contrast between the HoTT-based ontology and standard formal ontologies where propositions are supposed to be atemporal and aspatial whether or not other related entities are supposed to exist in space and time.

The assumption about the atemporal character of logic makes it difficult for traditional ontology to account for the simple idea that things typically change over time. The core First-order order logic has no means for making sense of this idea; the notion of identity through time and change presents for this logical framework an insurmountable problem. Modal extensions of this core logic aiming at building on its principles a version of *temporal* logic, as well as similar attempts towards building a *spatial* logic, do not involve a revision of the standard identity relation and thus do not solve the identity problem. By contrast, HoTT and DTT involve spatio-temporal structures and intuitions in a much deeper level of their constitution and organisation. As a consequence HoTT-based and particularly DDT-based ontologies qualify as *process ontologies*, i.e. ontologies built from elementary processes rather than substances, atoms or eternalised events. Elementary processes are represented in DTT as morphisms aka directed paths of form $p : X \to_T Y$ where terms X, Y represent, correspondingly, the starting and the ending points of process p and T represent the underlying type where X, Y belong. The fact that each process in this ontology comes with its endpoints is essential from an ontological point of view and deserves a serious consideration that we won't go into here. Notice, however, that there is no sense in which process p can be reduced in this framework to the pair of its endpoints since it belongs to a different type.

A critical reader may make here the following Platonic objection. Notwithstanding the fact that human knowledge is obviously fallible it essentially involves the idea of resisting the course of time and change. Whether one thinks of operation $7+5=12$ in terms of equality or in terms of non-invertible function $7+5 \to 12$ as suggested above, one should assume that numbers $7,5$ do not cease to exist after its performance. Expressions $7+5=12$ and $7+5 \to 12$ equally express patterns of mathematical reasoning that are supposed to be indefinitely reproducible. This implies that any reasonable ontology of arithmetics should stipulate numbers as atemporal (and by the same pattern aspatial) entities, no matter that each singular computation takes time and some space.

A similar argument applies to non-mathematical examples. The truthness of sentence "It rains" is contingent on changing weather conditions and on one's choice of a particular location on the Earth. This is a reason why this sentence does not qualify a well-formed logical proposition that may be possibly a piece of factual knowledge and be subject to logical rules. For our common concept of knowledge implies that known propositions do not change their truth-values over time (albeit our beliefs about these truth-values may eventually change). So in order to make the above sentence into (an expression of) a well-formed proposition on needs to eternalise it in Frege style by specifying time and place like "It rains in Nancy on July 7, 2024 afternoon". The truth-value of the latter proposition is no longer apt to any change. As a consequence, one must assume that the ontological counterpart aka a *truth-maker* [**?**] of this proposition — let us call it *fact* — is an entity that similarly does not admit for any kind of change.

Thus, so the argument goes, the whole idea of process ontology is not compatible with our best epistemic practices and thus should be abandoned. The Heraclitean world of perpetual chaotic flux cannot be possibly known. One needs to assume at least the existence of some stable structures and patterns in such a world in order to develop anything deserving to be called knowledge. This is sine qua non of doing science and mathematics. Using HoTT and DTT for describing processes can be a clever and effective mathematical trick but it doesn't cancel this fundamental point. So the ontological claims motivated by these formal theories should be taken with a big pinch of salt.

Our defence of process ontology against the above Platonic wisdom does not involve challenging the assumption according to which our concept knowledge involves the idea of resisting the course of time and preserving its contents over time indefinitely. But we disagree with the next step made by our opponent when they assume that the indefinite preservation of epistemic contents requires a stipulation of some immutable entities. In order to explain what we have here in mind let us consider another mathematical example, which is only slightly more involved than the above arithmetical example.

Think of the Pythagorean theorem. Like any other piece of mathematical knowledge this theorem needs to be (and as a matter of fact, it is) *reproducible*. It is reproducible as a stable linguistic pattern (modulo allowable variations) and, more importantly as a pattern of cognitive activity performed by different persons in different places and at different times when they learn this theorem, teach it to other people or simply recall it and use for some purpose. Encoding mathematical contents into executable codes using proof-assistants greatly helps to enhance such a basic reproducibility of mathematical knowledge.

In the presence of the basic reproduction mechanism just described it makes perfect sense to think of the Pythagorean theorem as an abstract static atemporal pattern. This pattern can be described as an invariant of transformations between particular textual and cognitive representations of the Pythagorean theorem, which we shall call *tokens* for further references. Since we assume that all such tokens are interchangeable (in their role of representing a mathematical content) we think of these transformations as being *invertible*: given token X can replace any other token Y of the same type and/or *vice versa*. The invertibility condition is important because it allows one to turn the tables and develop the idea of atemporal invariant pattern by using a Frege-style abstraction. Indeed, since the invertibility of transformations between tokens is granted one can define an equivalence relation between tokens of mathematical reasoning by stipulating that tokens X, Y are equivalent (in symbols $X \sim Y$) when X is transformable into Y. The reflexivity and the transitivity of $\sim$ are granted by the very idea of transformation (one can always assume that there is a "null transformation" of any token into itself and that transformations are composable), and the symmetry of $\sim$ is granted by the invertibility condition. Then one may classify tokens into equivalence classes, pick up the equivalence class representing the pattern Pythagorean theorem and finally apply Frege abstraction to stipulate this pattern as an invariant abstract object. If the invertibility condition is lifted nothing like this reasoning goes through.

Let us now change the time scale and see that the symmetric reproduction of patterns just described is not the only way of how mathematical and scientific knowledge persists through time and develops. Compare the Pythagorean theorem as it is formulated and proved in Euclid's *Elements* (Proposition 1.47) [**?**, p. 46-47] and the Pythagorean theorem as it is formulated and proved in today's typical school textbook. Is this indeed the *same* theorem? Let us see. In Euclid the statement refers to three geometrical squares constructed on the three sides of a given rectangular triangle. The statement says that the biggest square is *equal* to the two smaller squares (taken together). What Euclid understands here by the equality of polygons (or sums of polygons) requires some explanation; it is not what the modern reader typically expects. Euclid's concept equality is specified with his Postulates. In the cases of polygons and their mereological sums it can be can be explained in modern terms as follows: two polygons A, B are *equal* just in case when they are either (i) congruent or (ii) equidecomposable (*scissor-congruent*), i.e., can be cut into a finite number of pairwise congruent smaller polygons or (iii) equicomplementable, i.e., can be complemented with congruent polygons, so that the obtained bigger polygons are also congruent [22]. The same notion of equality also applies to mereological sums of polygons in an obvious way (as in Proposition 1.47). Euclid's proof of the Pythagorean theorem is not direct in the sense

[22]It can be shown that when two polygons are equal in Euclid's sense then they are equidecomposable (and obviously vice versa). So (iii) is in a sense redundant. But it helps Euclid to streamline his geometrical proofs and organise his geometrical theory axiomatically.

that it proceeds via a number of intermediate problems and theorems proved and solved earlier in the deductive order of his *Elements* and thus comes down the the first principles of his geometrical theory.

In today's textbook one usually finds the statement of the Pythagorean theorem in form of algebraic expression $c^2 = a^2 + b^2$ where a, b, c are the *lengths* of a rectangular triangle, and where the square operation $\_^2$ applies to numbers but not to straight lines or some other geometrical objects. Notice that the relevant notion of number is that of *real* number, which has been forged in mathematics only in the late 19th century. Any reasonable modern proof of the above statement is, of course, also differs drastically from Euclid's proof of his Proposition 1.47.

In the history of mathematics there exists a line of argument that supports the view according to which the Pythagorean theorem in Euclid and the theorem which is called by the same in a modern textbook are in fact two different theorems; some historians claim, more generally, that Greek mathematics and modern mathematics are two very different kinds mathematics which should not be judged by the same epistemic criteria. Without entering into the discussion let us state without further ado that this is not a view that we defend here. We claim that there is a good sense in which the Pythagorean theorem preserves its identity from Euclid to the present, and that Euclid's Proposition 1.47 and the Pythagorean theorem found in recent textbooks is indeed the same theorem. But we also claim that the preservation of identity of this theorem through historical time does not need and does not even admit the common conception of Pythagorian theorem in form of an atemporal invariant pattern.

Indeed, the principal reason why Euclid's Proposition 1.47 qualifies as the Pythagorean theorem in the modern sense of the word is that the content of Proposition 1.47 is *translatable* into the modern textbook statement of this theorem. This is notwithstanding the fact that the first principles of Euclid's theory and involved concepts are indeed quite different from anything that is found in today's geometry textbooks. To construct such a translation carefully paying attention to details of Euclid's reasoning is a useful and not quite simple exercise, which we leave to an interested reader. It has more than one solution. Our main point here is that there is no *backward* translation: while all mathematical contents of Euclid's *Elements* and of Greek mathematics can be reasonably translated into modern mathematical terms the contents of today's mathematics cannot, generally, be translated into the mathematical language used by Euclid and his contemporaries. This implies that translations of Euclid's Proposition 1.47 into the modern mathematical language are non-invertible[23].

This show that the technique of forming equivalence classes of tokens and then using abstraction for stipulating things like "the pattern of Pythagorean theorem", which we used in the case of basic symmetric reproduction of mathematical knowledge, does not apply when we deal with tokens which are not interchangeable but nevertheless admit for one-way translations. In our example such tokens have been produced in different historical epochs but one can imagine that they can be also produced by contemporary mathematicians having different ideas about the "same" subject matters and using different foundations. Thus the symmetric reproduction of knowledge described is a kind of local stability mechanism while at larger scales knowledge reproduces itself differently allowing for irreversible conceptual changes and developments.

As the example of Pythagorean theorem demonstrates the invertibility of transformations between tokens is not a necessary condition for preserving identities of known contents. The fact that we are in a position to translate in a reasonable way Euclid's Proposition 1.47 into the Pythagorean theorem of today's textbook [24] is sufficient to qualify the two statements as the same theorem. The popular Platonic idea according to which the two statements express in different ways the same immutable pattern is irrelevant in this case.

[23]Like in the above arithmetical example it is essential for our argument that we think here of translation between different mathematical languages as a rule-based procedure that is applicable to sufficiently large fragments of mathematics but not only to singular concepts or statements.

[24]Which translation qualifies as "reasonable" in this case can be a matter of historical and mathematical debate that we leave here aside. The aforementioned radical view according to which no such translation can possibly qualify as reasonable also deserves more consideration that we may grant it in this paper.

The conceptual change involved in the transformation of Euclid's theorem into its modern version has no proper invariant, and we do not need to apply such an invariant for claiming that in both cases we deal with the same mathematical content. We can make this claim because we understand how the content of Euclid's theory translates into the content of its modern version without assuming that the backward translation is equally possible. Similarly, the claim that today's Andrei is the same person as a boy featuring at some old photographs does not require assuming that Andrei has an immortal soul with respect to which all turns of his life or even his birth and death are nothing but unimportant accidents. Perhaps we should leave behind the linguistic habit of talking about the *preservation* of identity in such cases, which suggest the idea of some invariant essence saved from the change. The talk of *constitution* of identity through change is more appropriate.

Let us stress once again that the proposed way of thinking about identity is very liberal in a sense but it is not trivialising; it does not allow one to qualify any geometrical theorem as Pythagorian. Which mathematical statement qualifies as the Pythagorean theorem and which does not depends on how exactly the valid translations between different versions of this theorem are specified. Such specifications can be called the *identity conditions* of this changing mathematical entity.

The non-symmetric mechanism of reproduction of knowledge via non-invertible translations of epistemic contents is evidently more general and more important at larger historical scales than the local symmetric mechanism described earlier. By pointing to this fact we refute the Platonic argument according to which the stipulation of atemporal immutable entities is necessary for developing science and knowing the world we live in. Science and knowledge does not need this assumption. Thus the process ontology does not make the world unknowable. DTT is a promising formalism for supporting such ontology. It goes without saying that the conceptual change in logic and mathematics should not be confused with processes that take place in the physical, biological or social worlds, which can be *accounted for* using logic and mathematics. Our point is that logical and mathematical theories that involve the concept of change at the basic levels of their construction as does

DTT are more apt for accounting for physical, biological, social and other processes than the traditional logical and mathematical theories, which use the Platonic idea of atemporal invariant structure resisting all sorts of change.

What social scientists tell us about changing ethnic, national and cultural identities, what biologists tell about the identity of organisms, biological species and ecosystems, what geologists tells about the identity of continents, etc., — all such talks hardly admit a fruitful analysis in terms of the standard theory of identity. From a logician's viewpoint these talks are so hopelessly inaccurate that a logician has here a choice between (i) ignoring such talks altogether assuming that scientists conventionally call by the name of identity something completely different and (ii) developing a counter-intuitive and computationally ineffective ontological reconstruction of these talks, which no working scientist can buy (as in the case of standard 4-dimensional ontologies). The notion of directed identity in DTT outlined in this paper appears to be sufficiently flexible and rich to be able to account for many identity talks in science in a more interesting and more charitable manner. We leave a study of such possible applications of DTT for a future research.

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