A Philosophical Introduction to Hidden Symmetries in Physics

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1 Introduction

The discovery of symmetries propels physics forward. The Galilean invariance of classical mechanics eventually led to the concept of a unified space-time; the Lorentz symmetries of electromagnetism spurred the development of special relativity; and the $SU(3) \times SU(2) \times U(1)$ gauge symmetry of the standard model gives rise to three of the four known fundamental interactions. Moreover, symmetries entail the conservation of certain quantities via Noether's theorem. Some symmetries are apparent from the geometry of a problem. Others, however, are hidden from view. A well-known example is the Kepler problem. Whereas the rotational symmetry is evident from the geometry of this system, which is invariant under rotations in three-dimensional space, the system also displays another, *hidden* symmetry.

Despite the fact that such hidden symmetries are frequently discussed in the physics literature, they are all but absent from philosophical discussions of symmetries. This is unfortunate, because we believe that hidden symmetries are highly relevant to topics such as invariant quantities, spacetime structure and conservation laws. The aim of this paper is to introduce philosophers to hidden symmetries and some of their philosophical consequences.

On the one hand, hidden symmetries may seem distinct from other symmetries merely because they are more difficult to discover. If this were the case, philosophical treatments of symmetries should easily extend to hidden ones. We will show that this is not the case. On the other hand, hidden symmetries may have *no* philosophical significance: they are merely accidental artefacts of the mathematics. But this view belies the fact that hidden symmetries are routinely used in the practice of physics, for example to derive planetary orbits or the energy levels of the hydrogen atom. We will chart a middle view on which hidden symmetries are a distinct yet philosophically significant type of symmetry.

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The plan is as follows. In §2, we discuss the definition of hidden symmetries, which is not settled in the physics literature. §3 presents four detailed examples of hidden symmetries drawn from a range of theories. In §4, we discuss hidden symmetries' philosophical significance with a consideration of invariant quantities, spacetime structure, and the nature of lawhood. §5 concludes with an outlook.

2 Definitions of Hidden Symmetries

The concept of hidden symmetries comes from over a century of physics practice.[1] It refers to certain physical systems in which 'non-obvious' symmetries appear. An example of such a system is the Kepler problem, which consists of two point masses in a gravitational field. It is easy to identify two symmetries of this system, namely its translational and rotational symmetry. These two symmetries correspond to two constants of motion: energy and angular momentum.

It turns out, however, that there exists an additional constant of motion in the Kepler problem, known as the Runge-Lenz vector, described in detail in section §3.1. With the development of the formalism of Lie algebras and the language of group theory, it became clear that this quantity corresponds to a 'hidden' symmetry. For negative energies, this is an $SO(4)$ symmetry, i.e. a rotation in four dimensions. In contrast to the rotational symmetry described earlier, this new symmetry does not seem to correspond to any physically possible process. While it is possible to rotate an orbit in three-dimensional space, it is unclear how to convey this in four dimensions. For this reason, physicists frequently refer to $SO(4)$ as a 'hidden symmetry' of the Kepler problem.

Physicists tend to refer to hidden symmetries in rather imprecise terms:

> . . . a 'hidden symmetry' of the problem, in that its existence is not immediately apparent from an inspection of the geometric symmetries of the force field. (Prince and Eliezer 1979, after Cisneros and McIntosh 1970)

It remains unclear what exactly is meant here by "not immediately apparent" or the "geometric symmetries" of the force field. However, this quote does illustrate the widespread tendency to think of hidden symmetries as, in some sense, non-geometric. Care is needed here: it is not the case that an $SO(4)$ rotation does not admit of any geometrical interpretation, since such a rotation is perfectly plausible in four-dimensional space. But it is hard to see how an $SO(4)$ symmetry relates to the three-dimensional Euclidean geometry on which the Kepler problem is set.

Frequently, this non-geometrical aspect of hidden symmetries is understood in terms of the Hamiltonian. In such an approach, symmetries are considered as properties of the Hamiltonian rather than of the "geometry of the system". From this perspective, hidden symmetries are those that are not "immediately clearly" the symmetries of the Hamiltonian (Györgyi and Révai 1965, p. 967). Indeed, it is easy to see from the r^2-factor that the Hamiltonian of the two-body problem has a rotational symmetry $SO(3)$, but it is not "immediately clear" that this Hamiltonian also has an $SO(4)$ symmetry. This seems,

[1] This and the next section are based on Bielińska (2022).

however, a mere matter of practice; someone unfamiliar with the Hamiltonian formalism would not be able to identify the rotational symmetry either.

However, there also exist several formal definitions of hidden symmetries in the physics literature. These are formulated in different mathematical frameworks, from Lie groups to Killing vectors. It remains unclear whether these definitions are equivalent. In the remainder of this section we present the most important ones.

Hidden symmetries are most often described from the perspective of Lie groups and symmetries. The formal definition of hidden symmetries in this framework reads as follows:

> A hidden symmetry is a Lie point symmetry which appears in the target differential equation after a change of order using a nonlocal transformation and which does not have a point counterpart in the source equation. (Leach *et al.* 2012, p. 2)

This definition requires a description of the system in terms of a set of differential equations. It has been advanced mainly by a group of physicsts and mathematicians working on such equations, including Barbara Abraham-Shrauner (1993, 1994; with Guo: 1992, 1993; with Leach: 1993). In the case of the Kepler system, this would mean that the hidden symmetries are found from the equations of motions, since they are differential equations. But it is harder to apply this definition to some other examples of hidden symmetries, such as the Carter constant discussed below.

An alternative definition in terms of Killing tensors is also possible. Tensors are defined over differentiable manifolds, and for this reason they are used mainly in relativistic theories; but they are consistent with classical mechanics, given that $\mathbb{R}^n$ is also a differentiable manifold. Killing tensors are generalisations of Killing vectors. Recall that a Killing vector K_μ is a vector field that preserves the metric; by definition, it obeys the Killing equation:

$$K_{(\mu;\nu)} = 0, \tag{1}$$

where the round parentheses on the indices refer to the symmetric part. A second rank Killing *tensor* $K_{\mu\nu}$ is defined as

$$K_{\mu\nu} = K_{(\mu\nu)}, \qquad K_{(\mu\nu;\lambda)} = 0. \tag{2}$$

It is a well-known fact that Killing vectors, being the infinitesimal generators of isometries, generate symmetries and, via Noether's theorem, constants of motion. But it is less well-known that Killing tensors *also* generate symmetries. As Crampin (1979) argues, such symmetries are hidden. For example, it is possible to obtain the aforementioned Runge-Lenz vector for a single particle moving under an inverse-square central force using Killing tensors, as well as the hidden symmetry of the harmonic oscillator discussed below. Thus, hidden symmetries can alternatively be defined as symmetries that are obtained from Killing tensors, but not from Killing vectors.

As far as we are aware, there is no proof that these definitions identify the same symmetries as hidden. The most extensive discussion of these two approaches is found in Cariglia (2014), who defines hidden symmetries as "transformations in the whole phase space of the system such that the dynamics is left invariant" (p. 2) but that are not "lifted from a set of simpler transformations defined on a configuration space" (p. 4).[2] In any case, it is unclear whether this difference is philosophically relevant. Instead, we observe that physicists much more often refer to hidden symmetries in the slightly imprecise fashion recalled earlier: as symmetries that are somehow less obvious, and not apparent from the geometry of the system or from the Hamiltonian. In what follows we will mainly refer to this informal characterisation of hidden symmetries, amplified by a number of key examples presented in the next section. These examples cover the broad spectrum of theories with hidden symmetries, from classical mechanics to quantum mechanics and special relativity. Some of them, such as the Runge-Lenz symmetry of the Kepler problem, are relatively well-known; others, such as the Fradkin tensor in the n-dimensional harmonic oscillator or the Cartan constant, are entirely new to the philosophical literature. We omit discussion of more 'exotic' hidden symmetries found in supersymmetric theories (cf. Cariglia 2014).

Before we present these examples, let us briefly mention what hidden symmetries are *not*. Firstly, they are not *symmetries of a state*. For example, the Schwarzschild solution to GR is rotationally symmetric, but this is a symmetry of a particular state rather than of the equations of motions. The Kepler system, to the contrary, is not highly symmetric in this sense, yet it has a hidden symmetry. Secondly, hidden symmetries are not *spontaneously broken symmetries*. The latter refer to cases in which a particular solution does not share the symmetries of the dynamics, such as when a magnet becomes polarised in a specific direction. But hidden symmetries pertain to the dynamics themselves. Finally, they are not the same as *internal symmetries*. The latter are simply non-spatiotemporal symmetries, but they may be hidden or not. (That said, it may seem that *local gauge symmetries* are hidden in the sense that they are non-obvious from the Hamiltonian. We will not further discuss this suggestion here).

3 Examples of Hidden Symmetries

3.1 The Kepler problem

The Kepler problem is the most extensively discussed example of a hidden symmetry. It is used in physics to derive the trajectories of a particle in a central potential. It is also the only example of a hidden symmetry which we are aware of that is mentioned in the philosophical literature. It is discussed by Belot (2013) as a counterexample to the principle that symmetry-related states are always equivalent; Wallace (2019) aims to defuse the counterexample by a consideration of subsystem symmetries (see also Luc (2022), fn. 32). We will discuss these in §4. Because of its central importance, we consider this example in more detail than the others.

As mentioned above, the Kepler problem has two 'obvious' constants of motion: energy and angular momentum. In July 1845, however, William Rowan Hamilton, in a paper delivered to the Royal Irish Academy (published in 1847), derived another constant of motion which he called the *eccentricity vector* (for the explicit form of this vector see §4.1). It found many applications in the problem of elliptical orbits, such as the derivation of planetary trajectories.

[2]Notice the contrast between hidden symmetries, then, and symmetries as defined in Dewar (2020), Jacobs (2021a), Gomes (2022).

Interestingly, the contemporary name of this vector was only introduced in 1926 by Wolfgang Pauli, who used it in his pioneering derivation of the spectrum of the hydrogen atom without the Schrödinger equation (for more details see §3.2). Pauli took this treatment from Heinrich Lenz (1924), who credits the German mathematician Carl Runge. Runge himself makes no claim of originality, however, and it is possible that one of his inspirations was *Vector Analysis* (1901) by Josiah Willard Gibbs and Edwin Bidwell Wilson, in which they derive the orbits in the two-body problem using a conserved vector. Another scientist who independently discovered an additional constant of motion in the Kepler problem was Pierre Simon de Laplace, in *Traté de mécanique celeste* published in 1798. Thus, somewhat ironically, the Runge-Lenz vector owes its name to two scientists neither of whom claimed its discovery.

We now turn to the Kepler problem itself. Recall that a Kepler system consists of two point masses m_1 and m_2 interacting in a gravitational field through a gravitational force $\mathbf{F}_g$. The Lagrangian of this system is given by

$$L = \frac{m_1|\dot{\mathbf{r}}_1|^2}{2} + \frac{m_2|\dot{\mathbf{r}}_2|^2}{2} - V(|\mathbf{r}_1 - \mathbf{r}_2|), \tag{3}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vector positions of the two particles and $V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$ is the gravitational potential.

This system has several invariants and, as such, can be simplified by reducing the Lagrangian. After moving to Jacobi coordinates and reducing the translational symmetry, we obtain a reduced Lagrangian of the one-body problem. In the Hamiltonian formalism it reads as follows:

$$H_{red} = \frac{p^2}{2\mu} + V(|\boldsymbol{r}|), \tag{4}$$

where $p := |\boldsymbol{p}|$ is momentum, $\boldsymbol{r}$, is the *separation vector* $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2$, and μ is a *reduced mass*:

$$\mu := \frac{m_1 m_2}{m_1 + m_2}. \tag{5}$$

In the Kepler problem one is concerned with a central potential that is inversely proportional to the distance $\boldsymbol{r}$ between the two bodies. Therefore,

$$V(r) = \frac{k}{r}, \tag{6}$$

where k is a constant that indicates whether the force is repulsive ($k > 0$) or attractive ($k < 0$), and $r := |\boldsymbol{r}|$. In the case of a gravitational potential, which is the subject of this section, the value of k is always negative.

The symmetry groups of the Kepler problem can be obtained by calculating the Poisson brackets of certain conserved quantities, which can be later compared to the Lie brackets defining certain symmetry groups. A closed system of Poisson brackets for this system involves a square root of the Hamiltonian in the denominator, however, so we need to consider separately three intervals of the energy: negative, positive, and zero.

On the phase space which corresponds to the negative energy states, the span of vectors in Poisson brackets correspond is isomorphic to a 6-dimensional algebra that is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, which itself is isomorphic to $\mathfrak{so}(4)$. Therefore, the Kepler problems for negative energies has an $SO(4)$ symmetry group, which consists of two independent rotations. See Lévy-Leblond (1971) for an explicit expression for the associated coordinate transformations.

In a similar way one can calculate Poisson brackets for $E > 0$ and $E = 0$. For positive energy values, the hidden symmetry is (rather surprisingly!) a group of four-dimensional non-compact rotations isomorphic to $SO(3, 1)$, which is familiar as the Lorentz symmetry group of Minkowski spacetime. For zero energies, it is an Euclidean $E(3)$ symmetry.

3.2 The Hydrogen Atom

The hidden symmetry of the hydrogen atom was first discussed by Wolfgang Pauli in 1926. He was familiar with the classical Kepler problem, in which one can derive the trajectories of the particle by means of the Runge-Lenz vector $\boldsymbol{A}$. Idealising the hydrogen atom as a two-body problem in which an electron revolves around a proton, Pauli used the same method to correctly derive the hydrogen atom's energy levels. Notably, Pauli derived his result *before* the same energy levels were obtained from the Schrödinger equation.

A model of the hydrogen atom consists of a single electron of mass m_e in the field of a nucleus with a single proton of mass m_p, which are taken to be point particles. This is another example of a quantum two-body problem, in which we are concerned with the electromagnetic force $\boldsymbol{F}_e$. We can neglect their gravitational interaction as it is significantly weaker than the electromagnetic force. Because $m_p \gg m_e$, the centre of the mass of the system is approximately identical to the proton location. Therefore we assume that the electron revolves around the proton.

As in the classical two-body problem, we can reduce the hydrogen atom to a single point mass moving in an external central potential. The reduced mass of this system is:

$$\mu = \frac{m_e m_p}{m_e + m_p}. \tag{7}$$

The Hamiltonian of the system is then identical to one of a point mass μ in an external central electric potential. In particular, we can make the same change of coordinates to obtain the centre of mass reduction. The Hamiltonian of the non-relativistic hydrogen atom in these new coordinates is

$$H = -\frac{1}{2\mu}\nabla^2\psi + V(|\boldsymbol{r}|). \tag{8}$$

where $\boldsymbol{r}$ now denotes the distance between the proton and the electron. Furthermore, the potential of this system is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}, \tag{9}$$

where ϵ_0 is the *permittivity of space*, and e is the charge of the proton; the electron has the same charge, but with opposite sign. Therefore, the potential (9) always has negative values.

The Lie algebra for the negative energy subspace of the hydrogen atom is $\mathfrak{so}(4)$. Just as in the case of negative-energy solutions to the Kepler problem, then, the hydrogen atom possesses a hidden $SO(4)$ symmetry on top of the expected $SO(3)$ symmetry. The latter is a consequence of the rotational invariance of the system in three-dimensional space, but the former does not seem to have a readily apparent physical interpretation; it represents a *four*-dimensional rotation of the system. Nevertheless, it is the $SO(4)$ symmetry that has allowed physicists to derive the degeneracy of the energy levels of the hydrogen atom with respect to the quantum numbers l and m. We will discuss in §4 how one can account for the occurrence of a four-dimensional rotational symmetry here. The reader may rightly suspect that there exists a conserved quantity associated with this symmetry; we introduce it in §4.1.

3.3 N-dimensional Harmonic Oscillator

Soon after Pauli demonstrated that the hydrogen atom has a hidden $SO(4)$ symmetry, Josef-Maria Jauch and Edward Lee Hill (1940) determined that such hidden symmetries are not unique to inverse square potentials, but exist for all central potentials. In particular, they noted that there exists a hidden symmetry for the isotropic multidimensional harmonic oscillator. This result was further elaborated by David Fradkin in 1966, and hence the associated constant of motion is known as the Fradkin tensor.

An N-dimensional quantum harmonic oscillator is defined by the Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{P_i^2}{2m} + \frac{m\omega^2}{2} X_i^2 \right), \tag{10}$$

where X_i is a position operator, P_i is a momentum operator, m is a mass and ω is the angular frequency of oscillations. From this Hamiltonian, one can see that the N-dimensional harmonic oscillator is mathematically equivalent to a system of N independent one-dimensional harmonic oscillators with the same mass and frequency of oscillations. If the i's are the labels of these point masses, then the X_i denote their positions and P_i their momenta (where $1 \leq i \leq n$). These operators satisfy the familiar commutation relations.

Symmetry groups of this Hamiltonian can be, again, calculated from the Lie brackets of certain conserved quantities of this system. These brackets correspond to the Lie algebra of $\mathfrak{su}(3)$, which in turn correspond to the special unitary group $SU(3)$. One can further show that the Hamiltonian is invariant under the symmetry group $U(3)$, which is a unitary group. Let's define ladder operators:

$$a_i = \frac{1}{\sqrt{2m\omega}}(m\omega X_i + iP_i), \tag{11}$$

$$a_i^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega X_i - iP_i). \tag{12}$$

It is easy to see that these operators are self-adjoint: $a_i^* = a_i^\dagger$. Suppose that U_{ij} is a unitary matrix. We can then define two new operators:

$$b_i^\dagger := U_{ij} a_j^\dagger, \qquad b_i := U_{ij}^\dagger a_j. \tag{13}$$

It then follows that for all $U_{ij} \in U(3)$:

$$b_i^\dagger b_i = U_{ij} a_j^\dagger U_{ik}^\dagger a_k = U_{ij} U_{ik}^\dagger a_j^\dagger a_k = \delta_{jk} a_j^\dagger a_k = a_i^\dagger a_i. \tag{14}$$

Therefore, the Hamiltonian is also invariant under the symmetry group $U(3)$, which is also called the *degeneracy group* of the system. This observation extends to the N-dimensional harmonic oscillator, which possesses the symmetry group $U(N)$. Therefore, $U(N)$ is a hidden symmetry of the N-dimensional harmonic oscillator.

3.4 Kerr Black Holes

In 1968, Brandon Carter found an additional constant of motion in the Kerr solution to the Einstein Field Equations, which after him was named the *Carter constant*. This discovery enabled Carter to derive the equations for the trajectory of a test particle in Kerr spacetime. His derivation was based on the observation that the geodesic equation of this system is separable in specific coordinates. Roger Penrose and Martin Walker believed that one should be able to derive such a significant result without appeal to particular coordinates, and in 1970 they obtained the same solution by means of Killing tensors, which correspond to symmetries of relativistic theories.

The Kerr solution represents the field of a rotating body of mass M in an asymptotically flat spacetime. For simplicity, we will assume that the body has no charge; a similar calculation for a charged mass is presented by Carter (1968). The Kerr metric g in so-called Kerr-Newman coordinates has the form:

$$g_{\mu\nu} = g_{tt} dt^2 + g_{t\varphi} dt d\varphi - 2a \sin^2 \theta dr d\varphi + 2 dr dt + \rho^2 d\theta^2 + g_{\varphi\varphi} d\varphi^2. \tag{15}$$

where the metric components are given by:

$$g_{tt} := -1 + \frac{Mr}{\rho^2}, \tag{16}$$

$$g_{t\varphi} := -\frac{Mra \sin^2 \theta}{\rho^2}, \tag{17}$$

$$g_{\varphi\varphi} := \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta, \tag{18}$$

with

$$\rho^2 := r^2 + a^2 \cos^2 \theta, \qquad \Delta := r^2 - 2Mr + a^2. \tag{19}$$

Because the Carter constant is not as well-known as other hidden constants, we will present it in slightly more mathematical detail than the previous examples. Consider a test particle of mass m in a Kerr spacetime. The Lagrangian of this particle is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{20}$$

where an overdot denotes differentiation with respect to some affine parameter λ. Imposing the normalising condition

$$m^2 = -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{21}$$

and setting $m^2 = \pm 1$ we obtain, respectively, timelike and spacelike geodesics. The four-momenta are given by

$$p_\mu = g_{\mu\nu}\dot{x}^\nu, \tag{22}$$

and we can write down the Hamiltonian in terms of them:

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu. \tag{23}$$

Because the Hamiltonian does not depend on the parameter λ, the particle's rest mass is a constant of motion:

$$m = \sqrt{-g^{\mu\nu}p_\mu p_\nu}. \tag{24}$$

The next two quantities conserved in this system are also straightforward and can be read off directly from the Kerr metric. Firstly, it is *axially symmetric*, which means that it remains the same when rotated around some axis. The associated Killing vector field is $\frac{\partial}{\partial\varphi}$. Secondly, it is *stationary*, which means that it admits a timelike Killing vector field $\frac{\partial}{\partial t}$. The associated constants of motion are usually denoted as:

$$L_z := p_\varphi, \qquad\qquad E := -p_t, \tag{25}$$

referred to as the orbital angular momentum parallel respectively the energy. The third constant of motion can be read off from the normalisation equation (21).

However, just like the classical Kepler problem, the hydrogen atom, and the multidimensional harmonic oscillator, it turns out that the Kerr solution of the Einstein Field Equations possesses some hidden symmetry. This symmetry doesn't have any specific name. Discovering it enabled Carter to obtain the trajectory equations for a test particle (1968).

4 Philosophical Significance of Hidden Symmetries

Having presented the physics behind hidden symmetries, we now turn to their philosophical significance. The aim of the remainder of the paper is two-fold: first, to show that hidden symmetries are neither just hard-to-find symmetries, nor merely accidental; and second, to illustrate how hidden symmetries problematise certain popular positions on symmetries in the philosophical literature.

As to the first point, we believe that the case is clear that hidden symmetries are not merely accidental symmetries of no particular interest. Firstly, hidden symmetries are formally similar to other symmetries, so at this level there is no difference. Moreover, because of this formal similarity they also yield conserved charges via Noether's theorem, such as the Runge-Lenz vector. These constants are not spurious: the Runge-Lenz vector, for example, is used by physicists to classify different orbits. Finally, hidden symmetries may play a role in scientific explanations. The most famous example of this is Pauli's derivation of the spectrum of the hydrogen atom on the basis of a hidden $SO(4)$ symmetry, which is explained in more detail below. We will therefore proceed from the assumption that hidden symmetries are not merely mathematical artefacts.

Given that hidden symmetries are not insignificant, we now ask the opposite: is there anything that distinguishes them from other symmetries *apart* from the fact that they are 'hidden'? We believe that this is indeed the case. In particular, various philosophical approaches to symmetries are not applicable to hidden symmetries. For example, we show below that the Invariance Principle—which, roughly, states that only symmetry-invariant quantities are physically real—does not apply to quantities that vary under hidden symmetries. This means that hidden symmetries are importantly different from the symmetries that philosophers typically discuss. In addition to the Invariance Principle, we discuss two other claims often made by philosophers: that a theory's dynamical symmetries should match its spacetime symmetries, and that a theory's symmetries can explain their conservation laws. In each case, hidden symmetries problematise common wisdom and are thereby shown to differ from 'ordinary' symmetries.

4.1 Hidden Symmetries and Invariant Quantities

The main role of symmetries, from a philosopher's perspective, is that they act as 'a guide to superfluous theoretical structure' (Ismael and Van Fraassen 2003). Firstly, it is often said that quantities that *vary* under a symmetry are not physically real. The reason is that such quantities are, in virtue of their variance, in principle undetectable; absolute velocity is a well-known example.[3] This is called the *Invariance Principle*, and is espoused by Saunders (2003), Baker (2010), Caulton (2015), Dasgupta (2016). The Invariance Principle justifies a so-called *symmetry-to-reality inference*: quantity Q is variant, therefore unreal. The term 'symmetry-to-reality inference', due to Dasgupta, is a misnomer: it is rather a *variance-to-unreality inference*, so we will use the latter expression in what follows.[4] Secondly, it is often said that models of a theory related by a symmetry merely represent the same state of affairs. For example, applying a Lorentz transformation to a solution of special relativity results in another model that represents the very same physical state. This principle is called *Leibniz Equivalence*, and is espoused by Saunders (2003), Greaves and Wallace (2014) and Weatherall (2018). For a discussion of the relation between these principles, see Jacobs (2021b).

We should emphasise that neither principle is uncontroversial. We will in fact show that those principles are inapplicable to hidden symmetries. Those already sceptical of the Invariance Principle and Leibniz Equivalence may find that the difference between hidden symmetries and ordinary symmetries here is relatively minor. Nevertheless, hidden symmetries provide a particularly stark case of the failure of those principles.

To show this, let us consider some examples of conserved quantities entailed by the hidden symmetries introduced earlier. Let's start with the Kepler problem. There are two constants of motion that can be immediately seen from the Hamiltonian (4), namely energy and angular momentum. The former corresponds to time translation and the latter is related to rotation in three dimensions, denoted by a symmetry group $SO(3)$. However, as we have mentioned earlier, there is another conserved quantity, known as the Runge-Lenz vector:

$$\boldsymbol{A} = \frac{\boldsymbol{p} \times \boldsymbol{L}}{\mu} - \frac{k\boldsymbol{r}}{r} \tag{26}$$

[3]For discussion on this point, see Middleton and Murgueitio Ramírez (2021), Jacobs (2022) and Luc (2023).

[4]We thank an anonymous reviewer for this suggestion.

with components $\boldsymbol{A} = (A_x, A_y, A_z)$. There is no straightforward way to see that the Runge-Lenz vector is a constant of motion from the Hamiltonian (4). Therefore, one must guess its explicit form and then verify that it is indeed conserved, i.e. demonstrate that the total time derivative vanishes. The Runge-Lenz vector is used to define eccentricity of the orbit $\boldsymbol{e}$ as follows:

$$\boldsymbol{e} := \frac{\boldsymbol{A}}{k}. \tag{27}$$

Both the Invariance Principle and Leibniz Equivalence seem to misfire when applied to the Kepler problem, since one of the quantities that varies under the hidden symmetry in question is the eccentricity of the planet's orbit. The eccentricity is constant over time for any solution to the Kepler problem, but it has a different value for solutions related by the symmetry that generates it; compare this to linear momentum, which is also a constant of motion but varies under boost transformations. But eccentricity is an *observable* quantity, so the conclusion that it is unreal is contradicted by the available evidence. The Invariance Principle delivers an incorrect verdict in this case. The same is true for Leibniz Equivalence: if symmetry-related models represent the same state, then models that differ over the eccentricity and orientation of the planet's orbit would represent identical states. This would also mean that eccentricity is not physically real, contrary to the evidence of our senses.

(The alert reader may have noticed a discrepancy, namely that eccentricity is a three-vector whereas the hidden $SO(4)$ symmetry is four-dimensional. Shouldn't we therefore really look for a higher-dimensional invariant? We postpone discussion of such an approach to the next sub-section.)

Indeed, Belot (2013) uses the example of the Kepler problem to disprove Leibniz Equivalence ('D2' in his paper). But Belot does not mention the connection to hidden symmetries, and his other counterexamples are drawn from elsewhere in physics. In the remainder of this section, we wish to analyse this issue as it pertains to hidden symmetries in particular. This is an important problem, because hidden symmetries share many important features of symmetries that *are* involved in successful variance-to-unreality inferences: they are systematic transformations that leave the laws the same; they are associated to conserved quantities via Noether's theorem; and they feature in physical explanations. What is needed to save variance-to-reality inferences is a criterion that determines which symmetries it applies to. The correct criterion should exclude hidden symmetries. We will show that current philosophical definitions of symmetries are not sufficiently discriminatory in this respect.

It is helpful here to consider Dasgupta's tripartite classification of definitions of symmetries into *formal*, *ontic*, and *epistemic*. These are not intended to recover the way the term 'symmetries' is used by physicists, but rather as characterisations of a particular class of symmetries that support variance-to-unreality inferences. The plan for the remainder of this section is to discuss these definitions one-by-one in order to show tot none of them can easily explain why hidden symmetries don't support variance-to-unreality inferences.

Formal Definition

On a formal account, symmetries are defined in terms of a theory's formalism shorn of interpretation. For example, Wallace (2022) defines symmetries as bijections of phase space that commute with the dynamics; Belot (2013) discusses a number of definitions, from variational symmetries to classical and generalised symmetries.

The hidden symmetries we consider count as symmetries on most of these definitions, as Belot and Wallace acknowledge in the case of the Kepler problem. It would seem, then, that formal definitions conflict with variance-to-unreality inferences, since they would lead to the conclusion that measurable quantities such as eccentricity are not real.

This may seem a reason to reject a formal account of symmetries, but Wallace (2022) has recently defended a formal account from similar counterexamples. Wallace labels symmetries such as the Runge-Lenz symmetry *subsystem-specific*: they apply only to a particular subsystem, but as soon as one couples this subsystem to an environment the symmetry disappears. The Runge-Lenz symmetry, for instance, only applies to the two-body problem; as soon as one introduces an external environment the overall system has no hidden symmetry anymore. But in order to measure the eccentricity of an orbit, one has to introduce an external state for the measurement device. So, in the context of measurement eccentricity is not symmetry-variant, and a variance-to-unreality inference is inapplicable.

While we believe that Wallace's analysis is formally correct, we think it does not explain *why* quantities that vary under hidden symmetries are not subject to variance-to-unreality inferences. For suppose that the universe were to consist solely of a Kepler system. Would quantities such as eccentricity then become unreal? This seems implausible to us, because eccentricity is an *intrinsic* property of a planet's orbit; it does not depend on the existence of any external systems. Granted: if one were to introduce a measurement device, one would destroy the hidden symmetry for the total system. Wallace could simply restrict variance-to-unreality inferences to symmetries to those symmetries that apply jointly to the subsystem and its environment. But we are unpersuaded by this type of modal response. If a planet's eccentricity is real, then surely it is not so because of any putative features the planet has with respect to a non-actual environment. Put differently, the mere *possibility* of an external system cannot explain the planet's actual features. Indeed, this applies even if there *is* an external system: in order to find out the real features of the planet's orbit, one should not have to appeal to its environment.[5] It should therefore not be necessary to justify the non-applicability of variance-to-unreality inferences to eccentricity by an appeal to the environment.[6]

To show that external systems are a distraction, we present the example of the hydrogen atom. The conserved quantity that corresponds to the hidden symmetry of this system

[5] We therefore don't require what Wallace calls the 'Cosmological Assumption', that models of a theory represent the state of the entire universe.

[6] Indeed, eccentricity is a *dimensionless* quantity: it does not depend on an arbitrary choice of length unit. This means that the two-body system can *itself* act as a ruler. If the minor axis is set as a 'unit' of distance, then one can measure the major axis (and hence eccentricity) in terms of it (cf. Gryb and Sloan 2021).

is the *quantum Runge-Lenz vector*:

$$\boldsymbol{M} := \frac{\boldsymbol{P} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{P}}{2\mu} - \frac{e^2 \boldsymbol{r}}{|\boldsymbol{r}|}, \tag{28}$$

where $\boldsymbol{M} = (M_x, M_y, M_z)$ and $\boldsymbol{L} = (L_x, L_y, L_z)$ is the orbital angular momentum operator, defined as: $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}$. The quantum Runge-Lenz vector plays a very similar role to its classical counterpart in the Kepler problem, determining atomic orbitals. More precisely, along with the orbital angular momentum operator, it can be used to define ladder operators that act on the azimuthal quantum number l which specifies the shape of the orbital and on the magnetic quantum number m which specifies the orientation of the orbital.

If we were to apply the Invariance Principle to this case, it would imply that the shape and orientation of the orbital are unreal. In this case, too, a Wallace-style response would emphasise that the hidden symmetries is broken when the atom is coupled to an environment, such as an observer who measures the orbital.

However, the quantum Runge-Lenz vector has a further significance, which makes a Wallace-style response implausible. Quantum numbers not only specify the shape and orientation of an orbital, they also define the fine structure of the hydrogen atom. This structure can be derived with the aforementioned ladder operators defined in terms of $\boldsymbol{M}$:

$$M_{\pm} = M_x \pm i M_y, \tag{29}$$

and their action on quantum numbers is as follows:

$$M_{\pm}|nll\rangle = -\frac{1}{n}\sqrt{\frac{2(l+1)}{(2l+3)}\,(n^2 - (l+1)^2)}|n(l+1)(\pm l \pm 1)\rangle. \tag{30}$$

Moreover, the second quantum number can be lowered by analogous ladder operators defined as $L_{\pm} = L_x \pm i L_y$, which act on the quantum numbers as follows:

$$L_{\pm}|nlm\rangle = \sqrt{l(l+1) - m(m \pm 1)}|nl(m \pm 1)\rangle. \tag{31}$$

Therefore M_+ is a *raising* operator, since it increases the value of the quantum number l, giving rise to hydrogen's fine structure. Hence, for a given energy level labelled by the quantum number n, one can use M_+ to obtain all possible values of the l states, and then use L_- to explore their degeneracy by finding all m states. An example for the energy level $n = 3$ is presented in Figure 1.

Now, reconsider Wallace's appeal to environment-states applied to the hidden symmetry of the hydrogen atom. The natural proposal of Wallace's is that the fine structure of the atom is only real in virtue of the fact that the hidden symmetry is broken when the hydrogen is coupled to an observer. This seems highly implausible, as the very existence of the fine structure is an intrinsic feature of the hydrogen atom: it would still exist even if some hydrogen atom were the only object in spacetime or if one were to consider the atom in isolation. Therefore, the failure of the Invariance Principle on a formal definition is not adequately explained by an appeal to subsystem symmetries.

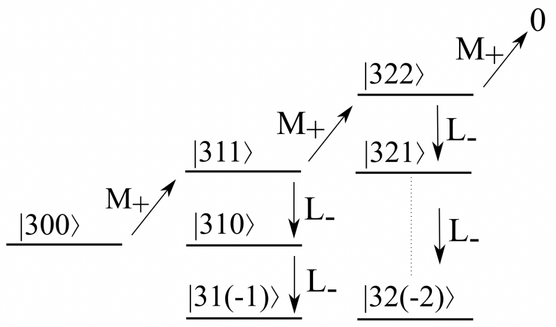

Figure 1: **The degeneracy of energy levels in the hydrogen atom for $n = 3$.** The ladder operator M_+ associated with the quantum Runge-Lenz operator raises by the value of one both quantum number l corresponding to the operator L^2 and quantum number m linked to the operator L_z. Then, the degeneracy of l is obtained using the lowering operator L_-.

In fact, a similar problem occurs even for the Kepler problem. Recall that for this system the eccentricity varies under a hidden symmetry. But in fact there are three different groups of hidden symmetries: which one the system exhibits depends on its energy value. Consequently, the shape of the orbit varies with the type of hidden symmetry it has, too. In polar coordinates, the trajectory equation is describes conic sections:

$$r(\phi) = \frac{r_0}{e \cos \phi + 1}. \tag{32}$$

In particular, if $e = 0$ then the motion is along a circle; for $0 < e < 1$ it is an ellipse; for $e = 1$ a parabola; and a hyperbola for $e > 1$. We see that non-negative energies correspond to hyperbolae or parabolae, and negative energies to ellipses.

Again, the shape of an orbit is clearly a feature that a system possesses even considered in isolation from the environment. While an ellipse and a circle are topologically equivalent, the very formulation of the dynamics of the Kepler problem requires a metric structure that distinguishes them (we return to the importance of metrical structure in the next sub-section). Moreover, the difference between open and closed curves is definable even topologically. Insofar as the shape of the planet's orbit varies under hidden symmetries, then, its reality should not depend on the way the system couples to an environment.

Therefore, hidden symmetries remain a problem for formal accounts of symmetries insofar as the Invariance Principle and Leibniz Equivalence do not apply to them. It is possible that some other formal feature sets hidden symmetries apart from other symmetries: we will discuss one possibility, based on higher-dimensional symmetries, in the next subsection. In the remainder of this subsection we discuss whether ontic and epistemic definitions can account better for hidden symmetries.

Ontic Definition

On an *ontic* account, one first specifies a privileged class of physical quantities, and then stipulates that a transformation that preserves the laws is a symmetry only if it also preserves those quantities. The main question for ontic accounts is: what are the privileged quantities?

Dasgupta (2016) suggests that they include "physical features like distance, mass, charge, spin, and so on". If distance is a quantity to be preserved by symmetry transformations, then the Runge-Lenz transformation is *not* a symmetry. This may seem to neatly solve our problem: while hidden symmetries are symmetries in the formal sense, they act on an irrelevant set of quantities, thereby failing to sustain variance-to-unreality inferences.

The problem with this response is that it falls prey to what Dasgupta calls an 'inferential circularity'. Why include distance as a physical quantity? If distances *are* physical, then the Runge-Lenz transformation is not a symmetry. But if the Runge-Lenz transformation *is* a symmetry, then distance is *not* physical. One person's modus ponens is another's modus tollens. Indeed, while the circularity is pointed out by Dasgupta himself, we believe that hidden symmetries provide the first concrete example of such circularity that we are aware of. What we would need is an antecedent reason to consider a quantity as physically real, such that the set of real quantities includes distance. As far as we are aware, no such account exists.

It may seem as if we have missed an obvious response: distance is physical because it is observable. This implies an epistemic definition of symmetry, to which we now turn.

Epistemic Definition

On an *epistemic* account, symmetries are defined as certain transformations that connect *empirically equivalent* states. Definitions of this type are proposed by Ismael and Van Fraassen (2003), Dasgupta (2016) and Read and Møller-Nielsen (2020).

The account of Ismael and Van Fraassen is particularly helpful. On their account, there is a standard of *qualitative identity* of states which is external to the theory in question: it is antecedently-known whether states are qualitatively distinct, that is, whether they are "distinguishable by even a gross discrimination of colour, texture, smell, and so on" (p. 376). The transformations that sustain variance-to-unreality inferences are those that *both* preserve the laws *and* preserve qualitative features. The quantities that vary under these transformations are not physically real—they are surplus structure. For example, Galilean boosts are symmetries that relate qualitatively identical states. Hence absolute velocity, which vary under boosts, is not real.

This definition of symmetries would exclude hidden symmetries if the latter were to alter the value of some qualitative quantity. In the first instance it seems that this is indeed the case: whether an orbit is circular or elliptical is a qualitative matter. However, the issue is more subtle. For on Ismael and Van Fraassen's account, distance is not qualitative. After all, it does not directly affect our sense of colour, smell and so on. The distinction between non-qualitative and qualitative features here is broadly parallel to Locke's distinction between primary and secondary qualities. For Locke, extension and shape are primary qualities, whereas colour and smell are secondary. Since hidden symmetries seem to leave such qualitative/secondary features invariant, they are in fact not excluded by Ismael and Van Fraassen's definition. In response, one may wonder whether the *shape* of an orbit isn't qualitative. If that were the case, then there is a qualitative difference between solutions with different eccentricity. But there is an important difference between the instantaneous shape of an object, and the shape traced out by the trajectory of an object over time. While the former is, plausibly, a qualitative feature, the latter is not. In the case of the Kepler problem, it is the shape traced out by the planet's orbit that varies, not the instantaneous shape of the two-body system. Therefore, the hidden symmetry of the Kepler problem is an empirical symmetry on Ismael and Van Fraassen's definition.

Ismael and Van Fraassen consider not only actual qualitative differences, but also *potential* differences. Consider a transformation that takes an empty bucket at rest to one that rotates. This transformation maps one physically possible state to another; it also preserves the state's qualitative features, since the empty bucket will look the same whether it rotates or not. Nevertheless, Ismael and Van Fraassen argue that this transformation is not a symmetry, because if one *were* to add water to the bucket, the (instantaneous) shape of the surface would make a qualitative difference. The same response could account for the Runge-Lenz symmetry: if one were to introduce an external measurement device, the transformation would make a qualitative difference.

We are not convinced by this response. Firstly, it is reminiscent of Wallace's subsystem response in that both appeal to the state of an environment external to the system in question in order to draw a difference between different states of it. We have already objected to such responses. Secondly, it introduces a significant *modal* element to the notion of symmetries. But the dissimilarity between orbits with different eccentricities is surely real not because of their modal profile, but because they are *actually* distinct. In this respect, the Kepler problem is different from the bucket, because the bucket does look the same whether it rotates or not.[7] Thirdly, once one considers potential qualitative differences the distinction between qualitative and measurable quantities becomes moot. If the measurable quantities are defined as those that *could* have an effect on the qualitative way the world looks, then it becomes circular to include such effects in one's definition of symmetries.

Of course, Ismael and Van Fraassen's is not the only empirical account of symmetries. Contrary to the former, Dasgupta (2016) explicitly classifies "appearing from my perspective to be two feet away" as observable. Read and Møller-Nielsen (2020) too eschew the qualitative-measurable distinction. Their proposal is to use various heuristics to find out which quantities are detectable, in a broad sense, and define symmetries as those transformations that preserve the detectable quantities. However, and as Read and Møller-Nielsen readily admit, this makes variance-to-unreality inferences redundant, since one already has to know which quantities are measurable in order to determine which transformations are symmetries.

There is thus no definition of symmetries that plays well with hidden symmetries. They either include hidden symmetries, which leads to erroneous variance-to-unreality inferences, or they beg the question by assuming from the outset that the quantities that vary under hidden symmetries are real. This shows that hidden symmetries are not just ordinary symmetries that are harder to find: their philosophical consequences are relevantly different from ordinary symmetries. Therefore, hidden symmetries pose a challenge to definitions of symmetries, and perhaps a helpful test case for future proposals.

4.2 Hidden Symmetries and (Spacetime) Structure

The initial motivation for principles such as Leibniz Equivalence and the Invariance Principle came from *spacetime* symmetries. The Galilean symmetry of Newtonian mechanics,

[7] Although, as Huggett (2006) notes, this ignores the deformation due to rotation.

for instance, entail that spatial translations represent a 'distinction without a difference', and hence that absolute position does not exist. Are hidden symmetries different simply because they are not spacetime symmetries?

We have already seen in §2 that physicists often define hidden symmetries as those that are not related to the geometry of the problem. If variance-to-unreality inference applies only to spacetime symmetries, there is no issue with hidden symmetries. But this is too quick. On the one hand, variance-to-unreality principles have also been applied successfully to non-spacetime symmetries, such as the local (gauge) symmetries of electrodynamics. On the other hand, hidden symmetries sometimes act on spacetime variables. The Runge-Lenz symmetry, for instance, alters the shape and orientation of a planet's orbit. The hidden conserved quantity in the Kerr solution, the Carter constant, is also related to spatiotemporal variables.

Nevertheless, the relation between hidden symmetries and spacetime structure deserves further discussion. It is helpful to introduce John Earman's (1989) famous symmetry principles:

SP1 Every dynamical symmetry is a spacetime symmetry

SP2 Every spacetime symmetry is a dynamical symmetry

Here, as a first approximation, a *dynamical symmetry* is a symmetry in the sense discussed so far; roughly, a transformation that preserves the satisfaction of the laws (but see below for an important caveat). On the other hand, a *spacetime symmetry* is an automorphism of the theory's spacetime structure. Formally: if $\langle M, A_1, ..., A_n \rangle$ is a spacetime, where M is a differentiable manifold and the A_i are geometric objects defined over M, then a diffeomorphism d of M is a spacetime symmetry iff $A_i = d_* A_i$ for all i, where d_* is the pushforward map of d.

The point of SP1 is to ensure that spacetime does not have *too much* structure. If a transformation is a dynamical symmetry, then it makes no difference to the laws. If it is nevertheless *not* a spacetime symmetry, then the theory's spacetime structure enables one to draw distinctions that are dynamically redundant. Consider a standard of absolute velocity: such a structure plays no role in the dynamics of Newtonian mechanics. Therefore, SP1 advocates its removal. The point of SP2, meanwhile, is to ensure that spacetime does not have *too little* structure. If a spacetime symmetry is not a dynamical symmetry, then certain transformations make a dynamical difference despite the fact that spacetime structure does not enable one to discern states related by them. These cases are rarer, so we will not discuss SP2 further here.

How are Earman's symmetry principles applied to hidden symmetries? For ease of discussion we will focus on the Kepler problem for negative energy and discuss other cases later on. To start with, it seems that the $SO(4)$ symmetry of the negative-energy Kepler problem seems to violate SP1: it is a dynamical symmetry but not a spacetime symmetry. The latter is clear from the fact that the action of this symmetry on the planet's orbit is discernible by the structure of Euclidean three-dimensional space. From a more abstract point of view, we can also point out that $SO(4)$ is not a subgroup of the Galilean symmetry group. It follows from SP1 that we would have to remove spacetime structure from Newtonian mechanics.

In particular, we would have to remove any structure that enables one to discern orbits with different eccentricities. This would result in a rather impoverished spacetime. It would have no spatial metric, since one cannot measure the length of the orbit's major axis. Neither would it have any structure that determines angles, since eccentricity is definable in terms of the angles of the minor and major axis. This rules out even shape space, a formulation of Newtonian mechanics in which no appeal to distance is made. But shape space is the weakest space for Newtonian mechanics. It is impossible to set Newtonian mechanics on a spacetime with neither distance nor shape; how would one even formulate Newton's laws? Therefore, a crude application of SP1 to hidden symmetries is clearly a dead end.

The reason this approach fails is that we have applied Earman's symmetry principles more liberally than intended. Earman's definition of a dynamical symmetry is stricter than just a transformation that preserves the dynamics. The dynamical symmetries to which SP1 and SP2 apply are transformations of a theory's models obtained as the 'lift' of a spacetime diffeomorphism. Formally: if $\langle M, A_i, O_i \rangle$ is a model of the theory, where the O_i are geometric objects that represent dynamical fields, then a diffeomorphism d is a dynamical symmetry iff $\langle M, A_i, O_i \rangle$ is a solution whenever $\langle M, A_i, d_* O_i \rangle$ is a solution for all of the theory's models. The Runge-Lenz transformations are not definable in this way; indeed, Cariglia (2014) took this as a defining feature of hidden symmetries. It is true that for any solution to the Kepler problem, one can define a diffeomorphism that takes it into another solution with an arbitrary orientation and eccentricity. But one cannot define a diffeomorphism that takes *any* solution to the Kepler problem into another solution with a different eccentricity. For example, a scale transformation fixed along the major axis of an orbit maps that solution onto another one with a different eccentricity; but if one translates the original orbit off the axis, the same transformation does not preserve solutionhood. Therefore, SP1 is inapplicable to the Runge-Lenz symmetries. We should note, however, that it is unclear whether the same is the case for the other examples of hidden symmetries we have presented; this remains a question for further research.

Despite the dead end, there is more to the connection between hidden symmetries and spacetime structure. On the one hand, the $SO(4)$ symmetry of the Kepler problem represents a rotation in *four*-dimensional space, so it was always awkward to try and recover this symmetry from a three-dimensional space. On the other hand, space *is* three-dimensional—so where does $SO(4)$ even come from? It turns out, rather unexpectedly, that one can reconceptualise the Kepler problem as a dynamical system set on the three-dimensional surface of a *four*-dimensional hypersphere. If one does so, the hidden symmetry $SO(4)$ becomes manifest: it is simply a consequence of the invariance of the hypersphere under rotations in four dimensions. In the remainder of this section, we will consider the implications of this fact.

Here is a brief account of this reformulation. Firstly, the n-sphere S^n is defined as the set

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \text{ such that } |x_1^2 + ... + x_{n+1}^2| = 1\}. \tag{33}$$

This is simply the generalisation of the definition of a three-dimensional sphere S^2. Next, we define the *stereographic projection* of the hypersphere S^3 from a hyperplane

$\{(x_1, x_2, x_3, x_0) \in \mathbb{R}^4\}$ as the map $\phi : \mathbb{R}^3 \to S^3$ such that

$$\phi(x_1, x_2, x_3, x_0) = \frac{2\mathbf{x} + (|\mathbf{x}|^2 - 1)\vec{n}}{|\mathbf{x}|^2 - 1}, \tag{34}$$

where $\vec{n} = (0, 0, 0, 1)$. The physical interpretation of this map is that ϕ maps a point $p = (\mathbf{x}, 0)$ of the hyperplane onto the point on S^3 intersected by the line from p to the north pole. Consider now a bound state $\mathbf{r}(t)$ of the Kepler problem. The momentum function $\mathbf{p} : \mathbb{R} \to \mathbb{R}^3$ is defined by $\mathbf{p}(t) = m\dot{\mathbf{r}}(t)$. The energy, E, is equal to $E = \frac{\mathbf{p}^2}{2m} - \frac{k}{r(t)}$. Define $p_E = \sqrt{-2mE}$. The space of momentum states of the system is isomorphic to $\mathbb{R}^3$. It is a remarkable fact, discovered by Hamilton, that the momentum vector of a planet with an elliptical orbit moves in a *circle* in momentum space. What is even more remarkable is that this circle is mapped, by a stereographic projection $u : \mathbb{R}^3 \to S^3$ defined from ϕ, onto a circular path on S^3:

$$u(\mathbf{p}, 0) = \phi\left(\frac{\mathbf{p}}{2p_E}, 0\right) = \frac{2p_E\mathbf{p} + (p^2 - p_E^2)\vec{n}}{p^2 + p_E^2}. \tag{35}$$

This is equivalent to the path of a *free* particle on S^3's surface! Therefore, one can reconceive the Kepler system as a free particle that moves uniformly across great circles of a four-dimensional hypersphere.

Moreover, the same reformulation is available for the hydrogen atom, as was discovered by Fock (1935). In brief, Fock showed that the Schrödinger equation in $\mathbb{R}^3$ is mathematically equivalent to the Laplace equation on S^3; and the space of square integrable functions on S^3 is a sum of irreducible representations of $SO(4)$.[8] In this way, Fock could explain the degeneracy of the energy levels of the hydrogen atom by means of hidden symmetries. However, it is unknown whether similar higher-dimensional formulations exist for our other examples of hidden symmetries.

More than a mere curiosity, this reformulation opens up another way to apply Earman's symmetry principles to the Kepler problem. The $SO(4)$ symmetries of the Kepler system are obtained by the lift of diffeomorphisms of S^3, so they fit Earman's definition of dynamical symmetries. Moreover, $SO(4)$ is also the automorphism group of S^3. Therefore, there is a match between the dynamical symmetries and the 'spacetime' symmetries. Of course, it is debatable whether S^3 is equal to spacetime here, but this is not too worrisome: several philosophers have extended SP1 and SP2 to non-spacetime symmetries (Hetzroni 2019, Dewar 2020, Jacobs 2021a). These generalisations state that a theory's dynamical symmetries are the same as its *kinematical* symmetries, which are defined as the automorphisms of those structures—spatiotemporal or not—posited by the theory. If S^3 is part of Newtonian mechanics' kinematical structure, then the Runge-Lenz symmetry satisfies SP1 and SP2 after all.

But *is* the four-dimensional space hidden in the Kepler problem one of the theory's kinematical structures? We can understand this question in two ways: is it part of the theory's *fundamental* structures, or is it part of the theory's *derivative* structures? Let's take these in turn. If the four-dimensional space is fundamental, it is either posited *instead of* the spacetime structure of Newtonian mechanics, or *in addition to*. Consider

[8] For more details, see Weinberg (2011).

the former option first. If we take the motion of our two-body system to take place on the surface of a four-dimensional hypersphere instead of a three-dimensional Euclidean space, then it would make sense to think of this four-dimensional space as the actual space within which events unfold. The three-dimensional world we see around us is then a mere 'shadow' of this space. This is akin to the idea that $6N$-dimensional phase space, rather than Euclidean 3D-space, is the fundamental arena of classical mechanics. This is a radical revision of our metaphysics. We see two problems with it. The first concerns the point that hidden symmetries only apply to particular subsystems. Although the two-body problem is conceivable as motion on S^3, general solutions of Newtonian mechanics are not. Therefore, this route would foreclose the possibility to model more complex systems accurately. The second problem is that S^3 does not seem to have enough structure to maintain empirical adequacy. Recall that different great circles on S^3 correspond to orbits with different eccentricities in ordinary spacetime. We started out with the assumption that these orbits are empirically distinct, because eccentricity is a measurable quantity. Because S^3 is $SO(4)$-invariant, however, these trajectories on S^3 are symmetry-related. By a variance-to-unreality inference, they represent the same state. This means that orbits with different eccentricities are physically equivalent, contrary to our initial assumption.

These problems do not beset the latter option, on which S^3 structure exists fundamentally in addition to spacetime structure. This is similar to a position that takes phase space to exist side-by-side with Euclidean space as theoretically equivalent descriptions. On this view, any solution to Newtonian mechanics is modelled within a classical spacetime. But those solutions—or parts of solutions—that have an $SO(4)$ symmetry additionally possess a formulation on S^3. This means that for any approximately isolated Kepler system within Euclidean space, E^3, there is a four-dimensional hypersphere on which it also lives. The E^3 and S^3 trajectories are, as it were, two sides of the same coin. Since the trajectories on S^3 are now projected down onto Euclidean space, they are discernible by the shape of their projected orbit. It is unclear whether Earman's principles are satisfied in this case, however; they are if one considers the trajectories as described on S^3, but not if one considers the same trajectories as described on E^3.

The previous picture mostly carries over if one takes the S^3-structure as derivative on ordinary space (to continue the analogy, if one takes phase space as less fundamental than Euclidean space). Recall that the planet's trajectories on S^3 are definable from their ordinary spacetime trajectories by a change of variables. Therefore, there is a sense in which these structures are 'already there'.[9] This option is more parsimonious than the previous ones, since it postulates less fundamental structure overall. Moreover, it explains a sense in which the Runge-Lenz symmetry is hidden: it concerns a metaphysically derivative structure, rather than the theory's fundamental spacetime. These considerations makes the final option the most desirable.

Let us take stock. We have seen that in the Kepler problem, hidden symmetries not only entail additional conserved quantities, but also reveal a hidden structure. This structure has an explicit $SO(4)$ symmetry to match the dynamical symmetries of the Kepler problem. The same is the case for the hidden symmetry of the hydrogen atom.

[9] For more on (implicit) definability as a criterion for ontological commitment, see Barrett (2015) and Jacobs (2022).

But it is not clear whether this carries over to our other examples. What, for example, is the hidden space in which the symmetry that ensures the conservation of the Carter constant becomes apparent? Nevertheless, there is a valuable heuristic here. Ismael and Van Fraassen construe the main role of symmetries in philosophy negatively: they are signs of redundant structure. The hidden symmetries discussed above, however, play a more positive role: they are guides toward novel theoretical structures. The fact that the Kepler system possesses a dynamical hidden $SO(4)$ symmetry is a hint that there is a hidden structure for that system with the same symmetries. In this way, hidden symmetries once more are shown to differ importantly other symmetries, in particular spacetime symmetries. For while the latter are connected to *redundant* structure, the former rather lead to the discovery of *novel* structure.[10]

4.3 Hidden Symmetries and Conservation Laws

In this section, we turn to our final consideration: the relation between hidden symmetries and *laws of nature*. In particular, we are interested in whether standard accounts of symmetries as laws carry over to hidden symmetries. We will see that here, too, the standard lessons from the philosophical literature on symmetries don't apply straightforwardly to hidden symmetries.

The relation between hidden symmetries and laws of nature is particularly relevant because hidden symmetries, like ordinary symmetries, entail conservation laws. It is sometimes claimed that symmetries also *explain* their associated conservation laws. This requires a particular account of symmetries. We will survey two such accounts: Lange's (2007, 2009) account of symmetries as meta-laws, and Humean accounts of symmetries. We will argue that neither is fully compatible with hidden symmetries.

On Lange's account, symmetries are *meta-laws*: higher-order laws that constrain other laws rather than first-order facts. Just as the regularities in nature are either accidental or lawlike, so are regularities in the *laws*. If, for example, the Galilean invariance of the laws is a meta-law, then this means that the regularity that all laws are Galilean invariant is lawlike rather than accidental. We can also put this in modal terms. If the Galilean symmetry of the laws is itself lawlike, then even if the laws were different, they would have the same symmetry; Galilean symmetries are 'nomically stable'. The details of Lange's position are more involved, but for our purposes this informal sketch suffices.

Lange further claims that symmetry laws explain conservation principles because the former, being meta-laws, are modally more robust: "The symmetry principle has greater modal force than the conservation law and so can explain it, but the conservation law lacks the symmetry principle's modal force and so cannot explain it" (Lange 2009, p. 114).

It is this claim that we wish to call into question. Ordinary symmetries are nomically stable: for example, any Lagrangian that does not explicitly depend on the time coordinate is time translation invariant. But hidden symmetries are highly dependent on the form

[10] Snell's law, which is Lorentz-invariant, is perhaps another example. One may take the invariance to indicate a hidden kinematical structure that is manifestly Lorentz-invariant, as argued by Janssen (2009). But it took science nearly a millennium from the initial discovery of Snell's law by Ibn Sahl in Baghdad to realise this! We thank Michel Janssen for suggesting this example.

of the Lagrangian. In the Kepler problem, for instance, the Runge-Lenz vector is only conserved when the particle moves in a central potential. For any other potential, there exists no hidden symmetry. Likewise, the constant of motion that corresponds to the hidden symmetry in the Kerr solution to general relativity, known as the *Carter constant* and defined as follows:

$$C := p_\theta^2 + \cos^2\theta \left(a^2(m^2 - E^2) + \left(\frac{L_z}{\sin\theta} \right)^2 \right), \tag{36}$$

is conserved in the Kerr solution, but not in every other solution to the Einstein Field Equations. In fact, historically the first discovery of this constant by Carter was based on an observation that the Hamilton-Jacobi equation is solvable by a separation of variables. Martin Walker and Roger Penrose [67] felt underwhelmed by the fact that Carter's method is based on such a "peculiar feature" of the class of Kerr solutions in a particular coordinate system. In 1970, they showed that the separability follows from the fact that there exists an irreducible second-rank Killing tensor $K^{\mu\nu}$ in terms of which one can define the constant which corresponds with the hidden symmetry of the Kerr black hole. Walker's and Penrose's discovery does not change the fact, however, that the hidden symmetry associated with the Carter constant depends on the form of the Lagrangian, namely it vanishes for even small modifications of the dynamics of the system.

Therefore, hidden symmetries are often highly unstable, for they may fail even if the laws are held the same. This rules out any explanation of hidden conserved quantities on the basis of their associated hidden symmetries' nomic stability. Nevertheless, these conserved quantities are far from spurious, as discussed at the start of §4. So, hidden symmetries pose a counterexample to Lange's account of symmetries as meta-laws.

In response, an advocate of Lange's account could claim that the conservation of the Runge-Lenz vector or the Carter constant is not a *law*, but a mere *fact*. But Lange's account was only intended to explain conservation laws, so it is untouched by our counterexamples.[11] Perhaps conservation facts are explained by hidden symmetry facts, but in a way that does not appeal to nomic stability; or conservation facts and hidden symmetry facts have a common explanation in the system's dynamics (i.e. the Hamiltonian); or they have no explanation at all. We find the first possibility implausible in light of the formal similarities between hidden and non-hidden conserved quantities. In both cases, one can derive the existence of conserved charges from the symmetries of the Lagrangian by Noether's theorem. Why would the explanation of conservation facts by symmetry facts proceed differently from that of conservation laws by symmetry laws? The second case falters on the same point: both hidden and non-hidden-symmetries are entailed by the relevant system's Hamiltonian, so why would the explanatory arrow run in different directions? Finally, to say that conservation facts have no explanation at all is simply an admission of defeat rather than a response to our puzzle. Of course, the advocate of Lange's account could provide some reason to ignore the formal similarities in favour of a different explanatory structure for hidden symmetries than for ordinary symmetries. Lange's explanation of conservation laws based on nomic stability would then only apply to the former. Even if we were to restrict Lange's account in this way, however, it would illustrate our broader point: hidden symmetries require a different treatment from the

[11] We thank a reviewer for pointing out the availability of this response.

non-hidden symmetries that philosophers ordinarily discuss. Whether such a treatment should replace Lange's in order to cover any conserved quantity, or whether it should merely supplement it is just a further question.[12]

Next, consider Humean accounts of laws. Humeanism espouses 'Humean Supervenience', the view that "all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another." (Lewis 1986, ix) The laws, for a Humean, are the axioms and theorems of the system that best systematises these matters of fact. This means that candidates for the elements of the best system must supervene on local states of affairs. For example, the proposition "copper is an electricity conductor" is a good candidate for an element of the best system because, to our best knowledge, whenever any piece of copper in the physical world is connected to an electrical current, it conducts electricity. Similarly, the proposition "the Kepler problem has an SO(3) symmetry" is a good candidate for an element of the best system because different two-body systems that are related to each other by a three-dimensional rotation have the same dynamics.

However, this looks very different for hidden symmetries such as those of the Kepler problem. Consider the proposition "the Kepler problem has an $SO(4)$ symmetry (when $E < 0$)". From a purely mathematical perspective, there is no difference between the $SO(3)$ and $SO(4)$ symmetry in this system. But there are no pairs of two-body systems in the actual world related to each other by a *four*-dimensional rotation (unless one reifies S^4 as discussed in §4.2), so it is hard to see how this proposition offers a decent systematisation of the Humean mosaic.

In light of this observation, a Humean can either claim that no symmetries (neither hidden nor 'non-hidden') are candidates laws, or claim that only 'non-hidden' symmetries are candidate laws, whereas hidden symmetries are not. The first option faces the problem that many Humeans want to admit at least ordinary symmetries as candidate laws (for recent discussions, see Townsen-Hicks (2019) and Friend (2023)). After all, symmetries are powerful generalisations that feature in physical explanations. If they are not laws, then whence their explanatory power?

The second option, meanwhile, is to claim an in principle difference between ordinary and hidden symmetries, such that only the former are candidate laws. The Humean view of laws offers a basis for this distinction: as we discussed above, ordinary symmetries are based on generalisations over different systems in the actual world, whereas hidden symmetries don't possess such a natural supervenience basis. However, while this option seems more natural for the Humean, it runs counter to the practice of physicists, who consider hidden symmetries as genuine symmetries and use them in physical explanations. As Emery (2022) argues, "when we are doing scientifically-informed metaphysics we ought to pay attention to the principles of scientific practice". The burden lies with the Humean to explain why the symmetries that account for the conservation of angular momentum are laws, whereas those that account for the conservation of the Runge-Lenz vector are not.

The existence of hidden symmetries therefore poses a fairly general problem for any ac-

[12] Another option is that hidden conserved quantities require no explanation at all. In that case, Lange's account does not even require supplementation. But such a position is unsatisfactory, given the law-like relation between hidden symmetries and conserved quantities.

count of laws. On the one hand, laws are usually accorded explanatory powers that accidental generalisations are not. To the extend that hidden symmetries are explanatory, they should count as laws. On the other hand, hidden symmetries are very different from other laws in that they only pertain to particular solutions; they seem to lack the required generality. Thus, one either has to give up the claim that hidden symmetries are explanatory; or the claim that a symmetry principle is explanatory only if it is a law; or the claim that laws are general. Neither option is entirely satisfactory; it remains to be seen which of them is preferable.

5 Outlook

We hope to have illustrated the philosophical interest of hidden symmetries in this paper. As we have seen, they are not merely accidental symmetries with no philosophical import. If that were the case, they would not entail the existence of physically significant conserved quantities or play a role in scientific explanations. Neither are they just symmetries that are harder to find, but otherwise exactly the same. For as we have seen, symmetry principles that are widespread in the philosophical literature often don't apply to hidden symmetries. In this way hidden symmetries present 'problem cases' for various popular views in the philosophical literature: variance-to-unreality inferences, Earman's symmetry principles and both Humean and non-Humean accounts of symmetries as law-like. In some cases, we have suggested potential solutions; in other cases, we pose these cases as open challenges to philosophers working on symmetries.

We should also note some limitations to our work. Firstly, a precise technical definition of hidden symmetries remains an open question, not answered by the physics literature. Even without such a definition, however, hidden symmetries are widely discussed in the practice of physics, so philosophers should not await one before they enter these discussions. Secondly, because of this it is not always clear whether our considerations apply unrestrictedly to every example of hidden symmetries. Finally, it would be worthwhile to discover further examples of hidden symmetries, both those that already exist in the physics literature and those that are still unknown.

Nevertheless, it is already clear that hidden symmetries are neither ordinary symmetries that are less obvious, nor purely accidental symmetries with no philosophical relevance. For this reason, we would like to encourage philosophers to carry out further research into this so far neglected branch of physics.

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