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Some Paradoxes of Infinity Revisited

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Abstract. In this article, some classical paradoxes of infinity such as Galileo's paradox, Hilbert's paradox of the Grand Hotel, Thomson's lamp paradox, and the rectangle paradox of Torricelli are considered. In addition, three paradoxes regarding divergent series and a new paradox dealing with multiplication of elements of an infinite set are also described. It is shown that the surprising counting system of an Amazonian tribe, Pirahã, working with only three numerals (one, two, many) can help us to change our perception of these paradoxes. A recently introduced methodology allowing one to work with finite, infinite, and infinitesimal numbers in a unique computational framework not only theoretically but also numerically is briefly described. This methodology is actively used nowadays in numerous applications in pure and applied mathematics and computer science as well as in teaching. It is shown in the article that this methodology also allows one to consider the paradoxes listed above in a new constructive light.

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1. Introduction

We use finite numbers every day and rarely think about *the nature* of the infinite using it mechanically in our math classes. However, infinity and infinitesimals are among the most fundamental notions in mathematics (and not only). They have attracted the attention of the most brilliant thinkers throughout the whole history of humanity. Arabic, Indian, and Babylonian mathematicians worked hard on these problems. Aristotle, Archimedes, Euclid, Eudoxus, Parmenides, Plato, Pythagoras, and Zeno dealt with these problems in antiquity. In the years 1500–1900, important contributions were made by such eminent researchers as Bolzano, Briggs, Cantor, Cauchy, Dedekind, Descartes, Dirichlet, Euler, Hermite, Leibniz, Lindemann, Liouville, Napier, Newton, Mercator, Peano, Stevin, Wallis, and Weierstrass. In the twentieth

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century, new exciting results have been obtained by Brouwer, Cohen, Frege, Gödel, Hilbert, Robinson, Scott, and Solovay.

Introduction of the ideas of the number line, positional number systems, negative numbers, zero, rational and irrational numbers, limits, cardinal and ordinal numbers, continuum hypothesis, problems of consistency and completeness, and non-standard analysis are among the major milestones of these impressive research efforts. Research on these topics continues to be very active nowadays, as well (see, e.g., [1,7,24,25,32,35,36,39–41,49,62] and references given therein).

However, it is well known that the ideas of infinities and infinitesimals lead to numerous paradoxes. Is it true that they are inevitable? Is it possible to propose a viewpoint allowing us to avoid some of them? In this paper, we try to answer these questions using counting systems of two tribes, Pirahã and Mundurukú, living in Amazonia nowadays (see [23,47]) together with a recent methodology working with finite, infinite, and infinitesimal numbers in a unique computational framework not only theoretically but also numerically on a patented supercomputer called the Infinity Computer (see a comprehensive technical survey [58], a brief survey in Italian [56], and a popular book [53] for its description).

2. Paradoxes of Infinity

Let us consider several classical paradoxes coming from different situations involving infinity. In many of them, the set, $\mathbb{N}$, of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\} \tag{1}$$

is involved. We informally define it as the set of numbers used to count objects. Notice that nowadays not only positive integers are taken as elements of $\mathbb{N}$, but also zero is frequently included in $\mathbb{N}$. However, since historically zero has been invented significantly later with respect to positive integers used for counting objects, zero is not included in $\mathbb{N}$ in this article.

2.1. Galileo's Paradox

In his book "Discourses and mathematical demonstrations relating to two new sciences" published in 1638, Galileo Galilei considered the set $\mathbb{N}$ together with the set, that we call I^2, of square natural numbers

$$I^2 = \{x : x \in \mathbb{N}, i \in \mathbb{N}, x = i^2\} = \{1, 4, 9, 16, 25, \dots\}. \tag{2}$$

He then established the following bijection among the sets I^2 and the set of natural numbers, $\mathbb{N}$, as follows:

$$\begin{array}{ccccccc} 1, & 2^2, & 3^2, & 4^2, & 5^2, & 6^2, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1, & 2, & 3, & 4 & 5, & 6, & \dots \end{array} \tag{3}$$

This bijection is paradoxical, since there are much more numbers than squares and still to any number there can be found the corresponding square and vice versa. Clearly, the same paradoxical result arises from considering a simpler bijection between $\mathbb{N}$ and the set $\mathbb{E}$ of even numbers being a proper subset of $\mathbb{N}$

$$\begin{array}{ccccccc} 2, & 4, & 6, & 8, & 10, & 12, & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1, & 2, & 3, & 4 & 5, & 6, & \dots \end{array} \tag{4}$$

From the modern point of view, these bijections mean that all the sets involved are countable. However, the perplexity noticed by Galileo remains, because in our every day life dealing with finite objects and sets, a part of a set is always less than the whole set. In his *Elements*, Euclid has expressed this property as Common Notion no. 5 'The whole is greater than the part', where Common Notions are evident assertions that are accepted without any proof.

Let us introduce now a new paradox that can be considered as a kind of inversion of (4) where we have started from $\mathbb{E}$ and established the bijection with the set $\mathbb{N}$. In the new paradox, that hereinafter will be called *set-multiplication paradox* we start from $\mathbb{N}$ and arrive to $\mathbb{E}$.

2.2. Set-Multiplication Paradox

Let us consider a finite even number n and the corresponding set of natural numbers

$$B = \{1, 2, 3, \dots, n-2, n-1, n\}.$$

Then, we multiply each of its elements by 2 and obtain the set

$$\bar{B} = \{2, 4, 6, \dots, n-4, n-2, n, n+2, n+4, \dots, 2n-4, 2n-2, 2n\}.$$

Notice the following three properties of the sets B and $\bar{B}$: (i) they have the same number of elements; (ii) $\bar{B} \not\subseteq B$; (iii) $n/2$ elements of $\bar{B}$, namely, $n+2, n+4, \dots, 2n-4, 2n-2, 2n$, do not belong to B.

Suppose now that we wish to multiply each element of the set of natural numbers, $\mathbb{N}$, by 2. Clearly, as a result, we obtain the set, $\mathbb{E}$, of even numbers. Let us see whether the properties (i)–(iii) of the sets B and $\bar{B}$ hold for $\mathbb{N}$ and $\mathbb{E}$. With respect to the property (i), we should say that, due to (4), the set obtained after multiplication has the same cardinality as the original set, i.e., it is countable. Then, we see a paradoxical situation, because, in contrast with the finite sets B and $\bar{B}$, the set $\mathbb{E}$ obtained after multiplication is a *proper subset* of the original set $\mathbb{N}$, i.e., the property (ii) does not hold. Once again due to (4), the property (iii) does not hold either.

2.3. Hilbert's Paradox of the Grand Hotel

This paradox proposed by David Hilbert in 1924 became popular thanks to the book "One, Two, Three, ... Infinity" of George Gamow (see [20]). It has the following formulation. We all know that in a hotel having a finite number of rooms, no more new guests can be accommodated if it is full. Hilbert's Grand Hotel has an infinite number of rooms (of course, the number of rooms is countable, because the rooms in the Hotel are numbered). If a new guest arrives at the Hotel where every room is occupied, it is, nevertheless, possible to find a room for the newcomer. To do so, it is necessary to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. In such way, room 1 will be ready for the new guest and, in spite of the assumption that there are no available rooms in the Hotel, an empty room is found.

The paradox consists in the fact that we have supposed that the hotel is full and, nevertheless, it becomes possible to accommodate a newcomer in it. There exist different generalizations of this paradox showing in a similar way how it is possible to accommodate a finite and even infinite number of new guests in it.

2.4. Three Paradoxes Regarding Divergent Series

Let us now present another kind of paradoxes dealing with divergent series. We shall show that a very simple chain of equalities including addition of an infinite number of summands can lead to a paradox. The first paradox is the following. Suppose that we have

$$x = 1 + 2 + 4 + 8 + \cdots \tag{5}$$

Then, we can multiple both parts of this equality by 2

$$2x = 2 + 4 + 8 + \cdots$$

By adding 1 to both parts of the previous formula, we obtain

$$2x + 1 = 1 + 2 + 4 + 8 + \cdots \tag{6}$$

It can be immediately noticed that the right-hand side of (6) is just equal to x and, therefore, it follows:

$$2x + 1 = x$$

from which we obtain

$$x = -1$$

and, as a final paradoxical result, the following equality follows:

$$1 + 2 + 4 + 8 + \cdots = -1. \tag{7}$$

The paradox here is evident: we have summed up an infinite number of positive integers and have obtained as the final result a negative number.

The second paradox considers the well-known divergent series of Guido Grandi $S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ By applying the telescoping rule, i.e., by writing a general element of the series as a difference, we can obtain two different answers using two general elements, $1 - 1$ and $-1 + 1$

$$S = (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0,$$
$$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 1.$$

In the literature, there exist many other approaches giving different answers regarding the value of this series (see, e.g., [33]). Some of them use various notions of average (for instance, Cesàro summation assigns the value 0.5 to S).

The third series we consider is the famous paradoxical result of Ramanujan

$$c = 1 + 2 + 3 + 4 + 5 + \cdots = -1/12. \tag{8}$$

To obtain this remarkable result, he multiplies the left-hand part of (8) by 4 and then subtracts the result from (8) as follows:

$$\begin{aligned} c &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\ 4c &= 4 + 8 + 12 + \cdots \\ -3c &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots \end{aligned} \tag{9}$$

Ramanujan then uses the result (considered in various forms by Euler, Cesàro, and Hölder) attributing to the alternating series $1 - 2 + 3 - 4 + \cdots$ the value $\frac{1}{4}$ as the formal power series expansion of the function $\frac{1}{(1+x)^2}$ for $x = 1$, that is

$$1 - 2 + 3 - 4 + \cdots = 1/4. \tag{10}$$

Thus, it follows from (9) and (10) that:

$$-3c = 1 - 2 + 3 - 4 + 5 - 6 + \cdots = 1/4,$$

from where Ramanujan gets (8). This result looks even stranger than (11), because the sum of infinitely many positive integers is not only negative but also fractional.

2.5. The Rectangle Paradox of Torricelli

This paradox proposed by Evangelista Torricelli (see, e.g., [1,44]) considers a rectangle ABCD that is not a square (see Fig. 1). Without loss of generality, let us suppose that the length $|AB|$ is two times smaller than $|BC|$. On one hand, it is evident that the diagonal AC splits the rectangle into two triangles ABC and CDA having equal areas. On the other hand, it is possible to propose the following reasoning using infinitesimals (Torricelli talks about indivisibles) that challenges this conclusion.

Let us cover the upper triangle ABC by an infinite number of horizontal line segments having an infinitesimal width. Analogously, the lower triangle CDA is covered by the corresponding equal number of vertical line segments also having an infinitesimal width. Figure 1 illustrates only six horizontal segments of this kind and the corresponding six vertical segments. By the construction, each horizontal line segment is two times greater in length than the corresponding vertical line segment, for instance, $|EF| = 2|FG|$. The area of the upper triangle ABC can be obtained by summing up the areas of the horizontal line segments covering it. Analogously, the area of the triangle ACD can be obtained by summing up the areas of the vertical lines covering it. Because each horizontal line segment has the length that is two times greater than the length of the corresponding vertical line segment, it follows that the triangle ABC has a greater area than the triangle ACD. Thus, the paradox arises, because the two triangles have areas that are equal and in the same time are not equal.

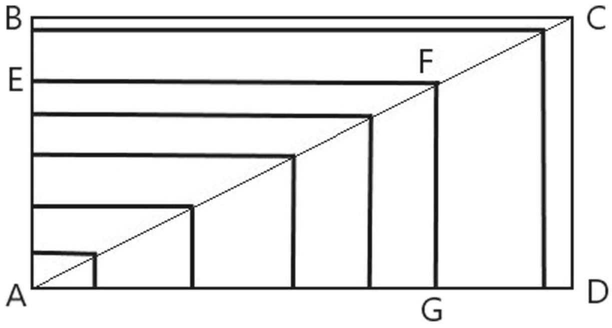

Figure 1. The rectangle paradox of Torricelli

2.6. Thomson's Lamp Paradox

The last paradox we consider here is the *Thomson Lamp Paradox* described in [63]. Suppose that we have a lamp and start to turn it on for $\frac{1}{2}$ minute, then turn it off for $\frac{1}{4}$ minute, then on again for $\frac{1}{8}$ minute, etc. At the end of one minute, the lamp switch will have been moved infinitely many times (to be precise, countably many times). Will then the lamp be *on* or *off* at the end of one minute? It is easy to see that this paradox is equivalent to the following question: Is the 'last' integer even or odd?

3. Numeral Systems of Pirahã and Mundurukú Prompt a New Point of View on Infinity

To understand how one can change his/her view on infinity, let us consider some numeral systems used to express finite numbers. Recall that a *numeral* is a symbol (or a group of symbols) that represents a *number* that is a concept. The same number can be represented by different numerals. For example, symbols '10', 'ten', 'IIIIIIIIII', and 'X' are different numerals, but they all represent the same number. Rules used to write down numerals together with algorithms for executing arithmetical operations form a *numeral system*. Thus, numbers can be considered as objects of an observation that are represented (observed) by instruments of the observation, i.e., by numerals and, more general, by numeral systems.

People in different historical periods used different numeral systems to count and these systems: (a) can be more or less suitable for counting; (b) can express different sets of numbers. For instance, Roman numeral system is not able to express zero and negative numbers and such expressions as II–VII or X–X are indeterminate forms in this numeral system. As a result, before the appearance of positional numeral systems and the invention of zero, mathematicians were not able to create theorems involving zero and negative numbers and to execute computations with them. The positional numeral system not only has allowed people to execute new operations but has led two new theoretical results, as well. Thus, numeral systems not only limit us in practical computations, they induce boundaries on theoretical results, as well.

It should be stressed that the powerful positional numeral system also has its limitations. For example, nobody is able to write down a numeral in the decimal positional system having 10^{100} digits (see a discussion on feasible numbers in [45,52]). In fact, suppose that one is able to write down one digit in one nanosecond. Then, it will take 10^{91} s to record all 10^{100} digits. Since in 1 year, there are $31.556.926 \approx 3.2 \times 10^7$ s, 10^{91} s are approximately 3.2×10^{83} years. This is a sufficiently long time, since it is supposed that the age of the universe is approximately 1.382×10^{10} years.

As we have seen above, Roman numeral system is weaker than the positional one. However, it is not the weakest numeral system. There exist very poor numeral systems allowing their users to express very few numbers and one of them is illuminating for our story. This numeral system is used by a tribe, Pirahã, living in Amazonia nowadays. A study published in *Science* in 2004 (see [23]) describes that these people use an extremely simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just 'many' and such operations as $2 + 2$ and $2 + 1$ give the same result, i.e., 'many'. Using their weak numeral system, Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers, because in their language, there are neither words nor concepts for that.

It is worthy of mention that the result 'many' is not wrong. It is just *inaccurate*. Analogously, when we observe a garden with 547 trees, then both phrases: 'There are 547 trees in the garden' and 'There are many trees in the garden' are correct. However, the accuracy of the former phrase is higher than the accuracy of the latter one. Thus, the introduction of a numeral system having numerals for expressing numbers 3 and 4 leads to a higher accuracy of computations and allows one to distinguish results of operations $2+1$ and $2+2$.

The poverty of the numeral system of Pirahã leads also to the following results:

$$\text{`many'} + 1 = \text{`many'}, \quad \text{`many'} + 2 = \text{`many'}, \quad \text{`many'} + \text{`many'} = \text{`many'} \qquad (11)$$

that are crucial for changing our outlook on infinity. In fact, by changing in these relations 'many' with ∞, we get relations used to work with infinity in the traditional calculus

$$\infty + 1 = \infty, \quad \infty + 2 = \infty, \quad \infty + \infty = \infty. \qquad (12)$$

Analogously, if we consider Cantor's cardinals (where, as usual, numeral $\aleph_0$ is used for cardinality of countable sets and numeral $\mathfrak{c}$ for cardinality of the continuum, see, e.g., [65]), we have similar relations

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + 2 = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0, \qquad (13)$$

$$\mathfrak{c} + 1 = \mathfrak{c}, \quad \mathfrak{c} + 2 = \mathfrak{c}, \quad \mathfrak{c} + \mathfrak{c} = \mathfrak{c}. \qquad (14)$$

It should be mentioned that the astonishing numeral system of Pirahã is not an isolated example of this way of counting. In [10], more than 20 languages having numerals only for small numbers are mentioned. For example, the same counting system, one, two, many, is used by the Warlpiri people, aborigines living in the Northern Territory of Australia (see [5]). The Pitjantjatjara people living in the Central Australian desert use numerals one, two, three, big mob (see [34]) where 'big mob' works as 'many'. It makes sense to remind also another Amazonian tribe–Mundurukú (see [47]) who fail in exact arithmetic with numbers larger than 5, but are able to compare and add large approximate numbers that are far beyond their naming range. Particularly, they use the words 'some, not many' and 'many, really many' to distinguish two types of large numbers. Their arithmetic with 'some, not many' and 'many, really many' reminds the rules Cantor uses to work with $\aleph_0$ and $\mathfrak{c}$, respectively. In fact, it is sufficient to compare

$$\text{'some, not many'} + \text{'many, really many'} = \text{'many, really many'} \tag{15}$$

with

$$\aleph_0 + \mathfrak{c} = \mathfrak{c} \tag{16}$$

to see this similarity.

Let us compare now the weak numeral systems involved in (11), (15) and numeral systems used to work with infinity. We have already seen that relations (11) are results of the weakness of the numeral system employed. Moreover, the usage of a stronger numeral system shows that it is possible to pass from records $1 + 2 =$ 'many' and $2 + 2 =$ 'many' providing for two different expressions the same result, i.e., 'many', to more precise answers $1 + 2 = 3$ and $2 + 2 = 4$ and to see that $3 \neq 4$. In these examples, we have the same objects—small finite numbers—but results of computations we execute are different in dependence of the instrument—numeral system—used to represent numbers. Substitution of the numeral 'many' by a variety of numerals representing numbers 3, 4, etc. allows us both to avoid relations of the type (11), (15) and to increase the accuracy of computations.

Relations (12)–(14), (16) manifest a complete analogy with (11), (15). Canonically, symbols ∞, $\aleph_0$, and $\mathfrak{c}$ are *identified* with concrete mathematical objects and (12)–(14), (16) are considered as intrinsic properties of these infinite objects (see e.g., [51,65]). However, the analogy with (11), (15) suggests that relations (12)–(14), (16) do not reflect the *nature* of infinite objects. They are just a result of weak numeral systems used to express infinite quantities. As (11), (15) show the lack of numerals in numeral systems of Pirahã, Warlpiri, Pitjantjatjara, and Mundurukú for expressing different finite quantities, relations (12)–(14), (16) show shortage of numerals in mathematical analysis and in set theory for expressing different infinite numbers. Another hint leading to the same conclusion is the situation with indeterminate forms of the kind III–V in Roman numerals that have been excluded from the practice of computations after introducing positional numeral systems.

Thus, the analysis made above allows us to formulate the following key observation that changes our perception of infinity:

Our difficulty in working with infinity is not a consequence of the *nature* of infinity but is a result of *weak numeral systems* having too little numerals to express the multitude of infinite numbers.

The way of reasoning where the object of the study is separated from the tool used by the investigator is very common in natural sciences where researchers use tools to describe the object of their study and the used instrument influences the results of the observations and determine their accuracy. The same happens in Mathematics studying natural phenomena, numbers, objects that can be constructed using numbers, sets, etc. Numeral systems used to express numbers are among the instruments of observation used by mathematicians. As we have illustrated above, the usage of powerful numeral systems gives the possibility to obtain more precise results in Mathematics in the same way as usage of a good microscope gives the possibility of obtaining more precise results in Physics. Traditional numeral systems have been developed to express finite quantities and they simply have no sufficiently high number of numerals to express different infinities (and infinitesimals).

4. A New Way of Counting

In this section, we briefly describe a recent numeral system that can be used to write down various infinite, finite, and infinitesimal numbers in a unique framework (see a comprehensive technical survey [58] and a popular book [53] for its description) concentrating ourselves on details that then will be used to reconsider the paradoxes. It should be emphasized immediately that the methodology to be presented is not a contraposition to the ideas of Cantor, Levi-Civita, and Robinson. In contrast, it is an applied evolution of their ideas. The new computational methodology introduces the notion of the accuracy of numeral systems and shows that different numeral systems can express different sets of finite and infinite numbers with different accuracies. The following clear analogy with Physics can be established in this context.

When a physicist uses a weak lens A and sees two black dots in his/her microscope he/she does not say: The object of the observation *is* two black dots. The physicist is obliged to say: the lens used in the microscope allows us to see two black dots and it is not possible to say anything more about the nature of the object of the observation until we replace the instrument—the lens or the microscope itself—with a more precise one. Suppose that he/she changes the lens and uses a stronger lens B and is able to observe that the object of the observation is viewed as ten (smaller) black dots. Thus, we have two different answers: (i) the object is viewed as two dots if the lens A is used; (ii) the object is viewed as ten dots by applying the lens B. Which of the answers is correct? Both. Both answers are correct but with the *different accuracies* that depend on the lens used for the observation. The answers are not in opposition one to another, and they both describe the reality (or whatever is behind the lenses of the microscope) correctly with the precision of the used lens. In both cases, our physicist discusses what he/she observes and does not pretend to say what the object *is*.

We shall do the same with infinite numbers and sets (objects of our study) and numeral systems used to observe them (our tools). Traditional approaches (Cantor, Robinson, etc.) and the methodology described here do not contradict one another: they are just different lenses having different accuracies for observations of mathematical objects.

Before we start a technical consideration, let us mention that a number of papers studying consistency of the new methodology and its connections to the historical panorama of ideas dealing with infinities and infinitesimals have been published (see [21,37,38,41,43,54,59,64]). In particular, in [59], it is stressed that it is not related to non-standard analysis. The methodology has been successfully applied in several areas of mathematics and computer science (more than 60 papers published in international scientific journals can be found at the dedicated web page [27]). We provide here just a few examples of areas where this methodology is useful. First of all, its successful applications in teaching mathematics should be mentioned (see, e.g., [3,28,30]). The dedicated web page [26] developed at the University of East Anglia, UK contains, among other things, a comprehensive teaching manual and a nice animation related to the Hilbert's paradox of the Grand Hotel. Then, we can indicate game theory and probability (see, e.g., [8,12,18,19,46,49,50]); local, global, and multiple criteria optimization (see [9,13–15,22,55,61,67]), hyperbolic geometry and percolation (see [31,42]), fractals (see [4,6,57]), infinite series (see [58,66]), Turing machines, cellular automata, and supertasks (see [11,48,50,60]), numerical differentiation and numerical solution of ordinary differential equations (see [2,16,17,29]), etc.

To start, let us mention that for thousands of years on the Earth, there exists a way of counting huge finite quantities that has not been formalized until the recent times. In traditional mathematics, after appearance of axioms of Peano, natural numbers are introduced starting from 0 by adding a unit to get 1, then adding another unit to 1 to obtain 2, and, by continuing in this way, other positive integers are introduced. Let us illustrate by an example that counting is a more complex procedure with respect to just adding 1 to 0 many times.

Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. Obviously, it is possible to answer that there are many seeds in the granary. This answer is correct, but its accuracy is low. To obtain a more precise answer, it would be necessary to count the grain seed by seed, but since the granary is huge, it is not possible to do this due to practical reasons.

To overcome this difficulty and to obtain an answer that is more accurate than 'many', people take sacks, fill them with seeds, and count the number of sacks. In this situation, we suppose that: (i) all the seeds have the same measure and all the sacks also; (ii) the number of seeds in each sack is the same and is equal to K_1, but the sack is so big that we are not able to count how many seeds it contains and to establish the value of K_1; (iii) in any case the resulting number K_1 would not be expressible by available numerals.

Then, if the granary is huge and it becomes difficult to count the sacks, trucks or even big train wagons are used. As it was for the sacks, we suppose that all trucks contain the same number K_2 of sacks, and all train wagons contain the same number K_3 of trucks; however, the numbers $K_i, i = 1, 2, 3$, are so huge that it becomes impossible to determine their values. At the end of this counting, we obtain a result in the following form: the granary contains 34 wagons, 27 trucks, 16 sacks, and 134 seeds of grain. Note, that if we add, for example, one seed to the granary, we can count it and not only see that the granary has more grain but also quantify the increment: from 134 seeds, we pass to 135 seeds. If we take out one wagon, we again are able to say how much grain has been subtracted: from 34 wagons, we pass to 33 wagons.

Let us make some considerations upon the way of counting described above. In our example, it is necessary to count large quantities. They are finite, but it is impossible to count them directly using the elementary unit of measure, u_0 (seeds), because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.

To solve the problem of 'infinite' quantities, new units of measure, u_1—sacks, u_2—trucks, and u_3—wagons, are introduced. The new units have an important feature: all the units u_{i+1} contain a certain number K_i of units u_i, but these numbers, $K_i, i = 1, 2, 3$, *are unknown*. Thus, quantities that it was impossible to express using only the initial unit of measure, u_0, are perfectly expressible in the new units $u_i, i = 1, 2, 3$. Notice that, in spite of the fact that the numbers K_i are unknown, the accuracy of the obtained answer is equal to one seed. In fact, if we add one seed, we are able to register and to quantify that we have more seeds, and if we subtract one wagon and two sacks, we again can quantify the decrease.

This key idea of counting by introduction of new units of measure with unknown, but fixed values K_i will be used in what follows to deal with infinite quantities together with the relaxation allowing one to use negative digits in positional numeral systems. It is necessary to extend the idea of the introduction of new units of measure from sets and numbers that are huge but finite to infinite sets and numbers. This can be done by extrapolating from finite to infinite the idea that n is both the number of elements of the set $\{1, 2, 3, \dots, n-1, n\}$ and the last element of this set. The infinite unit of measure is introduced as the number of elements of the set, $\mathbb{N}$, of natural numbers and expressed by the numeral ① called *grossone*. Using the granary example discussed above, we can offer the following interpretation: the set $\mathbb{N}$ can be considered as a sack and ① is the number of seeds in the sack. Following our extrapolation, the introduction of ① allows us to write down the set of natural numbers in the form

$$\mathbb{N} = \{1, 2, 3, \dots, ① - 3, ① - 2, ① - 1, ①\}, \tag{17}$$

where $① - 3, ① - 2, ① - 1, ①$ are infinite natural numbers. Thus, the set of natural numbers will be written in the form (17) instead of the usual record (1). We emphasize that in both cases, we deal with the same mathematical object—the set of natural numbers—that is observed through two different instruments. In the traditional case, usual numeral systems do not allow us to express infinite numbers, whereas the numeral system with grossone offers this possibility. Similarly, Pirahã are not able to see finite natural numbers greater than 2, but these numbers (e.g., 3 and 4) belong to $\mathbb{N}$ and are visible if one uses a more powerful numeral system. Notice also that in traditional statements (for example, in non-standard analysis), infinite numbers are not included in $\mathbb{N}$. However, if it is supposed that $\mathbb{N}$ is infinite and its elements are constructed, starting from 1 (or zero, as Peano did), according to the rule: the number n is followed by the number $n+1$, then each next number will be finite and, therefore, all natural numbers will be finite. Thus, by this construction, any set $\{1, 2, 3, \ldots, n\}$ will contain a finite number of elements. This would contradict the assumption that $\mathbb{N}$ is an infinite set.

Grossone is introduced by describing its properties postulated by the *Infinite Unit Axiom* (IUA) consisting of three parts: Infinity, Identity, and Divisibility. Similarly, to pass from natural to integer numbers, a new element—zero—is introduced, a numeral to express it is chosen, and its properties are described. The IUA is added to axioms for real numbers. Thus, it is postulated that associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, and existence of inverse elements with respect to addition and multiplication hold for grossone as they do for finite numbers.

Let us introduce the axiom and then give some comments upon it. Notice that in the IUA infinite sets will be described in the traditional form, i.e., without indicating the last element. For instance, the set of natural numbers will be written as (1) instead of the record (17) that will be used after the introduction of the axiom.

The Infinite Unit Axiom The infinite unit of measure is introduced as the number of elements of the set, $\mathbb{N}$, of natural numbers. It is expressed by the numeral ① called *grossone* and has the following properties:

Infinity Any finite natural number n is less than grossone, i.e., $n <$ ①.

Identity The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1, \quad 0^{①} = 0. \tag{18}$$

Divisibility For any finite natural number n sets $\mathbb{N}_{k,n}, 1 \le k \le n$, being the nth parts of the set, $\mathbb{N}$, of natural numbers have the same number of elements indicated by the numeral $\frac{①}{n}$ where

$$\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n, \ldots\}, \quad 1 \le k \le n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N}. \tag{19}$$

Let us comment upon this axiom. Its first part—Infinity—is quite clear. In fact, we want to describe an infinite number, and thus, it should be larger than any finite number. The second part of the axiom—Identity—tells us that ① interacts with identity elements 0 and 1 as all other numbers do. In the moment when we have stated that grossone is a number, we have fixed the usual properties of numbers, i.e., the properties described in Identity, associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, etc. The third part of the axiom—Divisibility—is the most interesting, since it links infinite numbers to infinite sets (in many traditional theories, infinite numbers are introduced algebraically, without any connection to infinite sets). It is based on Euclid's Common Notion no. 5 'The whole is greater than the part'. In the new methodology, it is applied to all quantities: finite, infinite, and infinitesimals.

Let us consider two examples for $n = 1$ and $n = 2$ in (19). If we take $n = 1$, then it follows that $\mathbb{N}_{1,1} = \mathbb{N}$ and Divisibility says that the set, $\mathbb{N}$, of natural numbers has ① elements. If $n = 2$, we have two sets $\mathbb{N}_{1,2}$ and $\mathbb{N}_{2,2}$, where

$$\begin{aligned} \mathbb{N}_{1,2} &= \{1, \quad 3, \quad 5, \quad 7, \dots \}, \\ \mathbb{N}_{2,2} &= \{\ \ 2, \quad 4, \quad 6, \quad \dots \} \end{aligned} \tag{20}$$

and they have $\frac{①}{2}$ elements each. Notice that the sets $\mathbb{N}_{1,2}$ and $\mathbb{N}_{2,2}$ have the same number of elements not because they are in a one-to-one correspondence but due to the Divisibility axiom. In fact, we are not able to count the number of elements of the sets $\mathbb{N}$, $\mathbb{N}_{1,2}$, and $\mathbb{N}_{2,2}$ one by one, because we are able to execute only a finite number of operations (we emphasize here the practical orientation of this methodology) whereas these sets are infinite. To define their number of elements, we use Divisibility and implement the principle 'The whole is greater than the part' in practice by determine the number of the elements of the parts using the whole.

In general, to introduce $\frac{①}{n}$, we do not try to count elements $k, k + n, k + 2n, k + 3n, \dots$ one by one in (19). In fact, we cannot do this due to the finiteness of our practical counting abilities. Using Euclid's principle, we construct the sets $\mathbb{N}_{k,n}, 1 \leq k \leq n$, by separating the whole, i.e., the set $\mathbb{N}$, in n parts and we affirm that the number of elements of the nth part of the set, i.e., $\frac{①}{n}$, is n times less than the number of elements of the entire set, i.e., than ①.

As was already mentioned, in terms of our granary example, ① can be interpreted as the number of seeds in the sack. In that example, the number K_1 of seeds in each sack was fixed and finite, but it was impossible to express it in units u_0, i.e., seeds, by counting seed by seed, because we had supposed that sacks were very big and the corresponding number would not be expressible by available numerals. In spite of the fact that K_1, K_2, and K_3 were inexpressible and unknown, using new units of measure (sacks, trucks, etc.), it was possible to count more easily and to express the required quantities. Now, our sack has the infinite but again *fixed* number of seeds. It is fixed, because it has a strong link to a concrete set—it is the number of elements of the set of natural numbers. Since this number is inexpressible by the existing numeral systems with the same accuracy afforded to measure finite small sets, we introduce a new numeral, ①, to express the required quantity. Then, we apply Euclid's principle and say that if the sack contains① seeds, then, even though we are not able to count the number of seeds of the nth part of the sack seed by seed, its nth part contains n times less seeds than the entire sack, i.e., $\frac{①}{n}$ seeds. Notice that the numbers $\frac{①}{n}$ are integer, since they have been introduced as numbers of elements of sets $\mathbb{N}_{k,n}$.

The new unit of measure allows us to express a variety of infinite numbers (including those larger than ① that will be considered shortly) and calculate easily the number of elements of the union, intersection, difference, or product of sets of type $\mathbb{N}_{k,n}$. Due to our accepted methodology, we do it in the same way as these measurements are executed for finite sets. Let us consider two simple examples showing how grossone can be used for this purpose (see [58] for a detailed discussion).

Let us determine the number of elements of the set $A_{k,n} = \mathbb{N}_{k,n} \backslash \{a\}$, $a \in \mathbb{N}_{k,n}, n \geq 1$. Due to the IUA, the set $\mathbb{N}_{k,n}$ has $\frac{①}{n}$ elements. The set $A_{k,n}$ has been constructed by excluding one element from $N_{k,n}$. Thus, the set $A_{k,n}$ has $\frac{①}{n} - 1$ elements. The granary interpretation can also be given for the number $\frac{①}{n} - 1$ as the number of seeds in the nth part of the sack minus one seed. For $n = 1$, we have $① - 1$ interpreted as the number of seeds in the sack minus one seed.

Let us consider the following two sets:

$$B_1 = \{4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, 79, \ldots\},$$
$$B_2 = \{3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, 135, \ldots\}$$

and determine the number of elements in the set $B = (B_1 \cap B_2) \cup \{3, 4, 5, 69\}$. It follows immediately from the IUA that $B_1 = \mathbb{N}_{4,5}$ and $B_2 = \mathbb{N}_{3,11}$. Their intersection

$$B_1 \cap B_2 = \mathbb{N}_{4,5} \cap \mathbb{N}_{3,11} = \{14, 69, 124, \ldots\} = \mathbb{N}_{14,55}$$

and, therefore, due to the IUA, it has $\frac{①}{55}$ elements. Finally, since 69 belongs to the set $\mathbb{N}_{14,55}$ and 3, 4, and 5 do not belong to it, the set B has $\frac{①}{55} + 3$ elements. The granary interpretation: this is the number of seeds in the 55th part of the sack plus three seeds.

The IUA introduces ① as the number of elements of the set of natural numbers and, therefore, it is the last natural number. We can also talk about the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\begin{aligned}\widehat{\mathbb{N}} = \{&\underbrace{1, 2, \ldots, ① - 1, ①}_{\text{Natural numbers}}, ① + 1, ① + 2, \ldots, 2① - 1, 2①, 2① + 1, \ldots \\ &①^2 - 1, ①^2, ①^2 + 1, \ldots 3①^{①} - 1, 3①^{①}, 3①^{①} + 1, \ldots\}.\end{aligned} \quad (21)$$

The extended natural numbers greater than grossone are also linked to infinite sets of numbers and can be interpreted in the terms of grain. For example, $① + 1$ is the number of elements of a set $B_3 = \mathbb{N} \cup \{a\}$, where a is integer and $a \notin \mathbb{N}$. In the terms of grain, $① + 1$ is the number of seeds in a sack plus one seed.

Let us give another example and determine the number of elements of the set

$$B_4 = \{(a_1, a_2) : a_i \in \mathbb{N}, i \in \{1, 2\}\},$$

being the set of couples of natural numbers. It is known from combinatorial calculus that if we have two positions and each of them can be filled in by one of l symbols, the number of the obtained couples is equal to l^2. In our case, since $\mathbb{N}$ has grossone elements, $l = ①$. Thus, the set B_4 has $①^2$ elements. This fact is illustrated below

$$\begin{array}{ccccc}
(1,1), & (1,2), & \dots & (1,①-1), & (1,①), \\
(2,1), & (2,2), & \dots & (2,①-1), & (2,①), \\
\dots & \dots & \dots & \dots & \dots \\
(①-1,1), & (①-1,2), & \dots & (①-1,①-1), & (①-1,①), \\
(①,1), & (①,2), & \dots & (①,①-1), & (①,①).
\end{array}$$

The introduced numeral system allows us to observe not only initial elements of certain infinite sets but also the final ones and some other infinite numbers in these sets. For example, we can write now the following records:

$$\mathbb{N} = \left\{1, 2, \dots, \dots \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \dots ① - 1, ①\right\},$$
$$\mathbb{O} = \left\{1, 3, 5, \dots, \dots \frac{①}{2} - 1, \frac{①}{2} + 1, \dots ① - 3, ① - 1\right\},$$
$$\mathbb{E} = \left\{2, 4, 6, \dots, \dots \frac{①}{2} - 2, \frac{①}{2}, \frac{①}{2} + 2, \dots ① - 2, ①\right\},$$
$$\mathbb{Z} = \{-①, -① + 1, -① + 2, \dots - 2, -1, 0, 1, 2, \dots, ① - 1, ①\}.$$

Due to the IUA, the set, $\mathbb{O}$, of odd numbers has $\frac{①}{2}$ elements, the set, $\mathbb{E}$, of even numbers also has $\frac{①}{2}$ elements. It is easy to calculate the number of elements of the set, $\mathbb{Z}$, of integers. It has ① positive elements, ① negative ones, and zero. Thus, the set $\mathbb{Z}$ has $2① + 1$ elements. For the purpose of this article, the introduced material is sufficient. As was already mentioned, more information about ① can be found in a comprehensive technical survey [58] and a popular book [53].

To conclude this section, it is necessary to emphasize that the introduced numeral system cannot give answers to *all* questions regarding infinite sets. As all numeral systems, it has its limitations. What can we say, for instance, about the number of elements of the set $\widehat{\mathbb{N}}$? Was this set described completely? The introduced numeral system based on ① is too weak to give answers to these questions, since it does not allow us to express the number of elements of this set. It is necessary to introduce in a reasonable way a more powerful numeral system by defining new numerals (for instance, ②, ③, etc).

5. Paradoxes of Infinity Revisited

In this section, we reconsider in the ①-based framework the paradoxes described above. It should be mentioned that several other paradoxes related to infinities and infinitesimals in probability theory and decision-making were considered using the ①-based methodology in [19, 48–50].

5.1. Galileo's Paradox

Before we start to consider the set I^2 of square natural numbers from (2), let us study bijection (4) between the sets of even and natural numbers. The traditional conclusion from (4) is that both sets are countable and they have the same cardinality $\aleph_0$.

Let us see now what we can say from the new methodological position, in particular, using Euclid's principle together with the separation of the objects of study form the tools used for this study. The objects of study here are two infinite sets, $\mathbb{N}$ and $\mathbb{E}$, and the instrument used to compare them is the bijection. Since we know that some elements of $\mathbb{N}$ do not belong to $\mathbb{E}$, the separation of the objects of our study from the tools suggests that another conclusion can be derived from (4): the accuracy of the used instrument is not sufficiently high to see the difference between the sizes of the two sets.

We have already seen that when one executes the operation of counting, the accuracy of the result depends on the numeral system used for counting. If one asked Pirahã to measure sets consisting of four apples and five apples, the answer would be that both sets of apples have many elements. This answer is correct, but its precision is low due to the weakness of the numeral system used to measure the sets.

Thus, the introduction of the notion of accuracy for measuring sets is very important and should be applied to infinite sets also. As was already discussed earlier, the similarity of Pirahã's rules (11) with the relations (13) and (14) holding for cardinal numbers suggests that the accuracy of the cardinal numeral system of Alephs is not sufficiently high to see the difference with respect to the number of elements of the two sets from (4).

To look at the record (4) using the new methodology, let us remind that due to the IUA the sets of even and odd numbers have ①/2 elements each and, therefore, ① is even. It is also necessary to recall that numbers that are larger than ① are not natural, they are extended natural numbers. For instance, ① + 2 is even but not natural, it is the extended natural, see (21). Thus, the last even natural number is ①. Since the number of elements of the set of even numbers is equal to $\frac{①}{2}$, we can write down not only the initial (as it is usually done traditionally) but also the final part of (4)

$$\begin{array}{cccccccccc} 2, & 4, & 6, & 8, & 10, & 12, & \dots & ①-4, & ①-2, & ① \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & 6, & \dots & \frac{①}{2}-2, & \frac{①}{2}-1, & \frac{①}{2} \end{array} \quad (22)$$

concluding so (4) in a complete accordance with the principle 'The part is less than the whole'. Both records, (4) and (22), are correct, but (22) is more accurate, since it allows us to observe the final part of the correspondence that is invisible if (4) is used. The new ①-based numerals allow us to see that the set $\mathbb{E}$ has two times less elements with respect to $\mathbb{N}$. Thus, the paradox is not present.

We are ready now to consider the set I^2 of square natural numbers from (2) and the corresponding bijection from (3). The traditional reasoning does not allow one to see that these two sets have different numbers of elements. Again, the answer that both sets are countable is correct, but its accuracy is low. The ①-based methodology allows us to see the difference in their number of elements and to express the final part of (3). The set I^2 can be defined now more accurately by emphasizing the fact that, by definition of ①, each square natural number should be less than or equal to grossone

$$I^2 = \{x : x \in \mathbb{N}, i \in \mathbb{N}, x = i^2, x \leq ①\}.$$

Then, the number of elements, J, of the set I^2 can be determined as

$$J = \max\{i : i^2 \leq ①\}.$$

By solving the required inequality $i^2 \leq ①$ and taking the maximal integer i satisfying this inequality we obtain that $J = \lfloor ①^{1/2} \rfloor$. Thus, as it was with the sets $\mathbb{N}$ and $\mathbb{O}$, we can re-write the bijection (3) indicating both its initial and finite elements

$$\begin{array}{ccccccccc} 1, & 2^2, & 3^2, & 4^2, & 5^2, & \dots & (\lfloor ①^{1/2} \rfloor - 2)^2, & (\lfloor ①^{1/2} \rfloor - 1)^2, & \lfloor ①^{1/2} \rfloor^2 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & \dots & \lfloor ①^{1/2} \rfloor - 2, & \lfloor ①^{1/2} \rfloor - 1, & \lfloor ①^{1/2} \rfloor. \end{array} \tag{23}$$

Since $\lfloor ①^{1/2} \rfloor < ①$, this paradox also vanishes.

5.2. Set-Multiplication Paradox

The introduction of the ①-based numeral system allows us to write down the sets of natural and extended natural numbers in the form (21). By definition, the number of elements of $\mathbb{N}$ is equal to ①. After multiplication of each of the elements of $\mathbb{N}$ by 2, the resulting set, that we call $\mathbb{E}^2$, will also have grossone elements, because multiplication of elements of a set by a constant that is not equal to zero does not change the number of elements of the set. In fact, the number $\frac{①}{2}$ multiplied by 2 gives us ① and $\frac{①}{2} + 1$ multiplied by 2 gives us $① + 2$ that is even extended natural number, see (21). Analogously, the last element of $\mathbb{N}$, i.e., ①, multiplied by 2 gives us 2①. Thus, the set $\mathbb{E}^2$ can be written as follows:

$$\mathbb{E}^2 = \{2, 4, 6, \dots ① - 2, ①, ① + 2, \dots 2① - 4, 2① - 2, 2①\},$$

and the corresponding bijection is

$$\begin{array}{cccccccccc} 2, & 4, & 6, & \dots & ① - 2, & ①, & ① + 2, & \dots & 2① - 4, & 2① - 2, & 2① \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & \dots & \frac{①}{2} - 1, & \frac{①}{2}, & \frac{①}{2} + 1, & \dots & ① - 2, & ① - 1, & ① \end{array}$$

where numbers $\{2, 4, 6, \dots ① - 4, ① - 2, ①\}$ are even and natural (they are $\frac{①}{2}$) and numbers $\{① + 2, ① + 4, \dots 2① - 4, 2① - 2, 2①\}$ are even and extended natural, they also are $\frac{①}{2}$. Thus, all the properties (i)–(iii) from Sect. 2.2 hold and the paradox does not occur.

5.3. Hilbert's Paradox of the Grand Hotel

Let us consider now the Grand Hotel in the ①-based framework. In the paradox, the number of rooms in the Hotel is infinite. In the new terminology, it is not sufficient to say this; it is required *to indicate explicitly* the infinite number of the rooms in the Hotel. Suppose that it has ① rooms. When a new guest arrives, it is proposed to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. At the end of this procedure, the guest from the last room having the number ① should be moved to the room ①+1. However, the Hotel has only ① rooms and, therefore, the poor guy from the room ① will go out of the Hotel (the situation that would occur in hotels with a finite number of rooms, if such a procedure would be implemented). A nice animation describing Hilbert's paradox of the Grand Hotel in the ①-based framework can be viewed at the didactic web page [26] developed at the University of East Anglia, UK.

Notice once again that there is no contradiction between the two ways to see the Grand Hotel. The traditional answer is that it is possible to put the newcomer in the first room. The ①-based way of doing confirms this result but shows something that was invisible traditionally—the guest from the last room should go out of the Hotel. Thus, the paradox is avoided.

5.4. Three Paradoxes Regarding Divergent Series

Let us consider the definition of x in (5). Thanks to ①, we have different infinite integers and, therefore, we can consider sums having different infinite numbers of summands. Thus, with respect to the new methodology, (5) is not well defined, because the number of summands in the sum (5) is not explicitly indicated. Recall that to say just that there are ∞ many summands has the same meaning of the phrase 'There are many summands' (cf. (11), (12)).

Thus, it is necessary to indicate explicitly an infinite number of addends, k, (obviously, it can be finite, as well). After this, (5) becomes

$$x(k) = 1 + 2 + 4 + 8 + \cdots + 2^{k-1},$$

and multiplying both parts by two and adding one to both the right and left sides of the equality gives us

$$2x(k) + 1 = 1 + 2 + 4 + 8 + \cdots + 2^{k-1} + 2^k.$$

Thus, when we go to substitute, we can see that there remains an addend, 2^k, that is infinite if k is infinite, that was invisible in the traditional framework

$$2x(k) + 1 = \underbrace{1 + 2 + 4 + 8 + \cdots + 2^{k-1}}_{x(k)} + 2^k.$$

The substitution gives us the resulting formula

$$x(k) = 2^k - 1$$

that works for both finite and infinite values of k giving different results for different values of k (exactly as it happens for the cases with finite values of k). For instance, $x(①) = 2^{①} - 1$ and $x(3①) = 2^{3①} - 1$. Thus, the paradox (11) does not take place.

Let us consider now Grandi's series. To calculate the required sum, we should indicate explicitly the number of addends, k, in it. Then it follows that:

$$S(k) = \underbrace{1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots}_{k \text{ addends}} = \begin{cases} 0, & \text{if } k = 2n, \\ 1, & \text{if } k = 2n + 1, \end{cases} \tag{24}$$

and it is not important whether k is finite or infinite. For example, $S(①) = 0$, since ① is even. Analogously, $S(① - 1) = 1$, because $① - 1$ is odd.

As it happens in the cases where the number of addends in a sum is finite, the result of summation does not depend on the way the summands are rearranged. In fact, if we know the exact infinite number of addends and the order the signs are alternated is clearly defined, we know also the exact number of positive and negative addends in the sum. Let us illustrate this point by supposing, for instance, that we want to rearrange addends in the sum $S(2①)$ as follows:

$$S(2①) = 1 + 1 - 1 + 1 + 1 - 1 + 1 + 1 - 1 + \cdots$$

Traditional mathematical tools used to study divergent series give an impression that this rearrangement modifies the result. However, in the ①-based framework we know that this is just a consequence of the weak lens used to observe infinite numbers. In fact, thanks to ①, we are able to fix an infinite number of summands. In our example, the sum has $2①$ addends, the number $2①$ is even and, therefore, it follows from (24) that $S(2①) = 0$. This means also that in the sum, there are ① positive and ① negative items. As a result, addition of the groups 1 + 1–1 considered above can continue only until the positive units present in the sum will not finish, and then, there will be necessary to continue to add only negative summands. More precisely, we have

$$S(2①) = \underbrace{1 + 1 - 1 + 1 + 1 - 1 + \cdots + 1 + 1 - 1}_{① \text{ positive and } \frac{①}{2} \text{ negative addends}} \underbrace{-1 - 1 - \cdots - 1 - 1}_{\frac{①}{2} \text{ negative addends}} = 0, \tag{25}$$

where the result of the first part in this rearrangement is calculated as $(1 + 1 - 1) \cdot \frac{①}{2} = \frac{①}{2}$ and the result of the second part summing up negative units is equal to $-\frac{①}{2}$ giving so the same final result $S(2①) = 0$. It becomes clear from (25) the origin of the Riemann series theorem. In fact, the second part of (25) containing only negative units is invisible if one works with the traditional numeral ∞.

Let us use now the ① lens to observe Ramanujan's paradoxical result (8). As it was in the summation discussed above, it is necessary to indicate explicitly an infinite number of addends, n, in the sum

$$c(n) = 1 + 2 + 3 + 4 + 5 + \cdots + n. \tag{26}$$

The ① methodology allows us to compute this sum for infinite values of n directly (see [58] for a detailed discussion) and to show that for infinite (and finite) values of n, it follows:

$$c(n) = 0.5n(1 + n), \tag{27}$$

and by taking $n = \text{①}$, we can easily compute the sum of all natural numbers

$$c(\text{①}) = 1 + 2 + 3 + 4 + 5 + \cdots + (\text{①} - 2) + (\text{①} - 1) + \text{①} \qquad (28)$$

that, obviously, is equal to $0.5\text{①}(1 + \text{①})$.

Let us now return to Ramanujan summation and consider the main trick of (9) consisting of displacement of addends in its second line. Since we work with natural numbers, we have $n = \text{①}$ addends in the sum (26). As a consequence, the displacement of (9) can be re-written more accurately with the observation of the last addends in each line of (9) as follows:

$$\begin{array}{rl} c(\text{①}) = & 1 + 2 + 3 + 4 + 5 + \cdots + \text{①} - 1 \quad + \text{①}, \\ 4c(\text{①}) = & \quad 4 \quad + 8 \quad + \cdots \qquad + 4\frac{\text{①}}{2} + 4\left(\frac{\text{①}}{2} + 1\right) + \cdots + 4\text{①}, \\ -3c(\text{①}) = & 1 - 2 + 3 - 4 + 5 + \cdots + \text{①} - 1 \quad - \text{①} - 4\left(\frac{\text{①}}{2} + 1\right) - \cdots - 4\text{①}. \end{array} \qquad (29)$$

Thus, to 0.5① even addends in (28), there will be added the first 0.5① numbers from (28) multiplied by 4, i.e., each even number i from the first line of (29) will be summed up with the number $4 \cdot \frac{i}{2}$ whereas 0.5① odd i from (28) are summed up with zeros. The displacement of the second line in (29) leads to the fact that only 0.5① summands of this line will participate in this addition and there will remain 0.5① more addends that were invisible in the traditional framework. They are

$$4 \cdot \left(\frac{\text{①}}{2} + 1\right) + 4 \cdot \left(\frac{\text{①}}{2} + 2\right) + \cdots + 4 \cdot \text{①}. \qquad (30)$$

Let us compute now the right-hand part of the third line of (29) using the fact that we can rearrange addends in the sum without changing the result. In such way, we can group the addends in three arithmetical progressions having 0.5① addends each

$$\begin{aligned} & 1 - 2 + 3 - 4 + 5 + \cdots + \text{①} - 1 - \text{①} - 4\left(\frac{\text{①}}{2} + 1\right) - 4\left(\frac{\text{①}}{2} + 2\right) - \cdots - 4\text{①} \\ & = \underbrace{(1 + 3 + 5 + \cdots + (\text{①} - 3) + (\text{①} - 1))}_{=(1+(\text{①}-1))\text{①}/4} - \underbrace{(2 + 4 + 6 + \cdots + (\text{①} - 2) + \text{①})}_{=(2+\text{①})\text{①}/4} \\ & \quad -4\underbrace{\left(\left(\frac{\text{①}}{2} + 1\right) + \left(\frac{\text{①}}{2} + 2\right) + \cdots + (\text{①} - 1) + \text{①}\right)}_{=((\frac{\text{①}}{2}+1)+\text{①})\text{①}/4} \\ & = (1 + (\text{①} - 1))\frac{\text{①}}{4} - (2 + \text{①})\frac{\text{①}}{4} - 4\left(\left(\frac{\text{①}}{2} + 1\right) + \text{①}\right)\frac{\text{①}}{4} = -3\frac{\text{①}}{2}(\text{①} + 1). \end{aligned}$$

Thus, we have obtained that

$$-3c(\text{①}) = -3\frac{\text{①}}{2}(\text{①} + 1).$$

As was expected, the obtained result shows that the third line of (29) is consistent with (27) for $n = \text{①}$.

Thus, it has been shown that Riemann's result on rearrangements of addends in series is a consequence of the fact that symbol ∞ used traditionally does not allow us to express quantitatively the infinite number of addends in

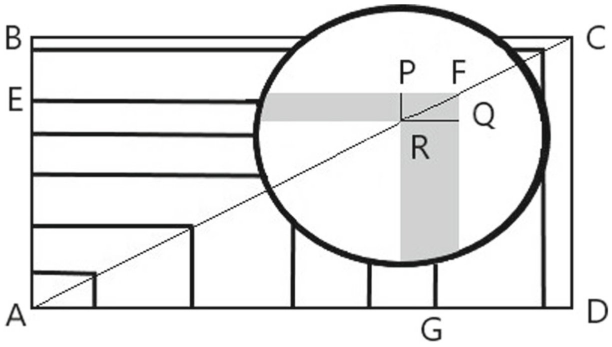

Figure 2. The rectangle paradox of Torricelli in the grossone-based framework

the series. The usage of the grossone methodology allows us to see that (as it happens in the case where the number of addends is finite) rearrangements of addends do not change the result for any sum with a fixed infinite number of summands. This happens, because if one knows the number of addends and the rule used to alternate their signs, he/she knows the number of positive and negative addends. Thus, the careful counting of the number of addends in infinite series allows us to avoid this kind of paradoxical results if ①-based numerals are used.

5.5. The Rectangle Paradox of Torricelli

This paradox involves infinitesimals. The numeral system based on ① allows us to express them easily and to execute arithmetical operations with them. For instance, numbers consisting of addends having negative finite powers of ① represent infinitesimals and the simplest infinitesimal number is $①^{-1} = \frac{1}{①}$. It is the inverse element with respect to multiplication for①

$$①^{-1} \cdot ① = ① \cdot ①^{-1} = 1. \tag{31}$$

The following two numbers are other examples of infinitesimals: $5.1①^{-2}$, $-6.1①^{-3} + 5.1①^{-32}$, etc. Note that all infinitesimals are not equal to zero. In particular, $\frac{1}{①} > 0$, because it is a result of division of two positive numbers. It also has a clear granary interpretation. Namely, if we have a sack containing ① seeds, then one sack divided by the number of seeds in it is equal to one seed. Vice versa, one seed, i.e., $\frac{1}{①}$, multiplied by the number of seeds in the sack, ①, gives one sack of seeds.

To consider Torricelli's paradox in the grossone framework, it is worthy to mention that sums with an infinite number of infinitesimal addends can give infinitesimal, finite, or infinite results in dependence of the number of summands and their value. As an example, let us consider this sum

$$T(k) = \underbrace{①^{-2} + ①^{-2} + ①^{-2} \cdots + ①^{-2} + ①^{-2}}_{k \text{ addends}}. \tag{32}$$

Then, for $k = 2①$, we obtain an infinitesimal result and for $k = 3①^2$ and $k = 4①^3$ finite and infinite results, respectively

$$T(2①) = ①^{-2} \cdot 2① = 2①^{-1}, \quad T(3①^2) = ①^{-2} \cdot 3①^2 = 3,$$
$$T(4①^3) = ①^{-2} \cdot 4①^3 = 4①.$$

This machinery allows us to compute directly the areas of the two triangles ABC and CDA using ①-based infinitesimals (see Fig. 2). To be able to execute numerical computations, let us suppose that the length $|AB| = 1$, $|BC| = 2$ and the line AC has no width (for instance, the rectangle was cut along this line). Obviously, it is also possible to consider the situation where AC has an infinitesimal width, but this point is not so important for the essense of the paradox. The procedure that has led to the paradox required to cover the triangles by segments having an infinitesimal width. Without loss of generality and for simplicity suppose that our horizontal segments have the width $h = ①^{-1}$ (it is easy to see that by taking $h = 2①^{-1}$ or $h = ①^{-3}$, the results will be analogous). Then, since $|AB| = 1$, it will take ① segments of the width $①^{-1}$ to cover the whole triangle ABC. By construction, the triangle CDA will also be covered by ① segments. Figure 2 considers one horizontal segment, EF, and the corresponding vertical segment, FG, and shows under a magnifying glass the situation in the neighborhood of the point F. The ①-based framework allows us to observe that both horizontal and vertical segments have triangular ends touching the line AC. Moreover, we can calculate easily the area of these triangles and both the width and the length of the vertical segments.

Since $|AB|/|BC| = |PR|/|RQ|$ and $|PR| = ①^{-1}$, it follows immediately that $|RQ| = 2①^{-1}$. As a result, areas of the triangles RPF and FQR (and of other $2① - 2$ similar triangles on the line AC) are equal to $①^{-1} \cdot 2①^{-1}/2 = ①^{-2}$. Thus, each horizontal segment $i, 1 \leq i \leq ①$, consists of a rectangle with the width $①^{-1}$ and the length $2 - 2①^{-1}i$, having so the area

$$①^{-1}(2 - 2①^{-1}i) = 2①^{-1} - 2①^{-2}i,$$

and of a triangle similar to RPF having the area equal to $①^{-2}$ (notice that for $i = ①$, the rectangle is absent). Therefore, the area S^i_{ABC} of the ith horizontal segment is

$$S^i_{ABC} = 2①^{-1} - 2①^{-2}i + ①^{-2}.$$

To obtain the area S_{ABC} of the whole coverage of the triangle ABC, it is sufficient just to sum up the areas of all ① small segments

$$S_{ABC} = \Sigma_{i=1}^{①} S^i_{ABC} = \Sigma_{i=1}^{①}(2①^{-1} - 2①^{-2}i + ①^{-2}) = 2 + ①^{-1} - 2①^{-2}\Sigma_{i=1}^{①} i$$
$$= 2 + ①^{-1} - 2①^{-2}(① + 1)①/2 = 2 + ①^{-1} - (1 + ①^{-1}) = 1.$$

The area of the triangle CDA is calculated by a complete analogy. Each vertical segment $i, 1 \leq i \leq ①$, consists of a rectangle with the width $2①^{-1}$ and the height $1 - ①^{-1}i$, having so the area

$$2①^{-1}(1 - ①^{-1}i) = 2①^{-1} - 2①^{-2}i,$$

and of a triangle similar to FQR having the area equal to ①$^{-2}$. Therefore, the area S^i_{CDA} of the ith vertical segment is

$$S^i_{CDA} = 2①^{-1} - 2①^{-2}i + ①^{-2} = S^i_{ABC}.$$

Since the number of horizontal and vertical segments is equal, this fact completes the consideration and shows that this paradox is also avoided.

5.6. Thomson's Lamp Paradox

To reconsider the Thomson lamp paradox, let us remind traditional definitions of infinite sequences and subsequences. An *infinite sequence* $\{a_n\}, a_n \in A, n \in \mathbb{N}$, is a function having as the domain the set of natural numbers, $\mathbb{N}$, and as the codomain a set A. A *subsequence* is obtained from a sequence by deleting some (or possibly none) of its elements. In a sequence $a_1, a_2, \ldots, a_{n-1}, a_n$, the number n is the number of elements of the sequence. Traditionally, only finite values of n are considered. Grossone-based numerals give us the possibility to observe infinite numbers and, therefore, to see not only the initial elements of an infinite sequence $a_1, a_2, \ldots$ but also its final part $\ldots, a_{n-1}, a_n$ where n can assume different infinite values.

The IUA states that the set of natural numbers, $\mathbb{N}$, has ① elements. Thus, by the above definition, any sequence having $\mathbb{N}$ as the domain has ① elements. Since any subsequence is obtained by deleting some (or possibly none) of the ① elements from a sequence, any sequence can have at most grossone elements.

Since the switches are executed in a sequence, the maximal number of switches that can be done is equal to ①. Remind also that we have already established that ① is even. Thus, after ① switches the lamp will be *off* if initially it was *on* and, vice versa, it will be *on* if initially it was *off*.

The ①-based methodology gives us the opportunity to calculate how much time will take this procedure of switching the lamp. Remind that it is on for $\frac{1}{2}$ minute, then it is off for $\frac{1}{4}$ minute, then again on for $\frac{1}{8}$ minute, etc. Thus we deal with the sum of ① addends of the form $\frac{1}{2^i}, 1 \le i \le$ ①. It is easy to show (see [58] for details) that

$$\Sigma_{i=1}^{①} \frac{1}{2^i} = 1 - \frac{1}{2^{①}},$$

i.e., this procedure of switches will not reach number one, it will be infinitesimally close to one.

6. A Brief Conclusion

It has been shown in this article that the surprising counting systems of Amazonian tribes, Pirahã and Mundurukú, open an interesting perspective on some classical paradoxes of infinity. The opportunity to use many different numerals to deal with infinities and infinitesimals offered by a recently introduced computational methodology has allowed us to switch from qualitative considerations of paradoxes to their quantitative analysis.

Declarations

Conflict of interest The author states that there is no conflict of interest and no funding has been used to execute this research.

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