Can We Recover Spacetime Structure from Privileged Coordinates?*

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Abstract

We make a few brief remarks on the exchange between Barrett and Manchak (2024a,b) and Gomes et al. (2024) concerning whether the structure of a relativistic spacetime can be recovered from its privileged coordinates.

1 Introduction

The 'Kleinian method' of presenting a geometric space begins by singling out a class of privileged coordinates for the space. One then looks to the transformations that carry us between these privileged coordinates and stipulates that the geometric space is comprised of those features that are 'invariant under' these transformations. One naturally wonders which geometric spaces can be given a Kleinian presentation. Norton attributes to Cartan (1927) the thought that moving to general relativity "threw into physics and philosophy the antagonism that existed between the two principle directors of geometry, Riemann and Klein. The spacetimes of classical mechanics and special relativity are of the type of Klein, those of general relativity are of the type of Riemann" (Norton, 1999, p. 128).

Barrett and Manchak (2024a) have recently provided one way to rigorously prove that Kleinian methods do not succeed in general relativity. Gomes et al. (2024) have responded by claiming that Barrett and Manchak (2024a) are not employing the correct Kleinian method. The aim of this paper is to make a few brief remarks on this exchange. Gomes et al. (2024) claim to have isolated an alternative Kleinian method that shows "clearly how an arbitrary Lorentzian metric can be recovered just from the full set of its local Lorentz charts in a manner clearly in the spirit of the Kleinian approach" (Gomes et al., 2024, p. 18). We show that this is not the case; the proposed Kleinian method does not allow one to recover the entire structure of a relativistic spacetime, only its conformal structure. Along the way we take the opportunity to provide some additional

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motivation for the Barrett and Manchak (2024a) results, and we catalogue some open questions for future work.

2 What are privileged coordinates?

We need to discuss what the 'privileged coordinates' of a spacetime might be. We begin with an instructive case. If (M, g_{ab}) is a flat spacetime, then it seems uncontroversial what the privileged coordinates of (M, g_{ab}) should be. For every point $p \in M$, there is an open set U containing p and a diffeomorphism $\phi : U \to V$ such that $\phi^*(\eta_{ab}) = g_{ab}$, where $V \subset \mathbb{R}^n$ is some open set and η_{ab} is the Minkowski metric on $\mathbb{R}^n$. One might call these charts (U, ϕ) the 'Minkowskian coordinates' on (M, g_{ab}). It is then natural to say that the Minkowskian coordinates of (M, g_{ab}) are its privileged coordinates. It is worth mentioning a few features that Minkowskian coordinates have.

Feature 1. They form a *locally G-structured space*, in the sense of Wallace (2019). In brief, a locally G-structured space can be thought of as a set S (in this case M) with a collection C of maps from subsets of S to $\mathbb{R}^n$ (in this case the maps ϕ) that satisfy a few basic conditions. That these coordinates form a locally G-structured space follows from (Barrett and Manchak, 2024a, Lemma 3.2.3).

Feature 2. They are *symmetry matching*. A collection of privileged coordinates on (M, g_{ab}) gives rise to a pseudogroup Γ on M; the elements of this pseudogroup are generated by the 'coordinate transformations' $\phi \circ \psi^{-1}$ for privileged coordinate maps ϕ and ψ. One can show that in this case Γ contains all and only those isometries between open subsets of M (Barrett and Manchak, 2024a, Proposition 3.2.1). In other words, Γ is the isometry pseudogroup of (M, g_{ab}). The 'coordinate transformations' between privileged coordinates are just the isometries between open subsets of M.

Feature 3. They are *adapted* to the metric g_{ab}. In the case of flat spacetimes, one can verify that if (U, ϕ) is a Minkowskian chart, then $g_{ab} = d_a u^1 d_b u^2 - \ldots - d_a u^n d_b u^n$ everywhere on U, where u^i are the coordinate maps. (Indeed, Barrett and Manchak (2024b) define Minkowskian charts in this manner.) In this sense, these privileged coordinates are ones in which the metric 'takes a simple form'; it looks exactly like the Minkowski metric everywhere on U.

We will see that there are different ways to generalize from the flat case depending on which of these features one takes to be salient. Different authors emphasize different ones. Wallace (2019) discusses Feature 1 at length. The atlas of a manifold M forms a locally G-structured space, and it is natural to think of privileged coordinates as retaining this property. (It is also what allows Barrett and Manchak (2024a) to prove that Feature 2 holds.) Regarding Feature 2, it is common to speak of the structures 'invariant under coordinate transformations' being the significant ones in a Kleinian privileged coordinate presentation. Norton (2002, p. 259) writes that under the Kleinian method a "geometric theory would be associated with a class of admissible coordinate systems and a group of transformations that would carry us between them. The cardinal rule was that physical significance can be assigned just to those features that were invariants of this group." Similarly, North (2021, p. 48) writes that "Klein suggested that any geometry can be identified by means of the transformations that preserve the structure, likewise by the quantities that are invariant under the group of those transformations." Wallace (2019, p. 135) remarks that the Kleinian method involves characterizing spaces "via the invariance groups of the geometry under transformations." One also often sees endorsements of Feature 3. For example, North (2021, p. 112) writes that "[a] preference for certain coordinates, in the sense that the laws take a simple or natural form in them, is indicative of, it is evidence for, underlying structure." Wallace (2019, p. 131) also emphasizes coordinate transformations that leave invariant "the form of the equations."

In addition to having Feature 1, Feature 2, and Feature 3, Minkowskian coordinates allow one to recover the structure of a flat spacetime (M, g_{ab}). Barrett and Manchak (2024b) point out that there are (at least) two different ways in which this recovery desideratum might hold of a geometric space X.

Determination. If another geometric space Y has the same privileged coordinates as X, then X and Y are the same.

Kleinian Presentability. X can be presented in the framework of locally G-structured spaces.

When Determination holds, the privileged coordinates of X 'fix' or 'determine' its structure, in the sense that no geometric space can have those privileged coordinates without being 'the same' as X. Of course, in order to carefully discuss Determination, one needs to make precise the sense in which the two spaces might have 'the same' privileged coordinates, along with the sense of 'sameness' between the spaces this entails. Kleinian Presentability is simple. Given a locally G-structured space (S, C), the maps in C determine manifold structure on S along with the coordinate transformation pseudogroup Γ on S. The natural way to recover geometric structure from this is to look to those tensor fields on S that are 'implicitly defined' or 'invariant' under this pseudogroup. Kleinian Presentability holds of X if there is some (S, C) that allows one to recover the structure of X using (something like) this method.

We will shortly discuss the argument from Barrett and Manchak (2024b) that Kleinian Presentability is strictly stronger than Determination; that argument is, in essence, the crux of the debate with Gomes et al. (2024). But it is first instructive to mention that both of these recovery desiderata hold of flat spacetimes and Minkowskian coordinates. First, Determination clearly holds since if two flat spacetimes admit the same Minkowskian coordinates, then they are equal (Barrett and Manchak, 2024b, Proposition 3). And second, one can show that all flat relativistic spacetimes are determined (up to homothety) by local isometry (Barrett and Manchak, 2024b, Proposition 4). This means that the only Lorentzian metrics implicitly defined on M by the coordinate transformation pseudogroup Γ are scalar multiples of g_{ab}. In this sense, the locally G-structured space of Minkowskian coordinates for (M, g_{ab}) allows one to recover in a Kleinian manner the structure of (M, g_{ab}) (up to homothety).

One wants an account of privileged coordinates for arbitrary spacetimes, not just flat ones. One can then ask whether it satisfies these recovery desiderata. We will shortly present the competing accounts of Barrett and Manchak (2024a) and Gomes et al. (2024). In brief, the former emphasizes Features 1 and 2, while the latter emphasizes Feature 3. Neither has all three features. It is therefore worth cataloguing the following question, which is closely related to the "Revision 1" question posed by Barrett and Manchak (2024a):

Question 1. *Is there an account of privileged coordinates for arbitrary relativistic spacetimes that satisfies Features 1, 2, and 3?*

In order to answer this question, one would first need a precise statement of Feature 3, and in particular, a clear account of what it is for a collection of coordinates to be 'adapted to' g_{ab}. The exact sense in which Minkowskian charts are adapted will clearly not do. For if $g_{ab} = d_a u^1 d_b u^2 - \ldots - d_a u^n d_b u^n$ everywhere on U, then g_{ab} must be flat on U. Both Barrett and Manchak (2024b) and Gomes et al. (2024) provide examples of different kinds of adapted coordinates. But we lack a general account of what it is for coordinates to be 'adapted' to g_{ab}. (See Jacobs (2024) and section 3 of Gomes et al. (2024) for further discussion of this point.) The basic idea is that (U, ϕ) is adapted to g_{ab} if on U (or perhaps on some part of U, for example, at the point p), the form that g_{ab} takes in (U, ϕ) coordinates renders it some structure that $\mathbb{R}^n$ 'naturally' has — for example, the Euclidean metric or the Minkowskian metric. The question, therefore, is whether or not one can make the relevant sense of 'naturality' precise. Without doing so, it seems that we lack a statement of Feature 3 that is clear enough to allow one to answer Question 1. It is worth cataloguing this as its own further question.

Question 2. *Can one make Feature 3 precise? What is it for coordinates to be adapted to a particular geometric structure on M?*

Supposing that one is able to answer Question 2 in the affirmative, there are some vague reasons to think that the answer to Question 1 will be "no." In particular, Features 2 and 3 seem to pull in opposite directions. Let (M, g_{ab}) be a Heraclitus spacetime, i.e. one with a trivial isometry pseudogroup (Manchak and Barrett, 2024). (That is, if U and V are open subsets of M, the only isometry $\phi : U \to V$ is the identity map.) Insofar as an account of privileged coordinates for (M, g_{ab}) satisfies Feature 2, it cannot admit 'too many' coordinates as privileged, since the coordinate transformations they determine must be few if they are to form the trivial isometry pseudogroup (which only contains identity maps). On the other hand, for the account to have Feature 3, it will likely have to countenance many coordinates as privileged. It strikes one as unlikely that there can be a 'small' collection of coordinates that reflect the 'form' of the metric g_{ab}, given how asymmetric g_{ab} is.

3 Two accounts

We turn to the account of privileged coordinates provided by Barrett and Manchak (2024a). Recall that to generate Minkowskian charts, one considers the class of isometries between open regions of the given flat spacetime and open regions of a fixed spacetime with underlying manifold $\mathbb{R}^n$, which in this case is Minkowski spacetime. One can generalize this exact idea to arbitrary spacetimes, so long as one no longer considers isometries to Minkowski spacetime. The basic idea behind the flat case will still hold; one will still be considering the privileged coordinates to be isometries between open regions of the spacetime and open regions of some spacetime with underlying manifold $\mathbb{R}^n$.

More precisely, one begins by showing that every relativistic spacetime has a representation (Barrett and Manchak, 2024a, Lemma 3.2.2). We say that a spacetime $(\mathbb{R}^n, g'_{ab})$ with underlying manifold $\mathbb{R}^n$ is a representation of (M, g_{ab}) if for every point $p \in M$, there are open sets $O \subset M$ and $O' \subset \mathbb{R}^n$ such that $p \in O$ and (O, g_{ab}) is isometric to (O', g'_{ab}). One easily verifies that Minkowski spacetime is a representation of every flat spacetime; it is in this sense that this account of privileged coordinates is generalizing from the flat case. The existence of representations allows one to construct a locally G-structured space from a relativistic spacetime (M, g_{ab}). Let (M, g_{ab}) be a relativistic spacetime with $(\mathbb{R}^n, g'_{ab})$ a representation of it. One then lets C be the collection of isometries between open subsets of (M, g_{ab}) and open subsets of $(\mathbb{R}^n, g'_{ab})$, i.e. diffeomorphisms $c : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $c^*(g'_{ab}) = g_{ab}|_U$. The resulting (M, C) is a locally G-structured space (Barrett and Manchak, 2024a, Lemma 3.2.3), and if one had picked a different representation, one would have constructed an isomorphic locally G-structured space (Barrett and Manchak, 2024a, Proposition 3.2.3). The manifold recovered by (M, C) is the manifold M, and the coordinate transformation pseudogroup that (M, C) induces is the isometry pseudogroup of (M, g_{ab}) (Barrett and Manchak, 2024a, Proposition 3.2.1). One therefore has an account of privileged coordinates that satisfies Features 1 and 2. It is a particularly natural account to adopt if one wants to assert (without caveat) that the significant structures of a spacetime are those 'invariant under coordinate transformation.' However, because the representations for some spacetimes will have metrics whose 'forms' are various — certainly not as clean as the Minkowski metric — it would seems that Feature 3 does not hold of this account, as Gomes et al. (2024) emphasize. (This kind of account nonetheless has precedent. Jacobs (2024), for example, also emphasizes the importance of symmetry over adaptedness when defining privileged coordinates.)

The privileged coordinates resulting from this account do not satisfy Kleinian Presentability. Indeed, the results of Barrett and Manchak (2024a,b) imply that no account with Feature 1 and (in particular) Feature 2 can. Given a Heraclitus spacetime $(\mathbb{R}^2, g_{ab})$, we know that if some locally G-structured space recovers g_{ab}, it must be that the coordinate transformation pseudogroup Γ it induces is trivial. For if not, Γ would contain a map that does not preserve g_{ab}, and hence Γ would not implicitly define (and hence recover) g_{ab}. But we know that there are non-isometric metrics on $\mathbb{R}^2$ that are invariant under all and only those maps in Γ. Indeed, one can show that there is another non-isometric Heraclitus spacetime $(\mathbb{R}^2, g'_{ab})$. The metric g'_{ab} will then be one such example. So the data provided by a symmetry matching account of privileged coordinates will not suffice to recover g_{ab} in any strong sense; Kleinian Presentability cannot hold. In addition, the locally G-structured spaces determined by $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ will be isomorphic; the two spacetimes have (up to isomorphism) the 'same' privileged coordinates, despite the spacetimes themselves being non-isometric. In this sense Determination fails too.

It takes some time to work through the technical details, but the basic idea behind this argument is intuitive and (as we will discuss later) has precedent in the literature. Gomes et al. (2024) suggest that "[d]espite the fifteen pages of meticulous setup through which Barrett and Manchak take their reader in order to get to this point, the result should not have come as a surprise. Characterising a spacetime geometry via its symmetry group is simply a non-starter when that spacetime lacks symmetries." This remark glosses over an important distinction. There is an ambiguity when one speaks of a "spacetime [that] lacks symmetries." One way in which a spacetime (M, g_{ab}) might lack symmetries is that it might be 'giraffe,' in the sense of Barrett et al. (2023) and Manchak and Barrett (2024). Giraffe spacetimes (M, g_{ab}) are such that the only isometry from (M, g_{ab}) to itself is the identity map. Such spacetimes lack 'global symmetries.' The existence of giraffe spacetimes is not surprising. (Indeed, it is also pointed to in discussions of Kleinian methods by North (2021, p. 117), Torretti (2016), and Norton (1999, p. 129–30).) But it is also not sufficient to establish the above results. As we mentioned earlier, all flat spacetimes satisfy a variety of Kleinian Presentability and Determination; some flat spacetimes are giraffe (Barrett et al., 2023, Example 2). This means that the existence of giraffe spacetimes does not by itself suffice to establish the above results. One needs the existence of a Heraclitus spacetime to do this. It is much more difficult to build Heraclitus spacetimes than giraffe spacetimes, as one can confirm by examining the example in Manchak and Barrett (2024). And until recently, the only 'no symmetry' idea in the philosophical discourse was the giraffe one. Only once one has a Heraclitus existence result does the above argument go through.

It is worth making one further remark about giraffe and Heraclitus spacetimes. Gomes et al. (2024) claim that "spacetimes lacking symmetries are well-known to be the generic case." It certainly seems likely that both giraffe and Heraclitus spacetimes are generic, but we are not aware of a full proof in either case. The results of Ebin (1968) and Fischer (1970) concern giraffe spacetimes in the Riemannian case with compact manifolds. Sunada (1985, Proposition 1) generalizes to the Heraclitus context, but the result still concerns the Riemannian case with compact manifolds. Mounoud (2015, Theorem 1) generalizes to the context that includes the Lorentzian case but the result still concerns giraffe spacetimes with compact manifolds. These results are certainly important. But they do not fully establish the genericity of giraffe or Heraclitus spacetimes with no restriction on the manifold topology. We therefore take this opportunity to pose the following question again (Manchak and Barrett, 2024):

Question 3. *Are giraffe spacetimes generic? Are Heraclitus spacetimes generic?*

One conjectures that both answers are "yes," but further work is required to be sure. Note that if the answer to the second part of Question 3 is "yes," then that (in conjunction with the discussion above) will imply that any account of privileged coordinates with Feature 1 and Feature 2 will fail to satisfy Kleinian Presentability and Determination for *almost all* relativistic spacetimes.

Barrett and Manchak (2024a) pose a few open questions about privileged coordinates. In particular, they wonder whether there is another account of privileged coordinates (closely related to what they call "Revision 1") or method of recovery (closely related to what they call "Revision 2") that might better allow one to recover spacetime structure. Gomes et al. (2024) take up both of these questions. Their account of privileged coordinates results from emphasizing Feature 3, rather than Features 1 and 2. It relies upon the existence of Lorentz coordinates. Let (M, g_{ab}) be a relativistic spacetime with $p \in M$. We will say that coordinates (U, ϕ) with $p \in U$ are **Lorentz coordinates** at p if $\phi(p) = (0, \ldots, 0) \in \mathbb{R}^n$ and the metric g_{ab} 'takes a simple form' at p in the sense that

$$g_{ab} = d_a u^1 d_b u^1 - d_a u^2 d_b u^2 - \ldots - d_a u^n d_b u^n$$

at p, where u^i are the coordinate maps associated with (U, ϕ). It is well known that for each point $p \in M$, there are Lorentz coordinates (U, ϕ) for g_{ab} at p.

Gomes et al. (2024, p. 17–18) take the Lorentz coordinates as the privileged coordinates of (M, g_{ab}). It is clear that there is a sense in which this account has Feature 3. Indeed, Lorentz coordinates about p are adapted to g_{ab}, since they are defined to be those in which the metric takes Minkowskian form at the point p. (In general this will not hold anywhere in U apart from at the one point p.) It is interesting to note that this account of privileged coordinates for (M, g_{ab}) does not have Features 1 or 2. One can easily see that it does not have Feature 2 by considering a Heraclitus spacetime (M, g_{ab}). In this case, coordinate transformations between Lorentz coordinates — that is, the maps in the collection Γ that these privileged coordinate induce — do not necessarily preserve g_{ab}. Let $p, q \in M$ be distinct points and suppose that we have Lorentz coordinates (U, ϕ) about p and (V, ψ) about q. Consider the 'coordinate transformation' map $\psi^{-1} \circ \phi$. Since $\psi^{-1} \circ \phi(p) = \psi^{-1}(0, \ldots, 0) = q$, we know that $\psi^{-1} \circ \phi$ is not the identity map. Since (M, g_{ab}) is Heraclitus, $\psi^{-1} \circ \phi$ cannot be contained in its isometry pseudgroup, and hence $(\psi^{-1} \circ \phi)^*(g_{ab}) \neq g_{ab}$. This point is made by Barrett and Manchak (2024b), and it is recognized by Gomes et al. (2024). Altogether this means that if one adopts Lorentz coordinates as the privileged coordinates of (M, g_{ab}), one can no longer assert without caveat that the significant structures of (M, g_{ab}) are those invariant under (privileged) coordinate transformations.

Moreover, in general the Lorentz coordinates for (M, g_{ab}) do not form a locally G-structured space. One can see this even in the case of two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta_{ab})$. One easily verifies that both of the following charts are Lorentz coordinates at $(0, 0)$: $(\mathbb{R}^2, 1_{\mathbb{R}^2})$, where $1_{\mathbb{R}^2}$ is the identity map on $\mathbb{R}^2$, and $(B_0, 1_{B_0})$, where B_0 is the open ball of radius 1 centered at $(0, 0)$

and 1_{B_0} is the identity map on this open set. Now let B_1 be the open ball of radius 1 centered at $(1, 0)$ and $c : B_1 \to B_0$ be the diffeomorphism defined by $c : (x, y) \mapsto (x-1, y)$. The chart (B_1, c) forms Lorentz coordinates at $(1, 0)$. Now one can ask whether the compatibility condition of locally G-structured spaces is satisfied. (See Barrett and Manchak (2024a) or Wallace (2019) for a precise definition.) Suppose that it is. Then since $(\mathbb{R}^2, 1_{\mathbb{R}^2})$ and (B_1, f) are privileged, this implies that the map $1 \circ c^{-1} : B_0 \to B_1$, which one can easily verify is just c^{-1} is in G. The compatibility condition implies that $s \circ c$ must be privileged for each $s \in G$, so we see that $c^{-1} \circ 1_{B_0}$ must be privileged. But this is just c^{-1}, and one can easily verify that (c^{-1}, B_0) is not Lorentz chart, since $(0, 0)$ is not in its range. In essence, it is the fact that Lorentz coordinates must map to open sets surrounding the origin that prevents them from forming a locally G-structured space. The fact that this account does not have Feature 1 addresses the question of Gomes et al. (2024) as to why Barrett and Manchak (2024a) "tacitly forego adapted coordinates." The aim of Barrett and Manchak (2024a, p. 3) was to "examine where the limits of [the framework of locally G-structured spaces] lie," so it was natural to restrict attention to account of privileged coordinates that have Feature 1. Since Lorentz coordinates do not form a locally G-structured space, they cannot lead to Kleinian Presentability in the exact way that Barrett and Manchak (2024b) state that recovery desideratum.

There is a weaker sense, however, in which this account of privileged coordinates has Features 1 and 2. We will return to this shortly. But first, it is important to mention that it satisfies Determination. This is a simple result: if two spacetimes admit the same Lorentz coordinates, then they must be equal. Barrett and Manchak (2024b, Proposition 6) show this for 'Lorentz normal coordinates,' and the same argument goes through here. Barrett and Manchak (2024b) use exactly this case to argue that Determination and Kleinian Presentability come apart. The idea is that Lorentz coordinates do not lead to a Kleinian presentation of (M, g_{ab}), since they do not form a locally G-structured space, and even if they did, the induced coordinate transformation pseudogroup would not implicitly define g_{ab} since Feature 2 does not obtain. Lorentz coordinates do nonetheless determine spacetime structure, in the sense that a spacetime can be exactly recovered from the data they provide. Indeed, consider how one would go about recovering the metric g_{ab} from the class of Lorentz coordinates of (M, g_{ab}). One would take a point $p \in M$, and find coordinates (U, ϕ) in this class in which $\phi(p) = (0, \ldots, 0)$. This guarantees that (U, ϕ) are Lorentz coordinates at p, rather than at some other point in U. One then would stipulate that the metric at p is $d_a u^1 d_b u^1 - d_a u^2 d_b u^2 - \ldots - d_a u^n d_b u^n$ in (U, ϕ) coordinates. One does this for each point $p \in M$ and thereby defines g_{ab}.

Gomes et al. (2024) object to this point. They argue that Lorentz coordinates lead to a different kind of Kleinian presentation of (M, g_{ab}). It will take a moment to make this idea precise; it is closely related to the idea that this account of privileged coordinates has weaker versions of Feature 1 and Feature 2. Let (U, ϕ) be Lorentz coordinates for (M, g_{ab}) about the point $p \in M$. Gomes et al. (2024) notice that since $\phi : U \to \phi[U]$ is a diffeomorphism, it induces global coordinates on the tangent space T_pM of p. These coordinates are given by the map $\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$. Since $T_{\phi(p)}\mathbb{R}^n$ is effectively just a copy of $\mathbb{R}^n$, one has induced coordinates on T_pM. One conjectures that indeed this collection of ϕ_* maps, for each (U, ϕ) that is Lorentz coordinates about p, form a G-structured space with underlying set T_pM. This would capture the weaker sense in which the Gomes et al. (2024) account has Feature 1. While Lorentz coordinates do not determine a locally G-structured space on M, they do on the tangent space T_pM for each $p \in M$.

One now uses these coordinatizations of T_pM to attempt to recover the metric g_{ab} on M. For each point $p \in M$ and Lorentz coordinates (U, ϕ) and (V, ψ) about p, one can consider the 'coordinate transformation' $\phi^* \circ \psi_* : T_pM \to T_pM$. One then looks to those structures that are left invariant by coordinate transformations of this kind. The collection of maps of the form $\phi^* \circ \psi_* : T_pM \to T_pM$ for Lorentz coordinates (U, ϕ) and (V, ψ) about p is just the collection of bijective linear maps $T_pM \to T_pM$ that preserve the generalized inner product $g_{ab}|_p$ on T_pM. (This follows from the discussion in the first paragraph of the proof of the Proposition below.) This captures a weaker sense in which this account of privileged coordinates has Feature 2. Lorentz coordinates do not specify symmetries of (M, g_{ab}), but when we restrict attention to a point, they do specify symmetries of T_pM with its associated inner product structure. At the very least, it is easy to see that these coordinate transformations all preserve the value of the metric g_{ab} at p, in the sense that $\phi^* \circ \psi_*(g_{ab}|_p) = g_{ab}|_p$. This follows immediately from the defining condition of Lorentz coordinates. In this sense, therefore, it would seem that the metric g_{ab} at p can be recovered in a Kleinian fashion: for each point $p \in M$ the value of the metric g_{ab} at p is implicitly defined in the sense described above. Gomes et al. (2024, p. 18) claim that this implies that their account of privileged coordinates satisfies a weaker kind of Kleinian Presentability. It is not that (M, g_{ab}) can be presented via one locally G-structured space, but rather the idea is that a family of G-structured spaces (one for the tangent space T_pM for each point p in M) can be used to present its structure. Gomes et al. (2024, p. 18) suggest that this demonstrates "clearly how an arbitrary Lorentzian metric can be recovered just from the full set of its local Lorentz charts in a manner clearly in the spirit of the Kleinian approach."

This unfortunately is not the case. One can see this by considering the following example. Let (M, g_{ab}) be a spacetime with α a smooth scalar field on M such that $\alpha > 0$ (everywhere on M). One can now easily see that the Lorentz coordinates of (M, g_{ab}) implicitly define the metric αg_{ab} in precisely the same way as they define the metric g_{ab}. Let $p \in M$ with (U, ϕ) and (V, ψ) Lorentz coordinates about p. Then we easily see that

$$\phi^* \circ \psi_*((\alpha g_{ab})|_p) = \alpha(p)(\phi^* \circ \psi_*(g_{ab}|_p)) = \alpha(p) g_{ab}|_p = (\alpha g_{ab})|_p$$

The first equality holds since the pushforward and pullback are both linear at p. The second equality holds since (U, ϕ) and (V, ψ) are Lorentz coordinates (for g_{ab}) at p. The idea behind this point is simple. The Lorentz coordinates only implicitly define g_{ab} at the point p up to scale factor since the generalized inner product $cg_{ab}|_p$ on T_pM admits precisely the same symmetries as $g_{ab}|_p$ does, for any positive constant c. This means that the Lorentz coordinates will also implicitly define the new metric αg_{ab} because, when one restricts attention to p, $(\alpha g_{ab})|_p$ is just a scalar multiple of g_{ab}. Altogether, this means that the method of Kleinian presentation that Gomes et al. (2024) propose can (at best) characterize only the conformal structure of (M, g_{ab}), not its entire structure. (Recall that metrics g_{ab} and g'_{ab} on M are conformally equivalent if there is a smooth positive scalar field α on M such that $g_{ab} = \alpha g'_{ab}$.) We note that this is exactly the kind of problem that Barrett and Manchak (2024a) had isolated for their account of privileged coordinates. Once again the proposed recovery procedure recovers non-isometric metrics on M.

It would nonetheless be interesting to know whether or not the conformal structure of (M, g_{ab}) can be presented in a Kleinian manner. We put forward the following simple question.

Question 4. *Let (M, g_{ab}) be a spacetime with $p \in M$ and g'_{ab} another metric on M of Lorentzian signature. Suppose that both $\phi^* \circ \psi_*(g_{ab}|_p) = g_{ab}|_p$ and $\phi^* \circ \psi_*(g'_{ab}|_p) = g'_{ab}|_p$ for all Lorentz coordinates (U, ϕ) and (V, ψ) for (M, g_{ab}) about p. Is there a positive constant c such that $g_{ab}|_p = cg'_{ab}|_p$?*

If so, then this will imply that the Gomes et al. (2024) method of recovering structure from Lorentz coordinates does recover the conformal structure of (M, g_{ab}). For suppose that g'_{ab} is implicitly defined on M by Lorentz coordinates in the same way as g_{ab} is. We would know that at each point p, $g_{ab}|_p = cg'_{ab}|_p$, for some positive scalar c, and hence we would know that g_{ab} and g'_{ab} are conformally equivalent.

The answer to Question 4 is "no" in two dimensions. The following example shows this. Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime where $\eta_{ab} = d_a t d_b t - d_a x d_b x$. Consider the spacetime (M, g_{ab}) where M is the $t > 0$ portion of $\mathbb{R}^2$ and $g_{ab} = t\eta_{ab}$. Since we are working in two dimensions, the metric $-g_{ab}$ has the same Lorentzian signature as g_{ab}. And it is now easy to see that $-g_{ab}$ will be implicitly defined by the Lorentz coordinates of (M, g_{ab}) in exactly the same way as g_{ab} is. At each point $p \in M$, the one is again a scalar multiple of the other. But (M, g_{ab}) and $(M, -g_{ab})$ are not conformally equivalent (and not isometric). Moreover, one can verify that the spacetimes are not even locally isometric since no neighborhood of a point in one spacetime can be isometrically mapped into any neighborhood of any point in the other. This follows since the Ricci scalar is everywhere positive for (M, g_{ab}) and everywhere negative for $(M, -g_{ab})$.

One might nonetheless conjecture that the answer to Question 4 is "yes" in dimensions higher than two. We have the following proposition due to David Malament.

Proposition. *Let (M, g_{ab}) be a spacetime with dim$(M) \geq 3$. Let $p \in M$ and g'_{ab} be another metric on M of Lorentzian signature. Suppose that both $\phi^* \circ \psi_*(g_{ab}|_p) = g_{ab}|_p$ and $\phi^* \circ \psi_*(g'_{ab}|_p) = g'_{ab}|_p$ for all Lorentz coordinates (U, ϕ) and (V, ψ) for (M, g_{ab}) about p. Then there is a positive constant c such that $g_{ab}|_p = cg'_{ab}|_p$.*

Proof. We first show that there is a close connection between Lorentz charts at p and orthonormal bases of T_pM. Let (U, ϕ) be a Lorentz chart at $p \in U$. One easily sees that the corresponding coordinate vectors $(\frac{\partial}{\partial u^1})^a, \ldots, (\frac{\partial}{\partial u^n})^a$ form an orthonormal basis with respect to g_{ab} at p. Conversely, given an orthonormal basis $\overset{1}{\xi}{}^a, \ldots, \overset{n}{\xi}{}^a$ for g_{ab} at p we can find a Lorentz chart (U, ϕ) with $p \in U$ such that $\overset{i}{\xi}{}^a = (\frac{\partial}{\partial u^i})^a$ for each i (O'Neill, 1983, p. 72). Now suppose that (U, ϕ) and (V, ψ) are Lorentz charts at p. One can easily verify that

$$(\psi^* \circ \phi_*)(\frac{\partial}{\partial u^i})^a = (\frac{\partial}{\partial v^i})^a$$

where here again $(\frac{\partial}{\partial u^i})^a$ and $(\frac{\partial}{\partial v^i})^a$ are the coordinate vectors for (U, ϕ) and (V, ψ). This means that $\psi^* \circ \phi_*$ simply takes the orthonormal basis associated with (U, ϕ) to the orthonormal basis associated with (V, ψ).

Let α^a and β^a be any two unit timelike (with respect to g_{ab}) vectors at p. And suppose that g'_{ab} is another metric on M that satisfies the condition in the statement of the proposition. Since we can build orthonormal bases $\alpha^a, \overset{2}{\xi}{}^a, \ldots, \overset{n}{\xi}{}^a$ and $\beta^a, \overset{2}{\xi}{}'^a, \ldots, \overset{n}{\xi}{}'^a$ for g_{ab} at p, the above discussion implies that there are Lorentz charts (U, ϕ) and (V, ψ) for g_{ab} at p such that $(\psi^* \circ \phi_*)(\alpha^a) = \beta^a$. We compute the following:

$$\begin{aligned} g'_{ab}|_p \beta^a \beta^b &= ((\psi^* \circ \phi_*)(g'_{ab}|_p))\beta^a \beta^b \\ &= g'_{ab}|_p((\phi^* \circ \psi_*)(\beta^a))((\phi^* \circ \psi_*)(\beta^b)) \\ &= g'_{ab}|_p \alpha^a \alpha^b \end{aligned}$$

The first equality follows from our assumption about g'_{ab}, the second from properties of the pushforward and pullback, and the third holds since $(\psi^* \circ \phi_*)(\alpha^a) = \beta^a$. Because α^a and β^a are unit vectors with respect to g_{ab}, this implies that

$$\frac{g'_{ab}\alpha^a\alpha^b}{g_{ab}\alpha^a\alpha^b} = \frac{g'_{ab}\beta^a\beta^b}{g_{ab}\beta^a\beta^b}$$

This means that we have established that the ratio

$$\frac{g'_{ab}\alpha^a\alpha^b}{g_{ab}\alpha^a\alpha^b}$$

is the same for all unit timelike (with respect to g_{ab}) vectors α^a. Since the ratio is uneffected if we rescale α^a (i.e. if we replace α^a by $k\alpha^a$ for some non-zero k), it follows that the ratio is the same for all timelike (with respect to g_{ab}) vectors α^a. Let this constant ratio be c. Altogether, this means that $(g'_{ab} - cg_{ab})\alpha^a\alpha^b = 0$ for all timelike (with respect to g_{ab}) vectors α^a. Malament (2012, Proposition 2.1.3) implies that $g'_{ab} = cg_{ab}$ at p. Since g_{ab} and g'_{ab} are of Lorentzian signature and non-degenerate, the fact that $\dim(M) \geq 3$ implies that $c > 0$. □

This proposition does not establish a method of recovering the entire structure of (M, g_{ab}) from its privileged coordinates, but it does take a step toward answering another question that Barrett and Manchak (2024a) suggest, closely related to what they call "Reservation 2": Can *part* of the structure of a relativistic spacetime (M, g_{ab}) be presented using privileged coordinates? This result shows that the conformal structure of a spacetime (of dimension higher than two) can be recovered from this kind of privileged coordinates by employing a kind of 'pointwise' Kleinian method. One gets the feeling that the Barrett and Manchak (2024a) account does not allow one to do that. Barrett and Manchak (2024b, Lemma 4) show that there are Heraclitus spacetimes that are not related by homothety; one conjectures that there are also Heraclitus spacetimes that are not conformally equivalent. If so, then any account of privileged coordinates with Feature 1 and Feature 2 will not allow one to recover conformal structure, for the same reason as it does not allow one to recover metric structure up to isometry (or homothety).

4 Conclusion

We have on hand two accounts of privileged coordinates in general relativity, one due to Barrett and Manchak (2024a) and another due to Gomes et al. (2024). Each represents a way to generalize an account of privileged coordinates from the case of flat spacetimes, where the situation was clear. Gomes et al. (2024) suggest that the account of privileged coordinates that Barrett and Manchak (2024a) provide is "cumbersome," "unnatural," and "involved." While we agree that the proof that every spacetime has a representative is "involved" (Barrett and Manchak, 2024a, Lemma 3.2.2), we have emphasized above that as a whole the Barrett and Manchak (2024a) account is natural from a geometrical perspective and directly motivated by the case of flat spacetimes. One nonetheless wonders which of these two accounts (if either) better captures what one has in mind when one speaks of privileged coordinates and 'Kleinian methods.' We conclude with a few brief remarks on this.

It is standard to understand Kleinian methods as 'globally' presenting the structure of a geometric space, by singling out a collection of global coordinates and associated group of global symmetries (which transform one between these coordinates). As evidence for this, one notes that it is frequently remarked that Kleinian methods struggle with giraffe spacetimes — those with trivial (global) symmetry groups. We return to this point below. For now, we note that both of the accounts considered above are extensions of Kleinian methods, at least as traditionally understood. The account considered by Barrett and Manchak (2024a) appeals to the 'local' coordinates and associated pseudogroup of (local) symmetries of the space under consideration, not just its 'global' coordinates and associated group of (global) symmetries. This idea is not novel; it is based upon the transition that Wallace (2019) describes from G-structured spaces to *locally* G-structured spaces. The Gomes et al. (2024) account attempts to go 'even more local,' by adopting a 'pointwise' Kleinian presentation. There one aims to present the structure of (M, g_{ab}) by individually presenting its structure at each p in M in a Kleinian manner. It is worth considering these pointwise

Kleinian methods further. There might be other ones that fare better than the one discussed above. Gomes et al. (2024, p. 21) gesture at some, but the details remain to be worked out. We note here, however, that pointwise Kleinian methods take a step back towards a traditional 'Riemannian' presentation of g_{ab}, in which one characterizes the metric by explicitly presenting its value at each point p in M. The success of a quasi-Kleinian method like this would nonetheless yield some interesting philosophical payoffs. Wallace remarks that "it seems too strong to say that geometry *simpliciter* in modern physics is Riemannian in character" and "rather than one conception of geometry having won out in modern physics, we actually have peaceful coexistence" (Wallace, 2019, p. 135). Barrett and Manchak (2024a, p. 22) agree when they write that "it does seem that there is an echo of these Kleinian methods lurking below the surface in general relativity." If some pointwise Kleinian method — which employs aspects of both traditional Riemannian and traditional Kleinian approaches — were successful, that would provide a way to make this thought precise.

Both the Barrett and Manchak (2024a) and Gomes et al. (2024) extensions of standard Kleinian methods have benefits and drawbacks. Recall the Norton and Cartan assertion that "the spacetimes of classical mechanics and special relativity are of the type of Klein." The Barrett and Manchak (2024a) account of privileged coordinates makes good on this remark. On that account, one can present all flat relativistic spacetimes up to homothety (Barrett and Manchak, 2024b, Corollary 1) and Minkowski spacetime up to isometry (Barrett and Manchak, 2024a, Proposition 4.1.1); one conjectures that analogous results hold of flat classical spacetimes (Barrett and Manchak, 2024b, p. 18). The Barrett and Manchak (2024a) account therefore has one of the benefits standardly attributed to Kleinian methods. As it currently stands, the Gomes et al. (2024) method does not. When one applies the Gomes et al. (2024) account of privileged coordinates to flat spacetimes, one recovers only the spacetime up to conformal factor. Since there are spacetimes conformally equivalent to Minkowski spacetime that are not flat, one cannot use the Gomes et al. (2024) method to even tell whether the spacetime we began with was flat, let alone to present its entire structure.

Recall also the Norton and Cartan assertion that the spacetimes "of general relativity are of the type of Riemann" and not of Klein. North (2021, p. 117) suggests too that there are geometric spaces that "lie beyond the scope of Klein's program." Torretti (2016) directly writes that

> Klein's conception is too narrow to embrace all Riemannian geometries, which include spaces of variable curvature. Indeed, in the general case, the group of isometries of a Riemannian n-manifold is the trivial group consisting of the identity alone, whose structure conveys no information at all about the respective geometry.

See Norton (1993, p. 832–3) and Norton (1999, p. 129–30) for similar remarks. The Gomes et al. (2024) account allows one to sidestep this kind of worry; that pointwise Kleinian method at least allows one to present the conformal structure of giraffe spacetimes. The same holds even of Heraclitus spacetimes. The kind of triviality problem for Kleinian methods that Norton, Cartan, North, Torretti, and others are pointing to is thus partially dodged; at least some of the structure of spacetimes with 'no symmetries' can be presented using pointwise Kleinian methods.

The Barrett and Manchak (2024a) account also partially dodges this worry. As our earlier discussion of the distinction between Heraclitus and giraffes illustrates, the Kleinian method considered by Barrett and Manchak (2024a) is more powerful than the one that (for example) Torretti has in mind in the above quote. One sees this by recalling again that there are flat relativistic spacetimes that are giraffe, i.e. whose isometry groups are trivial. Since they are flat, such spacetimes will be presentable up to homothety on the Barrett and Manchak (2024a) account. (For a closer approximation of the kind of Kleinian method Torretti has in mind, see the discussion of G-structured spaces — rather than *locally* G-structured spaces — in Barrett and Manchak (2024a) and Wallace (2019). The framework of G-structured spaces does struggle with mere giraffes.) In this sense, this account improves upon some of the shortcomings that are standardly attributed to Kleinian methods. But in addition, it allows one to extend and make precise the simple idea behind the triviality worry suggested by Norton, Cartan, North, Torretti, and others. It allows one to capture the common intuition that the Kleinian project breaks down for spaces with 'no symmetries.' For this more powerful Kleinian method, this breakdown occurs not at the giraffe level, but rather at the higher Heraclitus level. This point could not be appreciated until recently, since the only 'no symmetry' idea in the discourse was that of spacetimes with trivial isometry groups, i.e. giraffe spacetimes (Manchak and Barrett, 2024). Altogether, this means that the drawbacks that the Barrett and Manchak (2024a) account of privileged coordinates has are in the same vein as the ones standardly attributed to Kleinian methods, and this suggests that the Barrett and Manchak (2024a) account is a natural descendent of the traditional Kleinian method.

We conclude with the following suggestion. There are various desiderata that one wants an account of privileged coordinates and Kleinian methods to satisfy. Features 1, 2, and 3 provide a few examples. There is an account of privileged coordinates of flat spacetimes where all of these desiderata hold, but it is difficult to maintain all of them when generalizing to the arbitrary case (see Questions 1 and 2). As a result, it is important to acknowledge that there are different things one might mean by 'privileged coordinates' in general relativity. There is value in cataloguing the possibilities and then assessing their costs and benefits regarding whether, and to what extent, they allow one to recover spacetime structure.

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