



The exact (up to infinitesimals) infinite perimeter of the Koch snowflake and its finite area



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ABSTRACT

The Koch snowflake is one of the first fractals that were mathematically described. It is interesting because it has an infinite perimeter in the limit but its limit area is finite. In this paper, a recently proposed computational methodology allowing one to execute numerical computations with infinities and infinitesimals is applied to study the Koch snowflake at infinity. Numerical computations with actual infinite and infinitesimal numbers can be executed on the infinity Computer being a new supercomputer patented in USA and EU. It is revealed in the paper that at infinity the snowflake is not unique, i.e., different snowflakes can be distinguished for different infinite numbers of steps executed during the process of their generation. It is then shown that for any given infinite number n of steps it becomes possible to calculate the exact infinite number, N_n , of sides of the snowflake, the exact infinitesimal length, L_n , of each side and the exact infinite perimeter, P_n , of the Koch snowflake as the result of multiplication of the infinite N_n by the infinitesimal L_n . It is established that for different infinite n and k the infinite perimeters P_n and P_k are also different and the difference can be infinite. It is shown that the finite areas A_n and A_k of the snowflakes can be also calculated exactly (up to infinitesimals) for different infinite n and k and the difference $A_n - A_k$ results to be infinitesimal. Finally, snowflakes constructed starting from different initial conditions are also studied and their quantitative characteristics at infinity are computed.

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1. Introduction

Nowadays many fractals are known and their presence can be found in nature, especially in physics and biology, in science, and in engineering (see, e.g., [6,8,22,23] and references given therein). Even though fractal structures were ever around us, their active study started rather recently. The first fractal curves have been proposed at the end of XIXth century (see, e.g., historical reviews in [26,47]) and the word *fractal* has been introduced by Mandelbrot (see [16,17]) in the second half of the XXth century. The main geometric characterization of simple fractals is their self-similarity repeated infinitely many times: fractals are made by an infinite generation of an increasing number of smaller and smaller copies of a basic figure often called an initiator. More generally, fractal objects need not exhibit exactly the same structure at all scales, variations of initiators and generating procedures (see, e.g., L-systems in [25]) are conceded. Another important feature of fractal objects is that they often exhibit fractional dimensions.

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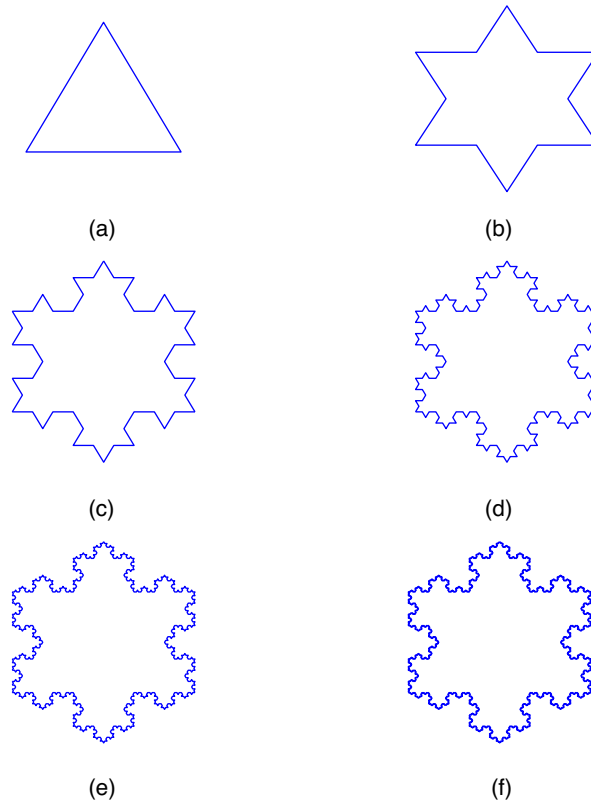


Fig. 1. Generation of the Koch snowflake.

Since fractals are objects defined as a limit of an infinite process, the computation of their dimension is one of a very few quantitative characteristics that can be calculated at infinity. After n iterations of a fractal process we can give numerical answers to questions regarding fractals (calculation of, e.g., their length, area, volume or the number of smaller copies of initiators present at the n -th iteration) only for *finite* values of n . The same questions very often remain without any answers when we consider an infinite number of steps because when we speak about limit fractal objects the required values often either tend to zero and disappear, in practice, or tend to infinity, i.e., become intractable numerically. Moreover, we cannot distinguish at infinity fractals starting from similar initiators even though they are different for any fixed finite value of the iteration number n .

In this paper, we study the Koch snowflake that is one of the first mathematically described fractals. It has been introduced by Helge von Koch in 1904 (see [13]). This fractal is interesting because it is known that in the limit it has an infinite perimeter but its area is finite. The procedure of its construction is shown in Fig. 1. The initiator (iteration number $n = 0$) is the triangle shown in Fig. 1(a), more precisely, its three sides. Then each side (segment) is substituted by four smaller segments as it is shown in Fig. 1(b) during the first iteration, in other words, a smaller copy of the triangle is added to each side. At the iteration $n = 2$ (see Fig. 1(c)) each segment is substituted again by four smaller segments, and so on. Thus, the Koch snowflake is the resulting limit object obtained at $n \rightarrow \infty$. It is known (see, e.g., [22]) that its fractal dimension is equal to $\frac{\log 4}{\log 3} \approx 1.26186$.

Clearly, for finite values of n we can calculate the perimeter P_n and the respective area A_n of the snowflake. If iteration numbers n and k are such that $n \neq k$ then it follows $P_n \neq P_k$ and $A_n \neq A_k$. Moreover, the snowflake started from the initiator (a) after n iterations will be different with respect to the snowflake started from the configuration (b) after n iterations. The simple illustration for $n = 3$ can be viewed in Fig. 1. In fact, starting from the initiator (a) after three iterations we have the snowflake (d) and starting from the initiator (b) after three iterations we have the snowflake (e).

Unfortunately, the traditional analysis of fractals does not allow us to have quantitative answers to the questions stated above when $n \rightarrow \infty$. In fact, we know only (see, e.g., [22]) that

$$\lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} \frac{4^n}{3^{n-1}} l = \infty, \quad \lim_{n \rightarrow \infty} A_n = \frac{8}{5} a_0, \quad a_0 = \frac{\sqrt{3}}{4} l^2, \quad (1)$$

where a_0 is the area of the original triangle from Fig. 1(a) expressed in the terms of its side length l .

In this paper, by using a new computational methodology introduced in [27,28,31] we show that a more precise quantitative analysis of the Koch snowflake can be done at infinity. In particular, it becomes possible:

- to show that at infinity the Koch snowflake is not a unique object, namely, different snowflakes can be distinguished at infinity similarly to different snowflakes that can be distinguished for different finite values of n ;
- to calculate the exact (up to infinitesimals) perimeter P_n of the snowflake (together with the infinite number of sides and the infinitesimal length of each side) after n iterations for different infinite values of n ;
- to show that for infinite n and k such that $k > n$ it follows that both P_n and P_k are infinite but $P_k > P_n$ and their difference $P_k - P_n$ can be computed exactly and it results to be infinite;
- to calculate the exact (up to infinitesimals) finite areas A_n and A_k of the snowflakes for infinite n and k such that $k > n$ and to show that it follows $A_k > A_n$, the difference $A_k - A_n$ can be also calculated exactly and it results to be infinitesimal;
- to show that the snowflakes constructed starting from different initiators (e.g., from initiators shown in Fig. 1) are different after k iterations where k is an infinite number, i.e., they have different infinite perimeters and different areas where the difference can be measured using infinities and infinitesimals.

2. A new numeral system expressing infinities and infinitesimals with high accuracy

In order to understand how it is possible to study fractals at infinity with an accuracy that is higher than that one provided by expressions in (1), let us remind the important difference that exists between numbers and numerals. A numeral is a symbol or a group of symbols used to represent a number. A number is a concept that a numeral expresses and the difference between them is the same as the difference between words written in a language and the things the words refer to. Obviously, the same number can be represented in a variety of ways by different numerals. For example, the symbols ‘11’, ‘eleven’, ‘IIIIIIIIII’, ‘XI’, and ‘≡’ are different numerals,¹ but they all represent the same number. A numeral system consists of a set of rules used for writing down numerals and algorithms for executing arithmetical operations with these numerals. It should be stressed that the algorithms can vary significantly in different numeral systems and their complexity can be also dissimilar. For instance, division in Roman numerals is extremely laborious and in the positional numeral system it is much easier.

Notice also that different numeral systems can express different sets of numbers. One of the simplest existing numeral systems that allows its users to express very few numbers is the system used by Warlpiri people, aborigines living in the Northern Territory of Australia (see [1]) and by Pirahã people living in Amazonia (see [7]). Both groups use the same very poor numeral system for counting consisting just of three numerals – one, two, and ‘many’ – where ‘many’ is used for all quantities larger than two.

As a result, this poor numeral system does not allow Warlpiri and Pirahã to distinguish numbers larger than 2, to execute arithmetical operations with them, and, in general, to say a word about these quantities because in their languages there are neither words nor concepts for them. In particular, results of operations $2+1$ and $2+2$ are not 3 and 4 but just ‘many’ since they do not know about the existence of 3 and 4. It is worthy to emphasize thereupon that the result ‘many’ is not wrong, it is correct but its accuracy is low. Analogously, when we look at a mob, then both phrases ‘There are 2053 persons in the mob’ and ‘There are many persons in the mob’ are correct but the accuracy of the former phrase is higher than the accuracy of the latter one.

Our interest to the numeral system of Warlpiri and Pirahã is explained by the fact that the poorness of this numeral system leads to such results as

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’}, \tag{2}$$

$$\text{‘many’} - 1 = \text{‘many’}, \quad \text{‘many’} - 2 = \text{‘many’}, \tag{3}$$

$$\text{‘many’} + \text{‘many’} = \text{‘many’} \tag{4}$$

that are crucial for changing our outlook on infinity. In fact, by changing in these relations ‘many’ with ∞ we get relations that are used for working with infinity in the traditional calculus:

$$\infty + 1 = \infty, \quad \infty + 2 = \infty, \quad \infty - 1 = \infty, \quad \infty - 2 = \infty, \quad \infty + \infty = \infty. \tag{5}$$

We can see that numerals ‘many’ and ∞ are used in the same way and we know that in the case of ‘many’ expressions in (2)–(4) are nothing else but the result of the lack of appropriate numerals for working with finite quantities. This analogy allows us to conclude that expressions in (5) used to work with infinity are also just the result of the lack of appropriate numerals, in this case for working with infinite quantities. As the numeral ‘many’ is not able to represent the existing richness of finite numbers, the numeral ∞ is not able to represent the richness of the infinite ones.

Notice that it is well known that numeral systems strongly bound the possibilities to express numbers and to execute mathematical operations with them. For instance, the Roman numeral system lacks a numeral expressing zero. As a consequence, such expressions as V–V and II–XI in this numeral system are indeterminate forms. The introduction of the positional numeral system has allowed people to avoid indeterminate forms of this type and to execute the required operations easily.

In order to give the possibility to write down more infinite and infinitesimal numbers, a new numeral system has been introduced recently in [27,29,31,43]. It allows people to express a variety of different infinities and infinitesimals, to perform numerical computations with them, and to avoid both expressions of the type (5) and indeterminate forms such as $\infty - \infty$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, etc. present in the traditional calculus and related to limits with an argument tending to ∞ or zero. The numeral system

¹ The last numeral, \equiv , is probably less known. It belongs to the Maya numeral system where one horizontal line indicates five and two lines one above the other indicate ten. Dots are added above the lines to represent additional units. So, \equiv means eleven and is written as $5 + 5 + 1$.

Table 1

Measuring infinite sets using $\textcircled{1}$ -based numerals allows one in certain cases to obtain more precise answers in comparison with the traditional cardinalities, \aleph_0 and C , of Cantor.

Description of sets	Cardinality	Number of elements
The set of natural numbers \mathbb{N}	Countable, \aleph_0	$\textcircled{1}$
$\mathbb{N} \setminus \{3, 5, 10, 23\}$	Countable, \aleph_0	$\textcircled{1}-4$
The set of even numbers \mathbb{E}	Countable, \aleph_0	$\frac{\textcircled{1}}{2}$
The set of odd numbers \textcircled{O}	Countable, \aleph_0	$\frac{\textcircled{1}}{2}$
The set of square natural numbers $\mathbb{G} = \{x : x = n^2, n \in \mathbb{N}, x \in \mathbb{N}\}$	Countable, \aleph_0	$\lfloor \sqrt{\textcircled{1}} \rfloor$
The set of integer numbers \mathbb{Z}	Countable, \aleph_0	$2\textcircled{1}+1$
The set of pairs of natural numbers $\mathbb{P} = \{(p, q) : p \in \mathbb{N}, q \in \mathbb{N}\}$	Countable, \aleph_0	$\textcircled{1}^2$
The set of numerals $\mathbb{Q}' = \{-\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	Countable, \aleph_0	$2\textcircled{1}^2$
The set of numerals $\mathbb{Q} = \{0, -\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	Countable, \aleph_0	$2\textcircled{1}^2 + 1$
The set of numerals A_2	Continuum, C	$2\textcircled{1}$
The set of numerals A'_2	Continuum, C	$2\textcircled{1} + 1$
The set of numerals A_{10}	Continuum, C	$10\textcircled{1}$
The set of numerals C_{10}	Continuum, C	$2 \cdot 10\textcircled{1}$

from [27,29,31,43] has allowed the author to propose a corresponding computational methodology and to introduce the infinity Computer (see the patent [34]) being a supercomputer working numerically with a variety of infinite and infinitesimal numbers. Notice that even though the new methodology works with infinite and infinitesimal quantities, it is not related to symbolic computations practiced in non-standard analysis (see [24]) and has an applied, computational character.

In order to see the place of the new approach in the historical panorama of ideas dealing with infinite and infinitesimal, see [12,14,15,18,21,33,35,43,44]. In particular, connections of the new approach with bijections is studied in [18] and metamathematical investigations on the new theory and its non-contradictory can be found in [15]. The new methodology has been successfully applied for studying percolation and biological processes (see [9,10,39,48]), infinite series (see [11,33,38,49]), hyperbolic geometry (see [19,20]), fractals (see [9,10,30,32,39]), numerical differentiation and optimization (see [2,37,50]), the first Hilbert problem, Turing machines, and lexicographic ordering (see [35,42,44–46]), cellular automata (see [3–5]), ordinary differential equations (see [40,41]), etc.

In this paper, the new numeral system and the respective computational methodology are used to study the Koch snowflake. Both the system and the methodology are based on the introduction in the process of computations of a new numeral, $\textcircled{1}$, called *grossone*. It is defined as the infinite integer being the number of elements of the set, \mathbb{N} , of natural numbers.² Symbols used traditionally to deal with infinite and infinitesimal quantities (e.g., ∞ , Cantor's ω , \aleph_0 , \aleph_1 , ... etc.) are not used together with $\textcircled{1}$. Similarly, when the positional numeral system and the numeral 0 expressing zero had been introduced, symbols I, IV, VI, XIII, and other symbols from the Roman numeral system had been substituted by the respective Arabic symbols.

The numeral $\textcircled{1}$ allows one to express a variety of numerals representing different infinities and infinitesimals, to order them, and to execute numerical computations with all of them in a handy way. For example, for $\textcircled{1}$ and $\textcircled{1}^{3.1}$ (that are examples of infinities) and $\textcircled{1}^{-1}$ and $\textcircled{1}^{-3.1}$ (that are examples of infinitesimals) it follows

$$\begin{aligned}
 0 \cdot \textcircled{1} &= \textcircled{1} \cdot 0 = 0, & \textcircled{1} - \textcircled{1} &= 0, & \frac{\textcircled{1}}{\textcircled{1}} &= 1, & \textcircled{1}^0 &= 1, & 1\textcircled{1} &= 1, & 0\textcircled{1} &= 0, \\
 0 \cdot \textcircled{1}^{-1} &= \textcircled{1}^{-1} \cdot 0 = 0, & \textcircled{1}^{-1} &> 0, & \textcircled{1}^{-3.1} &> 0, & \textcircled{1}^{-1} - \textcircled{1}^{-1} &= 0, \\
 \frac{\textcircled{1}^{-1}}{\textcircled{1}^{-1}} &= 1, & \frac{5 + \textcircled{1}^{-3.1}}{\textcircled{1}^{-3.1}} &= 5\textcircled{1}^{3.1} + 1, & (\textcircled{1}^{-1})^0 &= 1, & \textcircled{1} \cdot \textcircled{1}^{-1} &= 1, \\
 \textcircled{1} \cdot \textcircled{1}^{-3.1} &= \textcircled{1}^{-2.1}, & \frac{\textcircled{1}^{3.1} + 4\textcircled{1}}{\textcircled{1}} &= \textcircled{1}^{2.1} + 4, & \frac{\textcircled{1}^{3.1}}{\textcircled{1}^{-3.1}} &= \textcircled{1}^{6.2}, \\
 (\textcircled{1}^{3.1})^0 &= 1, & \textcircled{1}^{3.1} \cdot \textcircled{1}^{-1} &= \textcircled{1}^{2.1}, & \textcircled{1}^{3.1} \cdot \textcircled{1}^{-3.1} &= 1.
 \end{aligned} \tag{6}$$

It follows from (6) that a finite number b can be represented in this numeral system simply as $b\textcircled{1}^0 = b$, since $\textcircled{1}^0 = 1$, where the numeral b itself can be written down by any convenient numeral system used to express finite numbers. The simplest infinitesimal numbers are represented by numerals having only negative finite powers of $\textcircled{1}$ (e.g., the number $5.1\textcircled{1}^{-1.2} + 6.8\textcircled{1}^{-20.3}$ consists of two infinitesimal parts, see also examples above). Notice that all infinitesimals are not equal to zero. For instance, $\textcircled{1}^{-3.1} = \frac{1}{\textcircled{1}^{3.1}}$ is positive because it is the result of division between two positive numbers.

In the context of the present paper it is important that in comparison to the traditional mathematical tools used to work with infinity the new numeral system allows one to obtain more precise answers in certain cases. For instance, Table 1 compares

² Notice that nowadays not only positive integers but also zero is frequently included in \mathbb{N} . However, since zero has been invented significantly later than positive integers used for counting objects, zero is not include in \mathbb{N} in this text.

results obtained by the traditional Cantor's cardinals and the new numeral system with respect to the measure of a great dozen of infinite sets (for a detailed discussion regarding the results presented in Table 1 and for more examples dealing with infinite sets see [15,18,35,36,44]). Notice, that in \mathbb{Q} and \mathbb{Q}' we calculate different numerals and not numbers. For instance, numerals $\frac{3}{1}$ and $\frac{6}{2}$ have been counted two times even though they represent the same number 3. Then, four sets of numerals having the cardinality of continuum are shown in Table 1. Among them we denote by A_2 the set of numbers $x \in [0, 1)$ expressed in the binary positional numeral system, by A'_2 the set being the same as A_2 but with x belonging to the closed interval $[0, 1]$, by A_{10} the set of numbers $x \in [0, 1)$ expressed in the decimal positional numeral system, and finally we have the set $C_{10} = A_{10} \cup B_{10}$, where B_{10} is the set of numbers $x \in [1, 2)$ expressed in the decimal positional numeral system. It is worthwhile to notice also that $\textcircled{1}$ -based numbers present in Table 1 can be ordered as follows

$$\begin{aligned} \lfloor \sqrt{\textcircled{1}} \rfloor &< \frac{\textcircled{1}}{2} < \textcircled{1} - 4 < \textcircled{1} < 2\textcircled{1} < 2\textcircled{1} + 1 \\ &< \textcircled{1}^2 < 2\textcircled{1}^2 + 1 < 2^{\textcircled{1}} < 2^{\textcircled{1}} + 1 < 10^{\textcircled{1}} < 2 \cdot 10^{\textcircled{1}}. \end{aligned}$$

It can be seen from Table 1 that Cantor's cardinalities say only whether a set is countable or uncountable while the $\textcircled{1}$ -based numerals allow us to express the exact number of elements of the infinite sets. However, both numeral systems – the new one and the numeral system of infinite cardinals – do not contradict one another. Both numeral systems provide correct answers, but their answers have different accuracies. By using an analogy from physics we can say that the lens of our new ‘telescope’ used to observe infinite sets is stronger and where Cantor's ‘telescope’ allows one to distinguish just two dots (countable sets and the continuum) we are able to see many different dots (infinite sets having different number of elements).

3. Quantitative characteristics of the Koch snowflake at infinity

As was mentioned above, $\textcircled{1}$ can be successfully used for various purposes related to studying infinite and infinitesimal objects, in particular, for indicating positions of elements in infinite sequences (see, e.g., [43–45]) and for working with divergent series (see [11,33,38,49]). Both topics will help us to study the Koch snowflake at infinity. Let us first compare how infinite sequences are defined from the traditional point of view and from the new one.

The traditional definition is very simple: an infinite sequence $\{b_n\}$, where for all $n \in \mathbb{N}$ elements b_n belong to a set B is defined as a function having as its domain the set \mathbb{N} and as its codomain the set B . Let us see now how this definition can be reformulated using the new methodology and $\textcircled{1}$ -based numerals. Remember that $\textcircled{1}$ has been introduced as the number of elements of the set \mathbb{N} . Thus, due to the traditional definition given above, any sequence having \mathbb{N} as the domain has $\textcircled{1}$ elements.

In its turn, the notion of a subsequence is introduced traditionally as a sequence from which some of its elements have been deleted. In cases where both the original sequence and the obtained subsequence are infinite, in spite of the fact that some elements were excluded, the traditional fashion does not allow us to record in some way that the obtained infinite subsequence has less elements than the original infinite sequence. In the new fashion there is such a possibility. Having a sequence with $\textcircled{1}$ elements exclusion of k elements from it gives a subsequence having $\textcircled{1} - k < \textcircled{1}$ elements. For instance, in (7) the first infinite sequence has $\textcircled{1}$ elements and the second one $\textcircled{1}-2$ elements:

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}}_{\textcircled{1} \text{ elements}}, \underbrace{4, 5, 6, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}, \textcircled{1} + 1}_{\textcircled{1}-2 \text{ elements}}. \tag{7}$$

Thus, the numeral system using $\textcircled{1}$ allows us to observe not only the starting but also the ending elements of infinite processes, if the respective elements are expressible in this numeral system. This fact is important in connection with fractals because it allows us to distinguish different fractal objects after an infinite number of steps of their construction. Another useful observation consists of the fact that, since the number of elements of any sequence (finite or infinite) is less or equal to $\textcircled{1}$, any sequential process can have at maximum $\textcircled{1}$ steps (see [44]).

Let us see now what the new approach allows us to say with respect to divergent series. In particular, the situations where it is necessary to sum up an infinite number of infinitesimal numbers will be of our primary interest (for a detailed discussion see [11,33,38,49]). This issue is important for us since in (1) the perimeter P_n of the snowflake studied traditionally goes to infinity. Grossone-based numerals allow us to express not only different finite numbers but also different infinite numbers, so such expressions as $S_1 = b_1 + b_2 + \dots$ or $S_1 = \sum_{i=1}^{\infty} b_i$ become unprecise since the number of addends in S_1 is not indicated explicitly. If we use again the analogy with Warlpiri and Pirahã then we can say that the record $\sum_{i=1}^{\infty} b_i$ can be interpreted as $\sum_{i=1}^{\text{many}} b_i$. Notice that in the finite sums the situation is the same: it is not sufficient to say that the number of summands is finite, it is necessary to define explicitly their number.

The new approach gives the possibility to add finite, infinite, and infinitesimal values in a handy way, the number of summands can be finite or infinite, and results of addition can be finite, infinite, and infinitesimal in dependence on the sort and number of addends. To illustrate this assertion let us consider a few examples. First, it becomes possible to compute the sum of all natural numbers from 1 to $\textcircled{1}$ as follows

$$1 + 2 + 3 + \dots + (\textcircled{1} - 1) + \textcircled{1} = \sum_{i=1}^{\textcircled{1}} i = \frac{\textcircled{1}}{2} (1 + \textcircled{1}) = 0.5\textcircled{1}^2 + 0.5\textcircled{1}. \tag{8}$$

The following sum of infinitesimals where each summand is $\textcircled{1}$ times less than the corresponding item of (8) can be also computed easily

$$\begin{aligned} & \textcircled{1}^{-1} + 2\textcircled{1}^{-1} + \dots + (\textcircled{1} - 1) \cdot \textcircled{1}^{-1} + \textcircled{1} \cdot \textcircled{1}^{-1} \\ &= \sum_{i=1}^{\textcircled{1}} i\textcircled{1}^{-1} = \frac{\textcircled{1}}{2} (\textcircled{1}^{-1} + 1) = 0.5\textcircled{1}^1 + 0.5. \end{aligned} \tag{9}$$

As expected, the obtained number, $0.5\textcircled{1}^1 + 0.5$ is $\textcircled{1}$ times less than the result obtained in (8). Notice that this example shows, in particular, that sum of infinitely many infinitesimals can be infinite.

Then, in the same way as it happens in situations where the number of summands is finite, the following examples show that smaller or larger number of summands changes the result (cf. (8), (9))

$$\begin{aligned} \sum_{i=1}^{\textcircled{1}-1} i &= \frac{\textcircled{1}-1}{2} (1 + \textcircled{1} - 1) = 0.5\textcircled{1}^2 - 0.5\textcircled{1}, \\ \sum_{i=1}^{3\textcircled{1}^2} i\textcircled{1}^{-1} &= \frac{3\textcircled{1}^2}{2} (\textcircled{1}^{-1} + 3\textcircled{1}) = 4.5\textcircled{1}^3 + 1.5\textcircled{1}^2. \end{aligned}$$

Notice that sums can have more than $\textcircled{1}$ addends if it is not required to execute the operation of addition by a successive adding summands, i.e., the summation can be done in parallel. However, if in a particular application there exists a restriction that the required summation should be executed sequentially, then, since any sequential process cannot have more than $\textcircled{1}$ steps, the sequential process of the summation cannot have more than $\textcircled{1}$ addends.

We are ready now to return to the Koch snowflake and to study it at infinity using $\textcircled{1}$ -based numerals. It can be seen from Fig. 1 that at each iteration each side of the snowflake is substituted by four new sides having the length of one third of the segment that has been substituted. Thus, if we indicate as N_n , $n \geq 1$, the number of segments of the snowflake and as L_n their length at the n -th iteration then

$$N_n = 4N_{n-1} = 3 \cdot 4^n, \quad L_n = \frac{1}{3}L_{n-1} = \frac{l}{3^n}, \quad n > 1, \tag{10}$$

where l is the length of each side of the original triangle from Fig. 1. As a result, the perimeter P_n of the Koch snowflake is calculated as follows

$$P_n = N_n \cdot L_n = \frac{4}{3}N_{n-1} \cdot L_{n-1} = \frac{4}{3}P_{n-1} = \frac{4^n}{3^{n-1}}l. \tag{11}$$

Therefore, if we start our computations from the original triangle from Fig. 1, after $\textcircled{1}$ steps we have the snowflake having the infinite number of segments $N_{\textcircled{1}} = 3 \cdot 4^{\textcircled{1}}$. Each of the segments has the infinitesimal length $L_{\textcircled{1}} = \frac{1}{3^{\textcircled{1}}}l$. In order to calculate the perimeter, $P_{\textcircled{1}}$, of the snowflake we should multiple the infinite number $N_{\textcircled{1}}$ and the infinitesimal number $L_{\textcircled{1}}$. Thus the perimeter is

$$P_{\textcircled{1}} = N_{\textcircled{1}} \cdot L_{\textcircled{1}} = 3 \cdot 4^{\textcircled{1}} \cdot \frac{1}{3^{\textcircled{1}}}l = \frac{4^{\textcircled{1}}}{3^{\textcircled{1}-1}}l$$

and it is infinite. Analogously, in case we have executed $\textcircled{1} - 1$ steps we have the infinite perimeter $P_{\textcircled{1}-1} = \frac{4^{\textcircled{1}-1}}{3^{\textcircled{1}-2}}l$. Since the new numeral systems allows us to execute easily arithmetical operations with infinite numbers, we can divide the obtained two infinite numbers, $P_{\textcircled{1}}$ and $P_{\textcircled{1}-1}$, one by another and to obtain as the result the finite number that is in a complete agreement with (11)

$$\frac{P_{\textcircled{1}}}{P_{\textcircled{1}-1}} = \frac{\frac{4^{\textcircled{1}}}{3^{\textcircled{1}-1}}l}{\frac{4^{\textcircled{1}-1}}{3^{\textcircled{1}-2}}l} = \frac{4}{3}.$$

The difference of the two perimeters can also be calculated easily

$$P_{\textcircled{1}} - P_{\textcircled{1}-1} = \frac{4^{\textcircled{1}}}{3^{\textcircled{1}-1}}l - \frac{4^{\textcircled{1}-1}}{3^{\textcircled{1}-2}}l = \frac{4^{\textcircled{1}-1}}{3^{\textcircled{1}-2}}l \left(\frac{4}{3} - 1 \right) = \frac{4^{\textcircled{1}-1}}{3^{\textcircled{1}-1}}l$$

and it results to be infinite.

In case the infinite number of steps $n = 0.5\textcircled{1}$, it follows that the infinite perimeter is $P_{0.5\textcircled{1}} = \frac{4^{0.5\textcircled{1}}}{3^{0.5\textcircled{1}-1}}l$ and the operation of division of two infinite numbers gives us as the result also an infinite number that can be calculated precisely:

$$\frac{P_{\textcircled{1}}}{P_{0.5\textcircled{1}}} = \frac{\frac{4^{\textcircled{1}}}{3^{\textcircled{1}-1}}l}{\frac{4^{0.5\textcircled{1}}}{3^{0.5\textcircled{1}-1}}l} = \left(\frac{4}{3} \right)^{0.5\textcircled{1}}.$$

Thus we can distinguish now in a precise manner that the infinite perimeter $P_{\textcircled{1}}$ is infinitely times longer than the infinite perimeter $P_{0.5\textcircled{1}}$.

We can distinguish also at infinity the snowflakes having different initial generators. As we have already seen, starting from the original triangle after $\textcircled{1}$ steps the snowflake has $3 \cdot 4^{\textcircled{1}}$ segments and each of them has the infinitesimal length $L_{\textcircled{1}} = \frac{1}{3^{\textcircled{1}}}l$ and the perimeter of the snowflake is $P_{\textcircled{1}} = \frac{4^{\textcircled{1}}}{3^{\textcircled{1}-1}}l$. If we start from the initial configuration shown in Fig. 1(c) and also execute $\textcircled{1}$ steps then the resulting snowflake will have $N_{\textcircled{1}+2} = 3 \cdot 4^{\textcircled{1}+2}$ segments, each of them will have the infinitesimal length $L_{\textcircled{1}+2} = \frac{1}{3^{\textcircled{1}+2}}l$ and the perimeter of the snowflake will be $P_{\textcircled{1}+2} = \frac{4^{\textcircled{1}+2}}{3^{\textcircled{1}+1}}l$. Thus, this snowflake will have infinitely more segments than the one started from the original triangle. More precisely, this infinite difference is equal to

$$N_{\textcircled{1}+2} - N_{\textcircled{1}} = 3 \cdot 4^{\textcircled{1}+2} - 3 \cdot 4^{\textcircled{1}} = 3 \cdot 4^{\textcircled{1}}(4^2 - 1) = 45 \cdot 4^{\textcircled{1}}.$$

The lengths of the segments in both snowflakes are infinitesimal and, in spite of the fact that their difference is also infinitesimal, it can be calculated precisely as follows

$$L_{\textcircled{1}} - L_{\textcircled{1}+2} = \frac{1}{3^{\textcircled{1}}}l - \frac{1}{3^{\textcircled{1}+2}}l = \frac{1}{3^{\textcircled{1}}}l \left(1 - \frac{1}{3^2}\right) = \frac{8l}{3^{\textcircled{1}+2}}.$$

Let us see now what happens with the area of the snowflake at infinity. As it can be seen from Fig. 1, at each iteration n a new triangle is added at each side of the snowflake built at iteration $n - 1$ and, therefore, the number of new triangles, T_n , is equal to

$$T_n = N_{n-1} = 3 \cdot 4^{n-1}.$$

The area, a_n , of each triangle added at n th iteration is $\frac{1}{9}$ of each triangle added during the iteration $n - 1$

$$a_n = \frac{a_{n-1}}{9} = \frac{a_0}{9^n},$$

where a_0 is the area of the original triangle (see (1)). Therefore, the whole new area added to the snowflake is

$$T_n a_n = 3 \cdot 4^{n-1} \cdot \frac{a_0}{9^n} = \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot a_0 = \frac{a_0}{3} \cdot \left(\frac{4}{9}\right)^{n-1} \tag{12}$$

and the complete area, A_n , of the snowflake at the n th iteration is

$$\begin{aligned} A_n &= a_0 + \sum_{i=1}^n T_i a_i = a_0 \left(1 + \frac{1}{3} \sum_{i=1}^n \left(\frac{4}{9}\right)^{i-1}\right) = a_0 \left(1 + \frac{1}{3} \sum_{i=0}^{n-1} \left(\frac{4}{9}\right)^i\right) \\ &= a_0 \left(1 + \frac{1}{3} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}}\right) = a_0 \left(1 + \frac{3}{5} \left(1 - \left(\frac{4}{9}\right)^n\right)\right) = \frac{a_0}{5} \left(8 - 3 \left(\frac{4}{9}\right)^n\right). \end{aligned}$$

Traditionally, the limit of the area is considered and the result

$$\lim_{n \rightarrow \infty} A_n = \frac{8}{5} a_0$$

is obtained. Thanks to $\textcircled{1}$ -based numerals we are able now to work with infinitesimals easily (see [33,38,49] for more results on summing up infinitesimals, divergent series, etc.) and to observe the infinitesimal difference of the areas of the snowflakes at infinity. For example, after $\textcircled{1} - 1$ and $\textcircled{1}$ iterations the difference between the areas $A_{\textcircled{1}-1}$ and $A_{\textcircled{1}}$ is

$$A_{\textcircled{1}} - A_{\textcircled{1}-1} = \frac{a_0}{5} \left(8 - 3 \left(\frac{4}{9}\right)^{\textcircled{1}}\right) - \frac{a_0}{5} \left(8 - 3 \left(\frac{4}{9}\right)^{\textcircled{1}-1}\right) = \frac{a_0}{3} \left(\frac{4}{9}\right)^{\textcircled{1}-1} > 0.$$

This number is infinitesimal and it perfectly corresponds to the general formula (12).

4. Conclusion

Traditional approaches for studying dynamics of fractal processes very often are not able to give their quantitative characteristics at infinity and, as a consequence, use limits to overcome this difficulty. The Koch snowflake in fact is defined as the limit object and, in the limit, its perimeter goes to infinity and, at the same time, its limit area is finite. As a consequence, questions regarding quantitative characteristics of the snowflake at infinity very often remain without any answer if traditional mathematical tools are used. Moreover, we cannot distinguish at infinity snowflakes starting from different initiators even though they are different for any fixed finite value of the generation step n .

In this paper, it has been shown that recently introduced $\textcircled{1}$ -based numerals give the possibility to work with different infinities and infinitesimals numerically and to establish the presence of *infinitely many different* snowflakes at infinity instead of a unique snowflake observed traditionally. For different infinite values of generation steps n it becomes possible to obtain their exact quantitative characteristics instead of traditionally made qualitative declarations saying that limits under consideration go to infinity.

In particular, it becomes possible to compute the exact infinite number, N_n , of sides of the snowflake, the exact infinitesimal length, L_n , of each side for a given infinite n and to calculate the exact infinite value of the perimeter, P_n , of the Koch snowflake.

It has been shown that the sum of infinitely many infinitesimal sides gives as the result the infinite perimeter P_n of the Koch snowflake. As a consequence, it has been also shown that for infinite $k > n$ it follows that infinitesimal $L_k < L_n$, infinite $N_k > N_n$, $P_k > P_n$ and the exact difference $P_k - P_n$ can be calculated. The areas A_n and A_k can be also computed exactly for infinite values k and n . If $k > n$ then it follows that $A_k > A_n$, the difference $A_k - A_n$ is infinitesimal and it can also be calculated exactly.

Moreover, the importance of the initial conditions has been shown in the processes of the construction of the Koch snowflake. If we consider one process of the construction of the snowflake starting, e.g., from the original triangle and the initiator of the second process is a result of first k steps from the original triangle then after the same infinite number of steps, n , the two resulting snowflakes will be different and it is possible to calculate their exact perimeters, areas, etc.

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