DEFINABILITY IN PHYSICS

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Abstract. The concept of definability of physical fields in a set-theoretical foundation is introduced and an axiomatic set theory is proposed which provides precisely the tools necessary for a nonlinear sigma model. In this theory quantization of the model derives from a null postulate and becomes equivalent to definability. We also obtain scale invariance and compactification of the spatial dimensions effectively. The applicability of this foundation to quantum gravity is suggested.

We look to provide a deep connection between physics and mathematics by requiring that physical fields must be definable in a set-theoretical foundation. The well-known foundation of mathematics is the set theory called Zermelo-Fraenkel (ZF). In ZF, a set U of finite integers is definable if and only if there exists a formula $\Phi_U(\mathrm{n})$ from which we can unequivocally determine whether a given finite integer n is a member of U or not. That is, when a set of finite integers is not definable, then there will be at least one finite integer for which it is impossible to determine whether it is in the set or not. Other sets are definable in a theory if and only if they can be mirrored by a definable set of finite integers. Most sets of finite integers in ZF are not definable. Furthermore, the set of definable sets of finite integers is itself not definable in ZF. [1]

A physical field in a finite region of space is definable in a set-theoretical foundation if and only if the set of distributions of the field's energy among its eigenstates can be mirrored in the theory by a definable set of finite integers. This concept of definability is appropriate because, were there a field whose set of energy distributions among eigenstates corresponded to an undefinable set of finite integers, that field would have at least one energy distribution whose presence is impossible to determine, so the field could not be verifiable. Therefore, our task is to find a foundation in which it is possible to specify completely the definable sets of finite integers and which contains mathematics rich enough to obtain the fields corresponding to these sets.

The definable sets of finite integers cannot be specified completely in ZF because there are infinitely many infinite sets whose definability is undecidable. So we will start with a sub-theory containing no infinite sets of finite integers. Then all sets of finite integers are *ipso facto* definable. This will mean, of course, that the set of all finite integers, called ω, cannot exist in that sub-theory. The set ω exists in ZF directly in consequence of two axioms: an axiom of infinity and an axiom schema of subsets. Thus, we must delete one or the other of these axioms. If we delete the axiom of infinity we will then have no need for the axiom schema of subsets either since all sets are finite. However that theory is too poor to obtain the functions of a real variable necessary for physical fields. So the task reduces to whether or not, starting by deleting the axiom schema of subsets from ZF but retaining the axiom of infinity, we can get a theory which is rich enough to obtain physical fields corresponding to sets of finite integers.

In the appendix we show eight axioms. The first seven are the axioms of ZF except that the axiom schema of replacement has been modified. The usual replacement axiom (AR) asserts that for any functional relation, if the domain is a set, then the range is a set. That axiom actually combines two independent axioms: the axiom schema of subsets, which we wish to delete, and an axiom schema of bijective replacement (ABR), refers only to a one-to-one functional relation. Therefore, we can delete the axiom schema of subsets from ZF by substituting ABR for AR, forming the sub-theory ZF–AR+ABR.

We shall first discuss how ZF–AR+ABR differs from ZF. To do this, we look at the axiom of infinity. The axiom of infinity asserts the existence of at least one set ω^* that contains, in general, infinite as well as finite ordinals. There are actually infinitely many such sets. In ZF, we obtain the minimal ω^*, a set with just all the finite ordinals called ω, by using the axiom schema of subsets to provide the intersection of all the sets possible to create by the axiom of infinity. However, in ZF–AR+ABR, without the axiom schema of subsets, this minimal set ω cannot be obtained and therefore all provable statements must hold for any ω^*. A member of ω^* is an "integer". An "infinite integer" is a member that maps one-to-one ω^*. A "finite integer" is a member that is not an infinite integer. Also, in ZF-AR+ABR, any set of finite integers is finite. We denote finite integers by *i, j, k, ℓ, m or n.*

We now adjoin to ZF–ZR+ABR an axiom asserting all sets of integers are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Gödel has shown that an axiom asserting that all sets are constructible can be added to ZF, giving a theory usually called ZFC$^+$.[2] It has also been that no more than countably many constructible sets of integers can be proven to exist in ZFC^+.[3] This result will hold for the subtheory ZFC^+–AR+ABR. Therefore we can adjoin to ZF–AR+ABR a new axiom asserting all the subsets of ω^* are constructible and there are countably many such subsets. We call these eight axioms as theory T.

Cantor's proof or its equivalent cannot be carried out in T [4]; no uncountably infinite sets exist in T. Since all sets are countable, the continuum hypothesis holds. However, as the axiom schema of subsets is not available, we cannot prove the induction theorem, so not all countable sets that exist in ZF can exist in T. For example, we cannot sum infinite series, whereas in ZF infinite series play an important role in the development of mathematics. However, our axiom of constructibility provides a way to obtain at least some functions of a real variable.

Recall the definition of "rational numbers" as the set of ratios of any two members of the set ω, usually called **Q**. In T, we can likewise by the axiom of unions establish for any ω^* the set of ratios of any two of its integers, finite or infinite. This will be an "enlargement" of the rational numbers and we shall call this enlargement **Q***.

Two members of **Q*** are called "identical" if their ratio is 1. We employ the symbol "$\equiv$" for "is identical to". An "infinitesimal" is a member of **Q*** "equal" to 0, i.e., that is, letting y signify the member and employing the symbol "=" to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$. The reciprocal of an infinitesimal is "infinite". Any member of **Q*** that is not an infinitesimal and not infinite is "finite", $[y \neq 0 \wedge 1/y \neq 0] \leftrightarrow \exists k[1/k < y < k]$. We apply the concept of equality to the interval between two finite members of $\mathbf{Q}^*$; two finite members are either equal or the interval between them is finite. The constructibility axiom in T well-orders the power set of ω^*, creating a metric space composed of the subsets of ω^*. These subsets represent the binimals making up of a real line $\mathbf{R}^*$. [5]

Equality-preserving bijective mappings between finite intervals of $\mathbf{R}^*$ are homeomorphic, i.e., bijective mappings $\phi(x, u)$ of a finite interval X onto a finite interval U in which $x \in X$ and $u \in U$ such that
$\forall x_1,\ x_2,\ u_1,\ u_2[\phi(x_1, u_1) \wedge \phi(x_2, u_2) \rightarrow [x_1 = x_2 \leftrightarrow u_1 = u_2]]$ will produce biunique real function pieces that are continuous as either u(x) or x(u). These "biunique pieces" can now be joined continuously in x to obtain more general functions of a real variable u(x) which can be differentiated and integrated, have no singularities and are of bounded variation.

We define a "function of a real variable in T" in an interval (a,b) as a constant (which is obtained directly from ABR.) or a continuously connected sequence of biunique pieces such that its derivative with respect to x is also a function of a real variable in T. All these hereditarily defined functions are smooth and in fact restricted just to polynomials, since infinite series do not exist in T. We can show that every polynomial can be represented arbitrarily closely by the sum of a finite set of polynomials, generated from the following integral expression by minimizing λ for $\int_a^b ru^2dx$ constant:

$$\int_a^b \left[p \left(\frac{du}{dx} \right)^2 - qu^2 \right] dx \equiv \lambda \int_a^b ru^2 dx \tag{1}$$

where $a \neq b, u \left(\frac{du}{dx} \right) \equiv 0$ at a and b; and p, q and r are functions of x.

This integral expression provides an algorithm generating increasingly higher degree polynominals u_n, where n denotes the n^{th} iterations, such that $\forall k \exists n [\int_a^b [\left(\frac{du_n}{dx} \right)^2 - qu_n^2] dx - \lambda_n \int_a^r u_n^2 dx < 1k]$. We call polynomials of sufficiently high degree (say,$k > 10^{50}$) an "eigenfunction". Every eigenfunction, as it is a polynomial, is decomposable into "irreducible biunique eigenfunction pieces".

We now show this theory is a foundation for physical fields governed by a nonlinear sigma model. Let us first consider two eigenfunctions, $u_1(x_1)$ and $u_2(x_2)$; for each let $p \equiv 1, q \equiv 0$, and $r \equiv 1$ and we shall call x_1 "space" and x_2 "time". It is well known that $\left(\frac{\partial u_1 u_2}{\partial x_1} \right)^2 - a \left(\frac{\partial u_1 u_d}{\partial x_2} \right)^2$ is the Lagrange density for a one-dimensional string and, by minimizing the integral of this function over all space and time, i.e., by Hamilton's principle, we determine field equations. We can now generalize to a Lagrange density for separable fields in finitely many space-like(i) and time-like (j) dimensions. As they are functions for real variables in T, the fields obtained, or any finite sum of such fields, are locally homeomorphic, differentiable to all orders, of bounded variation and without singularities.

Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$ respectively. We assert a "field" is a sum of eigenstates: $\Psi_m = \Sigma_\ell \Psi_{\ell m} i_\ell, \Psi_{\ell m} = \Pi_i u_{\ell mi} \Pi_j u_{\ell mj}$ subject to the postulate that for every eigenstate m the value of the integral of the Langrange density over

$d\tau = dsdt = \Pi_i\, r_i\, dx_i\, \Pi_j\, r_j\, dx_j$ is *identically* null:

$$\sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] - \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_\ell^2 \right] \right\} \quad (2)$$
$$d\tau \equiv 0 \ for\ all\ m$$

In this integral expression the P and Q can be functions of any of the x_i and x_j, thus of any $\Psi_{\ell m}$ as well. This is a *nonlinear sigma model*. The $\Psi_{\ell m}$ can be determined by an algorithm.[6]

For expression (2), we can *prove quantization*. Since they are identical, we will represent both

$$\sum_m \sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} d\tau \ and$$
$$\sum_m \sum_\ell \int \left\{ \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau \ by\ \alpha$$

I. α is positive and must be closed to addition and to the absolute value of subtraction: In T we must have the α that is an integer times a constant which is infinitesimal or finite.

II. There is either no field, in which case $\alpha \equiv 0$, or otherwise in T the field is finite in which case $\alpha \neq 0$); thus $\alpha = 0 \leftrightarrow \alpha \equiv 0$.

III. Therefore $\alpha \equiv nI$ where n is an integer and I is a finite constant such that $\alpha = 0 \leftrightarrow n \equiv 0$

Expression (2) is in the form of a generalized Klein-Gordon equation. If we have infinitely many space dimensions but one time dimension, we also can obtain a generalized Schrödinger equation. Put $\int \Psi_m^2 ds \equiv 1$ and $\Psi_m^2 = A^2 \prod_i u_{mi}(x_i)[u_{1m}^2(t) + u_2^2(t)]$ into equation (2), then $\frac{d}{dt}[u_{1m^2}(t) + u_{2m}^2(t)]$ will give either $\frac{du_{1m}}{dt} = -\omega_m u_{2m}$ and $\frac{du_{2m}}{dt} = \omega_m$ or $\frac{du_{1m}}{dt} = \omega_m u_{2m}$ and $\frac{du_{2m}}{dt} = -\omega_m u_{2m}$. In both cases, we can immediately identify the basic constant I as the α determined for each irreducible biunique time-eigenfunction piece, thus $A^2 \omega_m^2 \frac{\pi}{2\omega_m} \equiv I or A^2 \equiv h/2\pi\omega_m$ where $h \equiv 4I$. If we now change to the notation $\Psi = \prod_i u_{mi}(x_i)[u_{1m}(t) + iu_{2m}(t)]$, where $i = \sqrt{-1}$, the time term in expression (2), which had been in the form $A^2 \left[\left(\frac{du_{1m}}{dt} \right)^2 + \left(\frac{du_{2m}}{dt} \right)^2 \right] \prod_i u_{mi}^2(x_i)$,

will take on the more familiar form for the Schrödinger equation,
$(h/4\pi i)\left[\Psi^*\left(\frac{\partial\Psi}{\partial t}\right)-\left(\frac{\partial\Psi^*}{\partial t}\right)\Psi\right]$

From the preceding discussion, $A^2\omega_m^2$, the energy in the m^{th} eigenstate, will occur only in quanta of $h\omega_m/2\pi$. The sum of energies in all of the eigenstates E_t is $\Sigma n_m h\omega_m/2\pi$ where n_m is the number of quanta in the m^{th} eigenstate. We can now offer a time-scale invariant argument regarding the definability in T of any field obtained from expression (2) in finitely many space-like dimensions and one time dimension in a finite region of space and with $\int\Psi_m^2 ds \equiv 1$. The field is definable in T if and only if the set of distributions of energy among its eigenstates is mirrored by a set (in T) of finite integers. Every ordered set of n_m corresponding to a distribution of energy E_t among eigenstates of the field maps to a unique finite integer and every finite integer maps to a unique set of n_m by the fundamental theorem of arithmetic, e.g.,

$$\{n_m|E_t\}\ \textit{maps with}\ \prod_m[P_m]^{n_m}\ \textit{where}\ P_m\ \textit{is the}\ m^{th}\ \textit{prime starting with}\ 2. \quad (3)$$

A set of these finite integers for all $E_t \le E$ exists in T. Thus quantization implies definability for any finite E. This holds in ZF as well as in T. However, in T we also can show the converse, that definability implies quantization. Given a finite E_t, if I were infinitesimal, then $\Sigma n_m\omega_m$ would have to be infinite and the set of all distributions of energy among the eigenstates cannot be mirrored by any set (in T) of finite integers. Thus *definability in T is equivalent to quantization*. By similar reasoning, definability in T can be shown equivalent to compactification of all the spatial dimensions.[7]

In addition to providing a foundation for definable fields, here are three examples of the applicability of theory T to physics. First, continuously connected biunique pieces are essential in T to construct fields. These pieces arise as a result of a homeomorphic mapping which is symmetric between range and domain. This construction necessitates that there are no discontinuities of the field and that space-time is relational. Second, the problem originally described by Dyson [8], that the power series employed in quantum electrodynamics are divergent (so can be only asymptotic expansions that in practice give an accurate approximation), is absent in this theory which all power series are finite. Moreover, other singularities that may appear at the Fermi scale will be resolved at the Planck scale, since fields can have no singularities. Therefore T offers a possible foundation for quantum gravity. Third, the metaphysical question raised by Wigner[9] about the unreasonable effectiveness of mathematics in physics is answered directly.

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References

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[2] Göedel, K., The consistency of the axiom of choice and of the generalized continuum hypothesis. *Annals of Math Studies*, 1940.

[3] Cohen, P. J., *Set Theory and the Continuum Hypothesis*, New York, 1966.

[4] The axiom schema of subsets is $\exists u[[u = 0 \vee \exists xx \in u] \wedge \forall xx \in u \leftrightarrow x \in z \wedge X(x)]$, where z is any set and $\boldsymbol{X}$(x) is any formula in which x is free and u is not free. The axiom enters ZF in AR but can also enter in the strong form of the axiom of regularity. (Note T has the weak form.) This axiom is essential to obtain the diagonal set for Cantor's proof, using $x \notin$f(x) for $\boldsymbol{X}$(x), where f(x) is an assumed one-to-one mapping between any ω^* and P(ω^*). The argument leads to the contradiction $\exists c \in z\boldsymbol{X}(c) \leftrightarrow \neg\boldsymbol{X}(c)$, where f(c) is the diagonal set. In ZF, this denies the mapping exists. In T, the same argument instead denies the existence of the diagonal set, whose existence has been hypothesized while the mapping was asserted as an axiom. What if we tried another approach for Cantor's proof, by using ABR to get a characteristic function? Let $\phi(x, y) \leftrightarrow [\boldsymbol{X}(x) \leftrightarrow y = (x, 1) \wedge \neg\boldsymbol{X}(x) \leftrightarrow y = (x, 0)] and z = \omega^*$. If c were a member of ω^*, t = (c,1) and t = (c,0) both lead to a contradiction. But, since the existence of the diagonal set f(c) is denied and since a

one-to-one mapping between ω^* and $P(\omega^*)$ is an axiom, as f(c) is not a member of $P(\omega^*)$, so c cannot be a member of ω^*. In T the characteristic function exists but has no member corresponding to a diagonal set.

[5] The axiom of constructibility generates sequentially all the subsets of ω^* in a set of ordered pairs. The left-hand member of each pair is a subset of ω^* and the right-hand member is an integer indicating the order in which it was generated. If we let the integers not present in each subset be "1" in the corresponding binimal and the integers that are present be a "0", then the right-hand member is the magnitude of that binimal and serves as a distance measure on the line R^*.

[6] The $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ are iterated using (1). The $p_{\ell mi(X_i)}, q_{\ell mi(x_i)}, p_{\ell mj}(x_j)$ and $q_{\ell mj}(x_j)$ will generally change at each iteration and are given by $p_{\ell mi} = \int \frac{P_{\ell mi}\Psi^2_{\ell m} d\tau}{u^2_{\ell mi} r_i dx_i} / \int \frac{\Psi^2_{\ell m} d\tau}{u^2_{\ell mi} r_i dx_i}$, etc. Since the field is continuous, differentiable to all orders, of bounded variation and thus free of singularities, iterations for all $u_{\ell mi(x_i)}$ and $u_{\ell mj}(x_j)$ will converge jointly within a finite region.

[7] The same reasoning can be applied to the spatial dimensions. The field of expression (2) is definable in T if and only if M is finite. In T, the range and domain of the irreducible biunique eigenfunction pieces in each of the spatial dimensions is finite (i.e., is not infinitesimal or infinite) and all functions are continuous. So, if any spatial dimension is infinite, M is infinite and the field is not definable. If all spatial dimensions are finite, we have shown that the field of expression (2) is quantized, hence definable T. The field is thus defineable in T if and only if all the spatial dimensions are finite. We have obtained compactification effectively. Note that this is achieved without invoking boundary conditions. Thus compactification of the spatial dimensions is equivalent to quantization.

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Appendix

ZF - AR + ABR + Constructibility

Extensionality	Two sets with just the same members are equal. $\forall x \forall y[\forall z[z \in x \leftrightarrow z \in y] \rightarrow x = y]$
Pairs	For every two sets, there is a set that contains just them. $\forall x \forall y \exists z[\forall w w \in z \leftrightarrow w = x \vee w = y]$
Union	For every set of sets, there is a set with just all their members. $\forall x \exists y \forall z \exists z[z \in y \leftrightarrow \exists u[z \in u \wedge u \in x]]$
Infinity	There is at least one set with members determined in infinite succession $\exists \omega^*[0 \in \omega^* \wedge \forall x[x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*]]$
Power Set	For every set, there is a set containing just all its subsets. $\forall x \exists P(x) \forall z[z \in P(x) \leftrightarrow z \subseteq x]$
Regularity	Every non-empty set has a minimal member (i.e. "weak" regularity). $\forall x[\exists y y \in x \rightarrow \exists y[y \in x \wedge \forall z \neg[z \in x \wedge z \in y]]]$
Replacement	Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement). Let ϕ(x,y) a formula in which x and y are free, $\forall z \forall x \in z \exists y[\phi(x, y) \wedge \forall u \in z \forall v[\phi(u, v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t[t \in r \leftrightarrow \exists s \in z \phi(s, t)]$
Constructibility	All the subsets of any ω^* are constructible. $\forall \omega^* \exists S[(\omega^*, 0) \in S \wedge \forall y \mathrm{E}! z[y \neq 0 \wedge y \subseteq \omega^* \wedge (y, z) \in S \rightarrow (y \cup m_y - \{m_y\}, z \cup \{z\}) \in S]]$ where m_y is the minimal member of y.