Decoherence and Probability

Richard Dawid

Stockholm University

Karim P. Y. Thébault

University of Bristol

Abstract. One cannot justifiably presuppose the physical salience of structures derived via decoherence theory based upon an entirely uninterpreted use of the quantum formalism. Non-probabilistic accounts of the emergence of probability via decoherence are thus unconvincing. An alternative account of the emergence of probability involves the combination of *quasi-probabilistic emergence*, via a partially interpreted decoherence model, with *semi-classical emergence*, via averaging of observables with respect to a positive-definite *quasi-probability* function and neglect of terms $O(\hbar)$. This approach avoids well-known issues with constructing classical probability measures in the context of the full set of states of a quantum theory. Rather, it considers a generalised *quasi-measure* structure, *partially interpreted* as weighting of possibilities, over a more general algebra, and delimits the context in which the combination of decoherence and a semi-classical averaging allows us to recover a classical probability model as a coarse-grained description which neglects terms $O(\hbar)$.

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E-mail addresses: richard.dawid@philosophy.su.se, karim.thebault@bristol.ac.uk.
Date: *October 2, 2024 Draft paper, comments welcome.*

1. Introduction

The interpretation of probability is a variously contested subject in both philosophy and the foundations of physics. There are, perhaps, two points of common ground, however. First, that *formally* a classical probability can be defined as a *mathematical structure* given by a normalized, positive, and σ-additive measure over a suitable algebra of events. Second, that in using such a structure to *represent* physical states of affairs, we are committing to a *partial interpretation* of the measure as (in some sense) a *weighting of possibilities*. Whether such weightings, and such possibilities, should be understood as epistemic or ontic; or, for that matter, subjective or objective, is still left open by such a *partial interpretation*.[1] Nevertheless, a probability is not purely a mathematical object when we are in the business of physical representation, even absent a full interpretation.

These remarks prove enlightening when considered in the context of discussions of probability and emergence in the Many Worlds or Everett approach to the interpretation of quantum theory. In particular, consider the *non-probabilistic* emergentist account of Wallace (2012); Saunders (2021b); Franklin (2023) in which features of the evolution equations of quantum theory are claimed to be sufficient, in some contexts to some degree, to justify a link between the Born weightings and *physical salience*, without these weightings being understood probabilistically. Decoherence, on this view, is a dynamical process under which probabilities emerge without the need for any prior probabilistic assumptions.[2]

In Section 2 we argue that the non-probabilistic emergentist account of probability via decoherence is unconvincing. One cannot justifiably presuppose the physical salience of structures derived via decoherence theory based upon an entirely uninterpreted use of the quantum formalism. We provide specific framing of this dialectic in terms of *similarity arguments*. Justifications of the physical salience of structures derived via decoherence theory purely based upon similarity to interpreted classical physics can never be satisfactory as an approach to the interpretation of physical theory. Some prior generalised concept of quantum measure or quantum probability as a weighting of possibilities must be assumed in the application of decoherence theory. Non-probabilistic emergentism about probability based on quantum decoherence fails. *Nothing comes from nothing.*

In response to this failure, we provide a novel account of the emergence of probability through the combined application of a partially interpreted decoherence

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[1] Following Carnap (1958), a partial interpretation is an assignment of meaning to theoretical terms such that there is a range of admissible interpretations in the complete language. A partial interpretation thus allows for the interpretation of theoretical terms to be strengthened by further postulates (Suppe 1971; Andreas 2021).

[2] It should be noted that Franklin (2023) only explicitly claims to have established the non-probabilistic emergence of 'classical structures' rather than probability.

model and semi-classical averaging leading to a coarse-grained description which neglects terms $O(\hbar)$. Our approach takes as its starting point a partially interpreted generalised quantum *quasi-probability* structure and uses decoherence together with semi-classical averaging to derive a classical probability model as a coarse-grained description. Our account of the emergence of probability thus involves the combination of a 'diachronic' *quasi-probabilistic emergence*, via a partially interpreted decoherence model, with a 'synchronic' *semi-classical emergence*, via averaging of observables and neglecting higher order terms. Emergence is understood to indicate the derivation of a novel and robust behaviour following the accounts of Butterfield (2011) and Palacios (2022).

There are well-known problems with constructing probability measures over the full set of states of a quantum theory. In particular, not only are there general grounds for thinking quantum interference is in tension with any probabilistic interpretation of the quantum amplitudes (Wallace 2014), but one can in fact show that there is a close formal relation between the absence of a well-defined joint probability distribution for non-commuting observables and violation of the Bell-CHSH inequalities (Fine 1982a,b; Pitowsky 1989; Suppes and Zanotti 1993; Hartmann 2015). Relatedly, in quantum theory the probabilities that come into the theory cannot be represented as measures induced by the integration of a genuine probability density function over phase space. Rather, they can at best be represented in terms of the marginal probability distributions for position and momentum considered separately, with the density function taking the form of Wigner function, which is formally a quasi-probability density function in a *quantum* phase space representation (Wigner 1932, 1971).

Our approach is based upon consideration of a generalised quasi-measure structure as induced by the Wigner function within the quantum phase space formalism. This structure is partially interpreted as weighting of possibilities such that decoherence can be understood as the suppression of certain possibilities. We show how *semi-classical averaging allows us to recover a classical probability model as a 'coarse-grained' description which neglects terms* $O(\hbar)$. This is not, of course, to offer a solution to the measurement problem in terms of a *full interpretation* of the relevant possibility spaces. Rather, what we offer is a conceptual framework for the analysis of classical and quantum probability within which any coherent interpretation must be expected to operate.

The key results of the paper are as follows. In Section 3 we reconstruct the quantum phase space formalism to provide a formalisation of quantum possibly space models that are directly comparable to the classical possibility space model given by a probability density function over a phase space. In Section 4, we demonstrate the sense in which classical possibility space models can be understood to emerge from quantum possibly space models. This demonstration depends upon two important results. First, that explicit models of decoherence in quantum phase space show the generic feature that that show the Wigner quasi-probability distribution is positive-definite after finite times of the order of the decoherence time. This is the quasi-probabilistic emergence with Wigner positivity the relevant novel and robust behaviour. Second, that the generalised Ehrenfest relations imply that the classical and quantum moment evolution equations are syntactically isomorphic with the Wigner function playing the role of a probability density function. A positive-definite Wigner function then displays localisation and conservation behaviour identical to that of a probability density function to the extent to which we can neglect terms $O(\hbar)$. This is the semi-classical emergence with localisation and conservation the relevant novel and robust behaviour. The combination of quasi-probabilistic emergence and semi-classical emergence thus allow us to derive a classical possibility space model from a quantum possibility space model.

On our analysis, an account the role of probability in quantum mechanics can most plausibly play out in only one of two ways. First, probability can be introduced as a fully formed classical probability in connection with an extra posit such as collapse, hidden variables, or observers. Second, one can abstain from extra posits, and establish the probabilistic nature of quantum mechanics as an approximate, emergent concept. In the latter case, there is no plausible way to avoid adding to pure wave mechanics a partial interpretation in terms of possibility weightings. In particular, there is no way to understand decoherence in general, or the suppression of small amplitudes in particular, absent a partially interpreted structure that weights possibilities. Formally, such weightings can be expected to have the structure of quasi-probabilities (or quasi-measures). On this approach, there are no true classical probabilities at a fine-grained level of description, only quantum probabilities that in some circumstances and to some extent resemble their classical counterparts.

2. Emergence and Everett

2.1. **Decoherence and Emergence.** The role of probability in the interpretation of quantum mechanics takes centre stage in the context of the relationship between the Everett interpretation and decoherence. In particular, according to what might be called the *Oxford approach* the Born rule can be extracted from the Many Worlds branching structure based on principles of reasoning that leave the application of the Born rule as the only rational way of betting on quantum outcomes open to an agent on an Everettian branch of the wave function who endorses the Everett interpretation.[3]

[3]See Deutsch (1999) and Wallace (2002) for the original proofs and Wallace (2012) for the systematic treatment in the context of the emergentist view. See also Saunders (2004, 2005); Wallace (2009); Greaves and Myrvold (2010). Various critical responses (and counter responses) are Price (2010); Rae (2009); Dizadji-Bahmani (2013); Adlam (2014); Dawid and Thébault (2014); Read

Zurek (2005) and Baker (2007) have criticized the Oxford approach by pointing at a circularity in its line of reasoning: decoherence must already be assumed to establish the branching of the wave function that provides the basis for identifying an agent who can consider betting along the lines of the decision theoretic argument. But decoherence already relies on a probabilistic interpretation of the process. In this context, Dawid and Thébault (2015) have argued that the the situation is even worse: the notion of probability required for understanding decoherence in the sense of a probabilistic suppression of off-diagonal elements of the density matrix is stronger than the decision theoretic notion of probability offered. This approach to extracting the Born rule therefore is not just circular but incoherent.

Against these critizisims, Saunders (2021b) and Franklin (2023) have sought to buttress the emergentist approach building on earlier discussions by Wallace (2010, 2012) and Saunders (2005). Franklin argues that "the neglect of terms with relatively small amplitudes can be justified non-probabilistically [...] in contexts where interference is rife, the probabilistic interpretation of the (mod-squared) amplitudes is ruled out [...] the Born rule, in such contexts, takes the form of an averaging measure rather than a probability measure. [...] we should think of the relation between small amplitudes and irrelevance as a dynamical phenomenon. The relative magnitude of the amplitudes encodes the dynamical contribution of each term."

Saunders argues using slightly different language towards the same central point. In particular, he claims that "Strongly peaked amplitude" does not, prior to defining the branching structure of the state, have to be interpreted as "highly probable." [...] the "average values of local densities" are defined not by averaging the densities, but as the values of the local densities on those trajectories on which the amplitudes are (very sharply) peaked. In the case of Ehrenfest's theorem, whilst it is possible to interpret $\langle x \rangle_\psi$ operationally, in terms of multiple measurements [...] it is also possible to interpret it realistically, as the location of the peak of the wave-function as it evolves over time, in accordance with classical equations...".

The original presentation of this form of similarity via dynamical irrelevance argument can be found within the highly influential emergentist defence and development of the Many Worlds interpretation due to Wallace (2012), who suggests that "[w]e can think of the significance of the Hilbert space metric as telling us when some emergent structure really is robustly present, and when it's just a 'trick of the light' that goes away when we slightly perturb the microphysics...What makes perturbations that are small in Hilbert-space norm 'slight', [is] not the probability interpretation of them. Ultimately, the Hilbert- space norm is just a natural

(2018); Brown and Porath (2020); Steeger (2022); March (2023). For other approaches to probability in Many Worlds Theory see, for example, Saunders (2021a); Short (2023). See also Saunders (2024) for a recent 'finite frequentist' approach to quantum probability.

measure of state perturbations in Hilbert space, and that naturalness follows from considerations of the microphysical dynamics, independent of higher-level issues of probability" (pp. 253–254).

What Wallace, Franklin and Saunders all seem to have in mind is that features of the evolution equations of quantum theory are sufficient, in some contexts to some degree, to justify a link between the Born weightings and *physical salience*, without these weightings being understood probabilistically in any sense. The problems with such a strong emergentist view will be considered in the following section in the context of the reliance on *uninterpreted similarity arguments*.

2.2. **Similarity and Interpretation.** The failure of non-probabilistic emergence based on uninterpreted similarity arguments can be understood to arise from a basic conflict with the principle that a scientific theory should allow for empirical testing on its own terms. The key problem is the assumption that the set of rules that specify an important part of the theory's empirical import, namely the decoherence of branches of the wave function, can be extracted from observing structural similarities to a theory that serves as a limiting case of that theory – the model where coherence terms are set to zero. In other words, a limiting theory serves as the basis for extracting empirical implications of the fundamental theory.

The problem with this line of reasoning is that it does not explain what measuring a certain value of an observable implies at the level of the full theory. As long as no such understanding is forthcoming at the level of the full theory, however, we have no basis to decide whether or not we are justified to call any other theory a limiting theory of our full theory. Mere similarity arguments are insufficient for making that decision for one reason: as long as the implications of measurements cannot be spelled out at the level of the full theory, we remain insensitive to the distinction between empirically relevant stable dynamics on the one hand and spurious dynamics of parameterization prescriptions on the other. In the limiting theory that sets coherence effects to zero, the set of allowed states are confined to states that show no coherence effects. Any discovery of coherence effects would therefore contradict the limiting theory. The question as to whether coherence should be considered probable or improbable thus does not arise. Coherence is ruled out. In the full theory, coherence is consistent with the theory. Coherence effects are represented in the theory's set of allowed states. To understand whether they are suppressed or not, it is not sufficient to point at a small dimensionless number that characterizes cohered states because small dimensionless numbers might, in principle, also be extracted from specific parameterizations of the theory that bear no physical significance. To rule out this possibility, one needs to find the basis for a probabilistic analysis of those states at the level of the full theory.

The similarity approach has a second, related problem. While a limiting theory can be deduced from a fundamental theory, the opposite is not true. A probabilistic interpretation of the non-cohered limiting theory (to the extent it can be given) does not formally imply the probabilistic characteristics of the fully quantum regime in terms of the full Born rule. In other words, we end up deploying two entirely different lines of reasoning to establish what formally looks like one coherent concept of quantum probability. All this is a far cry from the initial claim that Many Worlds quantum mechanics has the attractive feature to require no posits beyond the wave function equations. Indeed, an appeal to decoherence as a precondition of interpretational content would render the Many Worlds approach of a piece with precisely the pragmatic, neo-Bohrian outlook that the Many Worlds view motivated by rejecting. For example, such an approach would involve implementing the proscription on the use of the Born rule as a probabilistic rule due to Healey (2017). Pragmatic approaches to quantum theory are without doubt interesting and valuable in their own rights. However, we do not take a marriage with the Many Worlds view of quantum mechanics to be a union that would be to the profit of either party.

Viewing the similarity argument from a slightly different angle may contribute to understanding both the reason for its intuitive appeal and the point where it goes wrong. It is, of course, striking that the decohered limit of wave mechanics looks so similar to a model with a classical probability function. If quantum mechanics were new, no probabilistic interpretation of the mod-squared amplitudes were known, and there were no understanding of the theory's empirical implications, it would be plausible to infer from the stated similarity argument alone that quantum theory most probably has an interpretation that allows for neglecting small amplitudes. The similarity just looks too nice to be accidental. Heuristic reasoning of this kind is standard fare in physics and is often successful, which explains its intuitive appeal. But a similarity argument cannot replace a conceptual understanding as to how the theory of quantum mechanics provides the basis for the probabilistic character of its phenomenology. While the former amounts to a heuristics of theory selection, the latter is a matter of fully spelling out the theory.

In summary, returning to our principal argument: As long as no probabilistic interpretation of the wave function is provided at the level of quantum mechanics, it is not clear whether Born weights are a characteristic of physically relevant dynamics or of mere parameterization. Therefore, it is not justified to infer the empirical import of Many Worlds quantum mechanics from the fact that the resulting wave function in a given limit looks strikingly similar to the empirical results of a non-cohered theory. One might assert by fiat that the import of Many Worlds quantum mechanics matches the import of the corresponding non-cohered theory in a given limit. If one goes down that road, however, the non-cohered theory turns from a limiting theory of the full quantum theory into an essential element of quantum theory that is needed for providing the link between the theory's formal structure and its empirical import. The result is a confusing compound of mutually dependent theoretical posits. We cannot make valid inferences about the world based upon uninterpreted similarity arguments combined with the formal structure of a decoherence model.

3. PROBABILITY AND POSSIBILITY

The previous section demonstrated that a probabilistic understanding of quantum mechanics needs to be established at the level of the full theory. This conclusion stands in conflict, however, with a second step of reasoning put forward by Franklin (2023). Franklin writes: "[A] probabilistic interpretation of the mod-squared amplitudes is inapplicable before decoherence has occurred. In the presence of interference amplitudes may cancel each other out – thus, interpreting amplitudes in such contexts probabilistically will not do. It is only when interference is sufficiently suppressed that mod-squared amplitudes approximately conform to the probability axioms: any attempt to interpret mod-squared amplitudes as probabilities in the presence of interference will be empirically undermined [...]. Therefore, at least in some of the contexts where the Born rule measure is applied and expectation values are discussed these are not to be given a probabilistic interpretation." (Franklin 2023, pp. 13-14).

Franklin thus argues that it is misguided to even look for a probabilistic interpretation of the dynamics at the quantum level because quantum theory provides no basis for a quantum probability measure that satisfies the Kolmogorov axioms. On his reasoning, Zurek (2005), Baker (2007) and Dawid and Thébault (2015) are not just wrong in claiming that decoherence *needs* to be based on a probabilistic interpretation of quantum processes. They are already wrong in assuming that a probabilistic interpretation of the quantum regime is a meaningful goal. Franklin asserts that establishing a probabilistic account at the level of the limiting theory is the *only* way to get from quantum mechanics to empirical predictions.

In this section and the next we will carry out a detailed analysis of this issue and put forward a proposal for the precise sense in which probablistic concepts can be understood at the level of the full theory despite the fact that the full theory provides no quantum probability measure that satisfies the Kologorov axioms. We start by introducing two important types of probability structures: quasi-probability structures and classical probability structures. The first is a generalisation of the second. Each will be understood as uninterpreted formal structures. We will then show how the two structures can be augmented and *partially interpreted* to provide representations of *possibility space models* that implement quasi-probability and classical probability structures respectively. These representations correspond to quantum mechanics and classical statistical mechanics and respectively.

3.1. **Probability and Quasi-Probability Structures.** A quasi-probability model is a triple $(\Omega, \mathfrak{E}, \tilde{\mu})$ where the three elements are defined as follows:[4]

I **Sample Space**: Ω is a non-empty set;

II **Event Algebra**: $\mathfrak{E}$ is a non-empty collection of sub-sets of Ω such that:
  i $\Omega \setminus \alpha \in \mathfrak{E}$ for all $\alpha \in \mathfrak{E}$ (closed under comeplementation);
  ii $\alpha \cup \beta \in \mathfrak{E}$ for all $\alpha, \beta \in \mathfrak{E}$ (closed under finite union);

III **Quasi-Measure**: $\tilde{\mu}$ is a set function $\tilde{\mu} : \mathfrak{E} \to \mathbb{R}$ which is such that $\tilde{\mu}(\Omega) = 1$ (normalized).

By definition we have that $\emptyset \in \mathfrak{E}$, $\Omega \in \mathfrak{E}$, and $\mathfrak{E}$ is closed under-finite intersection.

Two important features that a quasi-probability model does not have are σ-additivity and positivity. The first is since we have not insisted that the event algebra $\mathfrak{E}$ is a σ-algebra; it need not be closed under countable unions.[5] The second is since we have not insisted that the quasi-measure $\tilde{\mu}$ is a measure; it need not be positive (nor indeed σ-additive). Strengthening the model to include these features results in the familiar formal structure of a classical probability model.

A classical probability structure is a triple (Ω, Σ, μ) where the three elements are defined as follows:

IV **Sample Space**: Ω is a non-empty set;

V **σ-Algebra**: Σ is a non-empty collection of sub-sets of Ω such that:
  i $\Omega \setminus \sigma \in \Sigma$ for all $\sigma \in \Sigma$ (closed under comeplementation);
  ii $\sigma_1 \cup \sigma_2 \cup \sigma_3 ... \in \Sigma$ for all $\sigma_1, \sigma_2, \sigma_3 ... \in \Sigma$ (closed under countable union);

VI **Probability Measure**: μ is a set function $\mu : \Sigma \to \mathbb{R}$ such that:
  i $\mu(\Omega) = 1$ (normalized)
  ii $\mu(\sigma) \geq 0$ for all $\sigma \in \Sigma$ (positive)
  iii $\mu(\sigma_1 \cup \sigma_2 \cup \sigma_3 ...) = \mu(\sigma_1) + \mu(\sigma_2) + \mu(\sigma_3)...$ for a countable collection of mutually disjoint algebra elements $\sigma_1, \sigma_2, \sigma_3 ... \in \Sigma$ (σ-additivite).

Evidently, on these definitions every classical probability model is a quasi-probability model. Moreover, whereas by design the probability measure in a classical probability model will satisfy the Kolmogorov probability axioms, a quasi-probability measure in general will not. However, in the sub-set of quasi-probability models where μ is positive and σ-additive will of course be representations of Kolmogvrovian probabilities.

3.2. **Classical Possibility Space Models.** A phase space representation of a classical possibility space model is a triple $(\Gamma, \mathfrak{D}, \rho)$ that takes the following form:

[4]Here we are using a slight generalisation of the framework set out in Dowker and Wilkes (2022).

[5]For a detailed discussion of relationship between forms of additivity and classical and quantum probabilities see Arageorgis et al. (2017).

VII **State Space**: $\Gamma = \mathbb{R}^{2N}$ represents the space of possible states of system as a $2N$-dimensional symplectic manifold equipped with the closed non-degenerate two form $\omega = dq \wedge dp$ and associated volume measure $dq \cdot dp$ in the Darboux chart;

VIII **Observable Algebra**: $\mathfrak{O}$ represents observables as a Poisson algebra given by the space of real-valued smooth functions over Γ with the Cartesian product $\cdot$ and Poisson bracket $\{,\}$, the relevant bilinear products. The distinguished function $H \in \mathfrak{O}$ induces a time evolution automorphism via the Poisson bracket: $\frac{d}{dt}A = \{A, H\}$ for all $A \in \mathfrak{O}$.

IX **Probability Density Function**: ρ is a phase space probability density function, $\rho(q,p) : \Gamma \to \mathbb{R}$, which is Lebesgue integrable with respect to the volume measure, $dq \cdot dp$, and induces a probability measure, μ, such that for any event with probability, $\mu(B)$, there is a corresponding PDF, $\rho(q,p)$, that satisfies the conditions:
   i $\mu(B) \geq 0$ for all $B \in \mathcal{B}$ (positive)
   ii $\int_\Gamma \rho(q,p) dq \cdot dp = 1$ (normalized)
   iii If $B_1, ..., B_n, ... \in \mathcal{B}$ with $B_i \cap B_j = \emptyset$ for $i \neq j$ then $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \int_{B_n} \rho(q,p) dq \cdot dp$ (σ-additive)

   where $B \in \mathcal{B}$ are the Borel sets $\mathcal{B}(\mathbb{R}^{2N})$.

X **Expectation Values**: $\langle A \rangle$ is the expectation value or mean of an observable defined as: $\langle A \rangle \equiv \int_\Gamma A(q,p) \cdot \rho(q,p) dq \cdot dp$ for all $A \in \mathfrak{O}$

A stochastic phase space model provides a partial interpretation of a classical probabilistic structure as follows: The state space Γ is the sample space Ω. The Borel sets given by sub-regions of phase space $\mathcal{B}(\mathbb{R}^{2N})$ are the σ-algebra (Feller 1991). The probability measure $p(B)$ is given by the integration of the probability density function $\rho(q,p)$ with respect to the volume measure $dq \cdot dp$ over a sub-region $B \subseteq \mathbb{R}^{2N}$. The connection between the conditions IXiii and VIiii is guaranteed by the definition of σ-algebra. The model includes a deterministic sub-set since a function that approximates a δ-function is an admissible PDF and thus the case in which the singleton of the Borel sets is measure (almost) one and (almost) all other points are measure zero is an admissible stochastic phase space model.

The conditions on the representation VII–X encode two features which will be important for the comparison with phase space representations of quantum possibly spaces. These are the *conservation* and *localisability* of probability density.

The conservation of probability density is a well known feature of a phase space representations of a classical possibly model. It is typically expressed via the Liouville equation:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0 \tag{1}$$

This equation guarantees that the additivity property of regions of phase space is preserved *over time*. If we think of the probability like a fluid, we can understand there to be a phase space 3-current given by the tuple $(\rho, \rho\dot{q}, \rho\dot{p})$. The Liouville equation then expresses the conservation of the 3-current and is equivalent to the statement that for *any* region $B \subset \Gamma$ the net 'efflux' of probability is zero. This is the characteristic property of a fluid with an *incompressible flow* and is a result of the absence sources or sinks of the probability 'fluid' (Pathria and Beale 2011, p. 28), cf. (Gibbs 1902, p.11).

The localisability of probability density is much less discussed but will be equally important for our discussions. The phase space representation given by conditions VII–X is such that the *essential support* of the probability density function $\rho(q, p)$ is given by phase space points $\{q, p\}$. The essential support of a function, $\texttt{ess sup}(f)$, is a measure theoretic concept and indicates the smallest closed subset in the domain of a measurable function such that the function can be zero 'almost' everywhere outside that subset. The 'almost' in this context is cashed out via the measure such that the points which are outside the essential support and where the function is non-zero are of measure zero. For any Lebesgue measurable function f we have that $\texttt{ess sup}(f) = \texttt{sup}(f)$ (Lieb and Loss 2001, p.13).

The important feature to hold in mind for our discussion is that essential support (and support) of $\rho(q, p)$ is given by the smallest possible phase space regions such that the function can be zero (almost) everywhere else. These are phase space points (the singleton elements of the Borel sets). This means that it is possible to consider probability density functions that are (almost) entirely concentrated at a single point which amounts to allowing the possibility that the probability density function approximates a δ-function. Correspondingly, since its integral over phase space is normalised, by concentrating a probability density function almost entirely at one point we must allow that the function is unbounded from above.

3.3. **Quantum Possibility Space Models.** A phase space representation of a quantum possibility space model is a triple $(\Gamma, \mathcal{O}, F)$ that takes the following form:

- XI **State Space**: $\Gamma = \mathbb{R}^{2N}$ represents the space of possible states of system as a $2N$-dimensional symplectic manifold equipped with the closed non-degenerate two form $\omega = dq \wedge dp$ and associated volume measure $dq \cdot dp$ in the Darboux chart;
- XII **Observable Algebra**: $\mathfrak{A}$ represents observables as a (non-commutative) Moyal algebra of real-valued smooth functions on phase space that are the Wigner transform of the algebra of (Weyl ordered) bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space of square integral functions $\mathcal{H} = L^2(\mathbb{R}^{2N})$. The binary operation is given by a $\star$-product operation which can be

expressed as a pseudo-differential operator in powers of $\hbar$ and the non-commutativity of the algebra is expressed via the fundamental relation that $[\hat{A}, \hat{B}] = \{\{A, B\}\} \equiv \frac{1}{i\hbar}(A \star B - A \star B)$ for all $A, B \in \mathfrak{A}$ and all $\hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})$. The distinguished function $H \in \mathfrak{A}$ induces a time evolution automorphism via the Moyal bracket such that $\frac{d}{dt}A = \{\{A, H\}\}$ for all $A \in \mathfrak{A}$;

XIII **Quasi-Probability Density Function**: is a possibility space weighting function $F(q,p) : \Gamma \to \mathbb{R}$ that induces a quasi-measure $\tilde{\mu}$ such that for any event α with quasi-measure, $\tilde{\mu}(\alpha)$, there is a corresponding quasi-density, F that satisfies the conditions:

i $\tilde{\mu}(\Gamma) = \lim_{n\to\infty} \int_{B_n} F(q,p) \star dq \cdot dp = 1$ (normalized)

ii $| F(q,p) | \leq \frac{1}{\epsilon}$ (bounded)

where $B_n = \{(q,p) \mid |q|^2 + |p|^2 \leq r_n\}$ and $\lim_{n\to\infty} r_n = \infty$ (Aniello 2016).

XIV **Expectation Values**: $\langle A \rangle$ is the expectation value or mean of an observable defined as: $\langle A \rangle \equiv \int_\Gamma A(q,p) \star F(q,p) dq \cdot dp$ for all $A \in \mathfrak{A}$.

The model provides a partial interpretation of a quasi-probability structure as follows: The state space Γ is the sample space Ω and the event algebra $\mathfrak{A}$ is given by regions of phase space of volume greater than or equal to some minimum volume which depends upon ϵ. The quasi-measure $\tilde{\mu}$ is given by the integral of the quasi-probability density function with respect to the volume measure. Since the quasi-probability density function is bounded it cannot be arbitrarily highly peaked, which means that not only are δ-functions not admissible quasi-probability density functions but, due to the unit norm, means that any function that leads to localisation of the quasi-probably mass of order ϵ are excluded. We thus get a direct connection between the essential support of the quasi-probability density function, the bound, and the smallest element of the event algebra. Thus, by design, we are guaranteed that for the smallest element of the event algebra there will be a quasi-probability density function that induces a measure such that the event is measure one and all other events are measure zero.

The contrast between classical and quantum possibly space models in their respective phase space representations is greatly clarified by examining the *failure* of *conservation* and *localisability* of quasi-probability density implied by the conditions XI–XIV. Let us demonstrate this failure explicitly for the choice of Wigner function, W as the quasi-probability density function.[6]

Failure of conservation of quasi-probability is a direct consequence of the non-commutativity of the Moyal algebra of quantum phase space observables in comparison to the Poisson algebra of classical phase space observables as encoded

[6]The Wigner function is the most important of the quasi-probability distributions on quantum phase space that can be defined via different operator ordering conventions. Our discussion principally draws upon details in Curtright et al. (2013) unless otherwise noted. See Section 4 for further details on the Wigner function.

in the relation $\{\{A,B\}\} = \{A,B\} + O(\hbar)$. We can show this explicitly by considering the quasi-probability flux for some arbitrary region of phase space $\mathcal{S}$ with volume greater than or equal to the minimum volume. This is given by the expression (Curtright et al. 2013, p. 57):

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}} dqdpW &= \int_{\mathcal{S}} dqdp\Big(\frac{\partial W}{\partial t} + \dot{q}\frac{\partial W}{\partial q} + \dot{p}\frac{\partial W}{\partial p}\Big) \\ &= \int_{\mathcal{S}} dqdp\Big(\{\{H,W\}\} - \{H,W\}\Big) \\ &= O(\hbar) \end{aligned} \tag{2}$$

where we have used the Wigner transform of the Heisenberg equations of motion $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$ and the Moyal equation $\frac{d}{dt}W = \{\{H,W\}\}$. The quasi-probability density associated with regions of phase space thus manifests a *violation of additivity over time* in marked contrast to the classical probability density function in phase space, cf. (Wallace 2021, p.23).

The failure of localisability can be understood as follows. A quasi-probability functions need not in general be Lebesgue integrable over the entire phase space. In the case of the Wigner quasi-probability density function, W, we find the possibility of failure of Lebesgue integrability (Aniello 2016) accompanied with a restriction of $\mathtt{ess\,sup}(W)$ to volumes of phase space greater than equal to one in units of $\hbar$ (Dell'Antonio 2016, p.19). By the Cauchy–Schwarz inequality the function is bounded such that $-\frac{2}{\hbar} \leq W(q,p) \leq \frac{2}{\hbar}$ and we thus have that $\epsilon = \frac{\hbar}{2}$. Correspondingly, as already anticipated above, the event algebra is given by regions of phase space with volume greater or equal to a minimum volume that depends upon ϵ. Thus, in contrast to the classical case, it is *not* possible to concentrate quasi-probability density almost entirely at a single point. This amounts to precluding the possibility that the quasi-probability density function approximates a δ-function in phase space (Leonhardt 2010, p.71). Phase space points are not in $\mathtt{ess\,sup}(W)$ and we cannot have a situation in which the Wigner function is non-zero at a point but zero (almost) everywhere else.

Physically speaking, this measure theoretic subtlety can be understood as a consequence of the Heisenberg uncertainty principle which, in turn, is a direct consequence of the non-commutative structure induced by the $\star$-product. See (Curtright et al. 2013, §5) and (Huggett et al. 2021, §5.1). That quasi-probability distributions are not localisable in phase space corresponds to the fact that the algebra of events does not include regions of phase space of arbitrarily small volume. That is, since we have insisted that for any event α with quasi-measure, $\tilde{\mu}(\alpha)$, there is a corresponding quasi-density, F, we must exclude events corresponding to regions order $\hbar$ since assigning quasi-measure (almost) one to such regions is not consistent with any quasi-probability density function. Thus $\mathfrak{A}$ is *inequivalent* to the Borel sets of $\mathbb{R}^{2N}$ which by definition form a σ-algebra which includes phase space points.

Furthermore, one can prove based upon the fact that a quasi-probability density is not localisable that the induced measure cannot be σ-additive. The relationship between the failure of σ-additivity and the failure of localisability is expressed in qualitative terms in (Curtright et al. 2013, p. 54) but has not, to our knowledge, previously been demonstrated explicitly. A short proof for the case of the Wigner function is provided in Appendix A.

4. DECOHERENCE AND CLASSICALITY

In this section we demonstrate that the combination of 'quasi-probabilistic emergence' and 'semi-classical emergence' allow us to derive a classical possibility space model from a quantum possibility space model. This is not, of course, to offer a solution to the measurement problem in terms of a full interpretation of the relevant possibility spaces. Rather, what we offer is a conceptual framework for the analysis of classical and quantum probability within which any coherent interpretation must be expected to operate.

4.1. **Wigner Negativity and Decoherence.** The Wigner function is at the centre of the phase space approach to quantum mechanics.[7] Representing the quantum state of a system via a density matrix, $\hat{\rho}$, the Wigner function, $W(q,p)$, takes the form:

(3) $$W(q,p) = \frac{1}{2\pi\hbar}\int dq' \, \langle q - q'|\, \hat{\rho}\, |q + q'\rangle \, e^{-iq'p/\hbar}$$

The transformation between the density matrix $\hat{\rho}$ and the Wigner function W can be generalised to an arbitrary operator $\hat{A}$ as:

(4) $$A(q,p) = \frac{1}{2\pi\hbar}\int dq' \langle q - q' \mid \hat{A} \mid q + q'\rangle e^{-iq'p/\hbar}$$

Where we understand $A(q,p)$ to be the *Wigner transform* for the operator $\hat{A}$. The Wigner transform coverts an operator on Hilbert space, with a preferred Weyl operator ordering, into a function on phase space.

An important property of the Wigner transform is that the trace of the product of two operators $\hat{A}$ and $\hat{B}$ is expressed in phase space in terms of the integral of the product of the relevant Wigner transforms:

(5) $$\mathrm{Tr}[\hat{A}\hat{B}] = \frac{1}{\hbar}\int\int A(q,p)B(q,p)dqdp$$

[7]A concise and very clear introduction to the Wigner function and the quantum phase space formalism is Curtright et al. (2013). Further useful discussions can be found in O'Connell and Wigner (1981); Hillery et al. (1984); Case (2008); De Gosson (2017); Leonhardt (2010). The small philosophical literature is principally comprised of the discussions found in Suppes (1961); Cohen (1966); Sneed (1970); Friederich (2021); Wallace (2021).

This immediately implies that we can express the expectation value of an operator as:

$$\langle A\rangle = \mathrm{Tr}[\hat{\rho}\hat{A}] = \frac{1}{\hbar}\int\int W(q,p)A(q,p)dqdp \tag{6}$$

The Wigner function behaves like a density in that we obtain the average value of a quantity by integrating over that quantity multiplied by the Wigner function.

The Wigner function has the important feature that it reproduces the marginal probability densities for position and momentum given by by the mod-squared amplitude since we have that:

$$\mu(q) = \int W(q,p)dp = \langle q|\,\hat{\rho}\,|q\rangle \tag{7}$$

$$\mu(p) = \int W(q,p)dq = \langle p|\,\hat{\rho}\,|p\rangle \tag{8}$$

Significantly, it can be proved that any quasi-probability distribution function of the form $F(q,p) = \langle\psi|\,\hat{A}(q,p)\,|\psi\rangle$ which and reproduces the marginal probability densities cannot also be positive semi-definite (Wigner 1971).

Wigner negativity has been variously recognised as the distinctive non-classical feature of the Wigner function and has been shown to have direct implications for both contextually and entanglement (Delfosse et al. 2017; Booth et al. 2022). The size of the regions of negativity in phase space are of order $\hbar$ which will be important in what follows. Significantly, the sub-set of Wigner functions that correspond to minimum uncertainty coherent states can be shown to be everywhere positive (and visa versa) (Hudson 1974; Mariño 2021).

Despite its negativity, the Wigner function has a number of attractive features that mark it out as privileged among the quasi-probability distribution functions. In particular, the density and marginal features noted above crucially depend upon the $\star$-product associated to the Wigner function being the Moyal $\star$-product. This is what allows one $\star$-product to be dropped inside an integral via integration by parts leading to formal behaviour that matches that of a genuine probability density function for the marginals and expectation values.[8]

Let us now consider the behaviour of the Wigner function within a simple model of decoherence with a focus on the role of Wigner negativity. The general framework for the study of decoherence is quantum master equations for the *reduced* density matrix of a quantum system. For our purposes it will suffice to consider the most basic master equation, that due to Joos and Zeh (1985). The Joos-Zeh equation can be derived based on a idealised decoherence model with recoilless scattering

[8]This feature is in contrast to the Husimi Q-function for which the associated $\circledast$-product cannot be integrated out and leads to marginals distributions that do not correspond to those of quantum mechanics (Curtright et al. 2013, §13). For further discussion of quasi-probability distributions and probability interpretations see Leonhardt (2010); Schroeck (2013); Friederich (2021); Stoica (2021); Umekawa et al. (2024).

that carries away information but not momentum of a quantum particle. It is a minimal model for position localisation of a quantum particle via the destruction of coherence. More realistic models include noise and dissipation terms but share the central formal feature of *Gaussian-smoothing*.

Explicitly, the Joos-Zeh master equation takes the form:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{2m}[\hat{p}^2, \hat{\rho}] - \frac{D}{2}[\hat{q}, [\hat{q}, \hat{\rho}]] \tag{9}$$

where we have assumed a free particle Hamiltonian and the decoherence time scale will be $t_0 = \sqrt{m/D}$. Physically, the localisation rate, D, measures how fast interference between different positions disappears for distances smaller than the wavelength of the scattered particles. It has units $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ and includes a factor of $\hbar^{-2}$ and a linear dependance on temperature (Joos et al. 2013, §3.2.1).

The quantum phase space equation corresponding to (9) is given by a Fokker-Planck equation for the Wigner function:

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial q} + \frac{D}{2}\frac{\partial^2 W}{\partial p^2} \tag{10}$$

Although it has the same functional form this equation must not be understood to be equivalent to a Fokker-Planck equation for a classical probability density function since the Wigner function is of course a quasi-probability density and has various non-classical features as per our earlier discussion.

Following Diósi and Kiefer (2002), the Fokker-Plank equation for the Wigner function can be demonstrated to be equivalent to a progressive Gaussian-smoothing of an initial Wigner function $W(\Gamma; 0)$. In particular, we can re-write the Equation (10) as a convolution of the form:

$$W(\Gamma; t) = g(\Gamma; \mathbf{C}_W(t)) * W(x - pt/m, p; 0) \tag{11}$$

where $g(\Gamma; \mathbf{C}_W(t))$ is a generalised Gaussian function with time dependent correlation matrix:

$$\mathbf{C}_W(t) = Dt \begin{pmatrix} t^2/3m^2 & t/2m \\ t/2m & 1 \end{pmatrix} \tag{12}$$

and we have used the $*$ symbol for the convolution operation to avoid confusion with the Moyal star product.

Convolution with a Gaussian function, as per the heat equation, has the general effect of *smoothing* the Wigner function.[9] The regions of Wigner negativity are of order $\hbar$ and a Gaussian smoothing can be shown to be such that it will

[9] More generally, we can understand decoherence in terms of convolution of the Wigner function with a Gaussian according to a *Weierstrass transform*. This is, in fact, precisely to transform a Wigner function into a Husimi Q-function (Curtright et al. 2013, §13). We should not expect the quantum mechanical marginal probabilities to be fully recoverable from the reduced state post-decoherence. Which is perhaps unsurprising.

progressively render any initial Wigner function positive-definite.[10] Indeed, Diósi and Kiefer (2002) show that by Equation (11), *any* initial state will be such that Wigner function will be strictly positive after a finite time t_D which is of the order of the decoherence timescale t_0 defined above. The result of Diósi and Kiefer (2002) demonstrates that even for the most simple model of decoherence the dynamical equations serve to smooth out structure of the Wigner function and eliminate Wigner negativity almost immediately. [11] Generically, we can expect that Wigner positivity is a novel and robust behaviour that emerges via decoherence based upon a partially interpreted quasi-probability structure.

4.2. **Probability and Semi-Classicality.** The previous section provided a simple illustration of how the non-classical feature of Wigner negativity can be eliminated via decoherence. These methods as a basis for describing the emergence of classicality can generalised to more realistic models. Perhaps most famously, this approach was extended to the study of non-linear models, such as that of the classically chaotic orbit of Hyperion, by Habib et al. (1998) leading to the iconic illustrations reproduced in Figure 1.

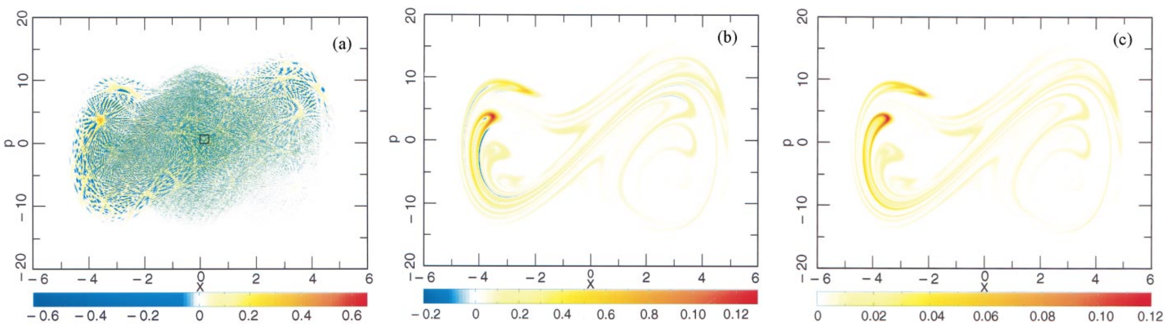

Figure 1. Illustrations of the results of numerical simulations from Habib et al. (1998) with warm (cold) colours marking regions of positive (negative) density. The figures (a) and (b) both show the Wigner distribution function from a solution a Fokker-Planck type equation for a non-linear system with a quartic term in the Hamiltonian. The difference between the figures corresponds to solutions to the model at a given time without (a) and with (b) the destruction of large scale quantum coherence. The figure (c) shows the solution of a classical Fokker-Planck equation for a classical probability density function. The box represents a phase space area of $4\hbar$.

The model of Habib et al. (1998) provides a vivid exemplification of what it means for classical dynamical behaviour to *emerge* from the quantum domain. In

[10] This is true for Gaussian smoothings but does not hold in general for any averaging (de Aguiar and de Almeida 1990).

[11] See (Brody et al. 2024) for a quantum dynamical model that achieves Wigner positivity in finite time without a von Neumann term, i.e. the first commutator on the right hand side of (9) in the Joos-Zeh-model, but with a different, but still simple, form of the dissipator term.

particular, for a quantum possibility space model evolving under an open quantum dynamics to exhibit dynamical behaviour that displays a remarkable close correspondence to that exhibited by a classical possibility space mode.

In a rich and insightful analysis of the Habib et al. (1998) model, Franklin (2023) proposes that we can frame an account of the emergence of macro-worlds with classical chaotic phenomenology based upon an underlying *non-chaotic* quantum state.[12] The claim runs as follows:

> ...we may think of the observed classically chaotic orbit of Hyperion as observable evidence of the effects of decoherence in suppressing quantum interference. Classically chaotic Hyperion counts as emergent because much of the structure of the underlying quantum state is conditionally irrelevant to the future dynamics of each classically chaotic Hyperion. In macroscopic terms, what's screened off are the interference terms that would describe interactions with the Hyperions in other branches – thus rendering the other branches irrelevant to each branch's evolution. And the classically chaotic dynamics is not instantiated in the quantum system absent environment induced decoherence. (p. 10)

There is much to recommend in Franklin's analysis as an account of emergence and the relationship between classical and quantum phenomenology. However, one must also bear in mind the foregoing detailed treatment of the structure of decoherence models based upon the Wigner function. Evidently, a partial interpretation of a quasi-probability structure via possibility space model is a necessary ingredient within any model of decoherence based upon the Wigner function (or, arguably, more generally). One is already implicitly applying a generalised form of probabilistic reasoning – in particular with regard to densities on possibility spaces – when one models decoherence via dynamical equations for the Wigner function as derived from quantum master equations.

Classical possibility space models do not emerge *ab initio* from a non-probabilistic and uninterpreted formalism, but rather are emergent in the relevant sense from a partially interpreted, quasi-probabilistic structure. The model of the emergence of classical phenomenology that decoherence models based upon the Wigner function provide is explicitly reliant on its role as a quasi-probability density function in inducing a *possibility space measure*. These models of decoherence make ineliminable use of *partially interpreted quasi-probabilistic structure* within the dynamical equations themselves. Decoherence must be understood as a basis

[12]Franklin's account builds upon on the account of emergence described in Franklin and Robertson (2021) which in turn builds on Ross (2000). See also Ladyman and Ross (2007), Wallace (2010) and Mulder (2024).

for quasi-probabilistic emergence rather than non-probabilistic emergence. Furthermore, as we saw earlier, the justification of the Wigner function as the privileged representation of quasi-probability relies upon its unique ability to recover the *experimentally confirmed* marginals given by Born rule probabilities.

Furthermore, it is evident based upon our analysis in Section 3.3 that Wigner positivity is *necessary but not sufficient* for us to interpret a model as a representation of a classical possibility space. In particular, the crucial features of conservation and localisation will still fail, notwithstanding the Wigner function being positive. On appropriate scales, we will still find the Gaussian-smoothed, positive Wigner function acting in a manner that is irrecognisable with it being a classical probability density. In particular, we will find the failure of localisability and conservation.

The *scale* at which non-classicality is relevant is all important. The failure of conservation and localisability are all of order $\hbar$. Thus one can expect an *approximation relation* to obtain between the Wigner function and a classical probability distribution to the extent to which terms $O(\hbar)$ are understood to make negligible quantitive contributions at the scale relevant to the description.[13] For example, in a simulation or plot where the grid or pixel size is big relative to $\hbar$. However, clearly in circumstances where terms $O(\hbar)$ are understood to be salient, such a similarity relation should not be understood to hold and in such circumstances decoherence should *not* be understood to lead to the emergence of a possibility space model with classical probabilistic structure.

We propose that one should understand classical probabilities to *semi-classically emerge* from a post-decoherence, positive Wigner function formalism. The sense of emergence we have in mind here is very close to the idea of *coarse-grained emergence* introduced by (Palacios 2022, p.39). On this account a coarse-grained description of a system emerges from a fine-grained description, if and only if the former has terms denoting properties or behaviour that are novel and robust with respect to the latter. In our case the 'fine-grained' description is the full quantum phase space model and the 'coarse-grained' description is the semi-classical phase space model which is such that the expectation values and expressions truncated $O(\hbar)$ are isomorphic to a classical phase space model.

We find emergence in the sense of Palacios (2022) account of coarse-grained emergence specifically since we have: (i) a fine-grained/coarse-grained distinction picked out by phase space areas at order $\hbar$/at order much bigger than $\hbar$; (ii) the coarse-grained description has features that are not features of the fine-grained description, specifically conservation and localisation of the (quasi)-probability density; (iii) the behaviour represented by the fine-grained description exists at the same time as the behaviour represented by the coarse-grained description (i.e. we

[13]See (Wallace 2021, p.23) for related remarks regarding an 'approximate isomorphism' between the dynamics of the quantum state and that of the classical probability distribution.

have synchronic emergence); (iv) the coarse-grained description refers to some behaviour that is insensitive to variation of the microphysical details that characterise a particular token (i.e. we have *robustness* in the sense of (Gryb et al. 2021)); (v) the coarse-grained level depends on the fine-grained level in the sense that every change in the coarse-grained level must imply a change in the fine-grained level (i.e. we have supervenience).[14]

We can demonstrate this sense of semi-classical emergence obtains in the case of classical and quantum possibility space models explicitly. Consider the correspondence between the semi-classical and quantum possibility space models via dynamical equations for the expectation values (first moments) of position and momentum. Assume a Hamiltonian of the standard form $H = \frac{p^2}{2m} + V(q)$. Explicit application of the star product as a pseudo-differential operation then gives the expression for the momentum expectation value:

$$\frac{d\langle p\rangle}{dt} = \langle\{\{p, V(q)\}\}\rangle \tag{13}$$

$$= -\langle\frac{dV(q)}{dq}\rangle \tag{14}$$

$$= -\int_\Gamma \frac{dV(q)}{dq} W dq dp \tag{15}$$

and for the position expectation value we get:

$$\frac{d\langle q\rangle}{dt} = \frac{1}{2m}\langle\{\{q, p^2\}\}\rangle \tag{16}$$

$$= \frac{1}{m}\langle p\rangle \tag{17}$$

$$= \int_\Gamma \frac{p}{m} W dq dp \tag{18}$$

Following Ballentine and McRae (1998), the corresponding formulas in the classical possibility space model is:

$$\frac{d\langle p\rangle}{dt} = -\int_\Gamma \frac{dV(q)}{dq}\rho dq dp \tag{19}$$

$$\frac{d\langle q\rangle}{dt} = \int_\Gamma \frac{p}{m}\rho dq dp \tag{20}$$

We thus have an isomorphism between the classical and quantum probabilistic phase space formalism. The classical and quantum moment evolution first equations are *syntactically isomorphic* with the Wigner function playing the role of a probability density function.

[14]We do not have *universality* in the sense of (Gryb et al. 2021), since we do not have that the coarse-grained description refers to some behaviour that is insensitive to variation of the macroscopic details that characterise the type of system considered. This is *weak autonomy* in the terminology of (Palacios 2022, p.39-40).

To the extent to which we can treat the Wigner function as a probability density function the equations will describe identical phenomenology. A positive-definite Wigner function then displays localisation and conservation behaviour identical to that of a probability density function to the extent to which we can neglect terms $O(\hbar)$. At such a scale a positive Wigner function is a real positive function that is conserved and localisable and can be treated as a genuine probability density. In particular, as shown in the Appendix A problems with σ-additivity are closely connected to the failure of localisability. It is therefore possible to understand the coarse-grained description as a classical possibility space model as per our earlier formalisation.

We therefore have emergence with localisation and conservation the relevant novel and robust behaviours. The coarse-grained description has terms denoting classical probability structure that is novel and robust with respect to the fine-grained description. The quasi-probabilistic emergence via decoherence was dependant on the decoherence time scale and is thus 'diachronic'. By contrast, the semi-classical emergence is dependent upon phase space areas in units of $\hbar$ and is thus 'synchronic'. A schematic diagram for the relevant pattern of interrelations is provided in Figure 2.

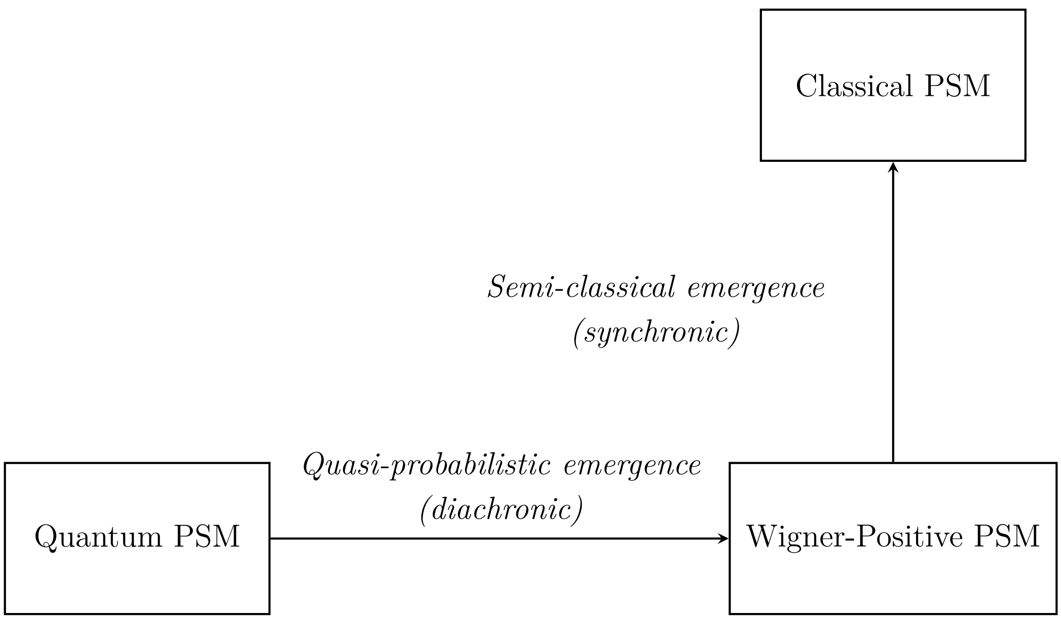

Figure 2. Schematic diagram showing relationship between Quantum and Classical Possibility Space Models (PSMs) quasi-probablitic emergence via decoherence, which is diachronic, and semi-classical emergence, which is synchronic. Inspired by (Palacios 2022, Fig.9).

5. Recapitulation and Outlook

Let us return to the original dialectic with which we started our analysis of probably and decoherence. Recall, in particular, that the arguments of Dawid and Thébault (2015) amounted the the implication that a certain package of interpretative moves concerning probability and the quantum formalism lead to an incoherent conclusion. The foregoing analysis allows us to consider the contraposition this argument. That is, we have now established a plausible framework for the analysis of classical and quantum probability within which any coherent interpretation must be expected to operate.

On our analysis, an account the role of probability in quantum mechanics can most plausibly play out in only one of two ways. First, probability can be introduced as a fully formed classical probability in connection with an extra posit such as collapse, hidden variables, or observers. Second, one can abstain from extra posits, and establish the probabilistic nature of quantum mechanics as an approximate, emergent concept. In the latter case, there is no plausible way to avoid adding to pure wave mechanics a partial interpretation in terms of possibility weightings. In particular, there is no way to understand decoherence in general, or the suppression of small amplitudes in particular, absent a partially interpreted structure that weights possibilities. The requirement for such a partial interpretation clearly does not render the Many Worlds interpretation incoherent in itself. It does, however, place strong constraints upon the way in which such an interpretation can be packaged together with an approach to probability and possibility. In particular, it shows that there is no coherent prospect for an interpretational package that seeks to combine an entirely non-probabilistic account of the emergence of 'words' with a post-decoherence decision theoretic derivation of probability. In this sense the claims of Dawid and Thébault (2015) can be understood to be vindicated against those of Saunders (2021b) and Franklin (2023).

More importantly, our analysis indicates that any *full interpretation* of quantum mechanics that does not seek to introduce probability via extra posits must grapple with the quasi-probabilistic nature of the theory. That is, if probability is not introduced as a fully formed classical concept in connection with an extra posit such as collapse, hidden variables, or observers, then we will need to find a way to attribute physical significance to quasi-probabilities (or quasi-measures) at the level of the fundamental theory. We have no specific suggestion as to how this can be achieved – although work on quantum measure theory is certainly interesting in this regard (Sorkin 2010; Clements et al. 2017). Arguments from similarity, however, do not provide a solution to this problem. As a conceptual basis for neglecting small amplitudes they fail; and using them as merely heuristic reasons for adding to quantum mechanics a prescription to neglect small amplitudes would subvert precisely the most attractive feature of Many Worlds interpretations: that of requiring no posits beyond the wave function equations. Indeed, an appeal to decoherence as a precondition of interpretational content would render the Many Worlds approach of a piece with precisely the pragmatic, neo-Bohrian outlook that the Many Worlds view motivated by rejecting. Things are possibly even worse for the Many Worlds advocate: *if* the emergence of branching structure without a partial interpretation is taken to be a necessary requirement for the justification of the Many Worlds interpretation to be plausible at all, then our work serves to undermine such a justification.

Many issues regarding decoherence and probability remain outstanding. We conclude by highlighting a small selection. First, it would be satisfying to extend the formalisation of Section 3 both to the history space formulations of classical and quantum theories including decoherent histories (Gell-Mann and Hartle 1996; Halliwell 2010), and, more generally, to towards an abstract and general characterisation of the emergence of classical from quantum possibility space models via decoherence. It can be proved that the diagonal elements of the decoherence functional are equivalent to a 'quantal-measure' which is a specific form of our quasi-measure that obeys a particular (non-classical) sum rule on the algebra of events (Sorkin 1994; Dowker and Wilkes 2022). The decoherent histories framework thus *is* a partial interpretation of a quasi-probability structure in precisely our terms. Since there is an explicit dependance coarse-graining in this approach to the emergence of classical probability there is plausible path for reconstructing our analysis in histories terms.

Second, and relatedly, it would be interesting to consider the connection between our account of the emergence of probability and quantum measures in terms a quasi-measure representation of the decoherence functional and the results of Feintzeig and Fletcher (2017). These results draw connections between non-contextual hidden variable interpretations and the existence of a finite null cover and this would appear to make difficult certain attempts to move from a partial to full interpretation of the quasi-measure over possibility space.

Third, it would be of significant physical and philosophical interest to more fully understanding the role of the $\hbar \to 0$ limit in the emergence of classical probability. In the $\hbar \to 0$ limit the bound on the Wigner function will be removed, since $W(q,p) \leq \frac{2}{\hbar}$ and thus localisability will obtain. However, the semi-classical limit of the Wigner function is not always well-behaved (Berry 1977; Mariño 2021) and, moreover, due to the dependance of the Wigner function on $\hbar$, we typically find that the distribution becomes 'spiky' and approximates a δ-function (Curtright et al. 2013). A particularly interesting question is the relation between our project and recent formal results regarding the relationship between decoherence, the semi-classical limit, and the emergence of classical probability (Layton and Oppenheim 2023; Hernández et al. 2023). We leave exploration of such issues to future work.

Acknowledgements

Work on this paper has drawn on a large number of discussions with a wide variety of helpful colleagues over the last few years. We are particularly grateful to Simon Saunders and Alex Franklin for continued highly enjoyable interactions. Further thanks are also due to Jim Al-Khalili, Kevin Blackwell, Dorje Brody, Adam Caulton, Fay Dowker, Sam Fletcher, Gui Franzmann, Simon Friedrich, Sean Gryb, Jonte Hance, Nick Huggett, Eddy Keming Chen, Jason Konek, James Ladyman, Tushar Menon, Richard Pettigrew, James Read, Christian Röken, Tony Short, Jer Steeger, Cristi Stoica, and Lev Vaidman (apologies if we have missed anyone out). Thanks also to audiences in Surrey, Bonn, Oxford, and Bristol. We are also appreciative to Cosmas Zachos for helpful written comments on a draft manuscript. Work on this paper was supported by the John Templeton Foundation grant number 62210. The views expressed may not represent the opinions of the Foundation.

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Appendix A. Localisability and σ-additivity

We prove that the quasi-measure induced by a set of Wigner functions cannot be σ-additive if it is assumed that for any event in the algebra we can define a quasi-probability density function which induces the quasi-measure that picks out the possibility associated with that region only. In this sense Wigner functions induce a measure that violates the Kolmogorov axioms will be violated notwithstanding Wigner negativity (as such, this proof would also apply to the Q-function).

I `Normalisation in terms of limit of phase space balls`. A phase space ball is a region of $\mathbb{R}^{2N}$ that can be defined as:

$$B_n = \{(q,p) \mid |q|^2 + |p|^2 \leq r_n\}$$

where $\lim_{n\to\infty} r_n = \infty$ and $\lim_{n\to\infty} B_n = \mathbb{R}^{2N}$ and we have suppressed an i index on the qs and ps that would run $i = 1, .., N$. We can express the phase space normalisation of Wigner function in terms of the limit of balls as:

$$\lim_{n\to\infty} \int_{B_n} W(q,p)dqdp = 1$$

(Aniello 2016, Eq. 22).

II `Induced quasi-measure and minimal volume`. By analogy with the probability measure μ associated with a probability density function we can introduce a quasi-measure $\tilde{\mu}$ associated with the Wigner quasi-probability density functions via the equation:

$$\tilde{\mu}(E) = \int_E W(q,p)dqdp$$

We assume that the quasi-measure is a real set-function on an algebra of sets closed under finite union and complementation and given by regions of phase space over which it is well defined. We also assume that for any event in the algebra, as picked out by a region, we can define a quasi-probability density function which induces the quasi-measure that picks out the possibility associated with that region only. By the condition on the essential support of W, events cannot be given by regions smaller than a characteristic volume of one in units of $\hbar$ (Dell'Antonio 2016).

III `Balls as union of disjoint annular regions`. Consider the family of annuli A_k defined as:

$$A_k = \{(q,p) \mid r_{k-1} < |q|^2 + |p|^2 \leq r_k\}$$

where $0 = r_0 < r_1 < r_2 ...$ and $\lim_{k\to\infty} r_k = \infty$. We will then have that each ball is equivalent to the union of n disjoint annuli:

$$B_n = \bigcup_{k=1}^{n} A_k$$

IV `Measurable balls`. The quasi-measure can be concentrated within regions of volume greater than order $\hbar$ thus assuming $r_n >> \hbar$:

$$\tilde{\mu}(B_n) = \int_{B_n} W(q,p)dqdp$$

V `Ball-Annulus Decomposition`. A ball radius r_n can be decomposed into a central ball radius r_m and an annular region $\tilde{A}$:

$$\tilde{A} = \{(q,p) \mid r_m < |q|^2 + |p|^2 \leq r_n\} = \bigcup_{k=m}^{n} A_k$$

Thus we have that:

$$B_n = \tilde{A} \cup B_m$$

VI `Proof by contradiction`. We can now use I-V to prove by contradiction $\tilde{\mu}$ cannot be σ-additive. The σ-additivity of $\tilde{\mu}$ implies:

$$\tilde{\mu}(B_n) = \tilde{\mu}\Big(\tilde{A} \cup B_m\Big) = \tilde{\mu}(\tilde{A}) + \tilde{\mu}(B_m)$$

This implies:

$$\tilde{\mu}(B_n) - \tilde{\mu}(B_m) = \tilde{\mu}(\tilde{A})$$

Now consider the this expression for $r_n^2 - r_m^2 \approx \hbar$. Since $r_n, r_m >> \hbar$, we can define quasi-measures, $\tilde{\mu}(B_n)$ and $\tilde{\mu}(B_m)$ that pick out the possibilities associated with each of the balls is isolation. We then have that $\tilde{\mu}(B_n) -$

$\tilde{\mu}(B_m)$ must be well defined from the basic property of the event algebra. However, the area of $\tilde{A}$ is order $\hbar$ so by II the right hand side of the expression cannot be well-defined since it picks out the possibility associated with a region order $\hbar$ is isolation and is thus not in the essential support of W. Thus, we have a contradiction.

□

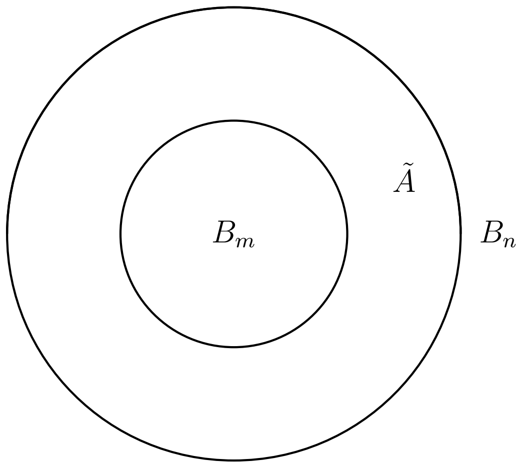

FIGURE 3. The decomposition of a ball B_n radius r_n into a central ball B_m radius r_n and an annulus $\tilde{A} = \{(q,p) \mid r_m < |q|^2 + |p|^2 \leq r_n\}$.