Quo Vadis Wheeler-DeWitt Time?

Challenging Emergent Time in Quantum Cosmology

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Abstract

This paper challenges the notion of emergent time in quantum cosmology by examining the reconciliation of the timeless Wheeler-deWitt equation with the Universe's dynamical evolution. It critically evaluates the analogy between the Wheeler-DeWitt and Klein-Gordon equations, highlighting challenges for the identification of an emergent time parameter. The paper concludes that refining this analogy may lead to a better understanding of emergent time in quantum cosmology, though still not free of complications.

Contents

1 Introduction

The problem of identifying time from a timeless description is one of the oldest and most fascinating puzzles in quantum gravity (QG) and, in particular, in quantum cosmology (QC). In this latter context, in particular, it takes the form of the problem of reconciling the apparent timelessness of the Wheleer-deWitt (WdW) equation with the dynamical evolution of our universe, particularly its (semi)classical description in terms of fields evolving over a smooth spacetime manifold.

Various strategies have been pursued in the attempt to resolve this problem, ranging from the use of a semiclassical approximation to justify the emergence of time in the semiclassical regime to different ways of identifying a time variable prior to quantization, which then makes the quantum regime temporal (Isham, 1992). The former strategy, realized by applying the WKB approximation to the WdW equation, has enjoyed significant popularity in the QC literature. Important discussions of this approach in the physics literature are Vilenkin (1989); Kuchař (2011); Kiefer (2012); in the philosophical literature Thebault (2021); Huggett and Thébault (2023) are recent in-depth discussions of the conceptual foundations of this strategy.

The emergent time strategy is often justified with a formal analogy between the WdW equation and the Klein-Gordon (KG) equation: in particular, one claims that, in the semiclassical limit, the volume of the universe α in the WdW equation is analogous to the time parameter in the KG equation. Hence, since upon taking two formally analogous equations and studying them in the appropriate regime, we find that the time parameter in KG is mapped to the α parameter in WdW, then it is natural to view α as an emergent time in WdW, or so the argument goes.

In this paper, we challenge this widespread view in QC by first arguing that the choice of the parameter α is underdetermined, and there are multiple equally good candidates for time (§2). We then discuss two possible strategies to isolate α as the "correct" time parameter, one based on the analogy with KG (§3), and the other based on a form of functionalism, put forward by Huggett and Thébault (2023) (§4): both strategies are found unsatisfactory. Finally, we discuss how to make the analogy between KG and WdW precise (§5). However, we point out that this analogy leads to a very different picture of time than what we might have expected, and it is not even clear whether, in this form, the analogy is sufficient to identify *time* in WdW (§6). We then conclude (§7).

2 On the Choice of α as Time

To apply canonical QG à la WdW to cosmology, we employ *minisuperspace*. Minisuperspace arises from the possibility of restricting the general problem of defining QG to simpler, highly symmetric spacetimes, reducing the dynamics to a finite-dimensional problem. The most relevant application of minisuperspace is to a homogeneous Universe, described by a class of different cosmologies known as the nine Bianchi models (Bianchi, 1897). In such spaces, the 3-geometries $\{h_{ij}\}$ are equivalent to the three scale factors $\{a(t), b(t), c(t)\}$ of the anisotropic universe, since the homogeneity condition makes the theory invariant under 3-diffeomorphisms. Consequently, the space of physical states is reduced to a finite-dimensional subspace of Wheeler's Superspace (the space of 3-geometries) and the WdW wavefunctional is replaced by a wavefunction. The construction of the Hamiltonian representation of Bianchi models is usually performed by using the variables $(\alpha(t), \beta_{\pm}(t))$, known as *Misner variables* (Misner, 1969): the variable $\alpha(t)$ is related to the spatial volume of the universe, while $\beta_{\pm}$ represent the spatial anisotropies and correspond to the two physical degrees of freedom of gravity. It is worth noting that the physical states $\psi(t, \alpha, \beta_{\pm})$ are defined in the space of the volume and anisotropies of the universe, which does not coincide with the physical spacetime, thus precluding any clear notion of causality.

In this new framework, the canonical formalism gives a weakly vanishing Hamiltonian, known as the Bianchi-Misner Hamiltonian H_{BM}, since it is a linear combination of first-class constraints. Therefore, as in the case of QG, the universe's wavefunction ψ does not depend on the time coordinate t. The WdW equation for the Bianchi models is:

$$H_{BM} |\psi\rangle = 2\chi e^{-3\alpha/2} \hbar^2 [\partial_\alpha^2 - \partial_{\beta_\pm}^2 + U_B(\alpha, \beta_\pm)] \psi(\alpha, \beta_\pm) = 0 \tag{1}$$

where $\chi = 8\pi G$ and $U_B(\alpha, \beta_\pm)$ is known as the Bianchi potential. Consistently with the finite-dimensional reduction of the general theory, it represents a single equation defining a 3-dimensional quantum system. One can see an immediate similarity with a KG equation with a varying mass, highlighting how the universe's volume, related to the variable α, has the features of a good internal time, while the $\beta_\pm$ variables represent the spatial coordinates. Hence, the evolution of the quantum universe resembles that of a relativistic particle moving in a 3-dimensional space (although the causal structure of the minisuperspace has no direct physical meaning).

However, α is not unique in having the features of a good time parameter: as is well known in the literature, in certain cosmological models, it is also legitimate to choose, *e.g.*, a scalar field as time parameter (Rovelli and Smolin, 1994; Isham, 1992). For instance, in the quantum FRLW cosmological model, it is possible to use the matter degrees of freedom as the physical clock of the theory.

Considering the Bianchi-Misner superHamiltonian (1) and taking the isotropic case, we obtain the WdW equation in Misner variables for the vacuum, flat FLRW cosmological model with zero cosmological constant:

$$2\chi e^{-3\alpha/2} \hbar^2 [\partial_\alpha^2 + U_B(\alpha)] \psi(\alpha) = 0 \tag{2}$$

Following Kiefer (1988), we take the matter content to be represented entirely by a homogeneous scalar field ϕ with a potential $V(\phi)$.

In such a way, the WdW equation of the model is:

$$\hat{H}_{FLRW} |\psi\rangle = 2\chi e^{-3\alpha/2}\hbar^2[\partial_\alpha^2 - \partial_\phi^2 + V(\alpha, \phi)]\psi(\alpha, \phi) = 0 \tag{3}$$

where $V(\alpha, \phi) = e^{3\alpha}V(\phi)$.

Under the inflationary assumption of a non-interacting field in the limit of the cosmological singularity (Kolb and Turner, 1994), necessary for QC, the potential $V(\phi)$ dependent on the derivatives of the scalar field ϕ can be neglected. Consequently, we recover the 2-dimensional KG-like equation:

$$2\chi e^{-3\alpha/2}\hbar^2[\partial_\alpha^2 - \partial_\phi^2]\psi(\alpha, \phi) = 0 \tag{4}$$

which describes a free, massless particle, where both the variable α and the scalar field ϕ can play the role of time (Bamonti et al., 2022). Hence, it is underdetermined which parameter plays the role of time in the (semi)classical regime since we have no straightforward way to decide which variable between ϕ and α should play this role. In the next section, we will discuss some attempts at establishing α as the correct emergent time parameter.

3 KG-WdW (dis)Analogy

The analogy between the KG and the WdW equations is fundamentally grounded in their structural similarity: both represent second-order differential equations that impose constraints on the wavefunction. This similarity lays the groundwork for the claim that the parameter α may fulfil an analogous role in each equation, despite the arguments of the preceding section. A crucial point for this analogy, in particular as concerns the temporal status of α, is that the *dynamics* of the two theories must be, in some sense, analogous. Since the temporal status of α emerges from dynamical considerations, as detailed in §2, a failure in a dynamical analogy between WdW and KG would imply that the two theories are disanalogous in the regime relevant to justifying the use of α as time; hence, for α to count as time, we need a deep similarity in the *dynamical behavior* of each theory.

In any quantum field theory, whether it describes gravitational or non-gravitational fields, the dynamic of the theory is determined by its inner product, as it underpins conservation laws and the unitary evolution of quantum states. Consequently, whether α can assume the role of time is the question of how analogous the inner products of the KG and WdW equations are. If these inner products were analogous, we would have a correspondence in the dynamics of both theories, thereby validating the interpretation of α as describing time. On the other hand, a lack of analogy between the inner products suggests that the dynamics differ significantly between the two theories. This divergence would indicate that the role of time could be distinctly conceptualized within each theory, cautioning against a straightforward equivalence of the role played by α in the two theories. In this second case, the justification was merely on the formal structural similarities of the equations without a deeper dynamical analogy. To understand the status of the (dis)analogy of the inner product in the two theories, we follow Witten (2022).

The KG inner product between two wavefunctions ϕ_1 and ϕ_2 both satisfying the KG equation in a spacetime M is given by:

$$\langle \phi_1 | \phi_2 \rangle_{KG} = i \int_\Sigma d\Sigma^\mu \ \overline{\phi}_1 \overleftrightarrow{\partial}_\mu \phi_2, \tag{5}$$

where Σ is any Cauchy hypersurface in M, $d\Sigma_\mu$ is the surface element over which the integral is performed, and $\overline{\phi}_1 \overleftrightarrow{\partial}_\mu \phi_2 = \overline{\phi}_1 \ \partial_\mu \phi_2 - \overline{\phi}_2 \ \partial_\mu \phi_1$, with $\overline{\phi}$ the complex conjugate of ϕ.

Defining an inner product for solutions to the WdW equation is significantly more involved than for the KG equation. Let us sketch the features of the WdW inner product that are most relevant to our discussion, though we will not give a complete construction. The general form of such an inner product is:

$$\langle \phi_1 | \phi_2 \rangle_{WdW} = \int_{\text{Met}/\text{3-Diff}} Dh \ \overline{\phi}_1(h) \prod_{\vec{x} \in S} \delta(\mathcal{H}(\vec{x})) \phi_2(h) \ . \tag{6}$$

where the integral spans the space of 3-metrics (Met) on a hypersurface S modulo 3-diffeomorphisms (Met/3-Diff), also known as the Wheeler's Superspace (see §2). This space encompasses the 3-metric configurations considered equivalent under spatial diffeomorphism transformations, highlighting the gauge invariance intrinsic to General Relativity. The wavefunctions ϕ_1 and ϕ_2 are functions of the 3-metric h, reflecting the quantum gravitational states as configurations of the geometry itself. The presence of the delta functions $\delta(\mathcal{H}(\vec{x}))$ within the integral enforces the Hamiltonian constraint at every point $\vec{x}$ on the hypersurface S, ensuring that only configurations satisfying the WdW equation contribute to the inner product. This mechanism guarantees that the inner product is computed over "physical" states—those that are solutions to the Hamiltonian constraint $\mathcal{H}\psi = 0$. This inner product also defines the following equivalence relation:[1]

$$\psi(h) \equiv \psi(h) + \sum_i \mathcal{H}(\vec{x}_i)\chi_i(h) \ , \tag{7}$$

which plays a crucial role in this context. It states that two quantum states are equivalent if their difference can be expressed as a sum of terms, each term being the application of the Hamiltonian operator $\mathcal{H}(\vec{x}_i)$ at various points $\vec{x}_i$ to arbitrary functions $\chi_i(h)$. This relation is a quantum analogue of the classical Hamiltonian constraint, extending its role from eliminating non-physical configurations to defining an equivalence class of quantum states. By integrating over (Met/3-Diff) and incorporating the product of delta functions, the inner product (6) inherently respects the equivalence relation, ensuring that the path integral quantization framework accommodates diffeomorphism invariance.

This revised interpretation of the constraint operators underlying the inner product (6) carries a significant advantage: this formulation allows the WdW theory to be quantized using the BRST formalism (Becchi et al., 1976; Henneaux and Teitelboim, 1992). In BRST quantization one has to introduce ghost fields, which transform like the generators of the gauge symmetry —diffeomorphisms in our framework — but with opposite statistics. Thus, in our case, we would have to introduce a new field $c^\mu(t, \vec{x})$ that

[1] The discrete sum over points $x_i \in S$ can also be replaced by a continuous integral.

transforms as the generator of diffeomorphisms but has a Fermi-Dirac statistics.[2] Having introduced the ghosts fields $c^\mu(t,\vec{x})$, one can then define the BRST charge Q as follows:

$$Q = \int_S d^{D-1}x\sqrt{h}\left(c^0(t,\vec{x})\mathcal{H}(\vec{x}) + c^i(t,\vec{x})P_i(\vec{x}) + \dots\right) \ , \tag{8}$$

where c^0 is the temporal component of the ghost field $c^\mu(t,\vec{x})$ and is coupled with $\mathcal{H}$, while $c^i(t,\vec{x})$ is its spatial component, and it is coupled with $P_i(\vec{x})$. The BRST charge Q obeys $Q^2 = 0$ and one can thus define a *cohomology structure*, i.e. a space of states that satisfy $Q\psi = 0$ modulo the equivalence relation $\psi \equiv \psi + Q\chi$ for any other state χ. Cohomology is key in BRST quantization because it defines the physical states of the theory: physical states belong to the cohomology generated by the BRST charge. This mathematical structure mirrors the Hamiltonian constraint in the WdW equation: the condition $Q\psi = 0$ gives both $\mathcal{H}\psi = 0$ and $P_i(\vec{x})\psi = 0$, which are, indeed, the traditional WdW constraints.[3]

Leaving aside the complex derivation of the inner product, one can rewrite the inner product (6) in the following form:

$$\langle\phi_1|\phi_2\rangle_{WdW} = \int Dh_0 \ Dc^i \ D\bar{c}^j \ \overline{\phi}_1 \left(\det\Xi\right)\phi_2 \ . \tag{9}$$

In this formulation, the WdW inner product is written as a path integral over the metric h and ghost fields. The crucial feature of this equation is the inclusion of $\det\Xi$, a

[2]For completeness, one also has to introduce antighosts and auxiliary fields in BRST quantization. However, these details are irrelevant to our argument; thus, we omit them for ease of exposition.

[3]For ease of exposition, we ignore the possibility that also c^μ can annihilate a state ψ. Generally speaking, this could be the case. The quantum theory generated by BRST quantization can have states that obey just one of the traditional WdW constraints, or even none. Thus, the traditional WdW theory is a subset of the theory developed in the BRST framework.

determinant resulting from integrating out additional ghost fields, just as is done when dealing with processes in QFTs, such as QCD.

Given equations (5) and (9) for KG and WdW respectively, we are now equipped to compare their inner products. Despite the structural similarity of the equation of the two theories, there are two crucial differences in their inner products:

- **Definiteness of the Inner Product:** It is well known that, unlike the positive-definite inner product found in non-relativistic quantum mechanics, the KG inner product can yield negative values for the norm of specific solutions. Indeed, the KG inner product can be negative, zero, or positive, reflecting the indefinite nature of the norm in the relativistic context. However, Witten (2022) shows that the inner product of the (revised version) of the WdW theory is positive-definite (as it is supposed to be for any theory of gravity).

- **Status of the Hamiltonian Constraint:** Another critical difference between KG and WdW lies in the status of the Hamiltonian constraint. In WdW, the constraint imposes $\mathcal{H}(\vec{x}) = 0$ for each point $\vec{x}$ in a Cauchy hypersurface S. However, the Hamilton constraint in KG theory imposes $\mathcal{H} = 0$, $\mathcal{H}$ being a single operator. Thus, the Hamilton constraint of the WdW theory can be associated with an *infinite family* of KG-like Hamiltonian constraints. The same crucial difference can be seen from another point of view: the KG inner product of equation (5) is defined on a codimension 1 hypersurface U, while the WdW inner product (9) is defined on a submanifold of infinite codimension.

4 Time and Functionalism

Having seen how attempts at justifying the use of α as a time parameter in the semiclassical approximation based on an analogy with the Klein-Gordon equation fail, let us now move to consider a different approach, i.e. the functional strategy developed in Huggett and Thébault (2023).

Without delving into unnecessary details, the point that Huggett and Thébault (2023) wish to make is that we can identify a time parameter by the role it plays in a certain physical theory, along the lines of how spacetime is analyzed in a functionalist approach (Knox, 2013, 2019; Baker, 2021). In particular, Huggett and Thébault (2023) suggest that time should be first divided into two aspects: the *chrono-metric* structure and the *chrono-directed* structure; the first one encodes a relation of betweenness between events, including a temporal metric to define temporal distances between such events, while the second one encodes a directionality for temporal relations, i.e. it tells whether an event is before or after another event to which is temporally related. For our purposes, it will be sufficient to discuss the status of the chrono-metric structure vis a vis emergent time, as this already is enough to see the problems that our discussion raises for this strategy to justify α as the time parameter.

Huggett and Thébault (2023) in general want to claim that any structure that plays the time role will satisfy the chrono-metric properties described above, i.e., it will fix a betweenness relation between events and duration for intervals between events. At the same time, (Huggett and Thébault, 2023, p. 4) also emphasize that:

> Now, when a set of objects possesses structures with the formal properties that we have described, it is still legitimate to ask whether they are in fact *temporal*: perhaps instead they refer to some other physical relations (spatial relations, for instance). We take the general attitude that a structure can be identified as temporal in virtue of the roles it plays in dynamical physical theories. We do not offer a general account of what these roles are; rather we appeal to the fact that time is already identified in existing physical theories, and in this paper aim to pinpoint the (emergent) structures playing those roles.

In other words, Huggett and Thébault (2023) take it that, given a certain physical theory, we should be able to identify the time parameter not by checking that it satisfies certain roles but rather that it corresponds, in the appropriate regime where time indeed emerges, to the time parameter of a theory whose temporal structure we antecedently understand. Huggett and Thébault (2023) then suggest that such a role, in the case of the WdW equation, is indeed played by the parameter α.

From this discussion, it is immediate to see the problem for this strategy vìs a vìs our discussion in §2. Our discussion above points out a fundamental ambiguity in what counts as a time parameter, in the sense that there are multiple parameters with a *prima facie* equally good claim to the title of time. Hence, the functionalist strategy that Huggett and Thébault (2023) use cannot resolve such a problem since the problem for us is not to identify a parameter playing the time role but rather that we have too many parameters playing such a role.

5 A Good Analogy between WdW and KG

Despite the crucial differences between the inner products of the KG and WdW theories analyzed in §3, an underlying analogy exists when considering the KG theory from an appropriate perspective. We find common ground by reformulating the KG theory as a covariant theory on a one-dimensional worldline, effectively introducing a gravity-induced gauge symmetry akin to that found in the WdW theory. In this reformulation, the metric on the worldline is represented by $e^2(t)$, allowing the KG action to be rewritten to mirror the gauge symmetry considerations of the WdW discussion. This approach bridges the conceptual gap, demonstrating that the two theories' inner products exhibit a crucial analogy rooted in their shared symmetry structures and gauge invariance principles. In this framework, the KG action can be rewritten as:

$$S = \frac{1}{2}\int_{\lambda} dt \, \left(-\frac{1}{e}\eta^{\mu\nu}\frac{dX_\mu}{dt}\frac{dX_\nu}{dt} - em^2 \right) \, . \tag{10}$$

This action, invariant under reparametrization of λ, captures the essence of KG's theory but incorporates a form of gauge symmetry through the dynamics of the one-dimensional worldline. The equivalence of this reparametrized action to the KG theory becomes evident through the Euler-Lagrange equation for the metric field e, which yields the KG Hamiltonian constraint. This result is derived as follows:

$$0 = -\frac{\partial S}{\partial e} = \frac{1}{e^2}\eta^{\mu\nu}\frac{dX_\mu}{dt}\frac{dX_\nu}{dt} + m^2 = \eta^{\mu\nu}\Pi_\mu\Pi_\nu + m^2 = -\eta^{\mu\nu}\frac{\partial}{\partial^\mu}\frac{\partial}{\partial^\mu} + m^2 \, , \tag{11}$$

where the conjugate momenta Π_μ are defined as $\Pi_\mu = \frac{1}{e}\eta_{\mu\nu}\frac{dX^\nu}{dt}$, and, upon quantisation, these momenta become $\Pi_\mu = -i\frac{\partial}{\partial X^\mu}$.

By establishing a covariant version of the KG theory on the worldline, we can extend the application of BRST symmetry techniques—previously utilized in reformulating the WdW theory—to this context. This process necessitates introducing ghost fields associated with the theory's symmetry generators. Specifically, for the covariant KG theory, we introduce a ghost field c tied to infinitesimal reparametrizations of the worldline. Given the need for this ghost field to possess opposite statistics compared to the field e, c is defined as a Grassmann number.[4]

Without entering into unnecessary details of this derivation, it is possible to show that the inner product resulting from the BRST derivation aligns precisely with the KG inner product in (5). This derivation, revealing the structural analogy between the KG and WdW theories through BRST quantization, is important in understanding how the ghost fields' integration, giving rise to the determinants, underpins both theories' inner products. Specifically, the KG's inner product $\overleftrightarrow{\partial}_\mu$ term and the WdW's $\det \Xi$ emerge as determinants resulting from integrating out ghost fields associated with their respective BRST symmetries. This structural similarity highlights a profound connection between the two theories, even as they manifest differently. The $\overleftrightarrow{\partial}_\mu$ in the KG theory and $\det \Xi$ in the WdW theory both function as ghost determinants but are tied to distinct BRST symmetries, reflective of their respective theories' underpinnings in relativistic quantum mechanics and QG. Consequently, these determinants contribute differently to the nature of each theory's inner product: $\overleftrightarrow{\partial}_\mu$ results in an indefinite inner product, aligning with the expectation for relativistic theories; on the other hand, $\det \Xi$ yields a positive definite inner product, consistent with the requirements for QG theories.

Having identified a plausible analogy between WdW and KG, this can *prima facie* justify the identification of α as a time variable, modulo the caveats just highlighted. In the following section, we will however discuss some complications for this approach.

[4]Grassmann numbers are used to describe fermionic variables or fermionic fields in QFT, where fermionic statistics require that operators corresponding to fermionic particles satisfy anticommutation relations (Peskin and Schroeder, 1995).

6 Where did Time Go?

We have seen in the previous section how one can justify using α as a time parameter, and that the analogy between the WdW and KG equations plays a crucial role. At the same time, such an analogy requires significant adjustments to justify the role of α as time. Let us briefly discuss some of the consequences and limitations of the argument in §5. First of all, however, let us emphasize what the argument of the previous section does indeed achieve. Insofar as it establishes a good formal analogy between the KG equation and the WdW equation, the argument of §5 ensures that we can at least think of α as the right parameter when we look for something that formally satisfies some of the properties of a time parameter. So, at the very least, the underdetermination problem raised in §2 is *prima facie* resolved by the discussion in §5. However, the analogy of §5 does not establish whether α behaves as we would expect time to behave, despite being formally analogous to the time parameter of the KG equation.

For example, it is still the case that the Hamiltonian constraint of the WdW equation is pointwise. This fact implies that, as the Hamiltonian constraint is the origin of the alleged time flow parametrized by α, this flow is pointwise, not the kind of global flow we usually associate with time. In other words, the emergent time that we are defining from the WdW equation does not describe the flow of a 3-dimensional surface along the time dimension but rather describes the flow of an infinite number of points, each flowing independently through their own time (fixed by the Hamiltonian constraint at that point).[5]

Another related but more fundamental problem with interpreting α as time lies with how we obtained the analogy with the KG equation in BRST quantization. In particular, we first moved to a worldline formulation of the theory to arrive at a BRST expression for the quantum KG equation. Then we noticed that in this context, there is a Hamiltonian constraint enforcing the invariance of the dynamics under arbitrary reparametrizations of the coordinate λ, enforcing that this coordinate does not encode physically relevant information. This Hamiltonian constraint then, upon BRST quantization, leads to an expression analogous to the WdW equation and hence allows

[5]A related notion is the idea that relativistic time is *many-fingered* (Anderson, 2012).

for the analogy between the two equations, particularly with respect to time, to go through.

However, what is unclear in this procedure is whether the Hamiltonian constraint on the worldline still encodes a time parameter. If this condition fails, the analogy cannot give us time in the WdW equation. Let us briefly comment on why one might be skeptical regarding the temporal status of the Hamiltonian constraint in the KG equation. As we mentioned above, the Hamiltonian constraint in this context enforces a kind of reparametrization invariance on the worldline; as such, it does not seem to have any straightforward relation to physical time. One can relate it to time by noticing that the coordinate λ, parametrizing the worldline, which extends only through time, keeps track of the time expired across two points on the worldline.

However, this interpretation only works because, in the KG equation, we have a background spacetime that gives us a physical definition of the time parameter. If we did not have such a background, then the Hamiltonian constraint would have told us that the theory is invariant under a choice of coordinate on the worldline, which either has no relation to time or, at best, enforces the same kind of disappearance of time that we found problematic in the first place when looking at the WdW equation. In other words, by moving to BRST quantization and fixing the analogy between KG and WdW, we have made KG more similar to WdW, and so harder to interpret, rather than simplifying the interpretation of WdW. Indeed, if we did not have a background, then KG would face the same problems as WdW; however, this is precisely the situation we find ourselves in with regard to WdW, which does not have a spacetime background, hence making the identification of α as time problematic, even if there is a broader analogy with KG. Indeed, once we have a well-defined analogy between KG and WdW, to identify time in KG we need to appeal to the spacetime background, which is the aspect in which the two theories are disanalogous. Hence, it is precisely in identifying the time parameter that the analogy between KG and WdW breaks down.

7 Conclusion

In this paper, we examined the WdW equation's implications for the concept of time in QC, especially the challenge of defining an emergent time in a timeless universe. Through a critical analysis of the analogy between the WdW and KG equations, we highlighted the difficulties in identifying a suitable time parameter in QC. Our discussion underscores the complexity and ambiguity in conceptualizing time in this domain, calling for a more sophisticated understanding of the nature of time and dynamics in the WdW context.

Indeed, the disanalogies in the inner products derived from KG and WdW underscore profound differences in their dynamics. While both theories exhibit structural similarities, the different nature of their inner products dictates fundamentally distinct dynamical behaviors. This disparity calls into question the straightforward use of analogy to justify treating the parameter α as time in QC. However, our discussion also reveals a profound analogy between the two theories, emphasizing the crucial role of quantizing gauge symmetries. While not immediately decisive for the question of α's role, this important connection points to a rich vein of theoretical insight that could enhance our understanding of QG and relativistic quantum field theories. Exploring these implications will be left for future works.

In conclusion, our investigation into the WdW equation reveals that the quest for an emergent time in QC faces significant conceptual hurdles, suggesting the development of frameworks that can accommodate the peculiar dynamical features of QC and deepen our understanding of our universe's quantum gravitational structure.

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