Entanglement by (anti-)symmetrisation does not violate Bell's inequalities: so what kind of entanglement does?

Adam Caulton
adam.caulton@philosophy.ox.ac.uk

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Abstract

The purpose of this short article is to build on the work of Ghirardi, Marinatto and Weber (Ghirardi, Marinatto & Weber 2002; Ghirardi & Marinatto 2003, 2004, 2005), in supporting a redefinition of entanglement for "indistinguishable" systems, especially fermions. According to the proposal, the non-separability of the joint state is insufficient for entanglement; rather, the joint state is entangled iff it cannot be represented as the (anti-) symmetrisation of a product state. The redefinition is justified by its physical significance, as enshrined in three biconditionals whose analogues hold of "distinguishable" systems. The proposed definition of entanglement also prompts a reconceptualisation of local operations and the reduced states of constituent subsystems.

Contents

1 Introduction

In this article I wish to give support for a redefinition of entanglement for "indistinguishable" systems; i.e. systems for which permutation invariance is imposed. To the best of my knowledge, this redefinition was first proposed by Ghirardi, Marinatto and Weber (Ghirardi, Marinatto & Weber 2002; Ghirardi & Marinatto 2003, 2004, 2005), and has more recently been endorsed by Ladyman, Linnebo and Bigaj (2013). My contribution here will be to prove that the proposed redefinition enjoys a physical significance that is not shared by the standard concept, according to which a joint state of a quantum assembly is entangled iff it is non-separable; i.e. inexpressible as a product state. I will also present a method for extracting subsystems' states from the joint state, in analogy with the familiar reduced trace procedure.

The physical significance of the concept, which I will call *GMW-entanglement*, is enshrined in three biconditionals, the analogues of which hold for the standard concept of entanglement for "distinguishable" systems; i.e. systems for which permutation invariance is not imposed. These three biconditionals are:

1. The joint state of any two-system assembly is entangled iff it violates a Bell inequality.
2. The joint state of any assembly is not entangled iff the constituent systems' states are pure.
3. The joint state of any assembly is not entangled iff the constituent systems' states determine the joint state.

Each of the three biconditionals may be construed in two ways: (i) as about the standard notion of entanglement, i.e. non-separability, as applied to "distinguishable" quantum systems; and (ii) as about GMW-entanglement, as applied to "indistinguishable" quantum systems. The biconditionals under (i) are well-known (the first is a theorem due to Gisin 1991); my aim here to show that the biconditionals under (ii) are also true. Furthermore, if we construe the biconditionals as about the standard notion of entanglement, as applied to "indistinguishable" quantum systems, then they typically fail. But they should be true under any good concept of entanglement. In consequence, GMW-entanglement is the right concept to apply when treating "indistinguishable" quantum systems.

Proving the first biconditional under construal (ii) is the main work of this paper. It will be crucial to this proof that a couple of other concepts are understood rather differently in the permutation-invariant setting than in the general setting. In particular, we need to revise our understanding of what counts as a *local* operation and how to extract the states of constituent systems from the joint state. Both of these revisions are necessary for the following reason: in the "distinguishable" case, these concepts make essential appeal to the factor Hilbert spaces that make up the assembly's joint Hilbert space; and our best understanding of permutation invariance is one in which factor Hilbert space indices—or, equivalently, the order in which they stand in the tensor product—have no physical meaning. The second and third biconditionals drop out as corollaries after this revision.

In Section 2, I briefly review the topic of permutation invariance in quantum mechanics, and argue that its best interpretation is one that treats the invariance as a symptom of representational redundancy in the standard quantum formalism. It is the fact of this redundancy which motivates the revisions in the concepts of local operation, constituents' states and entanglement. In Section 3, Gisin's Theorem and GMW-entanglement are both reviewed, and some potential confusions cleared up. Section 4 contains the proposed redefinitions of local operations, constituents' states and entanglement, and a proof of the first biconditional under its new construal. The remaining biconditionals are addressed in Section 5. Section 6 concludes with some morals regarding the viability of particle-talk in the permutation-invariant setting: I will argue there that successful talk about particles has its limits, and that these limits show themselves well before we enter the arena of quantum field theory.

2 Permutation invariance, symmetric operators and the wedge product

Permutation-invariant quantum mechanics is standard many-particle quantum mechanics with the additional condition of permutation invariance. We begin with the single-system Hilbert space $\mathcal{H}$ equipped with an algebra of quantities, which we may take as $\mathcal{B}(\mathcal{H})$, the full algebra of bounded operators defined on $\mathcal{H}$. From this we define the N-fold tensor product $\otimes^N\mathcal{H}$, the *prima facie* state space for N systems, and the associated algebra $\mathcal{B}(\otimes^N\mathcal{H}) \cong \otimes^N\mathcal{B}(\mathcal{H})$ (the symbol '$\cong$' denotes unitary equivalence). The "indistinguishability" of the particles is expressed by the fact that any two of the factor Hilbert spaces, each with their associated algebras, are unitarily equivalent.[1]

The joint Hilbert space $\otimes^N\mathcal{H}$ carries a natural unitary representation $U : S_N \to \mathcal{U}(\otimes^N\mathcal{H})$ of the group S_N of permutations on N symbols. For example, the permutation (ij), which swaps systems i and j, is represented by the unitary operator $U(ij)$ defined on basis states, having chosen an orthonormal basis $\{|\phi_k\rangle\}$ on $\mathcal{H}$, by

$$U(ij)|\phi_{k_1}\rangle\otimes\ldots\otimes|\phi_{k_i}\rangle\otimes\ldots\otimes|\phi_{k_j}\rangle\otimes\ldots\otimes|\phi_{k_N}\rangle \\ = |\phi_{k_1}\rangle\otimes\ldots\otimes|\phi_{k_j}\rangle\otimes\ldots\otimes|\phi_{k_i}\rangle\otimes\ldots\otimes|\phi_{k_N}\rangle \quad (1)$$

and then extended by linearity to the whole of $\otimes^N\mathcal{H}$.

Permutation invariance, otherwise known as the *Indistinguishability Postulate* (Messiah & Greenberg 1964, French & Krause 2006), is a condition placed on quantities defined for the joint system. Specifically, any operator $Q \in \mathcal{B}(\otimes^N\mathcal{H})$ is permutation-invariant iff it is *symmetric*;[2] i.e. for all permutations $\pi \in S_N$ and all states $|\psi\rangle \in \otimes^N\mathcal{H}$,

$$\langle\psi|U^\dagger(\pi)QU(\pi)|\psi\rangle = \langle\psi|Q|\psi\rangle\ . \quad (2)$$

The representation U is reducible, and decomposes into various irreducible representations, each kind of irreducible representation corresponding to a different *symmetry type.* The symmetry types include boson, fermion and (if $N \geqslant 3$) a variety of paraparticles (see e.g. Tung 1985, Ch. 5). If we consider only the information provided by the symmetric operators, we treat permutation invariance as a superselection rule, and each superselection sector corresponds to one of these symmetry types. The two symmetry types found

[1] The rather jarring use of scare-quotes around "indistinguishable" is an attempt to do justice to two facts at once. The first fact is that systems are often described as "indistinguishable" iff they are associated with unitarily equivalent algebras, perhaps with the addition that permutation invariance is then imposed. The second fact, as we shall see, is that the systems are typically distinguishable after all, in virtue of their state-dependent properties.

[2] This use of 'symmetric' is not to be confused with the condition that $\langle\psi|Q\phi\rangle = \langle Q\psi|\phi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathrm{dom}(Q)$.

in nature are bosons, corresponding to the totally symmetric subspace $\mathcal{S}_+(\otimes^N \mathcal{H})$, and fermions, corresponding to the totally antisymmetric subspace $\mathcal{S}_-(\otimes^N \mathcal{H})$.

What does it mean to *impose* permutation invariance? Isn't it rather that permutation invariance holds of some operators and not others? The key issue here is physical interpretation: specifically, which mathematically defined operators in the joint algebra represent genuinely physical quantities. So to *impose* permutation invariance is to lay it down as a necessary condition on any operator's receiving a physical interpretation. This may be justified by treating the factor Hilbert space labels—i.e. the order in which single-system operators and states lie in the tensor product—as nothing but an artefact of the mathematical formalism of quantum mechanics.

I prefer to think of it in reverse. We find, empirically, that the joint states of elementary particles always come in some symmetry type—in fact, they are always either bosonic or fermionic. This is *evidence* that factor Hilbert space labels have no physical significance. Not conclusive evidence, however: for there is an alternative explanation for the empirical finding. It *could* be rather that the joint state of any assembly of elementary particles remains in the fermionic or bosonic sector under all dynamical evolutions due only to the fact that the corresponding Hamiltonian happens to be permutation-invariant. And one would expect a permutation-invariant Hamiltonian to govern systems which share the same state-independent properties (i.e. mass, spin and charge). Indeed, this interpretative gloss is either explicitly propounded or implicitly assumed by most authors in the literature (e.g. French & Redhead 1988; Butterfield 1993; Huggett 1999, 2003; French & Krause 2006; Muller & Saunders 2008; Muller & Seevnick 2009; Caulton 2013).

However, this gloss comes unstuck when we consider where the theory of permutation-invariant many-particle quantum mechanics might come from. We can get to permutation-invariant many-particle quantum mechanics along a number of routes:

1. as above, by forming tensor products from single-particle Hilbert spaces and then imposing permutation invariance;
2. by quantifying some configuration space in a classical many-particle theory, appropriately quotiented under a classical analogue of permutation invariance;
3. by restricting to constant total-particle-number subspaces in some Fock space in some relevant quantum field theory (this is the most plausible route).

However, it is only along the first route that factor Hilbert space labels even come into the picture. Along the quantum-field-theoretic route, for example, we never see the "full" tensor product Hilbert space $\otimes^N \mathcal{H}$; rather we see only its totally symmetric or totally antisymmetric subspace $\mathcal{S}_\pm(\otimes^N \mathcal{H})$. This all suggests that the "full" tensor product Hilbert space and its associated factor Hilbert space labels are nothing but a convenient fiction—convenient, that is, so long as we are not misled. In truth, there is nothing that those factor Hilbert space labels name, and non-symmetric states are not just dynamically impossible, they don't even *represent* possibilities for the particles.

Of course, there are plenty of applications where the "full" tensor product Hilbert space is the appropriate one to work with. For example, the associated subsystems may be distinguished by their state-independent properties, such as their mass or charge. Also, the Hilbert space associated with a *single* system is of course taken as a tensor product of Hilbert spaces, each associated with an independent degree of freedom (e.g. a non-relativistic electron is usually associated with the Hilbert space $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, in which the spatial and spin degrees of freedom are represented as factor Hilbert spaces). My target here is not tensor products in general. My target is only the view that, when it comes to *elementary systems of the same species*—the sorts of systems which in quantum field theory are thought of as excitations in the same field, each of which is associated with the single-system Hilbert space $\mathcal{H}$ and an associated algebra—we should somehow think of the joint Hilbert space $\mathcal{S}_{\pm}(\otimes^N \mathcal{H})$ as embedded in the larger Hilbert space $\otimes^N \mathcal{H}$, where the latter has some cogent physical interpretation under which each copy of the single-system Hilbert space $\mathcal{H}$ in the tensor product corresponds to one of the N subsystems. That view, it seems to me, is unsustainable: the "full" tensor product Hilbert space $\otimes^N \mathcal{H}$ is nothing but a fiction, and with it the state-independent individuation of constituent systems by ordering of the factor Hilbert spaces.

In fact, as we shall see, tensor product structure, more generally, is still an important aspect of system individuation. But when it comes to elementary systems of the same species, for which permutation-invariance is imposed, the tensor product structure should not be thought of as being given once and for all, as in the "full" tensor product $\otimes^N \mathcal{H}$. The tensor product structure is rather more elusive and opportunistic. Appropriate decompositions into factor Hilbert spaces, where possible, are context-dependent, in the sense that they depend on the details of the particular joint state and do not capture the whole joint Hilbert space all at once. Moreover, appropriate decompositions are not always possible—except in the case of bipartite fermionic systems.

Much of my focus in the following will be on fermionic joint systems, particularly bipartite fermionic systems. So, following Ladyman, Linnebo and Bigaj (2013), I will make use of a harmless abuse of notation by referring to anti-symmetric states by their corresponding wedge product. In particular, given an orthonormal basis $\{|\phi_i\rangle\}$ on $\mathcal{H}$, the expression

$$|\phi_{i_1}\rangle \wedge |\phi_{i_2}\rangle \wedge \ldots \wedge |\phi_{i_N}\rangle \tag{3}$$

will be used as a shorthand for the following state in the fermionic joint Hilbert space $\mathcal{S}_{-}(\otimes^N \mathcal{H})$:

$$\frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^{\deg \pi} |\phi_{i_{\pi(1)}}\rangle \otimes |\phi_{i_{\pi(2)}}\rangle \otimes \ldots \otimes |\phi_{i_{\pi(N)}}\rangle \; . \tag{4}$$

In the fermionic case, joint states that can be expressed as wedge products in this way exactly correspond to those which are not GMW-entangled.

3 What is entanglement?

Let me first put an important consideration out of the way. This important consideration is that many sorts of things can appropriately be called entangled, and it matters which sort of thing we are talking about. I do not wish to enter into a detailed discussion here, so I will give three brief examples. First, even a single system can exemplify entanglement between its different degrees of freedom: for example, consider the single-particle state

$$\frac{1}{\sqrt{2}} \left(|L\rangle \otimes |{\uparrow}\rangle - |R\rangle \otimes |{\downarrow}\rangle \right) \; , \tag{5}$$

where $|L\rangle$ denotes some spatial state localised on the left of our lab, $|R\rangle$ denotes some spatial state localised on the right, orthogonal to $|L\rangle$, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are the familiar spin-up and spin-down eigenstates. In this state, the spatial and spin degrees of freedom are entangled.

Second, the simple superposition

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \ , \tag{6}$$

though clearly not entangled when construed as a state of the particle or any degree of freedom (there is only one particle here, and only one degree of freedom, spin), it nevertheless exhibits entanglement between *modes* associated with the spin-directions $\uparrow$ and $\downarrow$: this can be seen by expressing the state in terms of occupation numbers associated with $\uparrow$ and $\downarrow$:

$$\frac{1}{\sqrt{2}}\left(|1_\uparrow 0_\downarrow\rangle + |0_\uparrow 1_\downarrow\rangle\right) \ . \tag{7}$$

We might add that there is no entanglement between the modes $\leftarrow$ and $\rightarrow$, since the same state may be expressed as $|1_\leftarrow 0_\rightarrow\rangle$; so even different choices of modes will lead to different verdicts as to whether or not the state is entangled.

A third, and last, example: consider a system of two distinguishable, spinless particles on the real line, with coordinates r_1 and r_2. The joint system's Hilbert space is $\mathfrak{H} = \mathcal{L}^2(\mathbb{R})_{r_1} \otimes \mathcal{L}^2(\mathbb{R})_{r_2}$. But the same joint system may be construed instead in terms of their centre of mass and relative position coordinates, $R := \frac{1}{2}(r_1 + r_2)$ and $r := r_2 - r_1$, respectively, suggesting the factorisation $\mathfrak{H} = \mathcal{L}^2(\mathbb{R})_R \otimes \mathcal{L}^2(\mathbb{R})_r$. Joint states which are separable in terms of R, r may non-separable r_1, r_2 and *vice versa*. For example, let the joint state be such that the wavefunction $\Psi(R, r)$ is separable:

$$\Psi(R, r) = \phi(R)\chi(r) \ . \tag{8}$$

In r_1, r_2 coordinates, this is of course

$$\tilde{\Psi}(r_1, r_2) = \phi(\tfrac{1}{2}(r_1 + r_2))\chi(r_2 - r_1) \ , \tag{9}$$

which is expressible in the form $\tilde{\Psi}(r_1, r_2) = \varphi_1(r_1)\varphi_2(r_2)$ only in very special circumstances.[3]

To sum up: while entanglement is a feature that is uncontroversially objective in the sense of basis-independent, it is nevertheless relative to some specification of ontology. We can talk of entanglement between particles, between fictional constructs out of particles (such as the centre of mass, in the third example), between modes associated with some basis in the one-particle Hilbert space, or just between degrees of freedom of a single particle. Entanglement in terms of one ontology is not equivalent to entanglement in terms of another.

So let me make it clear now that my interest in this paper is solely entanglement *between particles*. That's because it's only this sort of entanglement that becomes obscure once permutation invariance is imposed, and which therefore prompts a redefinition. After all, permutation invariance is only ever imposed on the particles—or, rather, how

[3]E.g., when $\phi(\frac{1}{2}x) = \chi(x) = \frac{1}{N}\exp(-\alpha x^2)$, in which case $\varphi_1(x) = \varphi_2(x) = \frac{1}{N}\exp(-2\alpha x)$. A simple case where these special circumstances do not hold is where ϕ and χ are boxcar functions.

the particles are represented in the formalism—and not on the modes or on the degrees of freedom.[4]

Entanglement is standardly defined formally as the non-separability of the assembly's state; i.e. a state is entangled iff it cannot be written as a product state (see e.g. Nielsen & Chuang 2010, 96). The physical significance of this definition is underpinned by a biconditional, one half of which is Gisin's Theorem, which applies to assemblies of two ("distinguishable") subsystems:

Theorem 3.1 (Gisin 1991) *Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. If $|\psi\rangle$ is entangled (i.e. $|\psi\rangle$ is not a product state), then $|\psi\rangle$ violates a Bell inequality. That is, there is some state $|\chi\rangle \in \mathfrak{h}_1 \otimes \mathfrak{h}_2$, where $\mathfrak{h}_1 \leqslant \mathcal{H}_1, \mathfrak{h}_2 \leqslant \mathcal{H}_2$ and $\dim \mathfrak{h}_1 = \dim \mathfrak{h}_2$, accessible from $|\psi\rangle$ by a local operation, and a triplet of 2×2 matrices $\boldsymbol{\sigma}^{(1)} = (\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)})$ on $\mathcal{H}_1$ and a triplet of 2×2 matrices $\boldsymbol{\sigma}^{(2)} = (\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)})$, on $\mathcal{H}_2$, each satisfying*

$$[\sigma_a^{(i)}, \sigma_b^{(i)}] = 2i\epsilon_{abc}\sigma_c^{(i)} \ , \quad \{\sigma_a^{(i)}, \sigma_b^{(i)}\} = 2\delta_{ab} \ , \tag{10}$$

and four 3-vectors $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ such that

$$\mathcal{I} := |E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}')| + |E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')| > 2 \ , \tag{11}$$

where

$$E(\mathbf{a}, \mathbf{b}) := \langle\chi|\mathbf{a}.\boldsymbol{\sigma}^{(1)} \otimes \mathbf{b}.\boldsymbol{\sigma}^{(2)}|\chi\rangle \ , \tag{12}$$

etc.

Proof See Gisin (1991). □

So for the joint Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ to contain any entangled states, we must have $\dim \mathcal{H}_1, \dim \mathcal{H}_2 \geqslant 2$. The other half of the biconditional is the "easy half":

Proposition 3.2 *Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. If $|\psi\rangle$ is not entangled (i.e. $|\psi\rangle$ is expressible as a product state), then $|\psi\rangle$ satisfies any Bell inequality; that is, for any state $|\chi\rangle$ accessible from $|\psi\rangle$ by a local operation, and any triplet of 2×2 matrices $\boldsymbol{\sigma}^{(1)} = (\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)})$ on $\mathcal{H}_1$ and any triplet of 2×2 matrices $\boldsymbol{\sigma}^{(2)} = (\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)})$, on $\mathcal{H}_2$, each satisfying*

$$[\sigma_a^{(i)}, \sigma_b^{(i)}] = 2i\epsilon_{abc}\sigma_c^{(i)} \ , \quad \{\sigma_a^{(i)}, \sigma_b^{(i)}\} = 2\delta_{ab} \ , \tag{13}$$

and any four 3-vectors $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$, then

$$\mathcal{I} := |E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}')| + |E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')| \leqslant 2 \ . \tag{14}$$

[4] Imposing permutation invariance on the particles of course does induce a corresponding constraint on the fictional constructs one may define in terms of them. To take our third example above: imposing permutation invariance on the two particles leads to the result that any pure joint state is either symmetrised or anti-symmetrised in the particles' coordinates r_1, r_2: $\tilde{\Psi}(r_2, r_1) = \pm\tilde{\Psi}(r_1, r_2)$. The corresponding constraint in the centre-of-mass/relative position coordinates R, r is parity symmetry in r: i.e. $\Psi(R, -r) = \pm\Psi(R, r)$. While restriction to the symmetric or anti-symmetric sector means giving up a global tensor product structure in terms of r_1, r_2, a tensor product structure is preserved in terms of R and $|r|$, since e.g. $\mathcal{S}_-\left(\mathcal{L}^2(\mathbb{R})_{r_1} \otimes \mathcal{L}^2(\mathbb{R})_{r_2}\right) \cong \mathcal{L}^2(\mathbb{R})_R \otimes \mathfrak{h}_{|r|}$, where $\mathfrak{h}_{|r|}$ is the subspace of $\mathcal{L}^2(\mathbb{R}_+)_{|r|}$ corresponding to wavefunctions which vanish at $r = 0$.

Proof. Since $|\psi\rangle$ is non-entangled, then it has the form

$$|\psi\rangle = |\phi\rangle \otimes |\theta\rangle \tag{15}$$

for some $|\phi\rangle \in \mathcal{H}_1$ and $|\theta\rangle \in \mathcal{H}_2$. Any state accessible from $|\psi\rangle$ by a local operation also has this form, so we proceed with $|\chi\rangle = |\psi\rangle$. Any $E(\mathbf{a}, \mathbf{b})$ then takes the form

$$\begin{aligned} E(\mathbf{a}, \mathbf{b}) &:= \langle\psi|\mathbf{a}.\boldsymbol{\sigma}^{(1)} \otimes \mathbf{b}.\boldsymbol{\sigma}^{(2)}|\psi\rangle && (16)\\ &= \langle\phi|\mathbf{a}.\boldsymbol{\sigma}^{(1)}|\phi\rangle\langle\theta|\mathbf{b}.\boldsymbol{\sigma}^{(2)}|\theta\rangle && (17)\\ &=: \alpha\beta && (18)\end{aligned}$$

where $\alpha := \langle\phi|\mathbf{a}.\boldsymbol{\sigma}^{(1)}|\phi\rangle$ and $\beta := \langle\theta|\mathbf{b}.\boldsymbol{\sigma}^{(2)}|\theta\rangle$. If we similarly define α', β', then

$$\mathcal{I} = |\alpha(\beta - \beta')| + |\alpha'(\beta + \beta')| \ , \tag{19}$$

and since $|\alpha|, |\alpha'|, |\beta|, |\beta'| \leqslant 1$, there is no set of values for which $\mathcal{I}$ exceeds 2. □

Corollary 3.3 *Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. $|\psi\rangle$ is entangled iff it violates a Bell inequality.*

This biconditional gives entanglement physical meaning, since the Bell inequalities represent physically realisable results—at least *so long as* we assume that every bounded self-adjoint operator on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ represents a physical quantity.

However, when we turn to permutation-invariant quantum mechanics, the significance of this biconditional is called into question. Permutation invariance puts restrictions on the available algebra of quantities for the joint system, and some of those prohibited quantities are involved in the definition of the correlation functions $E(\mathbf{a}, \mathbf{b})$. In a permutation-invariant setting, $\mathcal{H}_1 = \mathcal{H}_2$ and the only symmetric correlation functions are of the form

$$E(\mathbf{a}, \mathbf{a}) := \langle\psi|\mathbf{a}.\boldsymbol{\sigma} \otimes \mathbf{a}.\boldsymbol{\sigma}|\psi\rangle \ . \tag{20}$$

Yet the Bell inequality requires us to independently vary the quantities on each system. Therefore, under permutation invariance the usual Bell inequality cannot even be constructed!

Two responses are available to us, only one of which is normally taken. The common response is to refrain from the interpretative strategy endorsed in Section 2, and to lift the restriction on the joint algebra placed by permutation invariance. Permutation-invariance is then construed as nothing more than a "dynamical inaccessibility": the prohibited quantities still have physical meaning; it is just that dynamical evolution under them is unavailable to the joint system. This is response is explicit, for example, in French & Krause (2006, §§4.1.3, 4.2). Any proponent of this response may still want to say that the biconditional linking entanglement to the violation of a Bell inequality can be taken seriously, and that therefore non-separability provides the right definition of entanglement.

However, as I argued in Section 2, we ought to take a stronger reading of permutation invariance. Under this reading, any element in the mathematical formalism that is not invariant under arbitrary permutation should not be given a physical interpretation. In that case, the non-symmetric quantities used in the definition of the correlation functions simply cannot be given any physical meaning. In that case, we must renounce the idea that non-separability provides a physically adequate definition of entanglement.

These doubts have been expressed by Ghirardi, Marinatto and Weber in a series of papers (Ghirardi, Marinatto & Weber 2002; Ghirardi & Marinatto 2003, 2004, 2005). They propose an alternative definition of entanglement, which have called *GMW-entanglement*. Although not their explicit definition, it turns out to be equivalent to following:

Definition 3.1 *A joint state is* GMW-entangled *iff it is not the anti-symmetrization of a product state.*

So, for example, the spin-singlet state $|\uparrow\rangle \wedge |\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$ counts as non-GMW-entangled.

It may come as a surprise, to say the least, that a state which we have all learned to think of as *maximally* entangled—indeed, the state most commonly used to illustrate the violation of a Bell inequality—should come out as *non*-entangled on what I am urging as the right definition. But there need be no confusion here. The singlet state is indeed entangled, *so long as* we have access to the full algebra of bounded operators. If we do not, as in the case of permutation invariance, then that attribution needs to be revised.

But aren't *electrons*, which are fermions, and therefore subject to permutation invariance, involved in physical violations of the Bell inequality? And don't those violations arise in particular when the electrons are prepared in the singlet state? The answer to both these questions is Yes, but we need to be careful about including all of the electrons' degrees of freedom. As Ghirardi, Marinatto & Weber (2003, 3) and Ladyman, Linnebo & Bigaj (2013, 216) point out, the full state in the standard EPRB experiment is

$$\frac{1}{2}\left(|L\rangle_1 \otimes |R\rangle_2 + |R\rangle_1 \otimes |L\rangle_2\right) \otimes \left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\right) \ , \tag{21}$$

where $|L\rangle$ and $|R\rangle$ represent spatial wavefunctions concentrated at the left-hand and right-hand sides of the lab respectively. Written using the wedge product, this state is

$$\frac{1}{\sqrt{2}}\left(|L,\uparrow\rangle \wedge |R,\downarrow\rangle - |L,\downarrow\rangle \wedge |R,\uparrow\rangle\right) \ , \tag{22}$$

which is *not* expressible as the anti-symmetrization of a product state. So it counts as GMW-entangled.

To sum up: when permutation invariance is imposed, we no longer have recourse to the full algebra of quantities associated with the "full" tensor product Hilbert space $\otimes^N \mathcal{H}$. This impoverishment of the joint algebra breaks the strong link between non-separability and the violation of a Bell inequality, as encapsulated in Gisin's Theorem. Specifically, there are non-separable states—entangled states, on the standard definition—whose non-separability cannot be made manifest by Bell-inequality-violating behaviour. These states count as *non*-entangled on the GMW definition.

That is the chief negative observation of this paper. But there is now positive work to be done. In particular, we currently lack any way of making sense of Bell inequality violation in the permutation-invariant case—one that agrees with the indisputable fact that state (21) violates a Bell inequality. It is the purpose of the next Section to provide such a way.

4 Bell inequalities and local operations under permutation invariance

In order to define a Bell inequality in a permutation-invariant setting, we need some way of picking out the subsystems that is permutation-invariant—in particular, we may not appeal to the factor Hilbert space labels. Our solution, inspired by Ghirardi, Marinatto & Weber (2002) and Dieks & Lubberdink (2011), is to appeal to the *states* of the subsystems. This may be seen as the quantum analogue of Russell's (1905) celebrated treatment of proper names as disguised definite descriptions. On this view, the named object is picked out in virtue of a property that it uniquely satisfies.

I illustrate the strategy for the case $N = 2$; its generalisation to $N > 2$ will be obvious. The quantum representation of a monadic property is a projector that acts on the single-system Hilbert space $\mathcal{H}$.[5] So to pick out two subsystems we select two projectors E_1, E_2 on $\mathcal{H}$ such that $E_1 \perp E_2$, i.e. $E_1E_2 = E_2E_1 = 0$; I call this condition *orthogonality.* The orthogonality of the projectors is crucial, since it is necessary and sufficient to ensure that, for any joint state, there is zero probability of the two projectors selecting the same subsystem.

However, there is still the danger that one of the projectors, E_1 say, will pick out both subsystems. Concentrating attention on fermionic subsystems, we can rely on Pauli exclusion to protect us from this if we impose $\dim E_1 = \dim E_2 = 1$. However, this condition is far too strong, since it will select subsystems only in the corresponding pure states, and we want to allow the subsystems to occupy mixed states. (In fact GMW-entangled states are precisely those for which we can ascribe the subsystems mixed states; see below.)

It is sufficient to demand that the joint state $|\psi\rangle$ be an eigenstate of the projector

$$E_1 \otimes E_2 + E_2 \otimes E_1 \tag{23}$$

with eigenvalue 1; I call this condition *exhaustion.* Note that this joint projector is permutation-invariant. I propose that we interpret it as picking out those joint states in which one subsystem is in a state in $\mathrm{ran}(E_1)$ and the other is in a state in $\mathrm{ran}(E_2)$. But it must be emphasized that (23) should *not* be interpreted as the quantum disjunction, 'Subsystem 1 is in a state in $\mathrm{ran}(E_1)$ and subsystem 2 is in a state in $\mathrm{ran}(E_2)$ QOR subsystem 1 is in a state in $\mathrm{ran}(E_2)$ and subsystem 2 is in a state in $\mathrm{ran}(E_1)$.' The individual disjuncts of this proposition are not permutation-invariant and so have no physical interpretation. Rather, we must interpret (23) *primitively* as the proposition 'Exactly one of the subsystems is in a state in $\mathrm{ran}(E_1)$ and exactly one of the subsystems is in a state in $\mathrm{ran}(E_2)$.' Interpreting the projector primitively in this way (i.e. not as a disjunction) is supported by the following fact: if $\dim E_1 = \dim E_2 = 1$, then (23) projects onto a single ray in the fermionic Hilbert space, and so could not be a non-trivial disjunction of other propositions.

Therefore our two conditions on what we might call *individuating projectors* E_1 and E_2 are that they be: (i) orthogonal; and (ii) exhaustive. A pair of individuating

[5] It is more customary to associate projectors with *propositions* rather than properties. But if we associate projectors on the Hilbert space associated with some system with propositions about that system, then we can think of projectors on the *generic* single-system Hilbert space $\mathcal{H}$ as systematically associated with propositions about *any* system whose states are represented in that Hilbert space. In this way, projectors on $\mathcal{H}$ correspond to properties that consituent systems may possess.

projectors can *always* be found for any given 2-fermion state (see below). The same is not true for bosonic or paraparticle states.

Once we have these individuating projectors, we can define operators associated with the corresponding subsystems. The proposal is simple: any operator A on $\mathcal{H}$ is associated with the subsystem individuated by E_i iff $E_i A E_i = A$. (Note that if we had demanded that $\dim E_1 = \dim E_2 = 1$, then the algebra of operators associated with each subsystem would be Abelian.) We can now define permutation-invariant operators on the joint system which act separately on each subsystem; i.e. they are the permutation-invariant analogues of $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes B$. They have the general form

$$E_1 A E_1 \otimes E_2 B E_2 + E_2 B E_2 \otimes E_1 A E_1 \ , \quad \text{where } A, B \in \mathcal{B}(\mathcal{H}) \ . \tag{24}$$

All this leads to the following proposal for what is for a fermionic joint state $|\psi\rangle$ to violate a permutation-invariant Bell inequality:

Definition 4.1 *Let $|\psi\rangle \in \mathcal{S}_-(\mathcal{H} \otimes \mathcal{H})$. $|\psi\rangle$ violates a permutation-invariant Bell inequality iff there is some state $|\chi\rangle$, accessible from $|\psi\rangle$ by a local operation, and two projectors E_1, E_2 on $\mathcal{H}$, such that $E_1 \perp E_2$ and*

$$(E_1 \otimes E_2 + E_2 \otimes E_1) \, |\chi\rangle = |\chi\rangle \ , \tag{25}$$

and two triplets of 2×2 matrices $\boldsymbol{\sigma}^{(1)} = (\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}), \boldsymbol{\sigma}^{(2)} = (\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)})$, satisfying

$$[\sigma_a^{(i)}, \sigma_b^{(i)}] = 2i\epsilon_{abc}\sigma_c^{(i)} \ , \quad \{\sigma_a^{(i)}, \sigma_b^{(i)}\} = 2\delta_{ab} \ , \quad E_i \sigma_a^{(i)} E_i = \sigma_a^{(i)} \ , \tag{26}$$

and four 3-vectors $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ such that

$$\mathcal{I}_{PI} := |F(\mathbf{a}, \mathbf{b}) - F(\mathbf{a}, \mathbf{b}')| + |F(\mathbf{a}', \mathbf{b}) + F(\mathbf{a}', \mathbf{b}')| > 2 \ , \tag{27}$$

where

$$F(\mathbf{a}, \mathbf{b}) := \langle\chi| \left(\mathbf{a}.\boldsymbol{\sigma}^{(1)} \otimes \mathbf{b}.\boldsymbol{\sigma}^{(2)} + \mathbf{b}.\boldsymbol{\sigma}^{(2)} \otimes \mathbf{a}.\boldsymbol{\sigma}^{(1)}\right) |\chi\rangle \ , \tag{28}$$

etc.

It is important to notice that the formal explication of a "local" operation, used in the definition above, must also change under permutation-invariance. The guiding physical idea is the same for both: just as, for "distinguishable" systems, a local operation is one that acts on each subsystem—i.e. each factor Hilbert space—independently, and so has product form, so too under permutation-invariance a local operation should act on each subsystem—as individuated by E_1 and E_2—independently. So under permutation invariance a local operation is one whose form is given in (24). Note that 'local' is now understood in terms of the state, rather than in terms of the tensor product—in particular, it could be in terms of the spatial location of the particles.

We are now ready to prove the biconditional linking GMW-entanglement to the violation of a permutation-invariant Bell inequality. Each direction of the biconditional will be proved separately.

Proposition 4.1 *Let $|\psi\rangle \in \mathcal{S}_-(\mathcal{H} \otimes \mathcal{H})$. If $|\psi\rangle$ is not GMW-entangled, (i.e. $|\psi\rangle$ is the anti-symmetrization of a product state) then $|\psi\rangle$ satisfies any Bell inequality for symmetric quantities. That is, for any state $|\chi\rangle$ accessible from $|\psi\rangle$ by a local operation, and any two projectors E_1, E_2 on $\mathcal{H}$ such that* *(i)* $E_1 \perp E_2$*; and*

(ii) $(E_1 \otimes E_2 + E_2 \otimes E_1)\,|\chi\rangle = |\chi\rangle$;

there is no pair of triplets of 2×2 *matrices* $\boldsymbol{\sigma}^{(1)} = (\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}), \boldsymbol{\sigma}^{(2)} = (\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)})$ *satisfying*

$$[\sigma_a^{(i)}, \sigma_b^{(i)}] = 2i\epsilon_{abc}\sigma_c^{(i)} \ , \quad \{\sigma_a^{(i)}, \sigma_b^{(i)}\} = 2\delta_{ab} \ , \quad E_i\sigma_a^{(i)}E_i = \sigma_a^{(i)} \tag{29}$$

for which, for some choice of four 3-vectors $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$*,*

$$\mathcal{I}_{PI} := |F(\mathbf{a}, \mathbf{b}) - F(\mathbf{a}, \mathbf{b}')| + |F(\mathbf{a}', \mathbf{b}) + F(\mathbf{a}', \mathbf{b}')| > 2 \ , \tag{30}$$

where

$$F(\mathbf{a}, \mathbf{b}) := \langle\chi| \left(\mathbf{a}.\boldsymbol{\sigma}^{(1)} \otimes \mathbf{b}.\boldsymbol{\sigma}^{(2)} + \mathbf{b}.\boldsymbol{\sigma}^{(2)} \otimes \mathbf{a}.\boldsymbol{\sigma}^{(1)}\right) |\chi\rangle \ , \tag{31}$$

etc.

Proof. I closely follow Gisin (1991). If $\dim \mathcal{H} < 4$, then no pair of triplets of 2×2 matrices satisfying the above conditions can be found. So we assume $\dim \mathcal{H} \geqslant 4$. Any two projectors E_1 and E_2 satisfying the above conditions must satisfy $E_1|\phi_1\rangle = |\phi_1\rangle, E_2|\phi_2\rangle = |\phi_2\rangle, E_1|\phi_2\rangle = E_2|\phi_1\rangle = 0$, where $|\psi\rangle$ can be written

$$|\psi\rangle = |\phi_1\rangle \wedge |\phi_2\rangle \ . \tag{32}$$

Any state accessible from $|\psi\rangle$ by a local operation (in the permutation-invariant sense) also has this form, so we proceed with $|\chi\rangle = |\psi\rangle$. Since $E_1 \perp E_2$ and $E_i\sigma_a^{(i)}E_i = \sigma_a^{(i)}$, $\sigma^{(i)}|\phi_j\rangle = 0$ if $i \neq j$ $(i, j \in \{1, 2\})$. Therefore

$$F(\mathbf{a}, \mathbf{b}) = \langle\phi_1|\mathbf{a}.\boldsymbol{\sigma}^{(1)}|\phi_1\rangle\langle\phi_2|\mathbf{b}.\boldsymbol{\sigma}^{(2)}|\phi_2\rangle \tag{33}$$

$$= \alpha\beta \tag{34}$$

where $\alpha := \langle\phi_1|\mathbf{a}.\boldsymbol{\sigma}^{(1)}|\phi_1\rangle$ and $\beta := \langle\phi_2|\mathbf{b}.\boldsymbol{\sigma}^{(2)}|\phi_2\rangle$. If we similarly define α', β', then

$$\mathcal{I}_{PI} = |\alpha(\beta - \beta')| + |\alpha'(\beta + \beta')| \ , \tag{35}$$

and since $|\alpha|, |\alpha'|, |\beta|, |\beta'| \leqslant 1$, there is no set of values for which $\mathcal{I}_{PI}$ exceeds 2. □

An important example of a non-GMW-entangled state is

$$|L, \uparrow\rangle \wedge |R, \downarrow\rangle := \frac{1}{\sqrt{2}}(|L, \uparrow\rangle \otimes |R, \downarrow\rangle - |R, \downarrow\rangle \otimes |L, \uparrow\rangle) \ . \tag{36}$$

No permutation-invariant Bell inequality is violated for this state.

For the second half of the biconditional, we will need a lemma (also used by Schliemann *et al* 2001 and Ghirardi & Marinatto 2004), which is the fermionic analogue of the Schmidt bi-orthogonal decomposition theorem; I merely report it here.

Lemma 4.2 *For any antisymmetric* $d \times d$ *complex matrix* A *(i.e.* $A \in \mathcal{M}(d, \mathbb{C})$ *and* $A^T = -A$*), there exists a unitary transformation* U *such that* $A = UZU^T$*, where* Z *is a block-diagonal matrix of the form*

$$Z = diag[Z_1, \ldots Z_r, Z_0], \quad \text{where } Z_i = \begin{pmatrix} 0 & c_i \\ -c_i & 0 \end{pmatrix} \text{ and } c_i \in \mathbb{C} \tag{37}$$

and Z_0 *is the* $(d - 2r) \times (d - 2r)$ *zero matrix.*

Proof. This is Theorem 7 in Hua (1944).

Proposition 4.3 *Let $|\psi\rangle \in \mathcal{S}_-(\mathcal{H}\otimes\mathcal{H})$. If $|\psi\rangle$ is GMW-entangled (i.e. $|\psi\rangle$ is not the anti-symmetrization of a product state), then $|\psi\rangle$ violates a Bell inequality for symmetric quantities. That is, there is some state $|\chi\rangle$, accessible from $|\psi\rangle$ by a local operation, and two projectors E_1, E_2 on $\mathcal{H}$, such that $E_1 \perp E_2$ and*

$$(E_1 \otimes E_2 + E_2 \otimes E_1)\,|\chi\rangle = |\chi\rangle \ , \tag{38}$$

and two triplets of 2×2 matrices $\boldsymbol{\sigma}^{(1)} = (\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}), \boldsymbol{\sigma}^{(2)} = (\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)})$, satisfying

$$[\sigma_a^{(i)}, \sigma_b^{(i)}] = 2i\epsilon_{abc}\sigma_c^{(i)} \ , \quad \{\sigma_a^{(i)}, \sigma_b^{(i)}\} = 2\delta_{ab} \ , \quad E_i\sigma_a^{(i)}E_i = \sigma_a^{(i)} \ , \tag{39}$$

and four 3-vectors $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ such that

$$\mathcal{I}_{PI} := |F(\mathbf{a},\mathbf{b}) - F(\mathbf{a},\mathbf{b}')| + |F(\mathbf{a}',\mathbf{b}) + F(\mathbf{a}',\mathbf{b}')| > 2 \ , \tag{40}$$

where

$$F(\mathbf{a},\mathbf{b}) := \langle\chi| \left(\mathbf{a}.\boldsymbol{\sigma}^{(1)} \otimes \mathbf{b}.\boldsymbol{\sigma}^{(2)} + \mathbf{b}.\boldsymbol{\sigma}^{(2)} \otimes \mathbf{a}.\boldsymbol{\sigma}^{(1)}\right) |\chi\rangle \ , \tag{41}$$

etc.

Proof. $|\psi\rangle$ has the general form

$$|\psi\rangle = \sum_{ij} a_{ij}|\theta_i\rangle \otimes |\theta_j\rangle \tag{42}$$

where $a_{ij} = -a_{ji}$. We can represent $|\psi\rangle$ as a complex $d\times d$ anti-symmetric matrix A. Any unitary transformation U on $\mathcal{H}$ corresponds to the transformation $A \mapsto UAU^T$. So, given Lemma 4.2, we can find a basis $\{|\phi_i\rangle\}$ such that

$$|\psi\rangle = \sum_{i=1}^{\lfloor\frac{d}{2}\rfloor} c_i|\phi_{2i-1}\rangle \wedge |\phi_{2i}\rangle \ . \tag{43}$$

If $|\psi\rangle$ is GMW-entangled, then we can order the basis vectors so that $c_1, c_2 \neq 0$. Now define

$$|\chi\rangle := \frac{c_1|\phi_1\rangle \wedge |\phi_2\rangle + c_2|\phi_3\rangle \wedge |\phi_4\rangle}{\sqrt{|c_1|^2 + |c_2|^2}} \ . \tag{44}$$

$|\chi\rangle$ may be obtained from $|\psi\rangle$ by a local, selective operation (where we use our new sense of 'local'). The idea now is to treat the state $|\chi\rangle$ analogously to the entangled state

$$c_1|\phi_1\rangle \otimes |\phi_2\rangle + c_2|\phi_3\rangle \otimes |\phi_4\rangle \ , \tag{45}$$

in the general setting (where permutation invariance is not imposed), which is subject to Gisin's Theorem. Now define the following individuation criteria

$$E_1 := |\phi_1\rangle\langle\phi_1| + |\phi_3\rangle\langle\phi_3| \ , \qquad E_2 := |\phi_2\rangle\langle\phi_2| + |\phi_4\rangle\langle\phi_4| \ . \tag{46}$$

Then it may be checked that $(E_1 \otimes E_2 + E_2 \otimes E_1)\,|\chi\rangle = |\chi\rangle$—in fact a selective measurement using the projector on the LHS is precisely the local operation which produces $|\chi\rangle$. The proof now closely follows Gisin (1991). We define Pauli-like matrices for the two 2-dimensional subspaces of $\mathcal{H}$ spanned by $\{|\phi_1\rangle, |\phi_3\rangle\}$ and $\{|\phi_2\rangle, |\phi_4\rangle\}$, respectively. Let

$$\sigma_x^{(1)} := |\phi_1\rangle\langle\phi_3| + |\phi_3\rangle\langle\phi_1| \tag{47}$$
$$\sigma_y^{(1)} := -i\left(|\phi_1\rangle\langle\phi_3| - |\phi_3\rangle\langle\phi_1|\right) \tag{48}$$
$$\sigma_z^{(1)} := |\phi_1\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_3| \tag{49}$$

and

$$\sigma_x^{(2)} := |\phi_2\rangle\langle\phi_4| + |\phi_4\rangle\langle\phi_2| \tag{50}$$
$$\sigma_y^{(2)} := -i\left(|\phi_2\rangle\langle\phi_4| - |\phi_4\rangle\langle\phi_2|\right) \tag{51}$$
$$\sigma_z^{(2)} := |\phi_2\rangle\langle\phi_2| - |\phi_4\rangle\langle\phi_4| \tag{52}$$

It may be checked that these operators satisfy the conditions above. Some calculation yields

$$\begin{aligned} F(\mathbf{a},\mathbf{b}) &:= \langle\chi|\left(\mathbf{a}.\boldsymbol{\sigma}^{(1)}\otimes\mathbf{b}.\boldsymbol{\sigma}^{(2)} + \mathbf{b}.\boldsymbol{\sigma}^{(2)}\otimes\mathbf{a}.\boldsymbol{\sigma}^{(1)}\right)|\chi\rangle \\ &= a_zb_z + \frac{2\Re\mathrm{e}(c_1c_2^*)}{|c_1|^2+|c_2|^2}(a_xb_x - a_yb_y) + \frac{2\Im\mathrm{m}(c_1c_2^*)}{|c_1|^2+|c_2|^2}(a_xb_y + a_yb_x) \qquad (53) \\ &= a_zb_z + \xi\cos\gamma(a_xb_x - a_yb_y) + \xi\sin\gamma(a_xb_y + a_yb_x) \qquad (54) \end{aligned}$$

where $\xi := \frac{2|c_1c_2|}{|c_1|^2+|c_2|^2}$ and $\gamma := \arg(c_1c_2^*)$. Note that $0 < \xi \leqslant 1$. We now choose $a_x = \sin\alpha, a_y = 0, a_z = \cos\alpha;\ b_x = \sin\beta\cos\gamma, b_y = \sin\beta\sin\gamma, b_z = \cos\beta$ to obtain

$$F(\mathbf{a},\mathbf{b}) = \cos\alpha\cos\beta + \xi\sin\alpha\sin\beta\ . \tag{55}$$

Making similar choices for $\mathbf{a}', \mathbf{b}'$, and selecting $\alpha = 0, \alpha' = \frac{\pi}{2}$, we obtain

$$\left|F(\mathbf{a},\mathbf{b}) - F(\mathbf{a},\mathbf{b}')\right| + |F(\mathbf{a}',\mathbf{b}) + F(\mathbf{a}',\mathbf{b}')| = \left|\cos\beta - \cos\beta'\right| + \xi\left|\sin\beta + \sin\beta'\right| \tag{56}$$

We may choose $\cos\beta = -\cos\beta' =: \eta,\ \sin\beta = \sin\beta' = \sqrt{1-\eta^2}$, for which

$$\left|F(\mathbf{a},\mathbf{b}) - F(\mathbf{a},\mathbf{b}')\right| + |F(\mathbf{a}',\mathbf{b}) + F(\mathbf{a}',\mathbf{b}')| = 2(\eta + \xi\sqrt{1-\eta^2}). \tag{57}$$

Given a value for ξ, this quantity is maximal for $\eta = \frac{1}{\sqrt{1+\xi^2}}$, for which it takes the value $2\sqrt{1+\xi^2}$, which is strictly greater than 2 for all $\xi > 0$; i.e. for any non-vanishing c_1 and c_2. $\square$

Corollary 4.4 *Let $|\psi\rangle \in \mathcal{S}_-(\mathcal{H}\otimes\mathcal{H})$. $|\psi\rangle$ is GMW-entangled iff it violates a permutation-invariant Bell inequality.*

As mentioned above, an important example of a GMW-entangled state is the EPRB state of two electrons:

$$\frac{1}{\sqrt{2}}\left(|L,\uparrow\rangle \wedge |R,\downarrow\rangle - |L,\downarrow\rangle \wedge |R,\uparrow\rangle\right)\ . \tag{58}$$

If we use the individuation criteria $E_1 = |L\rangle\langle L|\otimes\mathbb{1}_{spin}$ and $E_2 = |R\rangle\langle R|\otimes\mathbb{1}_{spin}$, and the subsystems do not change their location, then the 4D subspace $\mathfrak{S}$ of the joint Hilbert space spanned by

$$|L,\uparrow\rangle \wedge |R,\uparrow\rangle,\ |L,\uparrow\rangle \wedge |R,\downarrow\rangle,\ |L,\downarrow\rangle \wedge |R,\uparrow\rangle,\ |L,\downarrow\rangle \wedge |R,\downarrow\rangle \tag{59}$$

is physically equivalent (because unitarily equivalent) to the 4D tensor product Hilbert space spanned by

$$|\uparrow\rangle_L \otimes |\uparrow\rangle_R,\ |\uparrow\rangle_L \otimes |\downarrow\rangle_R,\ |\downarrow\rangle_L \otimes |\uparrow\rangle_R,\ |\downarrow\rangle_L \otimes |\downarrow\rangle_R\ . \tag{60}$$

in which the subsystems are indexed by their locations L and R, and permutation invariance is *not* imposed.[6] In particular, the state (58) is equivalent to

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_L \otimes |\downarrow\rangle_R - |\downarrow\rangle_L \otimes |\uparrow\rangle_R\right)\ , \tag{61}$$

which of course violates the standard Bell inequality.

To sum up this section: I have introduced a means to individuate systems according to their state-dependent properties. Individuation of two system proceeds by choosing two projectors E_1, E_2 on the single-system Hilbert space $\mathcal{H}$ which are: (i) orthogonal, i.e. $E_1 E_2 = E_2 E_1 = \mathbf{0}$; and (ii) exhaustive, i.e. the joint projector $E_1 \otimes E_2 + E_2 \otimes E_1$ acts as the identity on the joint state. By means of this individuation scheme, we may associate with the two systems a joint algebra of symmetric (i.e. permutation invariant) quantities, among which are joint operators with which we can define permutation-invariant analogues of correlation functions and local operations. With these, I have shown that a bipartite fermionic joint system is GMW-entangled iff it violates some Bell inequality formed from these permutation-invariant correlation functions, perhaps after some local operation. This result essentially follows from Gisin's (1991) original theorem and a unitary equivalence result that holds between a subspace of the anti-symmetrised joint Hilbert space $\mathcal{S}_-(\mathcal{H} \otimes \mathcal{H})$ and some tensor product Hilbert space.

5 Constituent states under permutation invariance

Two further biconditionals characterise entanglement for "distinguishable" systems, both of which can be extended to GMW-entanglement under permutation invariance. First we consider the biconditional that any joint state $|\psi\rangle$ is not entangled iff the constituent systems occupy pure states. In a permutation-invariant setting, we may say that constituent systems occupy pure states just in case individuation criteria E_1, E_2 may be found that satisfy our two conditions above (orthogonality and exhaustion) *and* $\dim E_1 = \dim E_2 = 1$ (maximal specificity). This conditional is obviously equivalent to $|\psi\rangle$'s being the anti-symmetrization of a product state, i.e. $|\psi\rangle$'s being non-GMW-entangled.

This brief consideration prompts a more in-depth look into how we might ascribe states to constituent systems in the permutation invariant setting. In the general setting, the usual prescription is to calculate reduced density operators by performing a partial trace over all factor Hilbert spaces except the one taken to correspond to the constituent system of interest (see e.g. Nielsen & Chuang 2010, §2.4.3). However, this procedure presumes exactly what I have here urged us to deny: namely, that the order of the factor Hilbert spaces carries any physical significance. If the order of the factor Hilbert spaces represents nothing, then what is the significance of a reduced density operator

[6] The relevant unitary is the restriction of $\sqrt{2}|L\rangle_1\langle L| \otimes \mathbb{1}^{(1)}_{spin} \otimes |R\rangle_2\langle R| \otimes \mathbb{1}^{(2)}_{spin}$ to $\mathfrak{S}$, which sends (58) to (61). This physical equivalence between permutation-invariant QM and permutation-non-invariant QM is essentially also pointed out in a more general discussion by Huggett & Imbo 2009.

obtained by privileging one factor Hilbert space over the others in the tensor product? (In fact, as we shall see below, it has *some* significance, as the *average state* of the constituent systems. But that is a far cry from what we're after, which is the *actual* state of *some particular* subsystem.)

What we need is some analogue of the partial trace procedure that lives up to our permutation-invariant scruples. Such an analogue is at hand. Recall from (24) that, on individuation criteria E_1, E_2 for a bipartite joint system, the permutation-invariant analogue of the operator $A \otimes B$ is $E_1 A E_1 \otimes E_2 B E_2 + E_2 B E_2 \otimes E_1 A E_1$. Now suppose that we interested in only one of the subsystems, say the one we wish to individuate using the projector E. Then let $E_1 = E$ and $E_2 = \mathbb{1} - E =: E_\perp$. These individuation criteria are of course guaranteed to be orthogonal, and have the best possible chance of being exhaustive. If we wish to act on the system individuated by E with the arbitrarily chosen quantity Q, and otherwise act trivially on the other subsystem, then the appropriate operator on the joint Hilbert space is

$$\pi_E(Q) := EQE \otimes E_\perp + E_\perp \otimes EQE \ , \tag{62}$$

where we hereby define a map π_E from the single-system algebra $\mathcal{B}(\mathcal{H})$ into the joint algebra. Let us now restrict: (i) to those *joint states* for which E succeeds in picking out a unique subsystem, i.e. those joint states for which $E \otimes E_\perp + E_\perp \otimes E$ acts as the identity; and (ii) to those *single-system quantities* Q which commute with the individuation criterion E. Then the map π_E so defined is a unital $*$-algebra homomorphism.[7]

The quantity $\pi_E(Q)$ will have an expectation value determined by the joint state. But it is properly associated with the subsystem individuated by the projector E, since we act trivially on the other subsystem. That is to say, the expectation value assigned by the joint state to the joint quantity $\pi_E(Q)$ is to be construed as the expectation value, which I will denote $\langle Q \rangle_E$, of the single-system quantity Q assigned by the reduced state of the E-subsystem. The idea now is to work backwards from the expectation values $\langle \pi_E(Q) \rangle$ for arbitrary Q to infer the explicit form of this reduced state. The relevant result is captured in the following Proposition.

Proposition 5.1 *Let $|\psi\rangle \in \mathcal{S}_\pm(\mathcal{H} \otimes \mathcal{H})$ be any symmetric or antisymmetric bipartite pure state and let the individuation criterion E, a projector on the single-system Hilbert space $\mathcal{H}$, succeed in picking out a unique subsystem; i.e.*

$$(E \otimes E_\perp + E_\perp \otimes E)\,|\psi\rangle \equiv \pi_E(\mathbb{1})|\psi\rangle = |\psi\rangle \ .$$

Then there is a unique density operator ρ_E such that

$$\mathrm{tr}(\rho_E Q) = \langle Q \rangle_E := \langle \psi | \pi_E(Q) | \psi \rangle = \langle \psi |\,(EQE \otimes E_\perp + E_\perp \otimes EQE)\,| \psi \rangle \ ,$$

where ρ_E is given by a Lüders' rule projection

$$\rho_E := \frac{E\overline{\rho}E}{\mathrm{tr}(\overline{\rho}E)} \tag{63}$$

[7] π_E's being a $*$-algebra homomorphism requires that $\pi_E(Q^*) = (\pi_E(Q))^*$, $\pi_E(\alpha Q) = \alpha \pi_E(Q)$, $\pi_E(Q_1 + Q_2) = \pi_E(Q_1) + \pi_E(Q_2)$ and $\pi_E(Q_1 Q_2) = \pi_E(Q_1)\pi_E(Q_2)$. For the last identity it is sufficient that $[Q_1, E] = [Q_2, E] = 0$. π_E's being unital requires in addition that $\pi_E(\mathbb{1}_\mathcal{H})$ act as the identity on the joint algebra, which holds if we restrict to joint states for which E succeeds to pick out a unique subsystem.

on the average single-system state $\overline{\rho} := tr_1(|\psi\rangle\langle\psi) \equiv tr_2(|\psi\rangle\langle\psi|)$*, obtained by a partial trace over either factor Hilbert space.*

Proof. Let us choose a basis $\{\ldots, |\phi_i\rangle, \ldots, |\theta_I\rangle, \ldots\}$ for the single-system Hilbert space $\mathcal{H}$ such that

$$E|\phi_i\rangle = |\phi_i\rangle \; ; \qquad E|\theta_I\rangle = 0 \; . \tag{64}$$

For the sake of clarity, I will use lowercase indices $i, j, k, \ldots$ to range over states in the range of E and uppercase indices to range over states in the range of $E_\perp$. Then the most general form for a bipartite joint state is

$$|\psi\rangle = \sum_{i,j} a_{ij}|\phi_i\rangle \otimes |\phi_j\rangle + \sum_{I,J} b_{IJ}|\theta_I\rangle \otimes |\theta_J\rangle + \sum_{i,J} c_{iJ}|\phi_i\rangle \otimes |\theta_J\rangle + \sum_{I,j} d_{Ij}|\theta_I\rangle \otimes |\phi_j\rangle \; . \tag{65}$$

The individuation criterion E succeeds iff $a_{ij} = b_{IJ} = 0$, for all i, j, I, J. So the joint states of interest are all of the form

$$|\psi\rangle = \sum_{i,J} c_{iJ}|\phi_i\rangle \otimes |\theta_J\rangle + \sum_{I,j} d_{Ij}|\theta_I\rangle \otimes |\phi_j\rangle \; . \tag{66}$$

This state yields the expectation value, for arbitrary operator Q on $\mathcal{H}$,

$$\langle Q\rangle_E := \langle\psi| \left(EQE \otimes E_\perp + E_\perp \otimes EQE\right) |\psi\rangle = \sum_{i,j,K} \left(c^*_{iK}c_{jK} + d^*_{Ki}d_{Kj}\right) \langle\phi_i|Q|\phi_j\rangle \; . \tag{67}$$

We now use the fact that $|\psi\rangle$ is symmetrised (bosons) or anti-symmetrised (fermions):

$$d_{Ki} = \pm c_{iK} \text{ for all } i, K,$$

to obtain

$$\langle Q\rangle_E = 2 \sum_{i,j,K} c^*_{iK}c_{jK}\langle\phi_i|Q|\phi_j\rangle \; . \tag{68}$$

At this point, we could appeal to Gleason's Theorem (Gleason 1957) to establish the existence of a density operator ρ_E which produces these expectation values. However, I will instead proceed by explicitly defining ρ_E, show that it produces the correct expectation values, and then show that it is unique.

A partial trace over (e.g.) second factor Hilbert space of the joint state (66), subject to the (anti-)symmetry condition, yields what I am calling the *average single-system state*

$$\begin{aligned} \overline{\rho} &= \mathrm{tr}_2\left(|\psi\rangle\langle\psi|\right) \\ &= \sum_{i,j,K} c_{iK}c^*_{jK}|\phi_i\rangle\langle\phi_j| + \sum_{I,J,k} c_{kI}c^*_{kJ}|\theta_I\rangle\langle\theta_J| \; . \end{aligned} \tag{69}$$

(The same result is obtained by a partial trace over the first factor Hilbert space.) It may be checked that $\overline{\rho}$ has unit trace so long as $|\psi\rangle$ is normalised, in which case

$$\sum_{i,J} |c_{iJ}|^2 = \frac{1}{2} \; . \tag{70}$$

We should expect to obtain $\frac{1}{2}$ here, since doing so is a necessary condition for the individuation criterion E succeeding in picking out a unique subsystem: equation (70) tells us that "half" of the single-systems' states lie in the range of E.

Given the definition of the $|\phi_i\rangle, |\theta_I\rangle$, it is clear that

$$E\overline{\rho}E = \sum_{i,j,K} c_{iK}c^*_{jK}|\phi_i\rangle\langle\phi_j| \ , \tag{71}$$

and so

$$\mathrm{tr}(\overline{\rho}E) = \mathrm{tr}(\overline{\rho}E^2) = \mathrm{tr}(E\overline{\rho}E) = \sum_{i,J}|c_{iJ}|^2 = \frac{1}{2} \ , \tag{72}$$

where in first and second identities we make use of the idempotence of E and cyclicity of the trace, respectively, and in the last identity we use (70). Combining our results (71) and (72), we now obtain an explicit form for ρ_E:

$$\rho_E := \frac{E\overline{\rho}E}{\mathrm{tr}(\overline{\rho}E)} = 2\sum_{i,j,K} c_{iK}c^*_{jK}|\phi_i\rangle\langle\phi_j| \ . \tag{73}$$

This density operator yields, for an arbitrary operator Q on $\mathcal{H}$, the expectation value

$$\mathrm{tr}(\rho_E Q) = 2\sum_{i,j,K} c^*_{iK}c_{jK}\langle\phi_i|Q|\phi_j\rangle = \langle Q\rangle_E \ , \tag{74}$$

which agrees with our result (68), as required.

This establishes the existence of ρ_E with the right properties. To establish uniqueness, first consider the general form of a density operator on $\mathcal{H}$:

$$\tilde{\rho} = \sum_{i,j} w_{ij}|\phi_i\rangle\langle\phi_j| + \sum_{i,J}(u_{iJ}|\phi_i\rangle\langle\theta_J| + u^*_{iJ}|\theta_J\rangle\langle\phi_i|) + \sum_{I,J} W_{IJ}|\theta_I\rangle\langle\theta_J| \ . \tag{75}$$

Now consider the expectation values that $\tilde{\rho}$ must yield for the operators $A_{IJ} := |\theta_J\rangle\langle\theta_I|$ if it is to agree with ρ_E. Using (74), we must have

$$\mathrm{tr}(\tilde{\rho}A_{IJ}) = W_{IJ} = 0 \tag{76}$$

for all I, J. Similar consideration of the operators $B_{iJ} := |\theta_J\rangle\langle\phi_i|$ establishes that $u_{iJ} = 0$ for all i, J. Finally, consideration of the operators $C_{ij} := |\phi_j\rangle\langle\phi_i|$ establishes that

$$\mathrm{tr}(\tilde{\rho}C_{ji}) = w_{ij} = 2\sum_K c^*_{iK}c_{jK} \ , \tag{77}$$

all of which serves to fix ρ_E uniquely. □

The reduced state ρ_E is obviously sensitive to its associated individuation criterion E. As one might expect (and so is a sanity check on the proposal), ρ_E has support only in the range of E: in other words, the system individuated by E can only be found in a state compatible with E. It immediately follows that if $\dim E = 1$ (and $\mathrm{tr}(\overline{\rho}E) > 0$), then $\rho_E = E$, and so the system individuated by E has a pure state. However, in general E may be non-minimal, and this allows ρ_E to be statistically mixed (even though the joint state is pure).

To take our permutation-invariant Bell state (58) again as an example, the following reduced states associated with single-system projectors may be found:

- $E = |L\rangle\langle L| \otimes \mathbb{1}_{spin}$. In this case, $\pi_E(\mathbb{1})$ indeed acts on the state as the identity, and $\rho_E = |L\rangle\langle L| \otimes \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$.

- $E = |R\rangle\langle R| \otimes \mathbb{1}_{spin}$. In this case, $\pi_E(\mathbb{1})$ indeed acts on the state as the identity, and $\rho_E = |R\rangle\langle R| \otimes \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$.

- $E = \mathbb{1}_{space} \otimes |\uparrow\rangle\langle\uparrow|$. In this case, $\pi_E(\mathbb{1})$ indeed acts on the joint state as the identity, and $\rho_E = \frac{1}{2}\left(|L\rangle\langle L| + |R\rangle\langle R|\right) \otimes |\uparrow\rangle\langle\uparrow|$.

- $E = \mathbb{1}_{space} \otimes |\downarrow\rangle\langle\downarrow|$. In this case, $\pi_E(\mathbb{1})$ indeed acts on the joint state as the identity, and $\rho_E = \frac{1}{2}\left(|L\rangle\langle L| + |R\rangle\langle R|\right) \otimes |\downarrow\rangle\langle\downarrow|$.

- $E = |L\rangle\langle L| \otimes |\uparrow\rangle\langle\uparrow|$ (a maximally specific projector). In this case, $\pi_E(\mathbb{1})$ fails to act as the identity on the joint state, and so we have a failure of individuation. Here it is because no system exists with the appropriate properties on *all* branches of the superposition. Nevertheless, using the definition (63), we do get a specification for ρ_E, which is $\rho_E = |L\rangle\langle L| \otimes |\uparrow\rangle\langle\uparrow| = E$.

- $E = \mathbb{1}_{space} \otimes \mathbb{1}_{spin}$ (the maximally *non*-specific projector). In this case, $\pi_E(\mathbb{1})$ fails to act as the identity on the joint state, and so again we have a failure of individuation. Here it is because more than once system exists with the appropriate properties. Nevertheless, using the definition (63), we do get a specification for ρ_E, which is $\rho_E = \frac{1}{4}\left(|L\rangle\langle L| + |R\rangle\langle R|\right) \otimes \left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \overline{\rho}$.

This last case vindicates my description of $\overline{\rho}$, obtained by the usual partial trace procedure, as the 'average single-system state'. $\overline{\rho}$ is the reduced state associated with the maximally non-specific individuation criterion, viz. the identity on $\mathcal{H}$, and it is no surprise that the maximally non-specific individuation criterion captures all constituent systems equally, thereby returning the average state of all of them.

These developments lead us directly to our third and final biconditional. Recall that this says that the joint (pure) state $|\psi\rangle$ is not entangled iff the constituents' states determine the joint state. (Or in metaphysicians' jargon: $|\psi\rangle$ is not entangled iff the joint state supervenes on the constituents' states.) This biconditional is linked to the second biconditional by the following two facts: (i) the joint state is always pure; and (ii) pure states are maximally specific (and so *a fortiori* more specific than mixed states). Since, by the second biconditional, the constituent states are pure iff the joint state is not entangled, the constituents' states carry enough information to collectively determine the joint state iff the joint state is not entangled. This reasoning carries over for GMW-entanglement in the permutation-invariant setting, so long as the symmetry type of the constituents is given. For example, for fermions: any collection of n mutually orthgonal single-system pure states serves to determine a unique, non-GMW-entangled joint state: namely, their anti-symmetric combination, or wedge product.

6 When particle-talk fails

Under the current proposals, attributing states—whether pure or mixed—to constituent systems relies on the two conditions (i) orthogonality, $E_1 \perp E_2$, and (ii) exhaustion, $(E_1 \otimes E_2 + E_2 \otimes E_1)|\psi\rangle = |\psi\rangle$, holding. Otherwise the individuation criteria either (i′) fail to be mutually exclusive, or (ii′) fail collectively to capture the entire joint state, respectively.

Either way, the projectors E_1, E_2 fail to serve as quantal definite descriptions for the two systems. Lemma 4.2 ensures that successful individuation criteria may *always* be found, no matter the joint state, for an assembly of two fermions. (However, these individuation criteria may be gruesomely miscellaneous.) But (i) and (ii) are impossible to satisfy in some bosonic states. In particular, and unsurprisingly, doubly occupied states $|\phi\rangle \otimes |\phi\rangle$ elude this treatment. Consider also the joint state

$$c_1|\phi_1\rangle \otimes |\phi_1\rangle + c_2 \frac{1}{\sqrt{2}} \left(|\phi_2\rangle \otimes |\phi_3\rangle + |\phi_3\rangle \otimes |\phi_2\rangle \right) \ , \tag{78}$$

where $c_1, c_2 \neq 0$. Intuitively, this joint state should come out as entangled under any physically salient definition—indeed, the state is GMW-entangled, since it is not the symmetrisation of any product state. But I have not discovered a satisfactory way to find individuation criteria or otherwise attribute constituent states to the two subsystems in joint states like these, where doubly occupied states have a non-vanishing amplitude.[8] Consequently, I cannot claim that the three biconditionals hold for bosons and GMW-entanglement.

This is perhaps to be expected: whenever there is a non-zero amplitude for the two bosons occupying the same state, one cannot expect to be able to perform measurements on—or even attribute properties to—one independently of the other. But this, of course, is a precondition on the possibility of a Bell-like experiment even being performed. So perhaps it is satisfactory, in the case of bosons, that an appropriate conditional claim hold instead. The appropriate conditional is: *if* the two bosons can be individuated, *then* the three biconditionals hold (where 'entanglement' is understood as GMW-entanglement). This claim is straightforwardly established. For, the two bosons can be individuated iff the conditions are met to set up exactly the same sort of unitary equivalence to the "ditinguishable" case that we made use of in the fermionic case. Under this unitary equivalence, entanglement becomes GMW-entanglement and our three biconditionals hold true.

I conclude with one further observation, which put the cogency of particle talk under severe strain. The observation is that one may choose rival sets of individuation criteria—where both sets satisfy (i) orthogonality and (ii) exhaustion—but there is no straightforward connection between criteria from the rival sets. A stark example is given at the end of Section 5 with regard to the familiar anti-symmetrised Bell state (58). The first and second criteria form an individuating set, as do the third and fourth; yet particles individuated under the first and second are in mixed states with respect to the properties associated with the third and fourth, and *vice versa*. There is therefore no identification of the systems between the rival sets of criteria: we can talk of '*the* particle on the left' and '*the* particle on the right', or '*the* spin-up particle' or '*the* spin-down particle'; but neither of the first pair is to be associated with either of the second pair (there is, for example, no such thing as '*the* spin-up particle on the left', as the failure of the individuation criterion in the fifth example given there shows).

[8] The analogue of Lemma 4.2, also noted by Schliemann et al (2001), is that any totally symmetric joint state may be "diagonalised": i.e., for any state in $\mathcal{S}_+(\mathcal{H}\otimes\mathcal{H})$, there is some orthonormal basis of $\mathcal{H}$ in which the joint state can be expressed as a superposition of doubly occupied states from that basis. This is entirely unhelpful for individuating the subsystems, since we require orthogonal individuation criteria. And if a bosonic joint state has a non-vanishing amplitude for some doubly occupied state in *some* basis, then for *any* basis there will be a non-vanishing amplitude for some doubly occupied state. That is: double occupation cannot be transformed away. Yet double occupation is fatal for finding quantal definite descriptions.

This individuation-criterion-relativity of the constituent particles flies in the face of our ordinary conception of objects as "having their identities" (whatever that means) independently of how they are picked out by their properties. What we find is that the attempt to individuate particles by their state-dependent properties—our only option, if we are to take the resolute line on permutation-invariance—involves transmitting quantum contextual weirdness from the familiar setting of talk about a system's *properties* to talk about the *systems themselves*.[9]

This may lead the reader to doubt my resolute line on permutation invariance. Wouldn't it do less violence to our intuitions about constituent particles to instead adopt the line that permutation invariance is nothing but a initial condition on the joint state, combined with the (explainable) permutation-symmetry of the governing dynamics? The problem I see with this conservative stance is that, however counter-intuitive the results that may follow, the weaker line on permutation invariance is simply not available. The true origins of permutation-invariant many particle quantum mechanics lie in quantum field theory, where the formal resources do not even exist to individuate constituent particles except by means of their state-dependent properties. (We can recover the symmetric or anti-symmetric sector $\mathcal{S}_{\pm}(\otimes^N \mathcal{H})$ in the constant-total-particle-number limit of some corresponding Fock space, but never do we see the "full" tensor product $\otimes^N \mathcal{H}$.)

It seems to me, therefore, that we must find a way to make peace with these counter-intuitive results. It may well be that the only satisfactory way of doing that is to think of particle talk as at best approximate, and even then as legitimate only in certain contexts. (I hope that the work in this paper suggests at least the rough outlines of those contexts.) If that is right, then we don't require any further considerations from quantum field theory, except that particles have no state-independent means of being individuated, before we may conclude that particles are not fundamental.

7 References

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[9]This predicament is particularly vivid in the case of non-GMW-entangled fermionic joint states, where one can find uncountably many rival families of *maximally specific* individuation criteria, each of which adequately captures the joint state. In other words: while the states of the constituent subsystems suffice to determine the joint state (since it is non-GMW-entangled), the joint state itself fails to determine *what the constituent systems are* (not just what their states are). I address this case, and its implications for any theory of fermionic composition, in Caulton (2015).

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